Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org
pp. **X–XX**

ON THE MOTION PLANNING OF THE BALL WITH A TRAILER

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(Communicated by the associate editor name)

Abstract. This paper is about motion planing for kinematic systems, and more particularly ϵ-approximations of non-admissible trajectories by admissible ones. This is done in a certain optimal sense.

The resolution of this motion planing problem is showcased through the thorough treatment of the *ball with a trailer* kinematic system, which is a non-holonomic system with flag of type $(2, 3, 5, 6)$.

1. **Introduction.** This article deals with motion planning for kinematic systems. In particular, we are interested in the *ball with a trailer, rolling on a plane*, associated with the following problem. A non-admissible path is specified in the configuration space, and we want the system to follow it as closely as possible. This is done in a certain optimal sense, detailed later in the exposition. We follow the methodology developed in the series of articles [9, 10, 11, 12, 13, 14, 15]. The interested reader is also invited to have a look at the seminal works [16, 17, 18, 19, 21, 24, 25]. In particular, in the present exposition the reader will find all the details and proofs that were left out of our previous paper [5].

The ball with a trailer is a follow up to the ball-plate problem (see [4, 6, 20]), and corresponds to the kinematic situation where a ball is rolling without slipping on a plane while pulling a trailer. As shown in Figure 1, it is described by:

1. the (x, y) position of the contact point between the ball and the plane,
2. the orientation of a frame attached to the center of the ball, given under the guise of a right orthonormal matrix $R \in SO(3, \mathbb{R})$,
3. the angle θ which provides the position of the trailer with respect to the ball (the length of the line used to tow the trailer is denoted by L).

2000 *Mathematics Subject Classification.* Primary: 58F15, 58F17; Secondary: 53C35.

Key words and phrases. ball with a trailer, motion planing, subriemannian geometry, robotics, optimal control.

J-P Gauthier is also with INRIA team GECO. This work is partly supported by ANR blanc GCM.

The corresponding kinematic equations are specified by means of a control system, linear in the controls:

$$\begin{aligned} \dot{x} &= u_1, \\ \dot{y} &= u_2, \\ \dot{R} &= \begin{bmatrix} 0 & 0 & u_1 \\ 0 & 0 & u_2 \\ -u_1 & -u_2 & 0 \end{bmatrix} R, \\ \dot{\theta} &= -\frac{1}{L}(\cos(\theta)u_1 + \sin(\theta)u_2). \end{aligned} \tag{1}$$

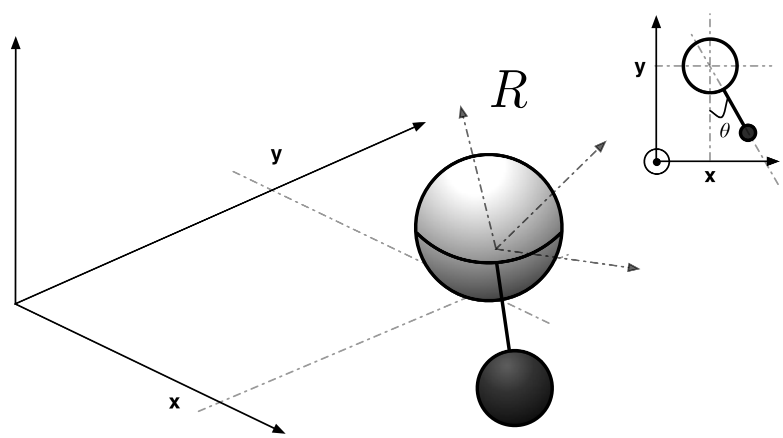


FIGURE 1. The ball with a trailer, rolling on a plane

This control system also writes $\dot{X} = F_1(X)u_1 + F_2(X)u_2$, where

- X belongs to the 6-dimensional manifold $M = \mathbb{R}^2 \times SO(3, \mathbb{R}) \times S^1$,
- $F_1 = \frac{\partial}{\partial x} + A_1 - \frac{1}{L}\cos(\theta)\frac{\partial}{\partial \theta}$, and $F_2 = \frac{\partial}{\partial y} + A_2 - \frac{1}{L}\sin(\theta)\frac{\partial}{\partial \theta}$ are smooth (C^∞) vector fields over M.

Therefore, F_1 and F_2 span a rank 2 distribution Δ over M. Henceforth, we denote $\Delta^1 = \Delta$, $\Delta^2 = [\Delta, \Delta]$, etc. The computation rules of the right invariant vector fields A_1, A_2 over $SO(3, \mathbb{R})$ are: $[A_1, A_2] = A_3$, $[A_1, A_3] = -A_2$, and $[A_2, A_3] = A_1$. Note that our convention for Lie bracket computations is: $[F_1, F_2] = \frac{\partial F_1}{\partial X}F_2 - \frac{\partial F_2}{\partial X}F_1$.

Let us now compute the Lie algebra generated by F_1 and F_2:

- $H = [F_1, F_2] = A_3 - \frac{1}{L^2}\frac{\partial}{\partial \theta}$, and $\dim(\Delta^2) = 3$,
- $I = [F_1, H] = -A_2 - \frac{1}{L^3}\sin(\theta)\frac{\partial}{\partial \theta}$, $J = [F_2, H] = A_1 + \frac{1}{L^3}\cos(\theta)\frac{\partial}{\partial \theta}$, and $\dim(\Delta^3) = 5$,
- $[F_1, I] = [F_2, J] = -A_3 - \frac{1}{L^4}\frac{\partial}{\partial \theta}$, $[F_1, J] = [F_2, I] = 0$, and $\dim(\Delta^4) = 6$.

Hence, the flag of distributions of System (1) is of type $(2, 3, 5, 6)$, Δ is *completely non-integrable*, and any smooth finite path $\Gamma : [0, T] \to M$ can be approximated by an admissible path $\gamma : [0, \tau] \to M$. Since we are dealing with a local problem in a neighborhood of Γ, M is identified with $\mathbb{R}^6$. Also, along the paper, we are interested in *generic* problems only, see [5] for details. In particular, it means that the curve Γ is always transversal to Δ.

In order to perform this approximation, it is natural to try to minimize a cost of the form:

$$J(u) = \int_0^\tau \sqrt{u_1^2 + u_2^2}\, dt.$$

This choice is motivated by several reasons:

1. the optimal curves do not depend on their parametrization,
2. the minimization of such a cost produces a metric space (the associated distance is called the subriemannian distance, or the Carnot-Caratheodory distance),
3. minimizing such a cost is equivalent to minimize the following quadratic cost, denoted $J_E(u)$ and called the *energy* of the path, in fixed time θ:

$$J_E(u) = \int_0^\tau \left(u_1^2 + u_2^2\right) dt.$$

Another way to interpret this problem is to consider the dynamics as being specified by the rank 2 distribution Δ (i.e. not by the vector fields F_i, but their span only). The cost is then determined by an Euclidean metric g over Δ, specified here by the fact that F_1 and F_2 form an orthonormal frame field for the metric.

The distance between two points, which is denoted by d, is defined as the minimum length of admissible curves connecting these two points. The length of the admissible curve corresponding to the control $u : [0, \theta] \to M$ is simply $J(u)$.

For a small parameter $\epsilon > 0$, we are searching for approximating trajectories that lie ϵ-close to Γ. As such, we need the following notions.

Definition 1.1.
1. We say that two quantities $f(\epsilon)$ and $g(\epsilon)$ *are equivalent* ($f \simeq g$) if $\lim_{\epsilon\to 0} \frac{f(\epsilon)}{g(\epsilon)} = 1$. The quantities $MC(\epsilon)$ and $E(\epsilon)$ below tend to $+\infty$ when $\epsilon \to 0$. As such they are considered up to equivalence.
2. $T_\varepsilon = \{x \in M \mid d(x, \Gamma) \leq \varepsilon\}$ is the *subriemannian tube* around Γ.
3. $C_\varepsilon = \{x \in M \mid d(x, \Gamma) = \varepsilon\}$ is the *subriemannian cylinder* around Γ.
4. The *Metric Complexity* $MC(\epsilon)$ is $\frac{1}{\epsilon}$ times the minimum length of an admissible curve γ_ϵ connecting the endpoints $\Gamma(0)$, and $\Gamma(T)$ of Γ, and remaining in T_ϵ.
5. The *Interpolation Entropy* $E(\epsilon)$ is $\frac{1}{\epsilon}$ time the minimum length of an admissible curve γ_ϵ connecting $\Gamma(0)$, and $\Gamma(T)$ such that in any segment of γ_ϵ of length larger than ϵ, there is a point of Γ (i.e. Γ is ϵ-interpolated).
6. An *Asymptotic Optimal Synthesis* is a one-parameter family γ_ϵ of admissible curves, that realizes the metric complexity or the entropy.

In the remainder of this article, we develop an asymptotic optimal synthesis for the ball with a trailer. In Section 2 we derive the normal form for a kinematic system with a flag of distribution of the type $(2, 3, 5, 6)$. We also present the form that is specific to the ball with a trailer system, and finally we define and display the corresponding nilpotent approximation. In Section 3, the invariants of the problems are discussed. Finally, in Section 4, the asymptotic optimal synthesis is developed. In the concluding Section 5, we use our optimal synthesis to solve the parking problem for the ball with a trailer. Finally, an appendix section that contains technical results removed from the body of text for clarity reasons, closes the paper.

2. **Normal Form.** Let us first consider a 4-dimensional **parametrized** surface S, transversal to Δ:

$$S = \{q(s_1, s_2, s_3, t) \in \mathbb{R}^6\}, \text{ with } q(0,0,0,t) = \Gamma(t). \tag{2}$$

Remind that Γ denotes a non-admissible curve non-tangent to Δ, thus, it is always possible to find such a surface S. Actually, since we want to remain ϵ-close to Γ, S may be a *germ* only defined in a neighborhood of Γ.

Second, we pick a "normal coordinate" system $\xi = (x, y, w)$ as it is done in the lemma below[1] (see e.g. [1, 2, 13, 14]):

Lemma 2.1. *(Normal coordinates with respect to S).*

There are mappings $x : \mathbb{R}^6 \to \mathbb{R}^2$, $y : \mathbb{R}^6 \to \mathbb{R}^3$, $w : \mathbb{R}^6 \to \mathbb{R}$, such that $\xi = (x, y, w)$ is a coordinate system on some neighborhood of S in $\mathbb{R}^6$, such that:

1. *$q(y, w) = (0, y, w)$, $\Gamma = \{(0, 0, w)\}$,*
2. *the restriction $\Delta_{|S} = \ker dw \cap_{i=1,..,3} \ker dy_i$, the metric $g_{|S} = (dx_1)^2 + (dx_2)^2$,*
3. *$\mathcal{C}_\varepsilon^S = \{\xi | x_1^2 + x_2^2 = \varepsilon^2\}$, where $\mathcal{C}_\varepsilon^S$ is the ε-cylinder around S,*
4. *the geodesics of the Pontryagin's maximum principle [22] meeting the transversality conditions w.r.t. S are the straight lines through S, contained in the planes $P_{y_0,w_0} = \{\xi | (y, w) = (y_0, w_0)\}$. Hence, they are orthogonal to S.*

These normal coordinates are unique up to changes of coordinates of the form

$$\begin{array}{cl} \tilde{x} & = T(y,w)x, \\ (\tilde{y}, \tilde{w}) & = (y, w), \end{array} \tag{3}$$

where $T(y, w) \in O(2)$, the orthogonal group over $\mathbb{R}^2$.

As a third step, in these normal coordinates and on the basis of the general normal form for kinematic systems in the 2-control case (see the proof below), we establish the normal form (4) of Theorem 2.2.

Theorem 2.2. *(Normal Form in the Generic $6-2$ case) There is a change of coordinates such that a trajectory of System (1) satisfies the following system of equations on the tube T_ϵ:*

$$\begin{aligned} \dot{x}_1 &= u_1 + O(\varepsilon^3), \\ \dot{x}_2 &= u_2 + O(\varepsilon^3), \\ \dot{y} &= (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2) + O(\varepsilon^2), \\ \dot{z}_1 &= x_2(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2) + O(\varepsilon^3), \\ \dot{z}_2 &= x_1(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2) + O(\varepsilon^3), \\ \dot{w} &= Q_w(x_1, x_2)(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2) + O(\varepsilon^4), \end{aligned} \tag{4}$$

where $Q_w(x_1, x_2)$ is a quadratic form in x depending smoothly on w.

Proof. Consider normal coordinates with respect to any surface $\mathcal{S}$, as in Lemma 2.1, and let the triple (y_1, y_2, y_3) be denoted by $\bar{y}$. There are smooth functions,

[1] $\mathcal{C}_\varepsilon^S$ denotes the cylinder $\{\xi;\, d(S, \xi) = \varepsilon\}$, and $S(y, w)$ is a short notation for the surface (2).

$\beta(x,\bar{y},w), \gamma_i(x,\bar{y},w), \delta(x,\bar{y},w)$, such that, on a neighborhood of Γ, System (1) can be written as:

$$\begin{aligned} \dot{x}_1 &= \left(1 + x_2^2\beta(x,\bar{y},w)\right) u_1 - x_1x_2\beta(x,\bar{y},w)u_2, \\ \dot{x}_2 &= \left(1 + x_1^2\beta(x,\bar{y},w)\right) u_2 - x_1x_2\beta(x,\bar{y},w)u_1, \\ \dot{y}_i &= \gamma_i(x,\bar{y},w)\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \\ \dot{w} &= \delta(x,\bar{y},w)\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \end{aligned} \tag{5}$$

where, moreover, $\beta(x,\bar{y},w)$ vanishes on Γ. This result has been proven first in [1] for the corank 1 case only. However, it still holds for any corank, see [2].

Let us now perform a series of changes of coordinates in $\mathcal{S}$, on the tube T_ε, such that the fact that $\Gamma(t) = (0,...,0,t)$ is always preserved.

Since x has order 1 (cf. Lemma 2.1), and $\beta_{|\Gamma} = 0$, we have on T_ε: $\dot{x}_i = u_i + O(\varepsilon^3)$ for $i \in \{1,2\}$. One of the γ_i's (say γ_1) has to be nonzero for Γ not to be tangent to Δ^2. Since $\gamma_{1|\Gamma} = \gamma_1(0,0,w) = \gamma_1(w)$, then, $\dot{y}_1$ writes $(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2)\gamma_1(w) + O(\varepsilon^2)$ on T_ε, and y_1 has order 2.

For $i = 2,3$, we now set $\tilde{y}_i = y_i - \frac{\gamma_i}{\gamma_1}y_1$. The differentiation gives $\frac{d\tilde{y}_i}{dt} = \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right)L_i(w).x + O(\varepsilon^3)$, where $L_i(w).x$ denotes a linear map w.r.t. x. The $\tilde{y}_i$'s have both order 3. We set $y = y_1$, and $z_1 = \tilde{y}_2$, $z_2 = \tilde{y}_3$. We also set $w = w - \frac{\delta}{\gamma_1}y_1$. Up to now, we achieved the form:

$$\begin{aligned} \dot{x} &= u + O(\varepsilon^3), \\ \dot{y} &= (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2)\gamma_1(w) + O(\varepsilon^2), \\ \dot{z}_i &= (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2)L_i(w).x + O(\varepsilon^3),\ i = 1,2 \\ \dot{w} &= (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2)\delta(w).x + O(\varepsilon^3), \end{aligned}$$

where $L_i(w).x$, and $\delta(w).x$ are linear in x. The function $\gamma_1(w)$ can be put to 1 by setting $\tilde{y} = \frac{y}{\gamma_1(w)}$.

Now let $T(w)$ be an invertible 2×2 matrix, and set $\tilde{z} = T(w)z$. It is easy to see that we can choose $T(w)$ such that:

$$\begin{aligned} \dot{x} &= u + O(\varepsilon^3), \\ \dot{\tilde{y}} &= (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2) + O(\varepsilon^2), \\ \dot{\tilde{z}}_i &= (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2)x_i + O(\varepsilon^3),\ i = 1,2 \\ \dot{w} &= (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2)\delta(w).x + O(\varepsilon^3). \end{aligned}$$

Next, we perform a change of the form $\tilde{w} = w + L(w).\tilde{z}$, where $L(w).\tilde{z}$ is linear in $\tilde{z}$, and chosen such as to kill $\delta(w)$. It yields $\dot{\tilde{w}} = (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2)O(\varepsilon^2)$. We simplify the notations by replacing the symbols $\tilde{y}$ and $\tilde{z}_i$ by y and z_i.

The $O(\varepsilon^2)$ that appears in the above equation of $\tilde{w}$, has to be of the form $Q_{\tilde{w}}(x) + h(\tilde{w})y + O(\varepsilon^3)$, where $Q_{\tilde{w}}(x)$ is quadratic in x. If we kill $h(\tilde{w})$, we get the expected result. This is done with a change of coordinates of the form: $w = \tilde{w} + \varphi(\tilde{w})\frac{y^2}{2}$. □

Remark 1. Note that all the changes of coordinates under consideration in the previous proof preserve the fact that coordinates are "normal coordinates" w.r.t. the original surface: mostly, these changes are changes of parametrization of the surface S.

Definition 2.3.
1. According to Normal form (4), we say that x_1 and x_2 *have weight* 1, y has weight 2, z_1 and z_2 have weight 3, and w has weight 4. Therefore, the vector fields $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ have weight -1, $\frac{\partial}{\partial y}$ has weight -2, and so on.
2. The *nilpotent approximation* of a kinematic system in the (generic) $6-2$ case is obtained from System (4) by keeping all the terms of order -1 only:
$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{y} &= (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2), \\ \dot{z}_1 &= x_2(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2), \\ \dot{z}_2 &= x_1(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2), \\ \dot{w} &= Q_w(x_1,x_2)(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2). \end{aligned} \tag{6}$$
3. Given a one-parameter family of (absolutely continuous, arclength parametrized) admissible curves $\gamma_\varepsilon : [0, T_{\gamma_\varepsilon}] \to \mathbb{R}^6$, **an ε-modification of** γ_ε is another one-parameter family of (absolutely continuous, arclength parametrized) admissible curves $\tilde{\gamma}_\varepsilon : [0, T_{\tilde{\gamma}_\varepsilon}] \to \mathbb{R}^6$ such that for all ε and for some $\alpha > 0$, if $[0, T_{\gamma_\varepsilon}]$ is split into subintervals of length ε (i.e. $[0,\varepsilon]$, $[\varepsilon, 2\varepsilon]$, $[2\varepsilon, 3\varepsilon]$, ...), then:
   (a) $[0, T_{\tilde{\gamma}_\varepsilon}]$ is split into corresponding intervals, $[0,\varepsilon_1]$, $[\varepsilon_1, \varepsilon_1+\varepsilon_2]$, $[\varepsilon_1+\varepsilon_2, \varepsilon_1+\varepsilon_2+\varepsilon_3], \ldots$ with $\varepsilon \le \varepsilon_i < \varepsilon(1+\varepsilon^\alpha)$, $i = 1, 2, \ldots$,
   (b) for each couple of an interval $I_1 = [\tilde{\varepsilon}_i, \tilde{\varepsilon}_i + \varepsilon]$, (with $\tilde{\varepsilon}_0 = 0$, $\tilde{\varepsilon}_1 = \varepsilon_1$, $\tilde{\varepsilon}_2 = \varepsilon_1 + \varepsilon_2, \ldots$) and the respective interval $I_2 = [i\varepsilon, (i+1)\varepsilon]$, $\frac{d}{dt}(\tilde{\gamma})$ and $\frac{d}{dt}(\gamma)$ coincide over I_2, i.e.:
$$\frac{d}{dt}(\tilde{\gamma})(\tilde{\varepsilon}_i + t) = \frac{d}{dt}(\gamma)(i\varepsilon + t), \text{ for almost all } t \in [i\varepsilon, (i+1)\varepsilon].$$

Theorem 2.4. *Consider a kinematic system with a flag of the form* $(2, 3, 5, 6)$, *without singularities. An asymptotic optimal synthesis (relative to the entropy) for System (4) is obtained as an ε-modification of an asymptotic optimal synthesis for the Nilpotent Approximation (6). As a consequence the entropy $E(\varepsilon)$ of System (4) is equal to the entropy $\hat{E}(\varepsilon)$ of System (6).*

The proof of this theorem can be found in [13].

3. **Invariants.** Let us consider a one-form ω that vanishes on Δ^3, and set $\alpha = d\omega_{|\Delta}$, the restriction of $d\omega$ to Δ. As in Section 1, we denote $H = [F_1, F_2]$, $I = [F_1, H]$, $J = [F_2, H]$. Let us now consider the following 2×2 matrix:
$$A(\xi) = \begin{pmatrix} d\omega(F_1, I) & d\omega(F_2, I) \\ d\omega(F_1, J) & d\omega(F_2, J) \end{pmatrix},$$

where $\xi = (x, y, z, w)$. In restriction[2] to Δ^3, $\omega([X,Y]) = d\omega(X,Y)$, which yields:

$$A(\xi) = \begin{pmatrix} \omega([F_1, I]) & \omega([F_2, I]) \\ \omega([F_1, J]) & \omega([F_2, J]) \end{pmatrix}.$$

Due to Jacobi Identity, $A(\xi)$ is a symmetric matrix. Let us now consider a gauge transformation, i.e. a feedback that preserves the metric, see e.g. [7], i.e. a change of orthonormal frame (F_1, F_2) obtained by setting

$$\tilde{F}_1 = \cos(\theta(\xi))F_1 + \sin(\theta(\xi))F_2\ ,$$
$$\tilde{F}_2 = -\sin(\theta(\xi))F_1 + \cos(\theta(\xi))F_2\ .$$

It is just a matter of tedious computations to check that the matrix $A(\xi)$ is changed for $\tilde{A}(\xi) = R_\theta A(\xi) R_{-\theta}$, where R_θ stands for the rotation of angle θ. On the other hand, the one-form ω is defined modulo multiplication by a nonzero function $f(\xi)$, and the same holds for α, since $d(f\omega) = fd\omega + df \wedge \omega$, and ω vanishes over Δ^3. Therefore the following lemma holds true:

Lemma 3.1. *The ratio $r(\xi)$ of the (real) eigenvalues of $A(\xi)$ is an invariant of the structure.*

Let us now consider the Normal form (4), and compute the form $\omega = \omega_1 dx_1 + ... + \omega_6 dw$ along Γ (that is, where $x, y, z = 0$). The computation of all the brackets shows that $\omega_1 = \omega_2 = ... = \omega_5 = 0$. This also shows that in fact, along Γ, $A(\xi)$ is just the matrix of the quadratic form Q_w.

Lemma 3.2. *The invariant $r(\Gamma(t))$ of the problem that consists of System (1) and the curve Γ, is the same as the invariant $\hat{r}(\Gamma(t))$ of the Nilpotent Approximation (6) along Γ.*

Let us now compute the ratio r for the ball with a trailer. The computations of Section 1 give:

$$F_1 = \frac{\partial}{\partial x_1} + A_1 - \frac{1}{L}\cos(\theta)\frac{\partial}{\partial \theta}, \qquad F_2 = \frac{\partial}{\partial x_2} + A_2 - \frac{1}{L}\sin(\theta)\frac{\partial}{\partial \theta},$$
$$H = A_3 - \frac{1}{L^2}\frac{\partial}{\partial \theta},$$
$$I = -A_2 - \frac{1}{L^3}\sin(\theta)\frac{\partial}{\partial \theta}, \qquad J = A_1 + \frac{1}{L^3}\cos(\theta)\frac{\partial}{\partial \theta},$$
$$[F_1, I] = [F_2, J] = -A_3 - \frac{1}{L^4}\frac{\partial}{\partial \theta}, \qquad [F_1, J] = [F_2, I] = 0.$$

Lemma 3.3. *For the ball with a trailer, the ratio $r(\xi) = 1$.*

The lemmas obtained in the present section are a key point in the developments of Section 4. In particular they imply, as we shall prove, that the system of geodesics of the nilpotent approximation is integrable in Liouville sense.

4. **Optimal synthesis.** We start by using Theorem 2.4, to reduce to the nilpotent approximation along Γ given in Equation (6). According to Lemma 3.3, we can consider that

$$Q_w(x_1, x_2) = \delta(w)\left((x_1)^2 + (x_2)^2\right) \tag{7}$$

[2]We recall the classical relation $d\omega(X,Y) = \omega([X,Y]) + L_X(\omega(Y)) - L_Y(\omega(X))$.

where $\delta(w)$ is **the main invariant**. In fact, it is the only invariant for the nilpotent approximation along Γ. Moreover, if we reparametrize Γ by setting $dw = \frac{dw}{4.\delta(w)}$, we can consider that $\delta(w) = 1/4$. This new system, denoted by $\dot{\xi} = Fu_1 + Gu_2$, is:

$$\begin{aligned}
\dot{x}_1 &= u_1, \\
\dot{x}_2 &= u_2, \\
\dot{y} &= (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2), \\
\dot{z}_1 &= x_2(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2), \\
\dot{z}_2 &= x_1(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2), \\
\dot{w} &= \frac{1}{4}\left(x_1^2 + x_2^2\right)(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2).
\end{aligned} \tag{8}$$

Next in order to compute the interpolation entropy, we need to maximize $\int \dot{w}dt$ in fixed time ε, subject to the interpolation conditions: $x(0) = 0$, $y(0) = 0$, $z(0) = 0$, $w(0) = 0$, $x(\varepsilon) = 0$, $y(\varepsilon) = 0$, $z(\varepsilon) = 0$.

The following lemma is crucial for our final result. The basic idea for the proof has been given to us by Andrei Agrachev.

Lemma 4.1. *Let us denote $\xi = (x, y, z, w) = (\varsigma, w)$. The trajectories of (8) that maximize $\int \dot{w}dt$ in fixed time ε, with interpolating conditions $\varsigma(0) = \varsigma(\varepsilon) = 0$, have a periodic projection over ς (i.e. $\varsigma(t)$ is smooth and periodic of period ε).*

Proof. The proof uses the transversality conditions of the Pontryaguin maximum principle in the case of mixed boundary conditions.

First, we need to work on the structure of System (8): it is a right invariant system on $\mathbb{R}^6$ with coordinates $\xi = (\varsigma, w) = (x, y, z, w)$, for a certain nilpotent Lie group structure over $\mathbb{R}^6$ (denoted by G). The group law is of the form $(\varsigma_2, w_2)(\varsigma_1, w_1) = (\varsigma_1 * \varsigma_2, w_1 + w_2 + \Phi(\varsigma_1, \varsigma_2))$, for a certain function Φ and where $*$ is the multiplication of another Lie group structure over $\mathbb{R}^5$, with coordinates ς (denoted by G_0).

We propose a proof of this claim under the guise of Lemma 6.1 in the appendix: the group laws in the $(2, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$ cases are already known, and given in [8], but we could not find an explicit computation in the case $(2, 3, 5, 6)$ in the litterature.

As a second step, let $(\varsigma, w_1), (\varsigma, w_2)$ be initial and terminal points of an optimal solution of our problem with the relaxed boundary conditions $\varsigma(0) = \varsigma(\epsilon)$ only. A right translation by $(\varsigma^{-1}, 0)$, maps this trajectory into another trajectory of the system, with initial and terminal points $(0, w_1 + \Phi(\varsigma, \varsigma^{-1}))$ and $(0, w_2 + \Phi(\varsigma, \varsigma^{-1}))$. The cost $\int \dot{w}(t)dt$ for this new trajectory has the same value. Actually, as one can see, the optimal cost is independent of the ς-coordinate of the initial and terminal conditions.

Therefore, our problem is the same as maximizing $\int \dot{w}(t)dt$ with the (larger) endpoint condition $\varsigma(0) = \varsigma(\varepsilon)$ (free).

We can now apply the general transversality conditions of Theorem 12.15, page 188 of [3]. It tells us that the initial and terminal covectors (p^1_ς, p^1_w) and (p^2_ς, p^2_w) are such that $p^1_\varsigma = p^2_\varsigma$. This is enough to show periodicity. □

Let us observe that the optimal trajectory must also be a length minimizer, then we can consider the usual Hamiltonian for length. It is easy to see that the abnormal extremals do not come into the picture, since they cannot be optimal due to the additional interpolation conditions. This observation leaves us with the normal case, where the Hamiltonian is $H = \frac{1}{2}((PF)^2 + (PG)^2)$, where

- $P = (p_1, ..., p_6)$ is the adjoint vector,
- $PF = u_1$, and $PG = u_2$.

In fact, we will show that **the Hamiltonian system corresponding to the Hamiltonian H is integrable**. Note that this fact holds for the ball with a trailer only.

As usual, we work in Poincaré coordinates, i.e. we consider level $\frac{1}{2}$ of the Hamiltonian H, and we set:

$$PF = \sin(\varphi), \quad PG = \cos(\varphi).$$

Differentiating twice, we get

$$\dot{\varphi} = P[F, G], \text{ and } \ddot{\varphi} = -PFFG.PF - PGFG.PG,$$

where $FFG = [F, [F, G]]$ and $GFG = [G, [F, G]]$. We set $\lambda = -PFFG$, $\mu = -PGFG$, and we get the equation:

$$\ddot{\varphi} = \lambda \sin(\varphi) + \mu \cos(\varphi). \tag{9}$$

Now, we compute $\dot{\lambda}$ and $\dot{\mu}$. We get, with similar notations as above for the brackets[3]:

$$\begin{aligned} \dot{\lambda} &= PFFFG.PF + PGFFG.PG, \\ \dot{\mu} &= PFGFG.PF + PGGFG.PG, \end{aligned}$$

and computing the brackets, we see that $GFFG = FGFG = 0$. Also, since the Hamiltonian does not depend on y, z, w, we get that p_3, p_4, p_5, and p_6 are constants. Computing the brackets FFG and GFG , we get that

$$\lambda = \frac{3}{2}p_5 + p_6 x_1, \quad \mu = \frac{3}{2}p_4 + p_6 x_2, \tag{10}$$

and then, $\dot{\lambda} = p_6 \sin(\varphi)$ and $\dot{\mu} = p_6 \cos(\varphi)$. Then, by (9), $\ddot{\varphi} = \frac{\lambda\dot{\lambda}}{p_6} + \frac{\mu\dot{\mu}}{p_6}$, and finally:

$$\begin{aligned} \dot{x}_1 &= \sin(\varphi), \\ \dot{x}_2 &= \cos(\varphi), \\ \dot{\varphi} &= K + \frac{1}{2p_6}(\lambda^2 + \mu^2), \\ \dot{\lambda} &= p_6 \sin(\varphi), \\ \dot{\mu} &= p_6 \cos(\varphi). \end{aligned} \tag{11}$$

We can normalize p_6 to 1 by a change of coordinates and time reparametrization, thus yielding:

[3] i.e. $FFFG = [F, [F, [F, G]]]$

$$\begin{aligned} \dot{x}_1 &= \sin(\varphi), \\ \dot{x}_2 &= \cos(\varphi), \\ \dot{\varphi} &= K + \frac{1}{2}(\lambda^2 + \mu^2), \\ \dot{\lambda} &= \sin(\varphi), \\ \dot{\mu} &= \cos(\varphi). \end{aligned} \tag{12}$$

It means that the curvature of the plane curve $(\lambda(t), \mu(t))$ is a quadratic function of the distance to the origin, while the optimal curve $(x_1(t), x_2(t))$ projected to the horizontal plane of the normal coordinates has a curvature which is a quadratic function of the distance to some point. A main fact is that this kind of system of equations is in general integrable as is proven in the appendix, Lemma 6.2.

Summarizing all the results obtained so far, we get the following theorem.

Theorem 4.2. ***(asymptotic optimal synthesis for the ball with a trailer)*** *The asymptotic optimal synthesis is an ε-modification of the one of the nilpotent approximation. The latter has the following properties in normal coordinates, in projection to the horizontal plane (x_1, x_2):*

1. *it is a closed smooth periodic curve, whose curvature is a function of the square distance to some point,*
2. *the area and the 2^{nd} order moments $\int_\Gamma x_1(x_2dx_1 - x_1dx_2)$ and $\int_\Gamma x_2(x_2dx_1 - x_1dx_2)$ are zero,*
3. *the entropy is given by the formula: $E(\varepsilon) = \frac{\sigma}{4\varepsilon^4}\int_\Gamma \frac{dw}{\delta(w)}$, where $\delta(w)$ is the main invariant from (7), and σ is a universal constant.*

Remark 2. *Item 2* is given by the interpolation conditions, and the fact that the integrands in the formulas of the second order moments are $\dot{z}_1$ and $\dot{z}_2$.

Item 3 comes from the fact that if $\delta(.) \equiv \frac{1}{4}$, then the formula for entropy is $E(\varepsilon) = \frac{\sigma}{\varepsilon^4}\int_\Gamma dw$. If $\delta(.) \not\equiv \frac{1}{4}$, we go to $\delta(.) \equiv \frac{1}{4}$ by the change of variables $d\tilde{w} = \frac{dw}{4\delta(w)}$ (cf. the proof of Theorem 2.2), thus giving the result.

Let us now go a little bit further to integrate explicitly System (12). Consider the reduced system

$$\begin{aligned} \dot{\lambda} &= \sin(\varphi) \\ \dot{\mu} &= \cos(\varphi) \\ \dot{\varphi} &= K + \frac{\rho^2}{2} \end{aligned} \tag{13}$$

where $\rho^2 = \lambda^2 + \mu^2$. From the Relations (10), we know that the curve $\Lambda = (\lambda, \mu)$ is a translation of the curve $X = (x_1, x_2)$, say for simplicity $\Lambda = (x_1 + a, x_2 + b)$.

Lemma 4.3. *The area and the two 2^{nd} order moments of the curve Λ vanish.*

Proof. The area swept by the curve Λ is

$$\int \left[\dot{\mu}\lambda - \dot{\lambda}\mu\right] d\tau = \int \left[\dot{x}_2 x_1 - \dot{x}_1 x_2\right] d\tau + \int \left[a\dot{x}_2 - b\dot{x}_1\right] d\tau$$

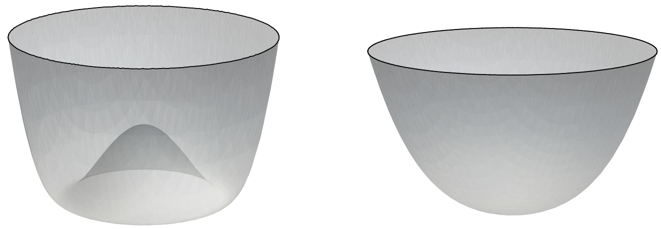


FIGURE 2. Graph of h with K negative (a), and K positive (b).

It is zero since the area of X is zero, and x_1 and x_2 are periodic.
Let us now consider, moment $\int \mu \left[\dot{\mu}\lambda - \dot{\lambda}\mu\right] d\tau$ (the same goes for the other one):

$$\int \mu \left[\dot{\mu}\lambda - \dot{\lambda}\mu\right] d\tau = \int x_2 \left[\dot{x}_2 x_1 - \dot{x_1} x_2\right] d\tau + a \int x_2 \dot{x}_2 d\tau + b \int x_2 \dot{x}_1 d\tau + b \int \left[\dot{\mu}\lambda - \dot{\lambda}\mu\right] d\tau$$

It is zero since,

1. the same moment, expressed in the X coordinates, is zero,
2. x_2 is periodic,
3. integration by parts and periodicity of x_1 and x_2 gives $\int \left[x_1\dot{x}_2 - x_2\dot{x}_1\right] d\tau = 2\int x_2\dot{x}_1 d\tau$, and
4. the area swept by the curve Λ is zero.

□

Next, the curve Λ is mapped onto a curve $\bar{\Lambda} = (\bar{\lambda}, \bar{\mu})$ as follows:

$$\bar{\lambda}(t) = \cos(\varphi(t))\lambda(t) - \sin(\varphi(t))\mu(t), \qquad \bar{\mu}(t) = \sin(\varphi(t))\lambda(t) + \cos(\varphi(t))\mu(t).$$

The equations for $\bar{\Lambda}$ are:

$$\begin{aligned} \dot{\bar{\lambda}} &= -\bar{\mu}\dot{\varphi}, \\ \dot{\bar{\mu}} &= 1 + \bar{\lambda}\dot{\varphi}. \end{aligned} \tag{14}$$

Set $r^2 = \bar{\lambda}^2 + \bar{\mu}^2$. Actually, $\bar{\Lambda}$ is a trajectory of the following quartic Hamiltonian

$$H = \bar{\lambda} + \frac{1}{2}\left(\frac{\bar{\lambda}^2 + \bar{\mu}^2}{2} + K\right)^2 = \bar{\lambda} + \frac{1}{2}\left(\frac{r^2}{2} + K\right)^2 \tag{15}$$

for a fixed parameter K, with dual variables $(\bar{\lambda}, \bar{\mu})$. We have the following relations for r and ρ:

$$r(t) = \rho(t), \text{ and } \frac{d}{dt}\left(\frac{1}{2}\rho^2(t)\right) = \bar{\mu}(t). \tag{16}$$

The following property of the area $S(t)$ swept by the curve Λ, between $\Lambda(0)$ and $\Lambda(t)$ is important:

$$S(t) = \int_0^t \left[\dot{\mu}\lambda - \dot{\lambda}\mu\right] d\tau = \int_0^t \left[\cos(\varphi)\lambda - \sin(\varphi)\mu\right] d\tau = \int_0^t \bar{\lambda}(\tau) d\tau. \tag{17}$$

The integral curves of the Hamiltonian (15) are the intersection of the graph of the quartic function $h(\bar{\lambda}, \bar{\mu}) = \frac{1}{2}\left(\frac{r^2}{2} + K\right)^2$ with a plane $\mathcal{P}_c = \left\{(\bar{\lambda}, \bar{\mu}, z) | c - \bar{\lambda} = z\right\}$ for some fixed parameters c and K. This situation is represented in Figure 2.

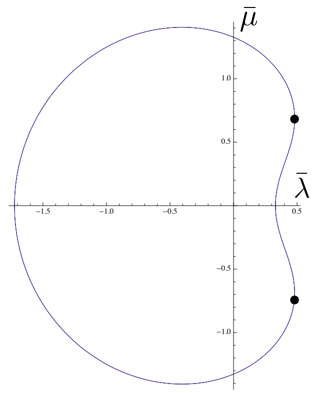


FIGURE 3. trajectory $\bar{\Lambda}$ of type II.

Therefore, integral curves are either convex (such a curve is said being *of type I*), or of the form shown in Figure 3 (i.e. *of type II*). In both cases, the curve $\bar{\Lambda}$ is symmetric with respect to the $\bar{\lambda}$ axis. Indeed, the graph of $h(\bar{\lambda}, \bar{\mu})$ has rotational symmetry w.r.t. the origin, and the planes $\mathcal{P}_c$ are symmetric with respect to the $(\bar{\lambda}, z)$ plane.

Let us remark that the solution curve Λ to System (13) can be considered as symmetric w.r.t. the λ axis (i.e. a change of the form $\tilde{\mu} = -\mu$, $\tilde{\varphi} = -\varphi$, and $\tilde{t} = -t$ gives the result), provided that $\varphi_0 = 0$ and $\mu_0 = 0$ which can be assumed. Indeed, it doesn't change System (14), and for any $\varphi_0 \neq 0$, an appropriate rotation of $\Lambda(t)$, and translation of $\varphi(t)$ shows that the solution trajectory of System (13) is just a rotation of the one obtained for $\varphi_0 = 0$.

Lemma 4.4. *The period P_Λ of Λ, is an integer multiple of the period $P_{\bar{\Lambda}}$ of $\bar{\Lambda}$: there is $n \in \mathbb{N}$, such that $P_\Lambda = nP_{\bar{\Lambda}}$.*

Proof. Let us first observe that $\rho^2(t)$ is periodic, of minimal period exactly $P_{\bar{\Lambda}}$. Indeed, from Equation (16), we know that $\rho^2(t)$ varies with $\bar{\mu}$, and $\left(\bar{\lambda}, \bar{\mu}\right)$ is periodic and symmetric with respect with the $\bar{\lambda}$ axis (see Figure 3). Up to a time shift, we can assume that the starting point is of the form $(-a, 0)$, $a > 0$. Since $\bar{\mu}$ is monotonic on each half period $P_{\bar{\Lambda}}/2$, the period is minimal.

Now, since $\Lambda(t)$ is periodic, and $r(t) = \rho(t)$, it must have a multiple period of $P_{\bar{\Lambda}}$. □

Remark 3. It appears clearly, from Systems (11) and (12) that $P_X = P_\Lambda$ where P_X denotes the period of X.

The next step is to work on the number of periods $P_{\bar{\Lambda}}$ needed to meet the interpolation conditions.

<u>Claim</u>: n must be strictly more than 1.

Below we provide a quite heuristic proof of this claim. We find it convincing and moreover we don't have better. The reader is kindly invited to inform us if able to get a precise proof.

Let us assume that $n = 1$, or in other words , $P_X = P_\lambda = P_{\bar{\lambda}}$. Because of the symmetry and periodicity of both curves Λ and $\bar{\Lambda}$, we shall study the problem on

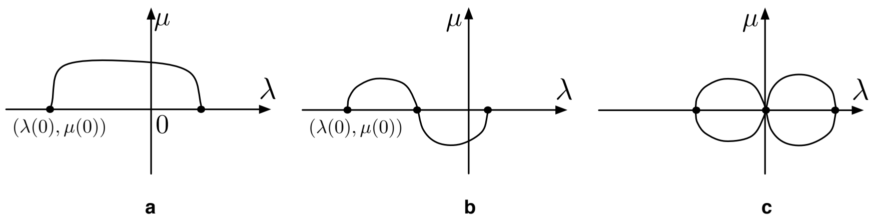


FIGURE 4. Form of the Λ trajectory for $\bar{\Lambda}$ trajectory of type II.

half a period $P_{\bar{\lambda}}$, starting from the point $\left(\bar{\lambda}, \bar{\mu}\right) = (-a, 0)$, $a > 0$. Let us remark that the inflection points of Λ correspond to the "bumps" of $\bar{\Lambda}$ (i.e. the dots on Figure 3).

In the case of a $\bar{\Lambda}$ curve of type I, the curve is convex, and there is no inflection point on Λ. Moreover, ρ is monotonic on the half-periods on behalf of Equation (16). Hence, the total area swept by λ cannot be zero, and such a curve is not suitable. This situation is illustrated in Figure 4(a).

In the case of a $\bar{\Lambda}$ curve of type II, there is only one inflection point on a half-period, and ρ is monotonic. The picture is of the form shown in 4(b), and the full curve Λ is a figure-eight.

It follows that the moment $m_\lambda = \displaystyle\int_X \dot{S}\lambda d\tau$ cannot be zero. As it is shown, in the proof of Lemma 4.3, a translation of Λ preserves the fact the 2^{nd} order moments vanish. After a translation of the central point of the figure-eight to the origin as in Figure 4(c), $\dot{S} < 0$ when $x < 0$, and $\dot{S} > 0$ when $x > 0$. Thus, the moment m_λ is non-vanishing, and such a curve is not suitable either.

Finally, n must be more than one.

It turns out that, for each $n > 1$, one can find a periodic curve with vanishing moments. With the help of a numerical software, it is possible to find the shortest one, shown on Figure 5 in the (x_1, x_2) coordinates. **It corresponds to $\mathbf{n = 2}$, and it is unique**.

We also display on Figure 6 a periodic trajectory corresponding to $n = 5$, with vanishing area.

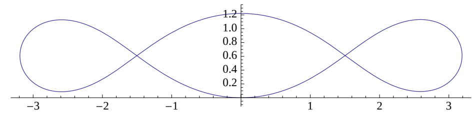


FIGURE 5. Projection of the solution for n=2, in the X coordinates.

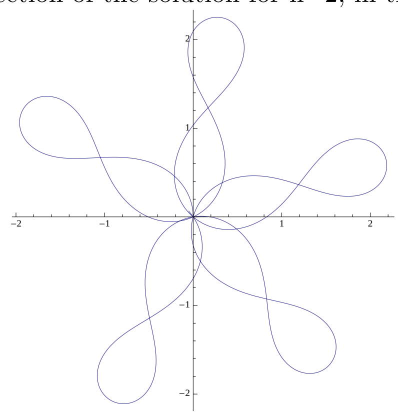


FIGURE 6. Projection of the solution for n=5, in the X coordinates.

5. **Conclusion.**

1. As a first conclusion, let us consider for instance the parking problem for the ball with a trailer. With the notations of System 1, it consists of approximating the non-admissible curve: $(x(t) = t, y(t) = 0, R(t) = Id, \theta(t) = 0)$. The ϵ-approximation of the optimal synthesis of Figure 5 gives the trajectory shown on Figure 7. The animated simulation is available on the website [26].

   Note that if we choose $\theta(t) = -\pi/2$, then $\dot{\Gamma} \in \Delta^3$, the problem is no more generic, and the solution is the one of the $(2, 3, 5)$ problem.

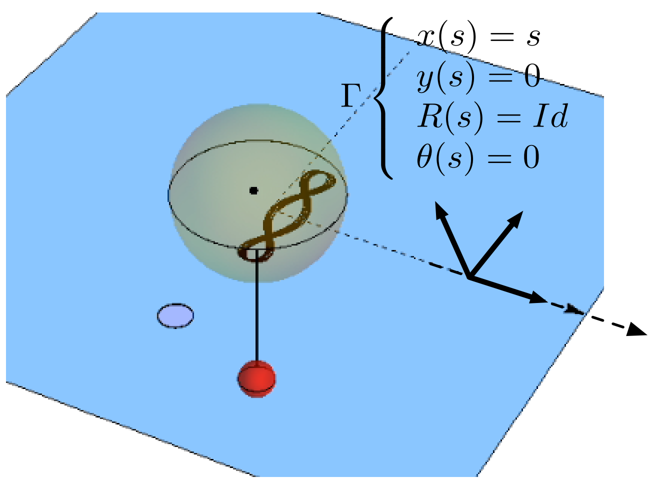
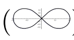
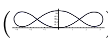


FIGURE 7. Parking the ball with a trailer. See also the simulation available on the website [26]

2. Let us go back to the general motion planning problem for two controls. Following the previous works [9, 10, 11, 12, 13], we know that in normal coordinates and in projection to the (x_1, x_2) plane, the optimal curves are:
   (a) in the two-step bracket generating case (the unicycle typically), periodic curves of constant curvature, i.e. circles ().
   (b) in the three-step bracket generating case (typically, the car with a trailer $(2,3,4)$, or the ball rolling on a plane $(2,3,5)$), periodic curves whose curvature is a linear function of the coordinates, the only periodic elasticae ().
   (c) in the four-step bracket generating case (typically our $(2,3,5,6)$ case, the ball wit a trailer), periodic curves whose curvature is a quadratic function of the position, our hyperelliptic curves ().

   This is the clear beginning of a certain series, and we are convinced that a strong purely topological fact holds when we want to realize (even approximately) successive brackets for a 2-control kinematic system.

6. **Appendix.**

6.1. **Group Law.**

Lemma 6.1. *System (8) is a right invariant system on $\mathbb{R}^6$ with coordinates $\xi = (\varsigma, w) = (x, y, z, w)$, for a certain nilpotent Lie group structure over $\mathbb{R}^6$ denoted by G. There exist a function Φ, and a Lie group structure over $\mathbb{R}^5$, with coordinates ς (denoted by G_0) and multiplication law $*$ such that the group law of G is of the form: $(\varsigma_2, w_2)(\varsigma_1, w_1) = (\varsigma_1 * \varsigma_2, w_1 + w_2 + \Phi(\varsigma_1, \varsigma_2))$.*

Proof. Let G denote the Lie group $\mathbb{R}^6$, with the Lie group structure determined by the fact that System (8) is right invariant. The elements of G are of the form $(x_1, x_2, y, z_1, z_2, w)$, and we want to find an expression for the group law of G.

The computation of the successive Lie brackets shows that $\frac{\partial}{\partial w}$ is right invariant, and belongs to the Lie algebra of G. Moreover, $\frac{\partial}{\partial w}$ commutes with all elements in this Lie algebra, hence $\mathbb{R} = \{(e, w)\}$ is in the center of G, with the elements of G denoted by (ς, w).

The group $G_0 = G/\mathbb{R} = \{(\varsigma, 0)\}$ is a subgroup of G which multiplication law is denoted by $*$. The group law of G is of the form

$$(\varsigma, w)\left(\varsigma^{'}, w^{'}\right) = \left(\varsigma * \varsigma^{'}, \psi\left(\varsigma, \varsigma^{'}, w, w^{'}\right)\right),$$

where ψ denotes an analytic function.

Let $x = (\varsigma_x, w_x)$ and $a = (\varsigma_a, w_a)$ denote two elements of G, $R_a(.)$ the right translation by element a, and $X(.)_w$ the last component of the vector field $X(.) = \frac{\partial}{\partial w}$. The right invariance of $\frac{\partial}{\partial w}$ writes: $d\left(R_a(x)\right) X(x) = X(xa)_w = 1$, which gives:

$$\frac{\partial \psi}{\partial \varsigma_x}\left(\varsigma_x, \varsigma_a, w_x, w_a\right).0 + \frac{\partial \psi}{\partial w_x}\left(\varsigma_x, \varsigma_a, w_x, w_a\right).1 = 1, \quad \Rightarrow \quad \frac{\partial \psi}{\partial w_x}\left(\varsigma_x, \varsigma_a, w_x, w_a\right) = 1,$$

$$\Rightarrow \psi\left(\varsigma_x, \varsigma_a, w_x, w_a\right) = w_x + \Phi\left(\varsigma_x, \varsigma_a, w_a\right), \tag{18}$$

where Φ denotes an analytic function.

This relation together with the fact that $(e, w_a)(\varsigma_x, w_x) = (\varsigma_x, w_x)(e, w_a)$, $\forall (\varsigma_x, w_x) \in G$, and $\forall w_a \in \mathbb{R}$, gives:

$$w_a + \Phi\left(e, \varsigma_x, w_x\right) = w_x + \Phi\left(\varsigma_x, e, w_a\right). \tag{19}$$

Let us now exploit the associativity of the law:

$$\begin{aligned}(\varsigma_x, w_x)(\varsigma_a, w_a)(\varsigma_b, w_b) = \\ &= (\varsigma_x\varsigma_a\varsigma_b, w_x + \Phi(\varsigma_x, \varsigma_a, w_a) + \Phi(\varsigma_x\varsigma_a, \varsigma_b, w_b)) \\ &= (\varsigma_x\varsigma_a\varsigma_b, w_x + \Phi(\varsigma_x, \varsigma_a\varsigma_b, w_a + \Phi(\varsigma_a, \varsigma_b, w_b))),\end{aligned}$$

which results in

$$\Phi(\varsigma_x, \varsigma_a, w_a) + \Phi(\varsigma_x\varsigma_a, \varsigma_b, w_b) = \Phi(\varsigma_x, \varsigma_a\varsigma_b, w_a + \Phi(\varsigma_a, \varsigma_b, w_b)). \tag{20}$$

The partial derivative of Relation (20), first with respect to w_a, and second with respect to w_b gives:

$$0 = \frac{\partial^2\Phi}{\partial w^2}(\varsigma_x, \varsigma_a\varsigma_b, w_a + \Phi(\varsigma_a, \varsigma_b, w_b)) \cdot \frac{\partial\Phi}{\partial w}(\varsigma_a, \varsigma_b, w_b).$$

Since the product of these two analytic functions vanishes, one of the functions has to be identically zero.

Let us suppose that $\frac{\partial\Phi}{\partial w} \equiv 0$, this means that $\Phi(\varsigma_a, \varsigma_b, w_b) = \Phi(\varsigma_a, \varsigma_b)$, and $\psi(\varsigma_a, \varsigma_b, w_a, w_b) = w_a + \Phi(\varsigma_a, \varsigma_b)$, for all $a, b \in G$. Relation (19) rewrites, for all (e, w_a), and $(\varsigma_x, w_x) \in G$

$$w_a + \Phi(e, \varsigma_x) = w_x + \Phi(\varsigma_x, e).$$

In particular, setting $w_a = w_x$ gives $\Phi(e, \varsigma_x) = \Phi(\varsigma_x, e)$, for all $\varsigma_x \in G_0$, and consequently $w_a = w_x$ for all $w_a, w_x \in \mathbb{R}$. A contradiction.
Consequently $\frac{\partial^2\Phi}{\partial w^2} \equiv 0$, which means that Φ is of the form:

$$\Phi(\varsigma_x, \varsigma_a, w) = \Phi_1(\varsigma_x, \varsigma_a)\, w + \Phi_2(\varsigma_x, \varsigma_a). \tag{21}$$

Associativity (20) now writes, for all $x, a, b \in G$:

$$\begin{aligned}&\Phi_1(\varsigma_x, \varsigma_a)\, w_a + \Phi_2(\varsigma_x, \varsigma_a) + \Phi_1(\varsigma_x\varsigma_a, \varsigma_b)\, w_b + \Phi_2(\varsigma_x\varsigma_a, \varsigma_b) = \\ &\qquad \Phi_1(\varsigma_x, \varsigma_a\varsigma_b)\,[w_a + \Phi_1(\varsigma_a, \varsigma_b)\, w_b + \Phi_2(\varsigma_a, \varsigma_b)] + \Phi_2(\varsigma_x, \varsigma_a\varsigma_b).\end{aligned}$$

By setting first $w_a = w_b = 0$, then $w_b = 0$, and finally $w_a = 0$, we obtain:

$$\Phi_1(\varsigma_x, \varsigma_a) = \Phi_1(\varsigma_x, \varsigma_a\varsigma_b) \tag{22}$$

$$\Phi_1(\varsigma_x\varsigma_a, \varsigma_b) = \Phi_1(\varsigma_x, \varsigma_a\varsigma_b)\Phi_1(\varsigma_a, \varsigma_b) \tag{23}$$

$$\Phi_2(\varsigma_x, \varsigma_a) + \Phi_2(\varsigma_x\varsigma_a, \varsigma_b) = \Phi_1(\varsigma_x, \varsigma_a\varsigma_b)\,\Phi_2(\varsigma_a, \varsigma_b) + \Phi_2(\varsigma_x, \varsigma_a\varsigma_b). \tag{24}$$

1. From (22), we deduce that $\Phi_1(\varsigma_x, \varsigma_a) = \Phi_1(\varsigma_x)$, for all $\varsigma_x \in G_0$.
2. From (23), and the above relation, we have $\Phi_1(\varsigma_x\varsigma_a) = \Phi_1(\varsigma_x)\Phi_1(\varsigma_a)$, for all $\varsigma_x, \varsigma_a \in G_0$.
3. From (24), with $\varsigma_a = \varsigma_b = e$, we have $\Phi_2(\varsigma_x, e) = \Phi_1(\varsigma_x)\,\Phi_2(e, e)$, which gives $\Phi_1(e) = 1$.

Relations (19) and (21), together with the above remarks yield $w_a + \Phi_2(e, \varsigma_x) = \Phi_1(\varsigma_x)\, w_a + \Phi_2(\varsigma_x, e)$. By setting $w_a = 0$, we obtain $\Phi_2(e, \varsigma_x) = \Phi_2(\varsigma_x, e)$ for all $\varsigma_x \in G_0$. Consequently $\Phi_1(\varsigma_x) = 1$, and we get the relation $\psi(\varsigma_x, \varsigma_a, w_x, w_a) = w_x + w_a + \Phi_2(\varsigma_x, \varsigma_a)$, for all $a = (\varsigma_a, w_a)$, and $x = (\varsigma_x, w_x)$ in G. $\square$

6.2. **Plane Curves Whose Curvature is a Function of the Distance to the Origin.**

Lemma 6.2. *Consider a plane curve $(x(t), y(t))$, whose curvature is a function of the distance from the origin, that is:*

$$\dot{x} = \cos(\varphi), \quad \dot{y} = \sin(\varphi), \quad \dot{\varphi} = k(x^2 + y^2), \tag{25}$$

for a certain smooth function $k(.)$. Then it is integrable.

Proof. Although this result is already known [23], the proof we provide here is very simple.

We first set

$$\bar{x} = x\cos(\varphi) + y\sin(\varphi), \text{ and } \bar{y} = -x\sin(\varphi) + y\cos(\varphi).$$

The derivatives are:

$$\frac{d\bar{x}}{dt} = 1 + \bar{y}k(\bar{x}^2 + \bar{y}^2), \quad \frac{d\bar{y}}{dt} = -\bar{x}k(\bar{x}^2 + \bar{y}^2), \tag{26}$$

and $k(\bar{x}^2 + \bar{y}^2) = k(x^2 + y^2)$.

Then, we only need to show that (26) is a Hamiltonian system. Indeed, since it is a two dimensional problem, it is always Liouville-integrable. Therefore, we are looking for solutions of the system of PDE's:

$$\frac{\partial H}{\partial \bar{x}} = 1 + \bar{y}k(\bar{x}^2 + \bar{y}^2), \quad \frac{\partial H}{\partial \bar{y}} = -\bar{x}k(\bar{x}^2 + \bar{y}^2).$$

They always do exist since the Schwartz integrability conditions are satisfied: $\frac{\partial^2 H}{\partial \bar{x}\partial \bar{y}} = \frac{\partial^2 H}{\partial \bar{y}\partial \bar{x}} = 2\bar{x}\bar{y}k'$. □

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Received xxxx 20xx; revised xxxx 20xx.

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