Practical and exact output feedback stabilizations at unobservable points

Marc-Aurèle Lagache[a,b], Ulysse Serres[b] and Jean-Paul Gauthier[a]

***Abstract*— In this paper, we analyze, via a simple academic example from quantum control, the problem of dynamic output feedback stabilization, when the point where we want to stabilize corresponds to a control value that makes the system unobservable.**

The stability analysis is performed in three steps. We have validated the method for a specific case, but the idea can be extended to more general problems. In particular, the method can be applied to the case of Killing systems that are injectable into bilinear skew-symmetric (or skew-adjoint) systems.

I. Introduction

It is usually not possible to measure the whole state of a system in practice, but only a part of it. This part is called the observation, and only this can be measured. Problems arise when one wants to stabilize a system to some target point since, in most cases, the knowledge of the whole state might be needed. In [1, Chapters 6 and 7] (see also [2]), the authors give different methods in order to reconstruct the whole state space and stabilize the system provided it is observable. However, for nonlinear systems, there exist in general inputs that make the system unobservable (see [3]). Here, we are interested in the case where the point we want to stabilize the system to corresponds to a control value that renders it unobservable. Although similar problems were already addressed in [4], [5], [6], [7], the literature on output feedback stabilization for non-uniformly observable systems is scarce. In [5] a sufficient condition for local stabilization in small time by means of a dynamic periodic time-varying output feedback law is given. Whereas the system treated in the present paper meets the conditions given in [5], our work aims to give (semi-global) asymptotic stability of the target by means of a smooth time-invariant dynamic output feedback.

The present paper deals with the same problem as the one exposed in [7]. However, [7, Theorem 2.10] includes a mistake since the result should not be stated as semi-global. Theorem 3.10 corrects this mistake. Moreover, while in [7] the authors gave a result on practical asymptotic stabilization of an arbitrarily small neighborhood of the target point, the present work adds a result about the asymptotic stability of the target point (exactly) and not only of a neighborhood of it.

The method we present here is illustrated on a specific example borrowed from quantum control that we treat in all details. Consider the following (three dimensional) typically observed closed quantum system (see e.g. [8], [9], [10]).

$$\begin{cases} \dot{x} = A(u)x \\ y = Cx, \end{cases} \tag{1}$$

where $(x_1, x_2, x_3) = x \in S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ is the state, $y \in \mathbb{R}$ is the measured output (or, the observation), $u = (u_1, u_2) \in \mathbb{R}^2$ is the control variable, $C = (0\ 0\ 1)$ is in $\mathbb{R}^{1\times 3}$, and $A(u)$ is a control-dependent matrix in $\mathbb{R}^{3\times 3}$ given by

$$A(u) = \begin{pmatrix} 0 & e & u_1 \\ -e & 0 & u_2 \\ -u_1 & -u_2 & 0 \end{pmatrix}.$$

In quantum mechanics, x corresponds to the state of a qubit and e is a constant energy gap which will be set to one. Even though the validity of the continuous measurement $y(t)$ is unclear in the quantum context, system (1) is a typical example of a more general situation that will be exposed in Section IV.

The problem is to stabilize system (1) at the target point $x_{\mathrm{t}} = (0, 0, -1)$, which is an equilibrium point corresponding to the control value $u = (u_1, u_2) = (0, 0)$. Notice that system (1) is unobservable for this control since indistinguishable points are those that are on the same parallel $\{x_3 = \mathrm{cst}\} \cap S^2$.

The paper is organized as follows. In Section II we characterize the set of trajectories that converge to an unstable equilibrium point. These preliminary results will be needed in Section III since the instability of an equilibrium point does not guarantee its repulsiveness.

Section III is devoted to the complete analysis of the problem. The main idea is to perturb a stabilizing control in order to preserve a small level of observability. Two approaches are studied. The first one does not stabilize system (1) exactly at the target point x_{t}, but at an equilibrium point arbitrarily close to it. In the second approach, the perturbation is chosen not only to preserve a small level of observability but also to stabilize (1) at the target point.

In Section IV we describe how the analysis done in Section III can be extended to more general situations. Subsection IV-A details how the method built through

[a]Université de Toulon, CNRS, LSIS, UMR 7296, F-83957 La Garde, France `gauthier at univ-tln.fr & marc-aurele-lagache at etud.univ-tln.fr`

[b]Univ. Lyon, Université Claude Bernard Lyon 1, CNRS, LAGEP UMR 5007, 43 bd du 11 novembre 1918, F-69100 Villeurbanne, France `marc-aurele.lagache & ulysse.serres at univ-lyon1.fr`

the example of system (1) can be extended to general bilinear skew-symmetric systems. Following [3], in Subsection IV-B, we give conditions under which more general systems can be immersed into skew-adjoint systems. The study of skew-adjoint systems is then only the first step towards the study of output feedback stabilization for a wider class of systems.

Due to space limitation, the detailed analysis of the two general situations presented in Section IV will be reported elsewhere.

Notation

We denote by $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^n$ and by $B(x,r)$ the open ball centered at x of radius r for this norm. If f is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$, the notation $Df(x)[v]$ stands for the differential at $x \in \mathbb{R}^n$ applied to the vector $v \in \mathbb{R}^n$ of the function f. Also, we use the symbol $'$ to denote the transposition operation.

II. Preliminary result

Consider the dynamical system

$$\dot{x} = X(x), \qquad x \in M, \tag{2}$$

where M is a m-dimensional smooth Riemannian manifold and X is a smooth vector field on M. Let x^* be an equilibrium point for system (2) and let (O,φ) be a local coordinate chart on M centered at x^*. Using the coordinate map φ, system (2) can be locally rewritten as

$$\begin{cases} \dot{\xi} = F\xi + f(\xi,\eta) \\ \dot{\eta} = G\eta + g(\xi,\eta), \end{cases} \tag{3}$$

where $F \in \mathbb{R}^{m_1\times m_1}$ has m_1 eigenvalues with non-positive real parts, $G \in \mathbb{R}^{m_2\times m_2}$ has m_2 eigenvalues with positive real parts and, f and g are smooth maps in a neighborhood N of $0 \in \mathbb{R}^m = \mathbb{R}^{m_1+m_2}$ such that $f(0,0) = 0$, $g(0,0) = 0$, $Df(0,0) = 0$ and $Dg(0,0) = 0$. Since the matrix F has eigenvalues with non-positive real parts, there may exist trajectories converging to the origin.

Kelley proved in [11] that for system (3) there exists an invariant manifold

$$W^{cs} = \{(\xi,\eta) \mid \|\xi\| < \delta\ ,\ \eta = h(\xi)\},$$

where, for $\delta > 0$ small enough, h can be chosen to be C^k for any arbitrarily but fixed $k \in \mathbb{N}$. Moreover, h satisfies $h(0) = 0$. W^{cs} is called a (local) center-stable manifold and is a priori not unique.

Although the idea of the proof of the following two lemmas was found in [11], we did not find these general results in the literature.

Lemma 2.1: Any trajectory of system (3) converging towards the origin belongs to a center-stable manifold.

Proof: Since W^{cs} is an invariant manifold for system (3), h satisfies

$$Dh(\xi)[F\xi + f(\xi,h(\xi))] = Gh(\xi) + g(\xi,h(\xi)). \tag{4}$$

Set $\tilde{\eta} = \eta - h(\xi)$. With equations (3) and (4) it yields

$$\begin{aligned} \dot{\tilde{\eta}} &= \dot{\eta} - Dh(\xi)[\dot{\xi}] = G\eta + g(\xi,\eta) - Dh(\xi)[F\xi + f(\xi,\eta)] \\ &= G\eta + g(\xi,\tilde{\eta}+h(\xi)) - Dh(\xi)[F\xi + f(\xi,\tilde{\eta}+h(\xi))] \\ &\qquad + Dh(\xi)[f(\xi,h(\xi))] - Dh(\xi)[f(\xi,h(\xi))] \\ &= G\eta - Gh(\xi) - g(\xi,h(\xi)) + g(\xi,\tilde{\eta}+h(\xi)) \\ &\qquad + Dh(\xi)[f(\xi,h(\xi)) - f(\xi,\tilde{\eta}+h(\xi))] \\ &= G\tilde{\eta} + \tilde{g}(\xi,\tilde{\eta}), \end{aligned}$$

where

$$\begin{aligned} \tilde{g}(\xi,\tilde{\eta}) &= -g(\xi,h(\xi)) + g(\xi,\tilde{\eta}+h(\xi)) \\ &\qquad + Dh(\xi)[f(\xi,h(\xi)) - f(\xi,\tilde{\eta}+h(\xi))]. \end{aligned}$$

Recall that, up to a linear change of coordinates, there exists $\mu > 0$ such that $\langle G\tilde{\eta},\tilde{\eta}\rangle \geqslant 2\mu\|\tilde{\eta}\|^2$ (see e.g. [11]). By construction, $\tilde{g}$ is a C^r map for any $r < \infty$ and

$$\tilde{g}(\xi,\tilde{\eta}) - \tilde{g}(\xi,0) = \int_0^1 D\tilde{g}(\xi,s\tilde{\eta})ds[(0,\tilde{\eta})].$$

Because $\tilde{g}(\xi,0) = 0$ for all $\xi \in N_\delta$, $D\tilde{g}(\cdot,\cdot)$ is continuous and $D\tilde{g}(0,0) = 0$, there exists a sufficiently small neighborhood of the origin such that $\|\tilde{g}(\xi,\tilde{\eta})\| \leqslant \mu\|\tilde{\eta}\|$. It follows that

$$\begin{aligned} \frac{d}{dt}\|\tilde{\eta}\|^2 &= 2\langle G\tilde{\eta},\tilde{\eta}\rangle + 2\langle \tilde{g}(\xi,\tilde{\eta}),\tilde{\eta}\rangle \\ &\geqslant 4\mu\|\tilde{\eta}\|^2 - 2\mu\|\tilde{\eta}\|^2 \geqslant 2\mu\|\tilde{\eta}\|^2. \end{aligned}$$

We conclude that any trajectory starting with an initial condition $\tilde{\eta}(0) \neq 0$ must leave any sufficiently small neighborhood of the origin. Equivalently, a trajectory converging to the origin must belong to a center-stable manifold. ∎

Lemma 2.2: Any trajectory of system (3) converging towards the origin belongs to the intersection of all center-stable manifolds.

Proof: Consider two center-stable manifolds W^{cs}_α and W^{cs}_β and a trajectory $(\xi(t),\eta(t)) \in W^{cs}_\alpha$ converging to the origin. Set $\tilde{\eta} = \eta - h_\beta(\xi)$. The same computations as in the proof of Lemma 2.1 yield

$$\frac{d}{dt}\|\tilde{\eta}\|^2 \geqslant 2\mu\|\tilde{\eta}\|^2. \tag{5}$$

Since $\xi(t)$ goes to zero as t goes to infinity, $\tilde{\eta}(t)$ must converge to zero. However, from (5), the norm of $\tilde{\eta}(t)$ cannot decrease. We conclude that $\tilde{\eta}(t)$ must be identically zero. In particular $\eta(t) = h_\alpha(\xi(t)) = h_\beta(\xi(t))$ for any $t \geqslant 0$, from which the lemma follows. ∎

Corollary 2.3: Any center-stable manifold contains all the trajectories converging to the origin.

Let ϕ_t denote the flow of system (2) and let E be the set defined by

$$E = \{x \in M \mid \phi_t(x) \to x^* \text{ as } t \to +\infty\}.$$

Lemma 2.4: If $m_2 > 0$, the set of all trajectories of system (2) converging to the unstable equilibrium point x^* have zero measure.

Proof: Let W^{cs} be a local center-stable manifold. Denote by d the distance on M. If $\delta > 0$ is chosen small enough, then $N = \varphi^{-1}(W^{cs}) \cap \varphi^{-1}(\partial B(0,\delta))$ is a codimension m_2+1 smooth submanifold of $\varphi^{-1}(\partial B(0,\delta))$ diffeomorphic to the sphere S^{m_1-1}. Set

$$N_q^R = \{x \in N \mid \forall t \in [q,0] \quad d(\phi_t(x), x^*) < R\}.$$

Notice that, because $N_q^R \subset N$, N_q^R has zero measure. We claim that

$$E \subset \left(\varphi^{-1}(W^{cs}) \cap \varphi^{-1}(B(0,\delta))\right) \cup \bigcup_Q \psi_q^R\left([q,0] \times N_q^R\right),$$

where $Q = \left\{(q,R) \in \mathbb{Q} \times \mathbb{N} : q < 0,\ N_q^R \neq \varnothing\right\}$ and, for a nonempty N_q^R, $\psi_q^R : [q,0] \times N_q^R \to M$ is the mapping given by $\psi_q^R(t,x) = \phi_t(x)$. We now prove the claim. Let $a \in E$. If $a \in \varphi^{-1}(B(0,\delta))$, it follows from Lemmas 2.1 and 2.2 that $a \in \varphi^{-1}(W^{cs})$. On the other hand, if $a \notin \varphi^{-1}(B(0,\delta))$, then there exists $t > 0$ such that $x_0 = \phi_t(a) \in \varphi^{-1}(\partial B(0,\delta))$. It follows from the existence theorem for flows that there exist $\eta, \varepsilon > 0$ and $q \in \mathbb{Q}$ such that ϕ is well defined on $[-\varepsilon - t, 0] \times \varphi^{-1}(B(x_0,\eta))$. Choose $q \in \mathbb{Q} \cap [-\varepsilon - t, -t]$ and let $R \in \mathbb{N}$ be such that $\|\phi_t(x)\| < R$ for all $t \in [q,0]$ and all $x \in \varphi^{-1}(B(x_0,\eta))$. By construction of q and R, $a \in \psi_q^R\left([q,0] \times N_q^R\right)$ and the claim follows.

Since $[q,0] \times N_q^R$ has zero measure and since ψ_q^R is Lipschitz, it follows that $\text{meas}\left(\psi_q^R\left([q,0] \times N_q^R\right)\right) = 0$. The lemma follows since E is included in the union of countably many such sets and $\varphi^{-1}(W^{cs}) \cap \varphi^{-1}(B(0,\delta)) \in \mathbb{R}^m$ whose measure is also zero as being a submanifold of positive codimension. ■

Remark 2.5: Lemma 2.4 is a general result. As soon as an equilibrium point has an unstable manifold, the set of all trajectories converging to the equilibrium point has zero measure.

III. Analysis of the example

A. Stabilizing state feedback and observer design

1) Feedback stabilization of system (1)*:* Consider the feedback control function $\lambda^s : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\lambda^s(x) = r_1(x_1, x_2), \tag{6}$$

where r_1 is an arbitrary positive constant.

Proposition 3.1: The target point x_t is an asymptotically stable equilibrium for the closed-loop system resulting from applying the feedback control $u = \lambda^s(x)$ (given by equation (6)) to system (1). Moreover, its basin of attraction is $S^2 \setminus \{-x_t\}$.

Proof: Let the mapping $V : S^2 \to \mathbb{R}$ given by

$$V(x) = x_3,$$

be the candidate Lyapunov function and

$$E = \{x \in S^2 \mid V(x) < 1\} = S^2 \setminus \{-x_t\}.$$

One has

$$\dot{V}(x) = \dot{x}_3 = -r_1(x_1^2 + x_2^2),$$

which is $\leqslant 0$ on E. Let I be the largest invariant set contained in

$$\{x \in S^2 \mid \dot{V}(x) = 0\}.$$

We have $I = \{-x_t, x_t\}$, where x_t is the target point. Since $I \cap E = \{x_t\}$ and since $E \subset S^2$ is bounded, according to LaSalle's principle (see e.g. [12, Corollary 2]), we get the desired result. ■

2) The observer: The chosen Luenberger-like observer for system (1) is

$$\dot{\hat{x}} = A(u)\hat{x} - r_2 C'(C\hat{x} - y), \quad \hat{x} \in \mathbb{R}^3, \tag{7}$$

where r_2 is a given positive constant. Hence, the observation error defined by $\varepsilon = \hat{x} - x$ satisfies

$$\dot{\varepsilon} = (A(u) - r_2 C'C)\,\varepsilon, \quad \varepsilon \in \mathbb{R}^3. \tag{8}$$

Remark 3.2: The constants r_1 and r_2 could be used as tuning parameters. However, in order to simplify the computations, from now these constants are set to one.

The Gram observability matrix relative to an input $u : [0,T] \to \mathbb{R}^2$ is defined by

$$G(u,T) = \int_0^T \Phi_u(t)' C' C \Phi_u(t) dt,$$

where $\Phi_u(t)$ is the resolvent matrix solution of equation (8), i.e.

$$\dot{\Phi}_u(t) = (A(u(t)) - CC')\,\Phi_u(t)$$
$$\Phi_u(0) = \text{Id},$$

and we denote by $\text{ind}(u,T)$ the observability index of u, that is, the smallest eigenvalue of $G(u,T)$. Clearly, $\text{ind}(u,T) \geqslant 0$. For $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^2)$, we denote by $u_{[\theta]}$ the shifted input defined by $u_{[\theta]}(t) = u(\theta + t)$. As in [3], we call an input $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^2)$ *persistent*, if there exists a time interval $[0,T]$, $T > 0$ such that

$$\limsup_{\theta \to +\infty} \text{ind}(u_{[\theta]}, T) > 0.$$

According to [3, Theorem 3 page 300], we have the following theorem.

Theorem 3.3: If $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^2)$ is a persistent input, then the observation error tends to zero, i.e.

$$\lim_{t \to +\infty} \varepsilon(t) = 0.$$

3) The coupled system controller-observer: The equations of the controller-observer system are, as usual

$$\begin{cases} \dot{\hat{x}} = A\left(\lambda^s(\hat{x})\right)\hat{x} - C'C\varepsilon \\ \dot{\varepsilon} = \left(A\left(\lambda^s(\hat{x})\right) - C'C\right)\varepsilon, \quad (\hat{x}, \varepsilon) \in M, \end{cases} \tag{9}$$

where λ^s is the stabilizing feedback (6) of Section III-A.1 and, since $x \in S^2$, $M = \{(\hat{x}, \varepsilon) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|\hat{x} - \varepsilon\| = 1\}$. It is clear that system (9) is not a stabilizing device since all the domains

$$\mathcal{D}_{a,b} = \{(\hat{x}, \varepsilon) \in M \mid \hat{x} = (0,0,a),\ \varepsilon \in C_b\}, \tag{10}$$

where C_b is any circle in $\mathbb{R}^3$ of the form $\{\varepsilon \in \mathbb{R}^3 \mid \varepsilon_1^2 + \varepsilon_2^2 = b^2, \varepsilon_3 = 0\}$ are invariant sets for the coupled system (9). Indeed, all the inputs that make system (1) unobservable are those for which $y = Cx$ is constant, i.e. such that

$$-u_1(t)x_1(t) - u_2(t)x_2(t) = 0, \quad \forall\, t \in \mathbb{R}_+.$$

In particular, the null input $u(\cdot) = 0$ makes system (1) unobservable. A natural and simple idea is then to choose a small perturbation of the feedback control that makes it observable.

B. Practical asymptotic stabilization

In order to make the input $\lambda^s(\hat{x}(\cdot))$ observable, a first simple choice is to add a (small) nonzero constant δ to the feedback, i.e. we consider

$$\lambda^s_\delta(\hat{x}) = \lambda^s(\hat{x}) + (\delta, \delta). \tag{11}$$

We shall prove the following lemma which is crucial for our purpose.

Lemma 3.4: All the inputs of the form (11) applied to the full coupled system (9) make system (1) observable on any time interval $[0, T]$, $T > 0$.

Proof: Suppose, contrary to our claim, that there exist a positive T and an input $t \mapsto u(t) = \lambda^s_\delta(\hat{x}(t))$ (with λ^s_δ given by equation (11)) that renders system (1) unobservable on $[0, T]$. Such an input is solution of system (9). This means that there exists a non-identically vanishing function $t \mapsto \omega(t) \in \mathbb{R}^3$ such that, for all $t \in [0, T]$, we have

$$\begin{aligned} \dot{\omega}(t) &= A(u(t))\omega(t) = A(\lambda^s_\delta(\hat{x}(t)))\omega(t), \\ y(t) &= C\omega(t) = \omega_3(t) = 0. \end{aligned} \tag{12}$$

Then, differentiating with respect to time equation (12), we get that

$$\begin{aligned} 0 &= \dot{\omega}_3 \\ &= -u_1\omega_1 - u_2\omega_2 \\ &= -(\hat{x}_1 + \delta)\omega_1 - (\hat{x}_2 + \delta)\omega_2. \end{aligned} \tag{13}$$

Differentiating once again with respect to time yields

$$\begin{aligned} 0 &= \dot{\hat{x}}_1\omega_1 + \hat{x}_1\dot{\omega}_1 + \dot{\hat{x}}_2\omega_2 + \hat{x}_2\dot{\omega}_2 + \delta(\dot{\omega}_1 + \dot{\omega}_2) \\ &= (e\hat{x}_2 + u_1\hat{x}_3)\omega_1 + \hat{x}_1 e\omega_2 \\ &\quad + (-e\hat{x}_1 + u_2\hat{x}_3)\omega_2 - \hat{x}_2 e\omega_1 + \delta(e\omega_2 - e\omega_1) \\ &= \delta e(\omega_2 - \omega_1). \qquad \text{(According to (13))} \end{aligned}$$

Hence, we have

$$\omega_2 - \omega_1 = 0, \tag{14}$$

which implies that $0 = \dot{\omega}_2 - \dot{\omega}_1 = -e(\omega_1 + \omega_2)$, i.e.

$$\omega_1 + \omega_2 = 0. \tag{15}$$

Therefore, according to equations (12), (14) and (15), $t \mapsto \omega(t)$ vanishes identically on $[0, T]$ which is a contradiction. Consequently, system (1) is observable for the chosen input $t \mapsto \lambda^s_\delta(\hat{x}(t))$ on any time interval. ■

With this lemma in hand, the proof of our stabilization result will follow after three steps.

Remark 3.5: The proof of Lemma 3.4 depends on the chosen perturbation of the stabilizing control, and in a general context, the chosen kind of perturbation is specific to each system. For example, if $A(u)$ is replaced by

$$\begin{pmatrix} 0 & e\varphi(u) & u_1 \\ -e\varphi(u) & 0 & u_2 \\ -u_1 & -u_2 & 0 \end{pmatrix}, \ \varphi : \mathbb{R}^2 \to \mathbb{R}, \ \varphi(0) = 0, \tag{16}$$

in system (1), a constant perturbation does not work while a perturbation $t \mapsto \delta(t)$ that satisfies $\ddot{\delta} = -\delta$ does.

1) Step 1: proof that the estimation error goes to zero: Using the perturbed feedback as an input, the fully coupled system (9) is rewritten as

$$\begin{cases} \dot{\hat{x}} = A\left(\lambda^s_\delta(\hat{x})\right)\hat{x} - r_2 C'C\varepsilon \\ \dot{\varepsilon} = \left(A\left(\lambda^s_\delta(\hat{x})\right) - r_2 C'C\right)\varepsilon \quad (\hat{x}, \varepsilon) \in M. \end{cases} \tag{17}$$

Let $T > 0$ be fixed. It follows straightforwardly from the continuous dependence on initial data of the solutions of an ordinary differential equation, that the map $(\hat{x}_0, \varepsilon_0) \mapsto \lambda^s_\delta(\hat{x}(\cdot))$ from M to $C^0([0, T], \mathbb{R}^2)$ is continuous. Then, so is the map $\lambda^s_\delta(\hat{x}(\cdot)) \mapsto \mathrm{ind}(\lambda^s_\delta(\hat{x}(\cdot)), T)$. Consequently, the mapping $F : M \to \mathbb{R}_+$ given by $F(\hat{x}_0, \varepsilon_0) = \mathrm{ind}(\lambda^s_\delta(\hat{x}(\cdot)), T)$ is continuous.

Lemma 3.6: All the inputs $\lambda^s_\delta\big(\hat{x}(\cdot)\big)$ are persistent.

Proof: According to (8), we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\varepsilon\|^2 &= \varepsilon'\dot{\varepsilon} \\ &= \varepsilon'(A(\lambda^s_\delta(\hat{x})) - r_2 C'C)\varepsilon \\ &= -r_2(C\varepsilon)^2, \qquad \text{(since } A(\cdot)' = -A(\cdot)) \end{aligned}$$

which shows that $t \mapsto \varepsilon(t)$ is bounded if $t \geqslant 0$. Consequently,

$$\begin{aligned} \|\hat{x}(t)\| &\leqslant \|\hat{x}(t) - x(t)\| + \|x(t)\| \\ &= \|\varepsilon(t)\| + 1. \qquad \text{(since } x(t) \in S^2) \end{aligned}$$

Therefore, $t \mapsto \hat{x}(t)$ is also bounded if $t \geqslant 0$. It follows that the solution of system (17) are defined for all $t \geqslant 0$ and that $\{(\hat{x}(t), \varepsilon(t)) \mid t \geqslant 0\} \subset K$, where $K = \bar{B}(0, \|\varepsilon_0\| + 1) \times \bar{B}(0, \|\varepsilon_0\|)$. In particular,

$$\{(\hat{x}(t), \varepsilon(t)) \mid t \geqslant \theta\} \subset K, \quad \forall\, \theta \geqslant 0.$$

The function F defined above being nonnegative, we infer that $\inf_K F \leqslant \inf\{F(\hat{x}(t), \varepsilon(t)) \mid t \geqslant \theta\}$, for all $\theta \geqslant 0$. It follows that for every fixed positive T,

$$\begin{aligned} \inf_K F &\leqslant \limsup_{\theta \to +\infty} F(\hat{x}(\theta), \varepsilon(\theta)) \\ &= \limsup_{\theta \to +\infty} \mathrm{ind}\big(\lambda^s_\delta\big(\hat{x}(\cdot + \theta)\big), T\big). \end{aligned}$$

By continuity, the infimum of F over K is reached, and is positive by the crucial Lemma 3.4. ■

We have the following corollary whose proof is an immediate consequence of Theorem 3.3

Corollary 3.7: For any trajectory of the coupled system (17), we have $\lim_{t \to +\infty} \varepsilon(t) = 0$.

2) Step 2 : Asymptotic stability of an equilibrium point close to x_{t}: In the present subsection, we prove that system (17) admits an asymptotically stable equilibrium point and an unstable equilibrium point that are respectively close to $(x_{\mathrm{t}}, 0)$ and $(-x_{\mathrm{t}}, 0)$.

Lemma 3.8: For δ small enough, system (17) admits a locally asymptotically stable equilibrium point $(x^s_\delta, 0)$ arbitrarily close to $(x_\mathrm{t}, 0)$.

Proof: Let (x^*, ε^*) be an equilibrium point of system (17). Then (x^*, ε^*) satisfies

$$\begin{cases} A\left(\lambda^s_\delta(x^*)\right) x^* - r_2 C'C\varepsilon^* = 0 \\ \left(A\left(\lambda^s_\delta(x^*)\right) - r_2 C'C\right)\varepsilon^* = 0. \end{cases}$$

A straightforward resolution of the previous system leads to three solutions : the zero solution (which is not admissible since $\|x\| = \|\hat{x} - \varepsilon\| = 1$) and two solutions arbitrarily close to $(x_\mathrm{t}, 0)$ and $(-x_\mathrm{t}, 0)$ respectively (as soon as δ is chosen small enough). In particular, the solution close to $(x_\mathrm{t}, 0)$ is $(x^s_\delta, 0) = (x^s_1, x^s_2, x^s_3, 0, 0, 0)$ where

$$\begin{aligned} x^s_1 &= -\frac{\delta\left((x^s_3)^2 - x^s_3\right)}{(x^s_3)^2 + 1} \\ x^s_2 &= -\frac{\delta\left((x^s_3)^2 + x^s_3\right)}{(x^s_3)^2 + 1} \\ x^s_3 &= -\sqrt{-\delta^2 + \sqrt{\delta^4 + 1}}, \end{aligned}$$

Let us write ε_3 as a function of ε_1, ε_2 and $\hat{x}$. Since $x \in S^2$, we have

$$\|x\| = \|\hat{x} - \varepsilon\| = 1 \tag{18}$$

which is a polynomial equation of degree two in ε_3. The resolution of equation (18) yields

$$\varepsilon_3 = \hat{x}_3 \pm \sqrt{1 - (\hat{x}_1 - \varepsilon_1)^2 - (\hat{x}_2 - \varepsilon_2)^2}. \tag{19}$$

In the following, we are going to study the stability of $(x^s_\delta, 0)$ by linearization. Hence, we assume that $(\hat{x}, \varepsilon)$ is close to $(x^s_\delta, 0)$ which is close to $(x_\mathrm{t}, 0)$. Consequently, in equation (19) we are only interested by the solution having the plus sign in its expression.

Now, substituting the expression given by (19) in system (17) yields a new system of the form

$$(\dot{\hat{x}}, \dot{\bar{\varepsilon}})' = F_\delta(\hat{x}, \bar{\varepsilon}), \quad F_\delta : \mathbb{R}^5 \to \mathbb{R}^5, \tag{20}$$

where $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2)$. The linearization of system (20) around $(x^s_\delta, 0)$ is

$$(\dot{\bar{x}}, \dot{\bar{\varepsilon}})' = DF_\delta(x^s_\delta, 0)[(\bar{x}, \bar{\varepsilon})], \tag{21}$$

where $\bar{x} = \hat{x} - x^s_\delta$. A standard analysis shows that, for δ small enough, the eigenvalues of $DF_\delta(x^s_\delta, 0)$ have negative real parts. Consequently, $(x^s_\delta, 0)$ is a locally asymptotically stable equilibrium point of (17). ■

Lemma 3.9: For δ small enough, system (17) admits an unstable equilibrium point $(x^u_\delta, 0)$ arbitrarily close to $(-x_\mathrm{t}, 0)$ and the set of all trajectories converging to $(x^u_\delta, 0)$ has zero measure.

Proof: Similar computations as in the proof of Lemma 3.8 yields that system (17) can be rewritten under the form

$$(\dot{\hat{x}}, \dot{\bar{\varepsilon}})' = F_\delta(\hat{x}, \bar{\varepsilon}), \quad F_\delta : \mathbb{R}^5 \to \mathbb{R}^5, \tag{22}$$

where $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2)$. The linearization of (22) around the equilibrium point $(x^u_\delta, 0)$ arbitrarily close to $(-x_\mathrm{t}, 0)$ is

$$(\dot{\bar{x}}, \dot{\bar{\varepsilon}})' = DF_\delta(x^s_\delta, 0)[(\bar{x}, \bar{\varepsilon})], \tag{23}$$

where $\bar{x} = \hat{x} - x^u_\delta$. A standard analysis shows that, for δ small enough, $DF_\delta(x^u_\delta, 0)$ has two eigenvalues with positive real parts and three with negative real parts. The result follows from Lemma 2.4. ■

3) Step 3 : The main result: We are now ready to state and prove our main result.

Theorem 3.10: The point $(x^s_\delta, 0) \in M$ is asymptotically stable for system (17), and its region of attraction is an open, dense and of full measure subset of M.

Proof: From Corollary 3.7 we know that the error converges to zero. The ω-limit points of system (17) are then of the form $(\hat{x}, 0)$, with $\hat{x} \in S^2$. In this case, a trajectory in the ω-limit set satisfies the differential equation

$$\begin{cases} \dot{\hat{x}} = A\left(\lambda^s_\delta(\hat{x})\right)\hat{x} \\ \dot{\varepsilon} = 0 \\ \dot{\delta} = 0. \end{cases} \tag{24}$$

Set $C^- = \{x \in S^2 \mid x_3 \leqslant 0\}$ and consider the function $L : C^- \to \mathbb{R}_+$ defined by

$$L(\hat{x}) = \frac{1}{2}\|\hat{x} - x^s_\delta\|^2.$$

Along the trajectories of system (24), we have (since $A(\cdot)$ is skew-symmetric)

$$\dot{L}(\hat{x}) = -(x^s_\delta)' A\left(\lambda^s_\delta(\hat{x})\right)\hat{x}.$$

The analysis of the maximum value of $\dot{L}$ on C^- indicates that for all $\hat{x} \in C^-$, $\dot{L}(\hat{x}) \leqslant 0$ and $\dot{L}(\hat{x}) = 0$ if and only if $\hat{x} = x^s_\delta$. From LaSalle's invariant principle, we get that x^s_δ is asymptotically stable for system (24) and its region of attraction contains C^-.

A similar reasoning with C^- replaced by $C^+ = \{x \in S^2 \mid x_3 \geqslant 0\}$, $\dot{L} \leqslant 0$ replaced by $\dot{L} \geqslant 0$, and x^s_δ replaced by x^u_δ proves that any trajectory of (24) starting in $C^+ \setminus \{x^u_\delta\}$ will enter C^-, i.e. the basin of attraction of x^s_δ.

It follows that the ω-limit set of any trajectory of system (17) is contained in the set $\{(x^s_\delta, 0), (x^u_\delta, 0)\}$. However, from Lemma 3.8, the point $(x^s_\delta, 0)$ is locally asymptotically stable. It follows that any trajectory goes either to $(x^s_\delta, 0)$ or to $(x^u_\delta, 0)$. Since, from Lemma 3.9, the set of trajectories converging to $(x^u_\delta, 0)$ has zero measure, the Lemma follows. ■

Theorem 3.10 immediately yields the following Corollary.

Corollary 3.11: For every neighborhood $\mathcal{N}$ of $(x_\mathrm{t}, 0)$, there exists $\delta_{\max} > 0$ such that for all $\delta \in (0, \delta_{\max})$, $\mathcal{N}$ contains an asymptotically stable attractor $\mathcal{A}$ (a point in our case) for system (17). Moreover, the region of attraction of $\mathcal{A}$ is an open, dense and of full measure subset of M.

Remark 3.12: In more general situations (for example, system (16) in Remark 3.5), Theorem 3.10 might not be true, but Corollary 3.11 remains valid. Note that in such a situation, the perturbation may depend on time but

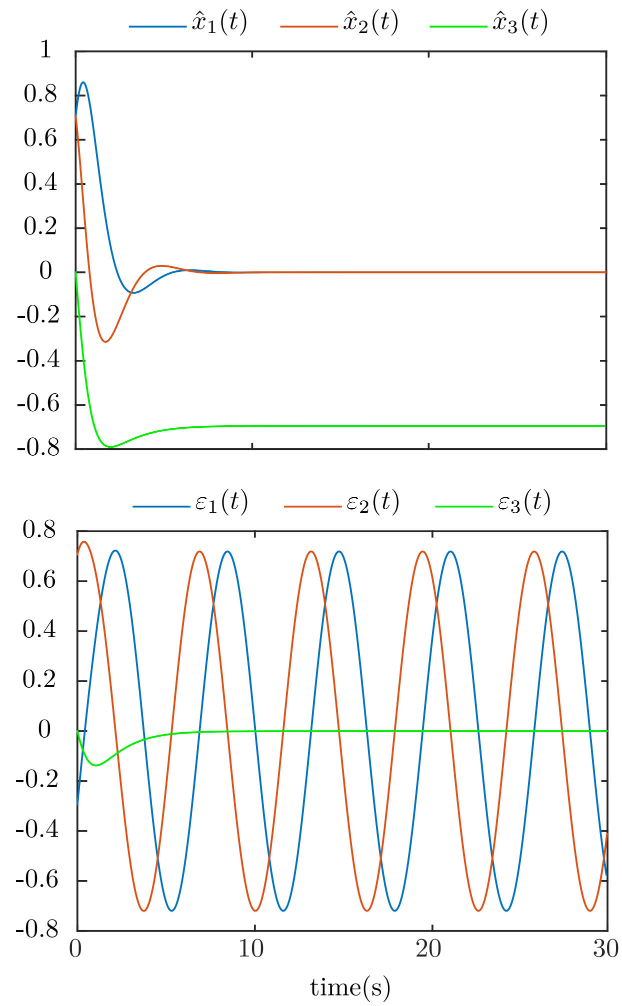

Fig. 1. State estimations and observation errors of system (9).

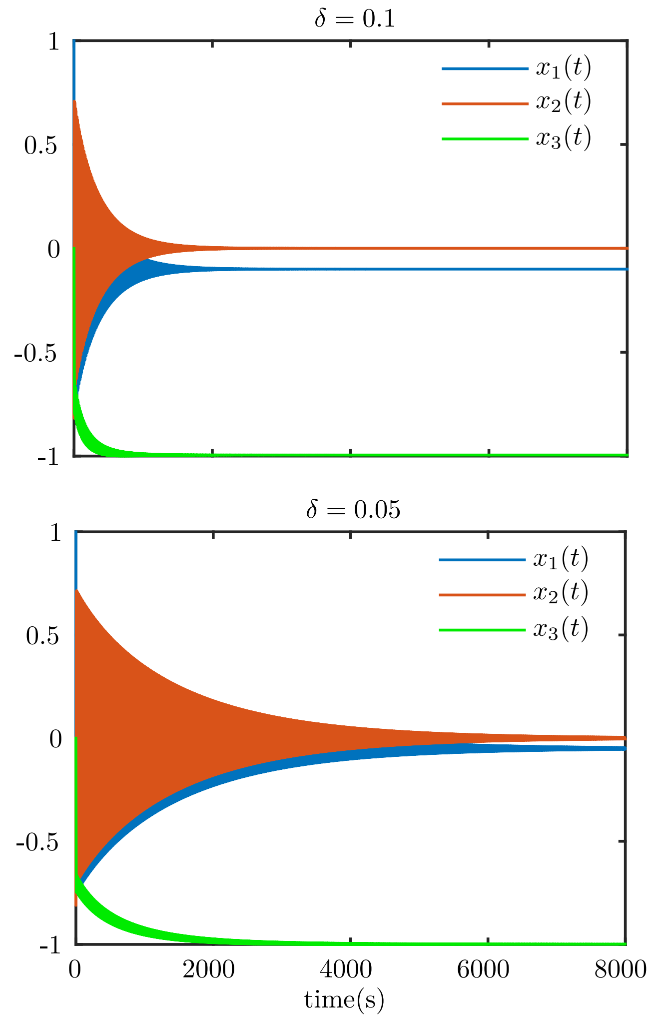

Fig. 2. State variables of system (1) with input $u = \lambda^s_\delta(\hat{x})$ and for $\delta = 0.1$ and $\delta = 0.05$.

system (24) has to remain autonomous in order to apply LaSalle's Principle [1].

4) Numerical simulations: The theoretical results presented in the previous section have been tested by means of a numerical simulation. The case of the quantum system (1) has been treated and the observer was constructed using equation (7). The system has been simulated with e, r_1 and r_2 set to 1. The initial conditions were arbitrarily set to $x_0 = (1, 0, 0)$ and $\hat{x}_0 = (1/\sqrt{2}, 1/\sqrt{2}, 0)$[2].

Fig. 1 shows the state variables of system (9) which is clearly not a stabilizing device because, we see that the error does not converge to zero but rather to a circle in $\mathbb{R}^3$, and the estimation converges to a point $(0, 0, a)$, i.e. $(\hat{x}, \varepsilon)$ lies indeed in a domain $\mathcal{D}_{a,b}$ (defined by (10)).

We now take a feedback control in order to make the system observable, that is, we choose a perturbed feedback control of the form (11). The simulations have been made for two different values of the perturbation of the stabilizing feedback control: $\delta = 0.1$ and $\delta = 0.05$. Fig. 2 shows, for both cases, the evolution of the state variables, and Fig. 3 shows the evolution of the observation errors. Notice that the thick aspect of the drawn curves is due to the observation of an oscillating system over a very long time.

In both cases, the simulations show that the observation errors converge to zero, which corroborates Corollary 3.7 as well as the idea that perturbing the stabilizing feedback makes the system observable. It is also clear that the state converges to a point close to x_{t} and that this point is closer to x_{t} for a smaller perturbation δ. This is exactly the statement of Theorem 3.10 which asserts that it is possible to asymptotically stabilize system (1) to a point arbitrarily close to the target point x_{t}.

However, notice that choosing a smaller perturbation leads the system to converge closer but slower to the target. The convergence can therefore be very slow.

C. Exact asymptotic stabilization

In order to stabilize (9) to the target point $(x_{\mathrm{t}}, 0)$, we add to the feedback a (small) perturbation that decreases

[1]This difficulty can be overcome by considering a perturbation solution of a finite dimensional ordinary differential equation.

[2]The choice of $\hat{x}_0 \in S^2$ is legitimated by the fact that the state of system (1) lies in S^2.

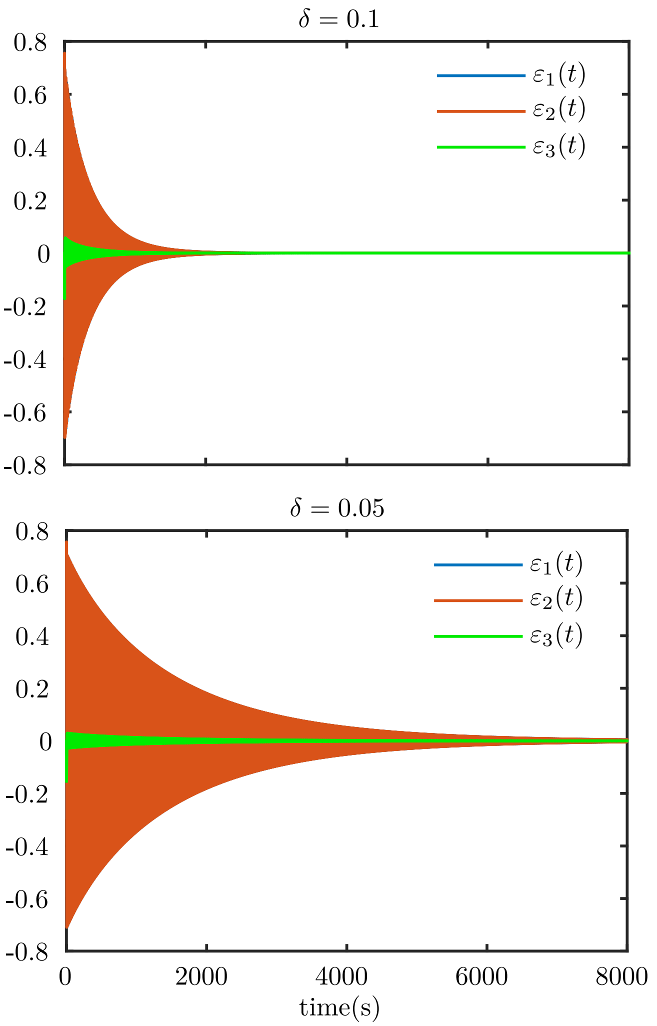

Fig. 3. Observation errors of system (17) for $\delta = 0.1$ and $\delta = 0.05$.

as $\hat{x}$ goes to x_{t}, i.e. we consider

$$\lambda^s_{\delta,\alpha}(\hat{x}) = \lambda^s(\hat{x}) + (\delta,\delta)((\hat{x}_3)^\alpha - 1), \tag{25}$$

where δ is a positive constant and α is a positive even number. We shall prove the following lemma which is crucial for our purpose.

Lemma 3.13: All the inputs of the form (25) applied to the full coupled system (9) make system (1) observable on any time interval $[0,T]$, $T > 0$.

Proof: Suppose, contrarily to our claim, that there exist a positive T and an input $t \mapsto u(t) = \lambda^s_{\delta,\alpha}(\hat{x}(t))$ (with $\lambda^s_{\delta,\alpha}$ given by equation (25)) that renders system (1) unobservable on $[0,T]$. Such an input is solution of system (9). This means that there exists a non-identically vanishing function $t \mapsto \omega(t) \in \mathbb{R}^3$ such that, for all $t \in [0,T]$, we have

$$\begin{aligned} &\dot{\omega}(t) = A(u(t))\omega(t) = A(\lambda^s_{\delta,\alpha}(\hat{x}(t)))\omega(t), \\ &y(t) = C\omega(t) = \omega_3(t) = 0. \end{aligned} \tag{26}$$

Differentiate equation (26) with respect to time to obtain

$$\begin{aligned} 0 &= \dot{\omega}_3 \\ &= -u_1\omega_1 - u_2\omega_2 && (27) \\ &= -(\hat{x}_1 + \delta((\hat{x}_3)^\alpha - 1))\omega_1 - (\hat{x}_2 + \delta((\hat{x}_3)^\alpha - 1))\omega_2 \\ &= -\hat{x}_1\omega_1 - \hat{x}_2\omega_2 - \delta((\hat{x}_3)^\alpha - 1)(\omega_1 + \omega_2). && (28) \end{aligned}$$

Differentiating once again yields

$$\begin{aligned} 0 &= \dot{\hat{x}}_1\omega_1 + \hat{x}_1\dot{\omega}_1 + \dot{\hat{x}}_2\omega_2 + \hat{x}_2\dot{\omega}_2 \\ &\quad + \delta\alpha(\hat{x}_3)^{\alpha-1}\dot{\hat{x}}_3(\omega_1 + \omega_2) \\ &\qquad + \delta((\hat{x}_3)^\alpha - 1)(\dot{\omega}_1 + \dot{\omega}_2) \\ &= (\hat{x}_2 + u_1\hat{x}_3)\omega_1 + \hat{x}_1\omega_2 + (-\hat{x}_1 + u_2\hat{x}_3)\omega_2 - \hat{x}_3\omega_1 \\ &\quad + \delta\alpha(\hat{x}_3)^{\alpha-1}(-u_1\hat{x}_3 - u_2\hat{x}_2 - \varepsilon_3)(\omega_1 + \omega_2) \\ &\qquad + \delta((\hat{x}_3)^\alpha - 1)(\omega_2 - \omega_1) \\ {}^3 &= \delta\alpha(\hat{x}_3)^{\alpha-1}(-\hat{x}_1^2 - \hat{x}_2^2 - \varepsilon_3 - (\hat{x}_1 + \hat{x}_2)\delta((\hat{x}_3)^\alpha - 1)) \\ &\quad \times (\omega_1 + \omega_2) + \delta((\hat{x}_3)^\alpha - 1)(\omega_2 - \omega_1) && (29) \end{aligned}$$

Equations (28) and (29) are rewritten as $B\bar{\omega} = 0$, where $\bar{\omega} = (\omega_1, \omega_2)$ and $\mathbb{R}^{2\times 2} \ni B$ depends on ε and $\hat{x}$. Solving $\det(B) = 0$ being cumbersome, it is not detailed here. However, one gets that $\det(B) = 0$ if and only if $(\varepsilon, \hat{x}) = (0, x_{\mathrm{t}})$ or $(\varepsilon, \hat{x}) = (0, -x_{\mathrm{t}})$. This means that provided that $(\varepsilon, \hat{x}) \neq (0, \pm x_{\mathrm{t}})$, $\omega(\cdot)$ vanishes identically on $[0,T]$ which is a contradiction.

We conclude that all the inputs of the form $\lambda^s_{\delta,\alpha}(\hat{x}(\cdot))$ make system (1) observable provided that $(\varepsilon, \hat{x}) \neq (0, \pm x_{\mathrm{t}})$. If $(\varepsilon, \hat{x}) = (0, \pm x_{\mathrm{t}})$, $x_3 = \pm 1$ and since $\|x\| = 1$ we get $x = \pm x_{\mathrm{t}}$. ■

With this lemma in hand, the proof of our stabilization result will follow after three steps.

Remark 3.14: Although the computations are more complicated, one can choose any $\alpha > 1$ and consider $|\hat{x}_3|$ instead of $\hat{x}_3$ in (25).

1) Step 1: proof that the estimation error goes to zero: Using the perturbed feedback as an input, the fully coupled system (9) is rewritten as

$$\begin{cases} \dot{\hat{x}} = A\left(\lambda^s_{\delta,\alpha}(\hat{x})\right)\hat{x} - C'C\varepsilon \\ \dot{\varepsilon} = \left(A\left(\lambda^s_{\delta,\alpha}(\hat{x})\right) - C'C\right)\varepsilon, \quad (\hat{x}, \varepsilon) \in M. \end{cases} \tag{30}$$

Fix $T > 0$. The same reasoning as in Subsection III-B.1 yields that the mapping $F : M \to \mathbb{R}_+$ given by $F(\hat{x}_0, \varepsilon_0) = \mathrm{ind}(\lambda^s_{\delta,\alpha}(\hat{x}(\cdot)), T)$ is continuous.

Lemma 3.15: For any trajectory of the coupled system (30), $\lim_{t\to+\infty} \varepsilon(t) = 0$.

Proof: In much the same way as in the proof of Lemma 3.6, it follows that

$$\{(\hat{x}(t), \varepsilon(t)) \mid t \geqslant 0\} \subset \bar{B}(0, \|\varepsilon_0\| + 1) \times \bar{B}(0, \|\varepsilon_0\|).$$

Define the compact set

$$K_n = \bar{B}(0, \|\varepsilon_0\| + 1) \times \left(\bar{B}(0, \|\varepsilon_0\|) \setminus B(0, \|\varepsilon_0\|/2^n)\right)$$

and suppose that there exists $n \in \mathbb{N}$ such that

$$\{(\hat{x}(t), \varepsilon(t)) \mid t \geqslant 0\} \subset K_n.$$

[3] According to (27)

In particular,

$$\{(\hat{x}(t), \varepsilon(t)) \mid t \geqslant \theta\} \subset K_n, \quad \forall\, \theta \geqslant 0.$$

We infer that $\inf_{K_n} F \leqslant \sup\{F(\hat{x}(t), \varepsilon(t)) \mid t \geqslant \theta\}$, for all $\theta \geqslant 0$. It follows that for every fixed positive T,

$$\begin{aligned}\inf_{K_n} F &\leqslant \limsup_{\theta \to +\infty} F(\hat{x}(\theta), \varepsilon(\theta)) \\ &= \limsup_{\theta \to +\infty} \operatorname{ind}\big(\lambda^s_{\delta,\alpha}\big(\hat{x}(\cdot + \theta)\big), T\big).\end{aligned}$$

By continuity, the infimum of F over K_n is reached, and is positive by the crucial Lemma 3.13. This means that $\lambda^s_{\delta,\alpha}\big(\hat{x}(\cdot)\big)$ is a persistent input and by Theorem 3.3 that the error goes to zero, which contradicts the hypothesis that the trajectory stays in K_n. Consequently, there exists a strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ such that for all $t \geqslant t_n$, $\varepsilon(t) \in B(\|\varepsilon_0\|/2^n)$. Since $\cap_{n\in\mathbb{N}} B(\|\varepsilon_0\|/2^n) = \{0\}$, we conclude that $\lim_{t\to+\infty} \varepsilon(t) = 0$ ■

Remark 3.16: In Subsection III-B, the input $\lambda^s_\delta\big(\hat{x}(\cdot)\big)$ is persistent and the convergence of the error towards zero follows immediately from Theorem 3.3. In the present subsection, the input $\lambda^s_{\delta,\alpha}\big(\hat{x}(\cdot)\big)$ is not persistent and yet the error still goes to zero.

2) Step 2 : Local stability of $(x_{\mathrm{t}}, 0)$ *and instability of* $(-x_{\mathrm{t}}, 0)$*:* In the present subsection, we prove that $(x_{\mathrm{t}}, 0)$ is a locally asymptotically stable equilibrium point for system (30). We also prove that $(-x_{\mathrm{t}}, 0)$ is an unstable equilibrium point for system (30) and that the set of all trajectories converging to $(-x_{\mathrm{t}}, 0)$ has zero measure.

Lemma 3.17: The point $(x_{\mathrm{t}}, 0)$ is locally asymptotically stable for system (30).

Proof: Since $x \in M$, we have $\|x\| = \|\hat{x} - \varepsilon\| = 1$, which leads for $\hat{x}$ close (enough) to $(x_{\mathrm{t}}, 0)$,

$$\hat{x}_3 = \varepsilon_3 - \sqrt{1 - (\hat{x}_1 - \varepsilon_1)^2 - (\hat{x}_2 - \varepsilon_2)^2}.$$

Substituting $\hat{x}_3$ in (30) we obtain the system in $\mathbb{R}^5$:

$$\begin{cases} \dot{Z}_1 = F Z_1 + f(Z_1, Z_2) \\ \dot{Z}_2 = G Z_2 + g(Z_1, Z_2) \end{cases}, \tag{31}$$

where $Z_1 = (\hat{x}_1, \hat{x}_2, \varepsilon_3)$, $Z_2 = (\varepsilon_1, \varepsilon_2)$,

$$F = \begin{pmatrix} -1 & 1 & \alpha\delta \\ -1 & -1 & \alpha\delta \\ 0 & 0 & -1 \end{pmatrix},\ G = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and $f : \mathbb{R}^5 \to \mathbb{R}^3$ and $g : \mathbb{R}^5 \to \mathbb{R}^2$ are C^∞ functions such that $f(0) = 0$, $Df(0) = 0$, $g(0) = 0$ and $Dg(0) = 0$. Since the eigenvalues of G have zero real parts we cannot conclude directly about the local stability of the origin. However, the center manifold theory (see [13]) yields our result. After simplifications, one gets that the dynamic on the center manifold is given by

$$\begin{cases} \dot{\varepsilon}_1 = \varepsilon_2 + o(\|(\varepsilon_1, \varepsilon_2)\|^5) \\ \dot{\varepsilon}_2 = -\varepsilon_1 - 4\beta(\varepsilon_1 + \varepsilon_2)(\varepsilon_1^2 + \varepsilon_2^2)^2 + o(\|(\varepsilon_1, \varepsilon_2)\|^5) \end{cases}, \tag{32}$$

with $\beta = \alpha^2\delta^2/32$. Applying the change of variables

$$\begin{cases} z_1 = \varepsilon_1 - \varepsilon_2 \\ z_2 = \varepsilon_1 + \varepsilon_2 \end{cases},$$

one gets

$$\begin{cases} \dot{z}_1 = z_2 + \beta z_2 \|z\|^4 + o(\|z\|^5) \\ \dot{z}_2 = -z_1 - \beta z_2 \|z\|^4 + o(\|z\|^5) \end{cases}, \tag{33}$$

with $z = (z_1, z_2)$. Let the mapping $W : \mathbb{R}^2 \to \mathbb{R}$ given by

$$W(z) = z_1^2 + \frac{\beta}{2} z_1 z_2 \|z\|^4 + \left(1 + \beta\|z\|^4\right) z_2^2$$

be the candidate Lyapunov function. One has

$$\dot{W}(z) = -\frac{\beta}{2}\|z\|^4(z_1^2 + 3z_2^2) + o(\|z\|^6),$$

which is negative definite on a neighborhood of the origin. From LaSalle's principle, we conclude that the origin is a locally asymptotically stable equilibrium point for system (32). The result then follows using the center manifold theorem [13, Theorem 2]. ■

Lemma 3.18: The set of all trajectories of system (30) converging to $(x_{\mathrm{t}}, 0)$ has zero measure.

Proof: A similar computation as in the proof of Lemma 3.17 yields that in a neighborhood of the point $(x_{\mathrm{t}}, 0)$ system (30) can be written

$$\begin{cases} \dot{Z}_1 = F Z_1 + f(Z_1, Z_2) \\ \dot{Z}_2 = G Z_2 + g(Z_1, Z_2) \end{cases}, \tag{34}$$

where F has an eigenvalue with negative real part and two eigenvalues with zero real parts, G has eigenvalues with positive real parts, Z_1 and Z_2 are vectors in $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively and f and g are smooth maps in a neighborhood $\mathcal{N}$ of $0 \in \mathbb{R}^5$ such that $f(0,0) = 0$, $g(0,0) = 0$, $Df(0,0) = 0$ and $Dg(0,0) = 0$. Since the matrix G has eigenvalues with positive real parts, according to Lemma 2.4, the result follows. ■

3) Step 3 : The main result: We are now ready to state and prove our main result.

Theorem 3.19: Provided that $\delta \in (-\sqrt{2}/\alpha, \sqrt{2}/\alpha)$, the point $(x_{\mathrm{t}}, 0) \in M$ is asymptotically stable for system (30) and its region of attraction is an open, dense and of full measure subset of M.

Proof: According to Lemma 3.15 the error converges to zero. Hence, the ω-limit points of any trajectory of system (30) are of the form $(\hat{x}, 0)$, with $\hat{x} \in S^2$. In this case, a trajectory in the ω-limit set satisfies the differential system

$$\begin{cases} \dot{\hat{x}} = A\left(\lambda^s_{\delta,\alpha}(\hat{x})\right)\hat{x} \\ \dot{\varepsilon} = 0 \\ \dot{\delta} = 0. \end{cases} \tag{35}$$

Let the mapping $V : S^2 \to \mathbb{R}$ defined in Section III-A.1 be a candidate Lyapunov function for system (35). One has

$$\dot{V}(\hat{x}) = \dot{\hat{x}}_3 = -(\hat{x}_1^2 + \hat{x}_2^2) - \delta((\hat{x}_3)^\alpha - 1)(\hat{x}_1 + \hat{x}_2).$$

Using $\|\hat{x}\| = 1$ and Bernoulli's inequality one gets

$$(\hat{x}_3)^\alpha = (1 - \hat{x}_1^2 - \hat{x}_2^2)^{\alpha/2} \geqslant 1 - \frac{\alpha}{2}(\hat{x}_1^2 + \hat{x}_2^2),$$

which yields

$$\dot{V}(\hat{x}) \leqslant -(\hat{x}_1^2 + \hat{x}_2^2)\left(1 - \frac{\alpha\delta}{2}(\hat{x}_1 + \hat{x}_2)\right).$$

Since $-\sqrt{2} \leqslant \hat{x}_1 + \hat{x}_2 \leqslant \sqrt{2}$, there exists $\delta_{\max} \geqslant \sqrt{2}/\alpha$ such that for all $\delta \in (-\delta_{\max}, \delta_{\max})$, $\dot{V}(\hat{x}) \leqslant 0$ on S^2. The largest invariant set contained in $\{\hat{x} \in S^2, \dot{V}(\hat{x}) = 0\}$ is then $\{-x_{\mathrm{t}}, x_{\mathrm{t}}\}$. It follows from LaSalle's principle (see e.g. [12, Theorem 2]) that the ω-limit set of any trajectory of system (30) is contained in the set $\{(-x_{\mathrm{t}}, 0), (x_{\mathrm{t}}, 0)\}$. However, from Lemma 3.17, the point $(x_{\mathrm{t}}, 0)$ is locally asymptotically stable. It follows that any trajectory goes either to $(-x_{\mathrm{t}}, 0)$ or to $(x_{\mathrm{t}}, 0)$. Since, from Lemma 3.18, the set of trajectories converging to $(-x_{\mathrm{t}}, 0)$ has zero measure, the Lemma follows. ■

4) Numerical simulations: The theoretical results presented in the previous section have been tested by means of a numerical simulation. The case of system (1) has been treated and the observer was constructed using equation (7). The initial conditions were arbitrarily set to $x_0 = (1, 0, 0)$ and $\hat{x}_0 = (1/\sqrt{2}, 1/\sqrt{2}, 0)$[4]. We took a feedback control in order to make the system observable, that is, we chose a perturbed feedback control of the form (25). The simulations have been made for two different parameters of the perturbation of the stabilizing feedback control: $(\delta, \alpha) = (0.1, 2)$ and $(\delta, \alpha) = (0.1, 6)$. Fig. 4 shows, for both cases, the evolution of the state variables, and Fig. 5 shows the evolution of the observation errors. Notice that the thick aspect of the drawn curves is due to the observation of an oscillating system over a very long time.

In both cases, the simulations show that the observation errors converge to zero, which corroborates Corollary 3.15 as well as the idea that perturbing the stabilizing feedback enables the observer to fulfill his task. It is also clear that the state converges to x_{t}, which is exactly the statement of Theorem 3.19. Notice moreover that the convergence is faster as α increases.

A comparison between the present simulations and those given in Subsection III-B enlightens that the convergence is much slower in the present case. However, the influence of the tuning parameters r_1, r_2, α and δ on the convergence rate has not been explored and a thorough study could be interesting.

IV. Generalizations

In Section III we tested, through a particular example, our idea to perturb a feedback control in order to stabilize system (1) to the target point x_{t} by mean of an output feedback. We believe that this idea is valid for the more general situations exposed below.

[4]The choice of $\hat{x}_0 \in S^2$ is legitimated by the fact that the state of system (1) lies in S^2.

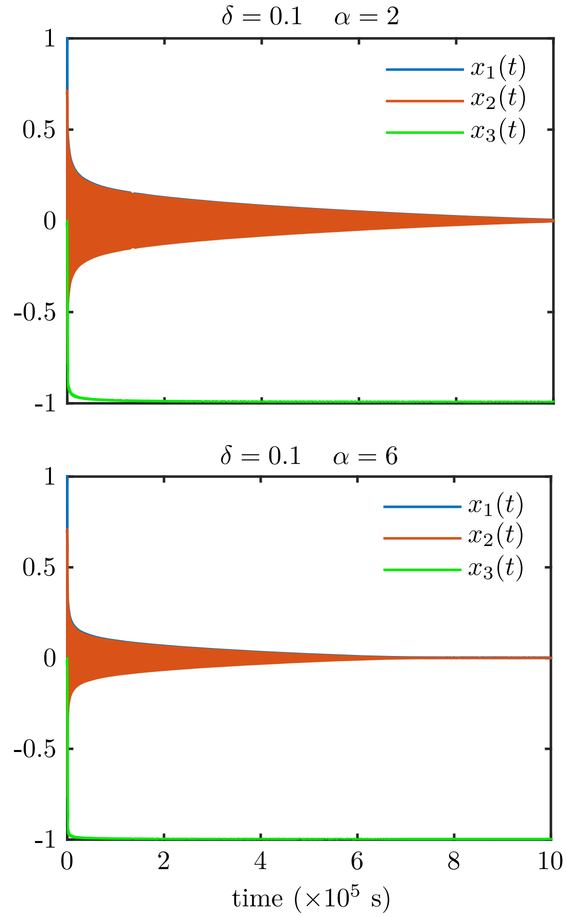

Fig. 4. State variables of system (1) with input $u = \lambda^s_{\delta,\alpha}(\hat{x})$ and for $(\delta, \alpha) = (0.1, 2)$ and $(\delta, \alpha) = (0.1, 6)$.

A. Bilinear skew-symmetric systems

The method exposed in Section III can be extended to all bilinear skew-symmetric systems of the form

$$\begin{cases} \dot{\chi} = A(u)\chi \\ \eta = C\chi, \end{cases} \tag{36}$$

where $\chi \in S^{n-1}$ is the state, $\eta \in \mathbb{R}^p$ is the measured output, $u \in \mathbb{R}^d$ is the control variable, $C \in \mathbb{R}^{1\times n}$ and $A(u)$ is a skew-symmetric matrix affine in u. The goal is to stabilize system (36) at an equilibrium point χ_{eq} whose corresponding control renders the system unobservable. Of course, system (36) may admit other invariant sets.

Using the same Luenberger-like observer as in Section III-A.2, the controller-observer system corresponding to system (36) is then

$$\begin{cases} \dot{\hat{\chi}} = A\left(\lambda^s_\delta(\hat{\chi})\right)\hat{\chi} - C'C\xi \\ \dot{\xi} = \left(A\left(\lambda^s_\delta(\hat{\chi})\right) - C'C\right)\xi, \quad (\hat{\chi}, \xi) \in M, \end{cases} \tag{37}$$

where $\hat{\chi}$ is the state estimate, ξ is the observation error and $M = \{(\hat{\chi}, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \|\hat{\chi} - \xi\| = 1\}$.

Suppose that there exists a state feedback $\lambda^s(\cdot)$ stabilizing system (36) at χ_{eq}. Denote by $\mathcal{A}$ the region of attraction of χ_{eq}. The general steps are the following.

1) Find a (small) perturbation $\tilde{\lambda}(\cdot)$ of $\lambda^s(\cdot)$ such that

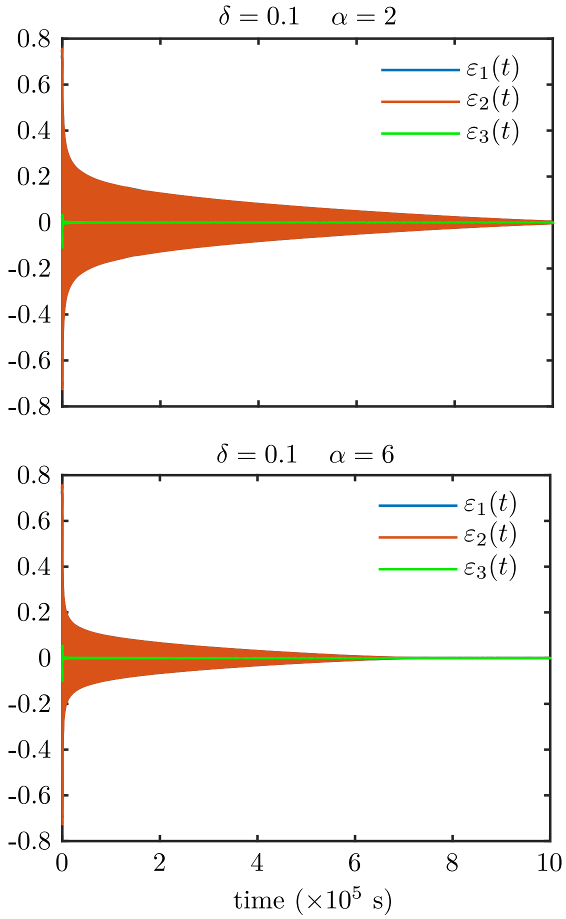

Fig. 5. Observation errors of system (30) for $(\delta, \alpha) = (0.1, 2)$ and $(\delta, \alpha) = (0.1, 6)$.

$\tilde{\lambda}(\cdot)$ used as a state feedback stabilizes system (36) at χ_{eq} and such that the input $\tilde{\lambda}(\hat{\chi}(\cdot))$ makes the observation error go to zero.

2) Prove that the equilibrium point $(\chi_{\mathrm{eq}}, 0)$ is (Lyapunov) stable for system (37) and prove that the invariant subsets of $\mathcal{B} = \{(\hat{\chi}, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \hat{\chi} - \xi \in \mathcal{A}\}$ have zero measure.
3) Using ω-limit arguments, prove that $(-\chi_{\mathrm{eq}}, 0)$ is asymptotically stable for system (37) and that its region of attraction is an open and dense subset of $\mathcal{B}$ that has full measure.

This generalization to bilinear skew-symmetric systems is only the very first step toward a far more general situation described below.

B. Immersion into bilinear skew-adjoint systems

Consider an analytic affine control system on a connected smooth manifold X

$$\begin{cases} \dot{x} = F_0(x) + \displaystyle\sum_{i=1}^{d} u_i F_i(x) \\ y = h(x), \end{cases} \tag{Σ}$$

where $x \in X$ is the state, $y \in \mathbb{R}^p$ is the observation, $u = (u_1, \cdots, u_d) \in \mathbb{R}^d$ is the control variable and the vector fields F_i, $i = 1, \ldots, d$ are complete. The problem is to stabilize system (Σ) at some equilibrium point x_{eq} (supposed to exist).

Set $\mathcal{F} = \{F_0 + \sum_{i=1}^d u_i F_i, u \in \mathbb{R}^d\}$. Denote by $\mathrm{Lie}(\mathcal{F})$ the Lie algebra generated by the family $\mathcal{F}$ and define $\mathrm{Gr}(\mathcal{F}) = \{e^{t_k f_k} \circ \cdots \circ e^{t_1 f_1} \mid t_i \in \mathbb{R}, f_i \in \mathcal{F}, k \in \mathbb{N}\}$ to be the *group of the system*. Also, following [3], let $\mathrm{Obs}(\Sigma)$ denote the observation space of (Σ), i.e. the smallest vector subspace of $C^\infty(X)$ containing the observation function h and closed under the Lie derivation along elements of $\mathcal{F}$.

Suppose that $\mathrm{Lie}(\mathcal{F})$ is a finite dimensional Lie algebra and that system (Σ) is observable and orbit minimal (i.e. $\mathrm{Gr}(\mathcal{F})$ acts transitively on X). Then, according to [3, Theorem 6], system (Σ) can be immersed into a finite dimensional bilinear skew-adjoint system if and only if $\mathrm{Obs}(\Sigma)$ is finite dimensional and a lift of h on $\mathrm{Gr}(\mathcal{F})$ is almost periodic (for precise definitions, see [14]).
There is another possible generalization when $\mathrm{Lie}(\mathcal{F})$ is finite dimensional but $\mathrm{Obs}(\Sigma)$ is not. In this case, if a lift of h on $\mathrm{Gr}(\mathcal{F})$ is a coefficient of a unitary representation of $\mathrm{Gr}(\mathcal{F})$ then system (Σ) can be immersed into an infinite dimensional skew-adjoint bilinear system. Alternatively, if a lift of h on $\mathrm{Gr}(\mathcal{F})$ is not a coefficient of a unitary representation of $\mathrm{Gr}(\mathcal{F})$, it is always possible to approximate the lift of h by a linear combination of pure positive-definite functions (see [14, Corollary 13.6.5]) and immerse the resulting system into an infinite dimensional skew-adjoint bilinear system. Notice that this approximation necessarily yields to practical stabilization and thus to a theorem having the taste of Corollary 3.11.
The generalization exposed in Subsection IV-A to infinite dimensional systems is more involved and its extension will not be exposed here (for an infinite dimensional version of the center manifold theorem, see e.g. [15]).

C. Immersion into bilinear non skew-adjoint systems

If the observation space is finite dimensional, then (Σ) can be immersed into a finite dimensional bilinear (not necessarily skew-adjoint) system (see [16]). For general bilinear control systems, there is another natural generalization of the above situation. The observer is (the deterministic version of) the (time dependent) Kalman filter, i.e.

$$\begin{cases} \dot{S} = -A'\left(u(\hat{x})\right) S - SA\left(u(\hat{x})\right) + C'R^{-1}C - SQS \\ \dot{\hat{x}} = A\left(u(\hat{x})\right)\hat{x} - S^{-1}C'C\varepsilon, \end{cases}$$

for some positive definite matrices R and Q.

Again, since S is bounded from below by the observability index, the same scheme of reasoning could be applied. Notice that the dimension of the observer, although finite, can be very big. A natural question then arises: if the observation space is finite dimensional and a lift of h on $\mathrm{Gr}(\mathcal{F})$ is a coefficient of a unitary representation of $\mathrm{Gr}(\mathcal{F})$, is it better to immerse (Σ) in an infinite dimensional bilinear skew-adjoint system or in a finite dimensional bilinear non skew-adjoint one ?

V. Conclusion and remarks

In this paper, we have analysed via an example the problem of output feedback stabilization when the point where we want to stabilize the system to corresponds to a control value that makes it unobservable.

Our main result (Theorem 3.10) show that adding a small well-chosen perturbation to the stabilizing feedback control allows to asymptotically stabilize the system to the target point.

We do believe that although the example treated in Section III satisfies the hypothesis of null-observability introduced by Coron (see [5, Definition 2.3] for the precise definition) which is necessary (for analytic systems) and sufficient for local stabilizability in small time by means of a continuous dynamic periodic time-varying output feedback law, our result should apply to a wider class of systems.

References

[1] J.-P. Gauthier and I. Kupka, *Deterministic observation theory and applications*. Cambridge University Press, Cambridge, 2001.

[2] V. Andrieu and L. Praly, "A unifying point of view on output feedback designs for global asymptotic stabilization," *Automatica J. IFAC*, vol. 45, no. 8, pp. 1789–1798, 2009. [Online]. Available: http://dx.doi.org.docelec.univ-lyon1.fr/10.1016/j.automatica.2009.04.015

[3] F. Celle, J.-P. Gauthier, D. Kazakos, and G. Sallet, "Synthesis of nonlinear observers: a harmonic-analysis approach," *Math. Systems Theory*, vol. 22, no. 4, pp. 291–322, 1989.

[4] G. Besançon and H. Hammouri, "Some remarks on dynamic output feedback control of non uniformly observable systems," in *Decision and Control, 2000. Proceedings of the 39th IEEE Conference on*, vol. 3, 2000, pp. 2462–2465 vol.3.

[5] J.-M. Coron, "On the stabilization of controllable and observable systems by an output feedback law," *Math. Control Signals Systems*, vol. 7, no. 3, pp. 187–216, 1994.

[6] B. Yang and W. Lin, *Emergent Problems in Nonlinear Systems and Control*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009, ch. Semi-global Output Feedback Stabilization of a Class of Non-uniformly Observable and Non-smoothly Stabilizable Systems, pp. 253–283.

[7] M. A. Lagache, U. Serres, and J. P. Gauthier, "Output stabilization at unobservable points: Analysis via an example," in *2016 IEEE Conference on Control Applications (CCA)*, Sept 2016, pp. 730–735.

[8] U. Boscain and R. Long, "Time minimal trajectories for two-level quantum systems with two bounded controls," in *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, Dec 2012, pp. 3626–3631.

[9] U. Boscain, J.-P. Gauthier, F. Rossi, and M. Sigalotti, "Approximate controllability, exact controllability, and conical eigenvalue intersections for quantum mechanical systems," *Comm. Math. Phys.*, vol. 333, no. 3, pp. 1225–1239, 2015.

[10] U. Boscain, F. Grönberg, R. Long, and H. Rabitz, "Publisher's Note: "Minimal time trajectories for two-level quantum systems with two bounded controls" [J. Math. Phys. 55, 062106 (2014)] [MR3390654]," *J. Math. Phys.*, vol. 55, no. 8, pp. 089 901, 1, 2014.

[11] A. Kelley, "The stable, center-stable, center, center-unstable, unstable manifolds," *J. Differential Equations*, vol. 3, pp. 546–570, 1967.

[12] J. P. LaSalle, "An invariance principle in the theory of stability," in *Differential Equations and Dynamical Systems (Proc. Internat. Sympos., Mayaguez, P.R., 1965)*. Academic Press, New York, 1967, pp. 277–286.

[13] J. Carr, *Applications of centre manifold theory*, ser. Applied Mathematical Sciences. Springer-Verlag, New York-Berlin, 1981, vol. 35.

[14] J. Dixmier, *Les C^*-algèbres et leurs représentations*, ser. Deuxième édition. Cahiers Scientifiques, Fasc. XXIX. Gauthier-Villars Éditeur, Paris, 1969.

[15] M. Haragus and G. Iooss, *Local bifurcations, center manifolds, and normal forms in infinite-dimensional dynamical systems*, ser. Universitext. Springer-Verlag London, Ltd., London; EDP Sciences, Les Ulis, 2011. [Online]. Available: http://dx.doi.org.docelec.univ-lyon1.fr/10.1007/978-0-85729-112-7

[16] M. Fliess and I. Kupka, "A finiteness criterion for nonlinear input-output differential systems," *SIAM J. Control Optim.*, vol. 21, no. 5, pp. 721–728, 1983. [Online]. Available: http://dx.doi.org.docelec.univ-lyon1.fr/10.1137/0321044

Marc-Aurèle Lagache received his master's degree in control theory and image processing from the Ecole Normale Supérieur de Cachan (France) in 2014. He is currently a Ph.D. student working in control theory at the University of Toulon (France), in the LSIS laboratory in partnership with the LAGEP laboratory (Lyon, France). His fields of study are optimal control, inverse optimal control and output feedback stabilization.

Ulysse Serres received the Ph.D. degree in mathematics from the Université de Bourgogne, Dijon, France, in 2006. Since 2009, he has been with the Department of Electrical and Chemical Engineering at the Université Claude Bernard Lyon 1 and with the LAGEP Laboratory, Lyon, France, where he is an Assistant Professor. His current research interests include geometric optimal control, switching systems, output feedback stabilization problems and (more recently) sub-Riemannian geometry.

Jean-Paul Gauthier was born in 1952. He is currently a Professor at the University of Toulon (France) in the Dpt of Electrical Engineering. He got his Ph.D. in physics in 1982. He got the medal of "Institut Universitaire de France" in 1992. He was a member of this institute from 1992 to 97, and is now honorary member. He got a Featured review of the American Mathematical Society in 2002, for his work on the subanalyticity of Carnot–Caratheodory distances. His current fields of interest are Automatic Control, Robotics, Signal and Image Processing, and Deterministic Observation Theory (he wrote a reference book at Cambridge University Press in 2001 on this last topic). Besides his academic activities, he is a member of "Federation Française de Go", "Ligue des Libres penseurs" and a strong supporter of "Toulon Rugby Club".