

# Motion Planning for Kinematic Systems

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**Abstract**—In this paper, we present a general theory of motion planning for kinematic systems. In particular, the theory deals with  $\epsilon$ -approximations of non-admissible paths by admissible ones in a certain optimal sense. The need for such an approximation arises for instance in the case of highly congested configuration spaces.

This theory has been developed by one of the authors in a previous series of papers. It is based upon concepts from subriemannian geometry. Here, we summarize the results of the theory, and we improve on, by developing in details an intricate case: the ball with a trailer, which corresponds to a distribution with flag of type 2, 3, 5, 6.

**Index Terms**—Optimal control, Subriemannian geometry, Robotics, Motion planning

## I. INTRODUCTION

WE PRESENT the main lines of a theory of motion planning for kinematic systems. This theory has been developed for several years in the papers [13], [14], [15], [16], [17], [18], [19]. One of the purposes of this article is to survey the whole theory disseminated in these papers. We improve on the theory with the exposition of a new case in which “the fourth order brackets are involved”. We also improve on several previous results (periodicity of our optimal trajectories for instance). Potential application of this theory is motion planning for kinematic robots. In order to illustrate the discussion, several academic examples are displayed. In particular, we invite the interested reader to have a look at our solution to the problem of approximating non-admissible motions for “the ball on a plate” and “the ball with a trailer” on the website [38].

Nonholonomic control problems have been studied from the early XIX<sup>th</sup> century on. Typical issues range from the determination of accessibility, to motion planning and feedback stabilization, see [6], [7], [27] for detailed reviews and insights into the field. The motion planning problem, which is the topic under consideration here, can be addressed in several different ways [30]: determination of simple trajectories, optimal motion planning, obstacle avoidance, etc.

Practical situations occur for which a path in the configuration space is precisely specified. It is the case for parking problems [22], and (for security reasons) for motion of robots around a shuttle in the spatial context, motion of certain maintenance robots in the pool of a nuclear plant, etc.

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Besides these particular cases, a reasonable general strategy to obtain an admissible path that avoids obstacles and complies with constraints, is as suggested in e.g. [21], [28], [32], [37]:

- 1) not taking into account admissibility issues, elaborate a path that copes with the configuration constraints,
- 2) use this first path as an Ariadne thread to compute an approximating admissible path (see [27] page 27 and references therein).

Actually, for highly congested configuration spaces, the admissible path has to stay  $\epsilon$ -close to the Ariadne thread. Such a situation may arise either because of a high density of obstacles in the physical space, or due to constraints imposed on the robot by the task, or mission, to be achieved. Of course, in our methodology where  $\epsilon$  is theoretically required to be small, in practice,  $\epsilon$  can be taken large as long as the performances remain acceptable.

The theory starts from the seminal work of F. Jean, in the papers [22], [23], [24]. At the root of this point of view in robotics, there are also more applied authors like J.P. Laumond [28]. See also [37]. The mathematical framework of the theory is subriemannian geometry. For a complete introduction, see the reference work [33], and the not yet published book [1], available online.

We consider kinematic systems that are given under the guise of a vector-distribution  $\Delta$  over a  $n$ -dimensional manifold  $M$ . The rank of the distribution is  $p$ , and the corank  $k = n - p$ . Motion planning problems will always be local problems in an open neighborhood of a given finite path  $\Gamma$  in  $M$ . Then, we can always consider that  $M = \mathbb{R}^n$ . From a control point of view, a kinematic system can be specified by a control system, linear in the controls, typically denoted by  $\Sigma$ :

$$(\Sigma) \quad \dot{x} = \sum_{i=1}^p F_i(x) u_i, \quad (1)$$

where the  $F_i$ ’s are smooth ( $C^\infty$ ) vector fields that span the distribution  $\Delta$ . The standard controllability assumption is always assumed, i.e. the Lie algebra generated by the  $F_i$ ’s is *transitive*<sup>1</sup> on  $M$ . Consequently, the distribution  $\Delta$  is *completely nonintegrable*, and any smooth path  $\Gamma : [0, T] \rightarrow M$ , can be uniformly approximated by an admissible path  $\gamma : [0, \theta] \rightarrow M$ , i.e. a Lipschitz path, which is almost everywhere tangent to  $\Delta$ , or in other words, a trajectory of (1).

This is precisely the *abstract answer* to the kinematic motion planning problem: *it is possible to approximate uniformly non-admissible paths by admissible ones*. The purpose of this paper is to present a general constructive theory that solves this problem in a certain *optimal* way.

<sup>1</sup>A lie algebra of vector fields over a manifold  $M$  is said *transitive* if it spans the whole tangent space at each point of  $M$ .

More precisely, in this class of problems, it is natural to try to minimize a cost of the following form:

$$J(u) = \int_0^\theta \sqrt{\sum_{i=1}^p (u_i)^2} dt.$$

This choice is motivated by several reasons:

- 1) the optimal curves do not depend on their parametrization,
- 2) the minimization of such a cost produces a metric space (the associated distance is called the subriemannian distance, or the Carnot-Caratheodory distance),
- 3) minimizing such a cost is equivalent to minimize the following quadratic cost, denoted  $J_E(u)$  and called the *energy* of the path, in fixed time  $\theta$ :

$$J_E(u) = \int_0^\theta \sum_{i=1}^p (u_i)^2 dt.$$

The distance between two points is defined as the minimum length of admissible curves connecting these two points. The length of the admissible curve corresponding to the control  $u : [0, \theta] \rightarrow M$  is simply  $J(u)$ .

In this presentation, another way to interpret the problem is as follows: the dynamics is specified by the distribution  $\Delta$  (i.e. not by the vector fields  $F_i$ , but their span only). The cost is then determined by an Euclidean metric  $g$  over  $\Delta$ , specified here by the fact that the  $F_i$ 's form an orthonormal frame field for the metric.

At this point we would like to make a philosophical comment: there is, in the world of nonlinear control theory, a permanent twofold criticism against the optimal control approach:

- 1) the choice of the cost to be minimized is in general rather arbitrary, and
- 2) optimal control solutions may not be robust.

Some remarkable conclusions of our theory show the following: in reasonable dimensions and codimensions, the optimal trajectories are extremely robust, and in particular, do not depend at all (modulo certain natural transformations) on the choice of the metric, but depend on the distribution  $\Delta$  only.

The following fact is even stronger: they depend only on the *nilpotent approximation along*  $\Gamma$  (a concept that will be defined later on, which is a good local approximation of the problem). For a lot of low values of the rank  $p$  and corank  $k$ , these nilpotent approximations are universal in a certain sense: they depend only on certain integer numbers, namely the dimensions of the successive bracket spaces generated by  $\Delta$ , and no functional or real parameter appears in the problem reduced to its nilpotent approximation. As a consequence, the *asymptotic optimal syntheses* (i.e. the phase portraits of the admissible trajectories that approximate up to a small  $\varepsilon$ ) are also universal.

Given a motion planning problem, specified by a (non-admissible) curve  $\Gamma$ , and a subriemannian structure (1), we will consider two distinct concepts, namely:

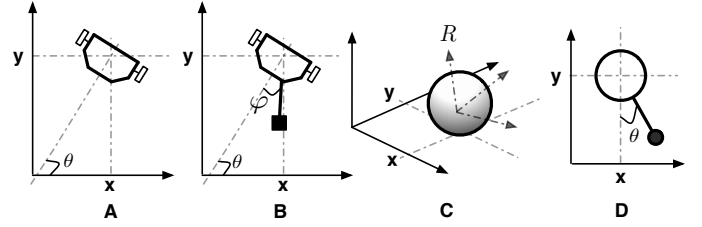


Fig. 1. Examples of kinematic systems

- 1) the *metric complexity*  $MC(\varepsilon)$  that measures asymptotically the length of the best  $\varepsilon$ -approximating admissible trajectories, and
- 2) the *interpolation entropy*  $E(\varepsilon)$ , that measures the length of the best admissible curves that interpolate  $\Gamma$  with pieces of length  $\varepsilon$ .

The first concept was introduced by F. Jean in his basic paper [22]. The second concept is closely related with the entropy of F. Jean in [23], which is more or less the same as the Kolmogorov's entropy of the path  $\Gamma$ , for the metric structure induced by the Carnot-Caratheodory metric of the ambient space.

Along this paper, we deal with *generic* problems only (generic has to be understood in a global sense, i.e. stable singularities are considered). That is, the set of motion planning problems on  $\mathbb{R}^n$  is the set of couples  $(\Gamma, \Sigma)$ , embedded with the  $C^\infty$  topology of uniform convergence over compact sets, and generic problems (or *problems in general position*) form an open-dense set in this topology. For instance, it means that the curve  $\Gamma$  is always transversal to  $\Delta$  (except maybe at isolated points, in the case  $k = 1$  only). Another example is the case of a surface of degeneracy of the Lie bracket distribution  $[\Delta, \Delta]$  in the case  $n = 3$ ,  $k = 1$ . Generically, this surface (the Martinet surface) is smooth, and  $\Gamma$  intersects it transversally at a finite number of points only.

In this paper, as is of common usage, we say that a system is "two-step bracket-generating" when  $\dim([\Delta, \Delta]) = n$ .

Also, along the paper, we illustrate our results with one among the following well known academic examples:

*Example 1: the unicycle, or two-driving wheel robot*, [28], [29], is described by the  $(x, y)$  position of the point at the middle of the axis of the wheels, and the orientation  $\theta$  of the mobile, as shown on Fig. 1.A. The kinematic model is:

$$\dot{x} = \cos(\theta)u_1, \quad \dot{y} = \sin(\theta)u_1, \quad \dot{\theta} = u_2. \quad (2)$$

*Example 2: the car with a trailer*, [28], [29], is a two-driving wheel robot with a trailer hooked to the middle point of the axis of the wheels. The distance between the robot and the trailer is assumed equal to 1. The position of the trailer is specified by the angle  $\varphi$  as in Fig. 1.B. The model is:

$$\dot{x} = \cos(\theta)u_1, \quad \dot{y} = \sin(\theta)u_1, \quad \dot{\theta} = u_2, \quad \dot{\varphi} = u_1 - \sin(\varphi)u_2. \quad (3)$$

*Example 3: the ball rolling on a plane* was also studied in [6], [9], [25]. As shown in Fig. 1.C, it is described by the  $(x, y)$  coordinates of the contact point between the ball and the plane,

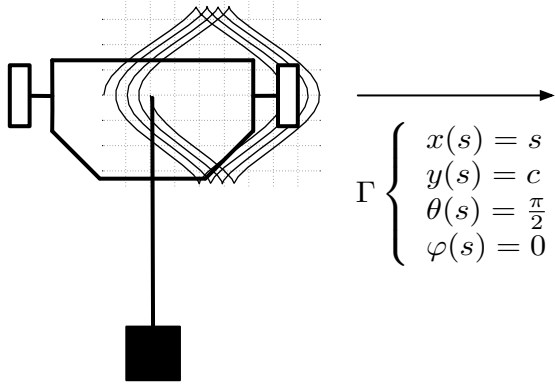


Fig. 2. Parking problem for the car with a trailer,  $c$  is a constant

and a right orthogonal matrix  $R \in SO(3, \mathbb{R})$  representing an orthonormal frame attached to the ball. The kinematic model is:

$$\dot{x} = u_1, \dot{y} = u_2, \dot{R} = \begin{bmatrix} 0 & 0 & u_1 \\ 0 & 0 & u_2 \\ -u_1 & -u_2 & 0 \end{bmatrix} R. \quad (4)$$

*Example 4: the ball with a trailer* is as in example 3, where the trailer's position is known from the angle  $\varphi$  as described in Fig. 1.D. The distance between the ball and its trailer is denoted by  $L$ .

$$\dot{x} = u_1, \dot{y} = u_2, \dot{R} = \begin{bmatrix} 0 & 0 & u_1 \\ 0 & 0 & u_2 \\ -u_1 & -u_2 & 0 \end{bmatrix} R, \quad (5)$$

$$\dot{\theta} = -\frac{1}{L}(\cos(\theta)u_1 + \sin(\theta)u_2).$$

Typical motion planning problems are:

- 1) for example (2), *the parking problem*: the non-admissible curve  $\Gamma$  is  $s \rightarrow (x(s), y(s), \theta(s), \varphi(s)) = (s, 0, \frac{\pi}{2}, 0)$ ,
- 2) for example (3), *the full rolling with slipping problem*,  $\Gamma : s \rightarrow (x(s), y(s), R(s)) = (s, 0, Id)$ , where  $Id$  is the identity matrix.

On Figs. 2 and 3 we show our approximating trajectories for both problems, that are in a sense universal. In Fig. 2, of course, the  $x$ -scale of the trajectory is much larger than the  $y$ -scale.

The basic academic kinematic problems have a lot of symmetries, and most of them have finite dimensional Lie algebras. Due to these symmetries, the associated subriemannian problems are often integrable in Liouville sense (roughly speaking minimizers can be explicitly computed up to quadratures). These explicit solutions, of course, could be used directly to solve the motion planning problem. The drawback of our method is that it forgets about these particular structures since in the nilpotent approximations along  $\Gamma$  the symmetries are not preserved. However:

- 1) when  $\epsilon$  tends to 0, the optimal trajectories of the original problem converge to those of their nilpotent approximation,

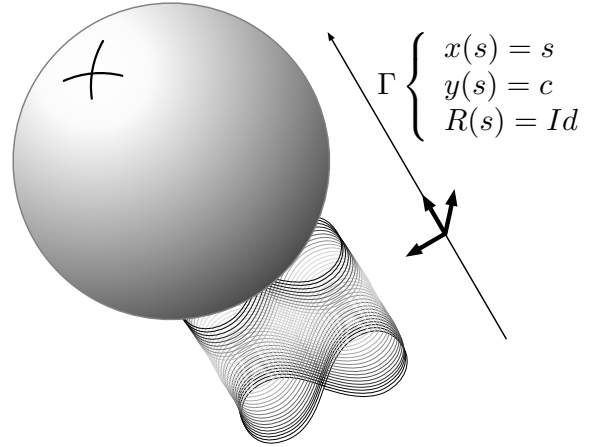


Fig. 3. Approximating rolling with slipping,  $c$  is a constant. See also the simulation available on the website [38]

- 2) the most important point: our methodology applies generically, not only for those academic examples with a lot of symmetries,
- 3) the problems reduced to their nilpotent approximation are themselves integrable while the original generic problems are not.

Up to now, our theory covers the following cases:

- (C1) The distribution  $\Delta$  is two-step bracket generating (i.e.  $\dim([\Delta, \Delta]) = n$ ) except maybe at generic singularities,
- (C2) The number of controls (i.e.  $\dim(\Delta)$ ) is  $p = 2$ , and  $n \leq 6$ .

The paper is organized as follows: in section II, we introduce the basic concepts underlying the theory, namely the metric complexity, the interpolation entropy, the nilpotent approximation along  $\Gamma$ , and the normal coordinates.

Section III summarizes the main results of our theory, disseminated in our previous papers, with some complements and details. Section IV is the detailed study of the case  $n = 6$ ,  $k = 4$ , which corresponds to example (4), the ball with a trailer. In Section V, we state several remarks, expectations and conclusions.

## II. BASIC CONCEPTS

In this section, we fix a generic motion planning problem  $\mathcal{P} = (\Gamma, \Sigma)$ . Also, along the paper  $\epsilon$  is a small parameter (we want to approximate up to  $\epsilon$ ), and there are quantities  $f(\epsilon), g(\epsilon)$  that go to  $+\infty$  when  $\epsilon$  tends to zero. We say that such quantities are equivalent ( $f \simeq g$ ) if  $\lim_{\epsilon \rightarrow 0} \frac{f(\epsilon)}{g(\epsilon)} = 1$ . Also,  $d$  denotes the subriemannian distance, and we consider the  $\epsilon$ -subriemannian tube  $T_\epsilon$  and cylinder  $C_\epsilon$  around  $\Gamma$ :

$$T_\epsilon = \{x \in M \mid d(x, \Gamma) \leq \epsilon\},$$

$$C_\epsilon = \{x \in M \mid d(x, \Gamma) = \epsilon\}.$$

### A. Entropy versus Metric Complexity

*Definition 5:* The metric complexity  $MC(\epsilon)$  of  $\mathcal{P}$  is  $\frac{1}{\epsilon}$  times the minimum length of an admissible curve  $\gamma_\epsilon$  connecting the endpoints  $\Gamma(0), \Gamma(T)$  of  $\Gamma$ , and remaining in the tube  $T_\epsilon$ .

*Definition 6:* The *interpolation entropy*  $E(\varepsilon)$  of  $\mathcal{P}$  is  $\frac{1}{\varepsilon}$  times the minimum length of an admissible curve  $\gamma_\varepsilon$  connecting the endpoints  $\Gamma(0), \Gamma(T)$  of  $\Gamma$ , and  $\varepsilon$ -interpolating  $\Gamma$ , that is, in any segment of  $\gamma_\varepsilon$  of length  $\geq \varepsilon$ , there is a point of  $\Gamma$ .

These quantities  $MC(\varepsilon), E(\varepsilon)$  are functions of  $\varepsilon$  which tends to  $+\infty$  as  $\varepsilon$  tends to zero. They are considered **up to equivalence**. The reason to divide by  $\varepsilon$  is that the second quantity counts the number of  $\varepsilon$ -balls needed to cover  $\Gamma$ , or the number of pieces of length  $\varepsilon$  needed to interpolate the full path. This is also the reason for the name ‘‘entropy’’.

*Definition 7:* An asymptotic optimal synthesis is a one-parameter family  $\gamma_\varepsilon$  of admissible curves, that realizes the metric complexity or the entropy.

Our main purpose in the paper is twofold:

- 1) We want to estimate the metric complexity and the entropy, in terms of certain invariants of the problem. Actually, in all the cases treated in this paper, we give explicit formulas.
- 2) We shall exhibit explicit asymptotic optimal syntheses realizing the metric complexity or/and the entropy.

## B. Normal Coordinates

Take a **parametrized**  $k$ -dimensional surface  $S$ , transversal to  $\Delta$  (which may be defined in a neighborhood of  $\Gamma$  only),

$$S = \{q(s_1, \dots, s_{k-1}, t) \in \mathbb{R}^n\}, \text{ with } q(0, \dots, 0, t) = \Gamma(t).$$

Such a *germ* exists if  $\Gamma$  is not tangent to  $\Delta$ . The exclusion of a neighborhood of an isolated point where  $\Gamma$  is tangent to  $\Delta$ , (that is  $\Gamma$  becomes ‘‘almost admissible’’), does not affect our estimates presented later on (it provides a term of higher order in  $\varepsilon$ ).

In the following,  $\mathcal{C}_\varepsilon^S$  denotes the cylinder  $\{\xi; d(S, \xi) = \varepsilon\}$ , and  $S(y, w)$  is a short notation for the surface defined above.

*Lemma 8:* (Normal coordinates with respect to  $S$ ).

There are mappings  $x : \mathbb{R}^n \rightarrow \mathbb{R}^p$ ,  $y : \mathbb{R}^n \rightarrow \mathbb{R}^{k-1}$ ,  $w : \mathbb{R}^n \rightarrow \mathbb{R}$ , such that  $\xi = (x, y, w)$  is a coordinate system on some neighborhood of  $S$  in  $\mathbb{R}^n$ , such that:

- 1)  $S(y, w) = (0, y, w)$ ,  $\Gamma = \{(0, 0, w)\}$ ,
- 2) The restriction  $\Delta|_S = \ker dw \cap_{i=1, \dots, k-1} \ker dy_i$ , the metric  $g|_S = \sum_{i=1}^p (dx_i)^2$ ,
- 3)  $\mathcal{C}_\varepsilon^S = \{\xi | \sum_{i=1}^p x_i^2 = \varepsilon^2\}$ ,
- 4) the geodesics of the Pontryagin’s maximum principle [34] meeting the transversality conditions w.r.t.  $S$  are the straight lines through  $S$ , contained in the planes  $P_{y_0, w_0} = \{\xi | (y, w) = (y_0, w_0)\}$ . Hence, they are orthogonal to  $S$ .

These normal coordinates are unique up to changes of coordinates of the form

$$\begin{aligned} \tilde{x} &= T(y, w)x, \\ (\tilde{y}, \tilde{w}) &= (y, w), \end{aligned} \quad (6)$$

where  $T(y, w) \in O(p)$ , the  $p$ -orthogonal group.

## C. Normal Forms, Nilpotent Approximation along $\Gamma$

1) *Frames:* Let us denote by  $F = (F_1, \dots, F_p)$  the orthonormal frame of vector fields generating  $\Delta$ . Hence, we will also write  $\mathcal{P} = (\Gamma, F)$ . If a global coordinate system  $(x, y, w)$ , not necessarily normal, is given on a neighborhood of  $\Gamma$  in  $\mathbb{R}^n$ , with  $x \in \mathbb{R}^p$ ,  $y \in \mathbb{R}^{k-1}$ ,  $w \in \mathbb{R}$ , then we write:

$$\begin{aligned} F_j &= \sum_{i=1}^p \mathcal{Q}_{i,j}(x, y, w) \frac{\partial}{\partial x_i} + \sum_{i=1}^{k-1} \mathcal{L}_{i,j}(x, y, w) \frac{\partial}{\partial y_i} \\ &\quad + \mathcal{M}_j(x, y, w) \frac{\partial}{\partial w}, \quad j = 1, \dots, p. \end{aligned} \quad (7)$$

Hence, the subriemannian metric is specified by the triple  $(\mathcal{Q}, \mathcal{L}, \mathcal{M})$  of smooth  $x, y, w$ -dependent matrices.

2) *The general normal form:* Fix a surface  $S$  as in Section II-B and a normal coordinate system  $\xi = (x, y, w)$  for a problem  $\mathcal{P}$ .

*Theorem 9:* (Normal form, [3]) There is an orthonormal frame  $F = (\mathcal{Q}, \mathcal{L}, \mathcal{M})$  for  $(\Delta, g)$  with the following properties:

1.  $\mathcal{Q}(x, y, w)$  is symmetric,  $\mathcal{Q}(0, y, w) = Id$  (the identity matrix),
2.  $\mathcal{Q}(x, y, w)x = x$ ,
3.  $\mathcal{L}(x, y, w)x = 0$ , and  $\mathcal{M}(x, y, w)x = 0$ .
4. Conversely if  $\xi = (x, y, w)$  is a coordinate system satisfying conditions 1, 2, 3 above, then  $\xi$  is a normal coordinate system for the subriemannian metric defined by the orthonormal frame  $F$  with respect to the parametrized surface  $\{(0, y, w)\}$ .

Clearly, this normal form is invariant under the changes of normal coordinates (6).

Let us write:

$$\begin{aligned} \mathcal{Q}(x, y, w) &= Id + Q_1(x, y, w) + Q_2(x, y, w) + \dots, \\ \mathcal{L}(x, y, w) &= 0 + L_1(x, y, w) + L_2(x, y, w) + \dots, \\ \mathcal{M}(x, y, w) &= 0 + M_1(x, y, w) + M_2(x, y, w) + \dots, \end{aligned}$$

where  $Q_r, L_r, M_r$  are matrices depending on  $\xi = (x, y, w)$ , the coefficients of which have order  $r$  w.r.t.  $x$  (i.e. they are in the  $r^{\text{th}}$  power of the ideal of  $C^\infty(x, y, w)$  generated by the functions  $x_r$ ,  $r = 1, \dots, p$ ). In particular,  $Q_1$  is linear in  $x$ ,  $Q_2$  is quadratic, etc.

Set  $u = (u_1, \dots, u_p) \in \mathbb{R}^p$ , and define  $L_{1,y,w}(x, u) = \sum_{j=1}^p L_{1,j}(x, y, w)u_j$ , where  $L_{1,j}(x, y, w)$  is the  $j^{\text{th}}$  column of  $L_1(x, y, w)$ . It is quadratic in  $(x, u)$ , and  $\mathbb{R}^{k-1}$ -valued. Its  $i^{\text{th}}$  component is the quadratic expression denoted by  $L_{1,i,y,w}(x, u)$ .

Similarly  $M_{1,y,w}(x, u) = \sum_{j=1}^p M_{1,j}(x, y, w)u_j$  is a quadratic form in  $(x, u)$ . The matrices of those several quadratic expression are denoted by  $L_{1,i,y,w}$ ,  $i = 1, \dots, k-1$ , and  $M_{1,y,w}$ .

The following was proved in [3], [10] for corank 1:

*Proposition 10:* 1.  $Q_1 = 0$ ,

2.  $L_{1,i,y,w}$ ,  $i = 1, \dots, k-1$ , and  $M_{1,y,w}$  are skew symmetric matrices.

A first useful very rough estimate in normal coordinates is given by the following proposition:

*Proposition 11:* If  $\xi = (x, y, w) \in T_\varepsilon$ , then:

$$\begin{aligned} \|x\|_2 &\leq \varepsilon, \\ \|y\|_2 &\leq \alpha\varepsilon^2, \end{aligned}$$

for some constant  $\alpha > 0$ .

From now on, we shall consider separately, first, the 2-step bracket-generating case, and second, the 2-control case that were mentioned in the introduction section.

3) *Two-step bracket-generating case:* In that case, we set, in accordance to Proposition (11), that  $x$  has weight 1, and the  $y_i$ 's and  $w$  have weight 2. Then, the vector fields  $\frac{\partial}{\partial x_i}$  have weight -1, and  $\frac{\partial}{\partial y_i}, \frac{\partial}{\partial w}$  have weight -2.

Inside a tube  $T_\varepsilon$ , we write our control system as a term of order -1, plus a residue, that has a certain order w.r.t.  $\varepsilon$ . Here,  $O(\varepsilon^k)$  has to be understood as a smooth term bounded by  $c\varepsilon^k$ ,  $c > 0$ . For a trajectory remaining inside  $T_\varepsilon$ , we have:

$$\begin{aligned} \dot{x} &= u + O(\varepsilon^2), & (a) & & (8) \\ \dot{y}_i &= \frac{1}{2}x' L^i(w)u + O(\varepsilon^2), & i &= 1, \dots, k-1, \\ \dot{w} &= \frac{1}{2}x' M(w)u + O(\varepsilon^2), \end{aligned}$$

where  $L^i(w), M(w)$  are skew-symmetric matrices depending smoothly on  $w$ .

*Remark 12:* In Equation (8.a), one would expect the term  $O(\varepsilon)$  instead of  $O(\varepsilon^2)$ . In fact, the order 2 is due to part (1) in Proposition 10.

In the present case, we define the **Nilpotent Approximation**  $\hat{P}$  along  $\Gamma$  of the problem  $\mathcal{P}$  by keeping the term of order -1 only:

$$\begin{aligned} \hat{P} \quad \dot{x} &= u, & (9) \\ \dot{y}_i &= \frac{1}{2}x' L^i(w)u, & i &= 1, \dots, p-1, \\ \dot{w} &= \frac{1}{2}x' M(w)u. \end{aligned}$$

Let us consider two trajectories  $\xi(t), \hat{\xi}(t)$  of  $\mathcal{P}$  and  $\hat{P}$  corresponding to the same control  $u(t)$ , issued from the same point on  $\Gamma$ , and both arclength-parametrized (which is equivalent to  $\|u(t)\| = 1$ ). For  $t \leq \varepsilon$ , we have the following estimates:

$$\|x(t) - \hat{x}(t)\| \leq c\varepsilon^3, \|y(t) - \hat{y}(t)\| \leq c\varepsilon^3, \|w(t) - \hat{w}(t)\| \leq c\varepsilon^3, \quad (10)$$

for a suitable constant  $c$ .

*Remark 13:* It follows that the distance (either  $d$  or  $\hat{d}$  – the distance associated with the nilpotent approximation) between  $\xi(t), \hat{\xi}(t)$  is smaller than  $\varepsilon^{1+\alpha}$  for some  $\alpha > 0$ .

This fact comes from the estimate just given, and the standard ball-box Theorem [20]. It will be the key point to reduce the motion planning problem to the one of its nilpotent approximation along  $\Gamma$ .

4) *The 2-control case:* In this second case, we have the following general normal form, in normal coordinates. It was proven first in [2], in the corank 1 case. Actually, the proof holds in any corank, without modification.

Consider Normal coordinates with respect to any surface  $\mathcal{S}$ . There are smooth functions,  $\beta(x, y, w), \gamma_i(x, y, w), \delta(x, y, w)$ , such that, on a neighborhood of  $\Gamma$ ,  $\mathcal{P}$  can be written as:

$$\begin{aligned} \dot{x}_1 &= (1 + (x_2)^2\beta)u_1 - x_1x_2\beta u_2, & (11) \\ \dot{x}_2 &= (1 + (x_1)^2\beta)u_2 - x_1x_2\beta u_1, \\ \dot{y}_i &= \gamma_i\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \\ \dot{w} &= \delta\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \end{aligned}$$

where moreover  $\beta$  vanishes on the curve  $\Gamma$ .

The following normal forms can be obtained, on the tube  $T_\varepsilon$ , by just changing coordinates in  $\mathcal{S}$  in an appropriate way. A trajectory  $\xi(t)$  of  $\mathcal{P}$  remaining in  $T_\varepsilon$  satisfies one of the following systems of equations.

a) *Generic 4 – 2 case (see [17]):*

$$\begin{aligned} \dot{x}_1 &= u_1 + O(\varepsilon^3), \\ \dot{x}_2 &= u_2 + O(\varepsilon^3), \\ \dot{y} &= \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right) + O(\varepsilon^2), \\ \dot{w} &= \delta(w)x_1\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right) + O(\varepsilon^3). \end{aligned}$$

We define the nilpotent approximation as:

$$\begin{aligned} (\hat{\mathcal{P}}_{4,2}) \quad \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{y} &= \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \\ \dot{w} &= \delta(w)x_1\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right). \end{aligned}$$

As before, we consider two trajectories  $\xi(t), \hat{\xi}(t)$  of  $\mathcal{P}$  and  $\hat{\mathcal{P}}$  corresponding to the same control  $u(t)$ , issued from the same point on  $\Gamma$ , and both arclength-parametrized (which is equivalent to  $\|u(t)\| = 1$ ). For  $t \leq \varepsilon$ , we have:

$$\|x(t) - \hat{x}(t)\| \leq c\varepsilon^4, \|y(t) - \hat{y}(t)\| \leq c\varepsilon^3, \|w(t) - \hat{w}(t)\| \leq c\varepsilon^4. \quad (12)$$

This estimates implies that, for  $t \leq \varepsilon$ , the distance ( $d$  or  $\hat{d}$ ) between  $\xi(t)$  and  $\hat{\xi}(t)$  is less than  $\varepsilon^{1+\alpha}$  for some  $\alpha > 0$ . Again, it will be the keypoint to reduce our problem to the nilpotent approximation.

b) *Generic 5 – 2 case (see [18]):*

$$\begin{aligned} \dot{x}_1 &= u_1 + O(\varepsilon^3), \\ \dot{x}_2 &= u_2 + O(\varepsilon^3), \\ \dot{y} &= \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right) + O(\varepsilon^2), \\ \dot{z} &= x_2\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right) + O(\varepsilon^3), \\ \dot{w} &= \delta(w)x_1\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right) + O(\varepsilon^3). \end{aligned}$$

We define the nilpotent approximation as:

$$\begin{aligned} (\hat{\mathcal{P}}_{5,2}) \quad \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{y} &= \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \\ \dot{z} &= x_2\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \\ \dot{w} &= \delta(w)x_1\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right). \end{aligned}$$

The estimates necessary to reduce to the nilpotent approximation are:

$$\begin{aligned} \|x(t) - \hat{x}(t)\| &\leq c\varepsilon^4, \|y(t) - \hat{y}(t)\| \leq c\varepsilon^3, \\ \|z(t) - \hat{z}(t)\| &\leq c\varepsilon^4, \|w(t) - \hat{w}(t)\| \leq c\varepsilon^4. \end{aligned} \quad (13)$$

c) *Generic 6 – 2 case (proven in Appendix):*

$$\begin{aligned} \dot{x}_1 &= u_1 + O(\varepsilon^3), \\ \dot{x}_2 &= u_2 + O(\varepsilon^3), \\ \dot{y} &= \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right) + O(\varepsilon^2), \\ \dot{z}_1 &= x_2\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right) + O(\varepsilon^3), \\ \dot{z}_2 &= x_1\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right) + O(\varepsilon^3), \\ \dot{w} &= Q_w(x_1, x_2)\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right) + O(\varepsilon^4), \end{aligned} \quad (14)$$

where  $Q_w(x_1, x_2)$  is a quadratic form in  $x$  depending smoothly on  $w$ .

We define the nilpotent approximation as:

$$\begin{aligned} (\hat{\mathcal{P}}_{6,2}) \quad \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{y} &= \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \\ \dot{z}_1 &= x_2\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \\ \dot{z}_2 &= x_1\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \\ \dot{w} &= Q_w(x_1, x_2)\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right). \end{aligned} \quad (15)$$

The estimates necessary to reduce to the nilpotent approximation are:

$$\begin{aligned} \|x(t) - \hat{x}(t)\| &\leq c\varepsilon^4, \|y(t) - \hat{y}(t)\| \leq c\varepsilon^3, \\ \|z(t) - \hat{z}(t)\| &\leq c\varepsilon^4, \|w(t) - \hat{w}(t)\| \leq c\varepsilon^5. \end{aligned} \quad (16)$$

In fact, the proof of the reduction to this normal form, given in Appendix A, also contains the 4-2 and 5-2 cases.

5) *Invariants in the 6-2 case, and the ball with a trailer:* Let us consider a one-form  $\omega$  that vanishes on  $\Delta'' = [\Delta, [\Delta, \Delta]]$ . Set  $\alpha = d\omega|_{\Delta}$ , the restriction of  $d\omega$  to  $\Delta$ . Set  $H = [F_1, F_2]$ ,  $I = [F_1, H]$ ,  $J = [F_2, H]$ , and consider the  $2 \times 2$  matrix  $A(\xi) = \begin{pmatrix} d\omega(F_1, I) & d\omega(F_2, I) \\ d\omega(F_1, J) & d\omega(F_2, J) \end{pmatrix}$ .

Due to Jacobi Identity,  $A(\xi)$  is a symmetric matrix. Since  $\omega([X, Y]) = d\omega(X, Y)$  in restriction to  $\Delta''$ , we also have  $A(\xi) = \begin{pmatrix} \omega([F_1, I]) & \omega([F_2, I]) \\ \omega([F_1, J]) & \omega([F_2, J]) \end{pmatrix}$ .

Let us consider a gauge transformation, i.e. a feedback that preserves the metric, see e.g. [11], i.e. a change of orthonormal frame  $(F_1, F_2)$  obtained by setting

$$\begin{aligned} \tilde{F}_1 &= \cos(\theta(\xi))F_1 + \sin(\theta(\xi))F_2, \\ \tilde{F}_2 &= -\sin(\theta(\xi))F_1 + \cos(\theta(\xi))F_2. \end{aligned}$$

It is just a matter of tedious computations to check that the matrix  $A(\xi)$  is changed for  $\tilde{A}(\xi) = R_\theta A(\xi) R_{-\theta}$ . On the other hand, the one-form  $\omega$  is defined modulo multiplication by a nonzero function  $f(\xi)$ , and the same holds for  $\alpha$ , since

$d(f\omega) = f d\omega + df \wedge \omega$ , and  $\omega$  vanishes over  $\Delta''$ . Therefore the following lemma holds true:

*Lemma 14:* The ratio  $r(\xi)$  of the (real) eigenvalues of  $A(\xi)$  is an invariant of the structure.

Let us now consider the normal form (14), and compute the form  $\omega = \omega_1 dx_1 + \dots + \omega_6 dw$  along  $\Gamma$  (that is, where  $x, y, z = 0$ ). The computation of all the brackets shows that  $\omega_1 = \omega_2 = \dots = \omega_5 = 0$ . This also shows that in fact, along  $\Gamma$ ,  $A(\xi)$  is just the matrix of the quadratic form  $Q_w$ .

*Lemma 15:* The invariant  $r(\Gamma(t))$  of the problem  $\mathcal{P}$  is the same as the invariant  $\hat{r}(\Gamma(t))$  of the nilpotent approximation along  $\Gamma$ .

Let us compute the ratio  $r$  for the ball with a trailer, Equation (5). We denote by  $A_1, A_2$  the two right-invariant vector fields over  $SO(3, \mathbb{R})$  appearing in (5). We have:

$$\begin{aligned} F_1 &= \frac{\partial}{\partial x_1} + A_1 - \frac{1}{L} \cos(\theta) \frac{\partial}{\partial \theta}, \\ F_2 &= \frac{\partial}{\partial x_2} + A_2 - \frac{1}{L} \sin(\theta) \frac{\partial}{\partial \theta}. \end{aligned}$$

$$[A_1, A_2] = A_3, \quad [A_1, A_3] = -A_2, \quad [A_2, A_3] = A_1.$$

Let us compute the brackets:  $H = A_3 - \frac{1}{L^2} \frac{\partial}{\partial \theta}$ ,  $I = -A_2 - \frac{1}{L^3} \sin(\theta) \frac{\partial}{\partial \theta}$ ,  $J = A_1 + \frac{1}{L^3} \cos(\theta) \frac{\partial}{\partial \theta}$ ,  $[F_1, I] = -A_3 - \frac{1}{L^4} \frac{\partial}{\partial \theta}$ ,  $[F_2, J] = -A_3 - \frac{1}{L^4} \frac{\partial}{\partial \theta}$ , and  $[F_1, J] = [F_2, I] = 0$ .

*Lemma 16:* For the ball with a trailer, the ratio  $r(\xi) = 1$ .

These two last lemmas are a key point in the section IV: they imply in particular that the system of geodesics of the nilpotent approximation is integrable in Liouville sense.

### III. RESULTS

In this section, we summarize and comment most of the results obtained in the papers [13], [14], [15], [17], [18], [19].

#### A. General Results

We need the concept of an  $\varepsilon$ -modification of an asymptotic optimal synthesis.

*Definition 17:* Given a one-parameter family of (absolutely continuous, arclength parametrized) admissible curves  $\gamma_\varepsilon : [0, T_{\gamma_\varepsilon}] \rightarrow \mathbb{R}^n$ , an  $\varepsilon$ -**modification** of  $\gamma_\varepsilon$  is another one-parameter family of (absolutely continuous, arclength parametrized) admissible curves  $\tilde{\gamma}_\varepsilon : [0, T_{\tilde{\gamma}_\varepsilon}] \rightarrow \mathbb{R}^n$  such that for all  $\varepsilon$  and for some  $\alpha > 0$ , if  $[0, T_{\gamma_\varepsilon}]$  is split into subintervals of length  $\varepsilon$  (i.e.  $[0, \varepsilon]$ ,  $[\varepsilon, 2\varepsilon]$ ,  $[2\varepsilon, 3\varepsilon]$ , ...), then:

1.  $[0, T_{\tilde{\gamma}_\varepsilon}]$  is split into corresponding intervals,  $[0, \varepsilon_1]$ ,  $[\varepsilon_1, \varepsilon_1 + \varepsilon_2]$ ,  $[\varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_2 + \varepsilon_3]$ , ... with  $\varepsilon \leq \varepsilon_i < \varepsilon(1 + \varepsilon^\alpha)$ ,  $i = 1, 2, \dots$ ,

2. for each couple of an interval  $I_1 = [\tilde{\varepsilon}_i, \tilde{\varepsilon}_i + \varepsilon]$ , (with  $\tilde{\varepsilon}_0 = 0$ ,  $\tilde{\varepsilon}_1 = \varepsilon_1$ ,  $\tilde{\varepsilon}_2 = \varepsilon_1 + \varepsilon_2$ , ...) and the respective interval  $I_2 = [i\varepsilon, (i+1)\varepsilon]$ ,  $\frac{d}{dt}(\tilde{\gamma})$  and  $\frac{d}{dt}(\gamma)$  coincide over  $I_2$ , i.e.:

$$\frac{d}{dt}(\tilde{\gamma})(\tilde{\varepsilon}_i + t) = \frac{d}{dt}(\gamma)(i\varepsilon + t), \text{ for almost all } t \in [i\varepsilon, (i+1)\varepsilon].$$

*Remark 18:* This concept of an  $\varepsilon$ -**modification** is for the following use: we will construct asymptotic optimal syntheses for the nilpotent approximation  $\hat{\mathcal{P}}$  of problem  $\mathcal{P}$ . Then, the asymptotic optimal syntheses have to be slightly modified in order to realize the interpolation constraints for the original



(non-modified) problem. This has to be done “slightly” for the length of paths to remain equivalent.

In this section it is always assumed, but not stated, that **we consider generic problems only**. One first result is the following:

*Theorem 19:* In the cases 2-step bracket generating, 4-2, 5-2, 6-2, (without singularities), an asymptotic optimal synthesis (relative to the entropy) for  $\mathcal{P}$  is obtained as an  $\varepsilon$ -modification of an asymptotic optimal synthesis for the nilpotent approximation  $\hat{\mathcal{P}}$ . As a consequence the entropy  $E(\varepsilon)$  of  $\mathcal{P}$  is equal to the entropy  $\hat{E}(\varepsilon)$  of  $\hat{\mathcal{P}}$ .

This theorem is proven in [17]. However, we can easily get an idea of the proof, using the estimates of formulas (10, 12, 13, 16).

All these estimates show that, if we apply an  $\varepsilon$ -interpolating strategy to  $\hat{\mathcal{P}}$ , and the same controls to  $\mathcal{P}$ , at time  $\varepsilon$  (or length  $\varepsilon$ , since it is always possible to consider arlength-parametrized trajectories), the endpoints of the two trajectories are at subriemannian distance (either  $d$  or  $\hat{d}$ ) of order  $\varepsilon^{1+\alpha}$ , for some  $\alpha > 0$ . Then the contribution to the entropy of  $\mathcal{P}$ , due to the correction necessary to interpolate  $\Gamma$ , will have higher order.

In the two-step bracket-generating case, the following equality holds:

*Theorem 20:* (two-step bracket-generating case, corank  $k \leq 3$ ) The entropy is equal to  $2\pi$  times the metric complexity:  $E(\varepsilon) = 2\pi MC(\varepsilon)$ .

The reason for this distinction between corank less or more than 3 is very important, and will be explained in the section III-C.

Another very important result is the following **logarithmic lemma**, that describes what happens in the case of a (generic) singularity of  $\Delta$ . In the absence of such singularities, as we shall see, formulas for the entropy (as for the metric complexity) always are of the following type:

$$E(\varepsilon) \simeq \frac{1}{\varepsilon^p} \int_{\Gamma} \frac{dt}{\chi(t)}, \quad (17)$$

where  $\chi(t)$  is a certain invariant along  $\Gamma$ . When the curve  $\Gamma(t)$  crosses transversally a codimension-1 singularity (of  $\Delta'$ , or  $\Delta''$ ), the invariant  $\chi(t)$  vanishes. This may happen at isolated points  $t_i$ ,  $i = 1, \dots, r$  only. In that case, we always have the following:

*Theorem 21:* (logarithmic lemma). The entropy (resp. the metric complexity) satisfies:

$$E(\varepsilon) \simeq -2 \frac{\ln(\varepsilon)}{\varepsilon^p} \sum_{i=1}^r \frac{1}{\rho(t_i)}, \quad \text{where } \rho(t) = \left| \frac{d\chi(t)}{dt} \right|.$$

On the contrary, there are also generic codimension-1 singularities where the curve  $\Gamma$ , at isolated points, becomes tangent to  $\Delta$ , or  $\Delta'$ , ... At these isolated points, the invariant  $\chi(t)$  of Formula 17 tends to infinity. In that case, **the formula (17) remains valid** (the integral converges).

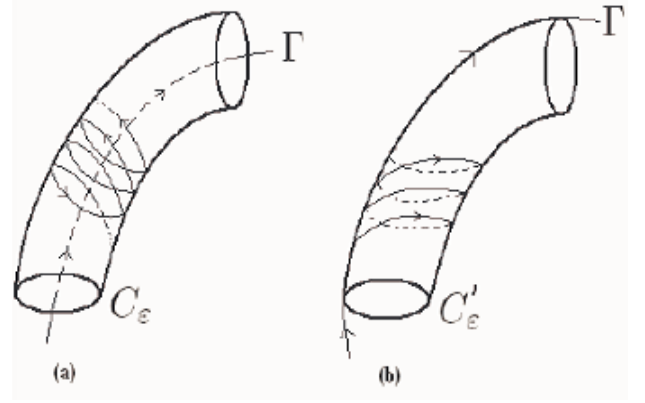


Fig. 4. 3-dimensional contact case

### B. Generic Distribution in $\mathbb{R}^3$

This is the simplest case, and it is important since many cases reduce to it. Let us describe it in details.

Generically, the 3-dimensional space  $M$  contains a 2-dimensional singularity (called the Martinet surface, denoted by  $\mathcal{M}$ ). This singularity is a smooth surface, and (except at isolated points on  $\mathcal{M}$ ), the distribution  $\Delta$  is not tangent to  $\mathcal{M}$ . Generically, the curve  $\Gamma$  crosses  $\mathcal{M}$  transversally at a finite number of isolated points  $t_i$ ,  $i = 1, \dots, r$ . These points are not the special isolated points where  $\Delta$  is tangent to  $\mathcal{M}$  (this would not be generic). They are called Martinet points. This number  $r$  can be zero. There are also other isolated points  $\tau_j$ ,  $j = 1, \dots, l$ , at which  $\Gamma$  is tangent to  $\Delta$  (which means that  $\Gamma$  is almost admissible in a neighborhood of  $\tau_j$ ). Out of  $\mathcal{M}$ , the distribution  $\Delta$  is a contact distribution (a generic property).

Let  $\omega$  be a one-form that vanishes on  $\Delta$  and that is 1 on  $\dot{\Gamma}$ , defined up to multiplication by a function which is 1 along  $\Gamma$ . Along  $\Gamma$ , the restriction  $d\omega|_{\Delta}$  of the 2-form  $d\omega$  can be made into a skew-symmetric endomorphism  $A(\Gamma(t))$  of  $\Delta$  (skew symmetric with respect to the scalar product over  $\Delta$ ), by duality:  $\langle A(\Gamma(t))X, Y \rangle = d\omega(X, Y)$ . Let  $\chi(t)$  denote the moduli of the eigenvalues of  $A(\Gamma(t))$ . We have the following:

*Theorem 22:* 1. If  $r = 0$ ,  $MC(\varepsilon) \simeq \frac{2}{\varepsilon^2} \int_{\Gamma} \frac{dt}{\chi(t)}$ . At points

where  $\chi(t) \rightarrow +\infty$ , the formula is convergent.

2. If  $r \neq 0$ ,  $MC(\varepsilon) \simeq -2 \frac{\ln(\varepsilon)}{\varepsilon^2} \sum_{i=1}^r \frac{1}{\rho(t_i)}$ , where  $\rho(t) = \left| \frac{d\chi(t)}{dt} \right|$ .

3.  $E(\varepsilon) = 2\pi MC(\varepsilon)$ .

Let us describe the asymptotic optimal syntheses. They are shown on Figs. 4, 5.

Fig. 4 concerns the case  $r = 0$  (everywhere contact type). The points where the distribution  $\Delta$  is not transversal to  $\Gamma$  are omitted (again, they do not change anything). Hence  $\Delta$  is also transversal to the cylinders  $C_\varepsilon$ , for  $\varepsilon$  small. Therefore,  $\Delta$  defines (up to sign) a vector field  $X_\varepsilon$  on  $C_\varepsilon$ , tangent to  $\Delta$ , that can be chosen of length 1.

The asymptotic optimal synthesis consists of:

- 1) reaching  $C_\varepsilon$  from  $\Gamma(0)$ ,
- 2) following a trajectory of  $X_\varepsilon$ , and

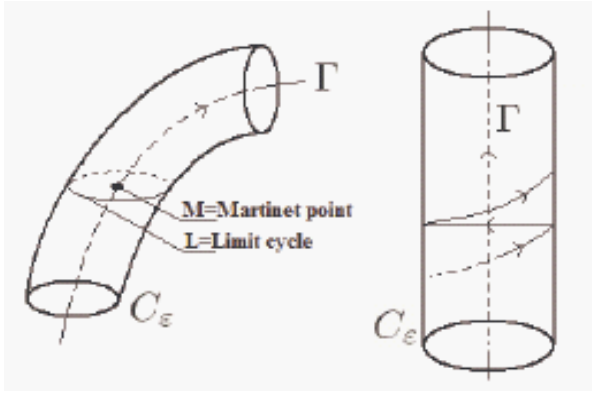


Fig. 5. 3-dimensional Martinet case

### 3) joining $\Gamma(t)$ .

The steps 1 and 3 cost  $2\varepsilon$ , which is negligible w.r.t. the full metric complexity. To get the optimal synthesis for the interpolation entropy, one has to make the same construction, but starting from a subriemannian cylinder  $C'_\varepsilon$  tangent to  $\Gamma$ .

In normal coordinates, in that case, the  $x$ -trajectories are simply circles, and the corresponding optimal controls are trigonometric functions, with period  $\frac{2\pi}{\varepsilon}$ .

Fig. 5 concerns the case  $r \neq 0$  (crossing Martinet surface). At a Martinet point, the vector-field  $X_\varepsilon$  has a limit cycle, which is not tangent to the distribution.

The asymptotic optimal strategy consists of:

- 1) following a trajectory of  $X_\varepsilon$  till reaching the height of the center of the limit cycle,
- 2) crossing the cylinder, with a negligible cost  $2\varepsilon$ , and
- 3) following a trajectory of the opposite vector field  $-X_\varepsilon$ .

The strategy for entropy is similar, but using the tangent cylinder  $C'_\varepsilon$ .

### C. The Two-Step Bracket-Generating Case

For the corank  $k \leq 3$ , the situation is very similar to the 3-dimensional case. It can be completely reduced to it. For details, see [15].

At this point, this strange fact appears: there is the limit corank  $k = 3$ . If  $k > 3$  only, new phenomena appear. Let us explain now the reason for this.

Let us consider the following mapping:

$$\begin{aligned} \mathcal{B}_\xi : \Delta_\xi \times \Delta_\xi &\rightarrow T_\xi M / \Delta_\xi, \\ (X, Y) &\rightarrow [X, Y] + \Delta_\xi. \end{aligned}$$

It is a well defined **tensor mapping**, which means that it actually applies to vectors (and not to vector fields, as expected from the definition). This is due to the following formula, for a one-form  $\omega$ :  $d\omega(X, Y) = \omega([X, Y]) + L_X\omega(Y) - L_Y\omega(X)$ . Let us call  $I_\xi$  the image by  $\mathcal{B}_\xi$  of the product of two unit balls in  $\Delta_\xi$ . The following holds:

**Theorem 23:** For a generic  $\mathcal{P}$ , and  $k \leq 3$ , the sets  $I_{\Gamma(t)}$  are **convex**.

This theorem is shown in [15], with the consequences that we will state just below.

This is no more true for  $k > 3$ , the first catastrophic case being the case 10-4 (a  $p = 4$  distribution in  $\mathbb{R}^{10}$ ). The

intermediate cases  $k = 4, 5$  in dimension 10 are interesting, since on some open subsets of  $\Gamma$ , the convexity property may hold or not. These cases are studied in the paper [18].

The main consequence of this convexity property is that everything reduces (out of singularities where the logarithmic lemma applies) to the 3-dimensional contact case. We briefly summarize the results.

Consider the one-forms  $\omega$  that vanish on  $\Delta$  and that are 1 on  $\dot{\Gamma}$ , and again, by duality w.r.t. the metric over  $\Delta$ , define  $d\omega|_\Delta(X, Y) = \langle AX, Y \rangle$ , for vector fields  $X, Y$  in  $\Delta$ . Now, we have along  $\Gamma$ , a  $(k-1)$ -parameter affine family of skew symmetric endomorphisms  $A_{\Gamma(t)}$  of  $\Delta_{\Gamma(t)}$ . Let us

write  $A_{\Gamma(t)}(\lambda) = A_{\Gamma(t)}^0 + \sum_{i=1}^{k-1} \lambda_i A_{\Gamma(t)}^i$ , and set  $\chi(t) = \inf_\lambda \|A_{\Gamma(t)}(\lambda)\| = \|A_{\Gamma(t)}(\lambda^*(t))\|$ .

Out of isolated points of  $\Gamma$  (that count for nothing both in the metric complexity and in the entropy), the  $t$ -one-parameter family  $A_{\Gamma(t)}(\lambda^*(t))$  can be smoothly block-diagonalized (with  $2 \times 2$  blocks), using a gauge transformation along  $\Gamma$ . After this gauge transformation, the 2-dimensional eigenspace corresponding to the largest (in moduli) eigenvalue of  $A_{\Gamma(t)}(\lambda^*(t))$ , corresponds to the two first coordinates in the distribution, and to the 2 first controls. In the asymptotic optimal synthesis, all other controls are put to zero (here the convexity property is used), and the picture of the asymptotic optimal synthesis is exactly that of the 3-dimensional contact case. We still have the formulas:

$$MC(\varepsilon) \simeq \frac{2}{\varepsilon^2} \int_{\Gamma} \frac{dt}{\chi(t)}, \quad E(\varepsilon) = 2\pi MC(\varepsilon).$$

The case  $k > 3$  was first treated in [17] in the 10-dimensional case, and was completed in general in [19]. In that case, the situation does not reduce to the 3-dimensional contact case: the optimal controls, in the asymptotic optimal synthesis for the nilpotent approximation are still trigonometric controls, but with different periods that are successive integer multiples of a given basic period. New invariants  $\lambda_{\theta(t)}^j$  appear, and the formula for the entropy is:

$$E(\varepsilon) \simeq \frac{2\pi}{\varepsilon^2} \int_0^T \frac{\sum_{j=1}^r j \lambda_{\theta(t)}^j}{\sum_{j=1}^r (\lambda_{\theta(t)}^j)^2} d\theta.$$

The optimal controls are of the form:

$$\begin{aligned} u_{2j-1}(t) &= -\sqrt{\frac{j \lambda_{\theta(t)}^j}{\sum_{j=1}^r j \lambda_{\theta(t)}^j}} \sin\left(\frac{2\pi j t}{\varepsilon}\right), \\ u_{2j}(t) &= \sqrt{\frac{j \lambda_{\theta(t)}^j}{\sum_{j=1}^r j \lambda_{\theta(t)}^j}} \cos\left(\frac{2\pi j t}{\varepsilon}\right), \quad j = 1, \dots, r \\ u_{2r+1}(t) &= 0 \text{ if } p \text{ is odd.} \end{aligned} \quad (18)$$

These last formulas hold in the free case only (i.e. the case where the corank  $k = \frac{p(p-1)}{2}$ , the dimension of the second homogeneous component of the free Lie-algebra with  $p$  generators). The non-free case is more complicated (see [19]). There is a paper by R. W. Brockett [8] closely related to these results.



To prove all the results in this section, one has to proceed as follows:

- 1) use the theorem of reduction to nilpotent approximation (19),
- 2) use the Pontriaguin's maximum principle on the normal form of the nilpotent approximation, in normal coordinates.

#### D. The 2-control case, in $\mathbb{R}^4$ and $\mathbb{R}^5$ .

These cases correspond respectively to the car with a trailer (Example 2) and the ball on a plate (Example 3).

We use Theorem 19 of reduction to nilpotent approximation, and we consider the normal forms  $\hat{\mathcal{P}}_{4,2}$ ,  $\hat{\mathcal{P}}_{5,2}$  of Section II-C4. In both cases, we change the variable  $w$  for  $\tilde{w}$  such that  $d\tilde{w} = \frac{dw}{\delta(w)}$ . We look for arclength-parametrized trajectories of the nilpotent approximation (i.e.  $(u_1)^2 + (u_2)^2 = 1$ ), that start from  $\Gamma(0)$ , and reach  $\Gamma$  in fixed time  $\varepsilon$ , maximizing  $\int_0^\varepsilon \dot{w}(\tau) d\tau$ .

Abnormal extremals do not come into the picture, and optimal curves correspond to the Hamiltonian

$$H = \sqrt{(PF_1)^2 + (PF_2)^2},$$

where  $P$  is the adjoint vector. It turns out that, in our normal coordinates, the same trajectories are optimal for both the 4-2 and the 5-2 cases (one has just to notice that the solution of the 4-2 case meets the extra interpolation condition corresponding to the 5-2 case).

Setting as usual  $u_1 = \cos(\varphi) = PF_1$ ,  $u_2 = \sin(\varphi) = PF_2$ , we get  $\dot{\varphi} = P[F_1, F_2]$ ,  $\ddot{\varphi} = -P[F_1, [F_1, F_2]]PF_1 - P[F_2, [F_1, F_2]]PF_2$ .

At this point, we have to notice that only the components  $P_{x_1}, P_{x_2}$  of the adjoint vector  $P$  are not constant (the Hamiltonian in the nilpotent approximation depends only on the  $x$ -variables), therefore,  $P[F_1, [F_1, F_2]]$  and  $P[F_2, [F_1, F_2]]$  are constant (the third brackets are also constant vector fields). Hence,  $\ddot{\varphi} = \alpha \cos(\varphi) + \beta \sin(\varphi) = \alpha \dot{x}_1 + \beta \dot{x}_2$  for appropriate constants  $\alpha, \beta$ . It follows that, for another constant  $k$ , we have, for the optimal curves of the nilpotent approximation, in normal coordinates  $x_1, x_2$ :

$$\begin{aligned} \dot{x}_1 &= \cos(\varphi), \\ \dot{x}_2 &= \sin(\varphi), \\ \dot{\varphi} &= k + \lambda x_1 + \mu x_2. \end{aligned}$$

- Remark 24:*
- 1) It means that we are looking for curves in the  $x_1, x_2$  plane, whose curvature is an affine function of the position,
  - 2) in the two-step bracket generating case (contact case), optimal curves were circles, i.e. curves of constant curvature,
  - 3) the conditions of  $\varepsilon$ -interpolation of  $\Gamma$  say that these curves must be periodic (there will be more details on this point in the next section), that the area of a loop must be zero ( $y(\varepsilon) = 0$ ), and finally (in the 5-2 case) that another moment must be zero.

It is easily seen that such a curve, meeting these interpolation conditions, must be an elliptic curve of elastica-type. The

periodicity and vanishing surface requirements imply that it can be the periodic elastic curve shown on Fig. 7.B only, and parametrized in a certain way [31].

The formulas are, in terms of the standard Jacobi elliptic functions:

$$\begin{aligned} u_1(t) &= 1 - 2dn\left(K\left(1 + \frac{4t}{\varepsilon}\right)\right)^2, \\ u_2(t) &= -2dn\left(K\left(1 + \frac{4t}{\varepsilon}\right)\right)sn\left(K\left(1 + \frac{4t}{\varepsilon}\right)\right)\sin\left(\frac{\varphi_0}{2}\right), \end{aligned}$$

where  $\varphi_0 = 130^\circ$  (following [31], p. 403) and  $\varphi_0 = 130.692^\circ$  (following Mathematica<sup>®</sup>), with  $k = \sin(\frac{\varphi_0}{2})$  and  $K(k)$  is the quarter period of the Jacobi elliptic functions. The trajectory in the  $x_1, x_2$  plane, shown on Fig. 7.B, has equations:

$$\begin{aligned} x_1(t) &= -\frac{\varepsilon}{4K} \left[ \frac{-4Kt}{\varepsilon} + 2(Eam(\frac{4Kt}{\varepsilon} + K) - Eam(K)) \right], \\ x_2(t) &= k \frac{\varepsilon}{2K} cn\left(\frac{4Kt}{\varepsilon} + K\right). \end{aligned}$$

On Figs. 2 and 3, we show the  $\varepsilon$ -approximated trajectories, which are ‘‘repeated small deformations’’ of the above basic trajectory both for the *car with a trailer* and the *ball rolling on a plane*. In the first example, the non-admissible trajectory to be approximated is: *going transversally to the car's orientation while keeping the trailer aligned with the car*. In the second example, the expected trajectory is: *going straight while keeping the same orientation* (i.e. slipping without rolling). An animated simulation of the *ball rolling on a plane* is available on the website [38].

The formula for the entropy is, in both cases:

$$E(\varepsilon) = \frac{3}{2\sigma\varepsilon^3} \int_\Gamma \frac{dt}{\delta(t)},$$

where  $\sigma$  is a universal constant,  $\sigma \approx 0.00580305$ .

The details of the computations for the 4-2 case can be found in [17], and in [18] for the 5-2 case.

#### IV. THE BALL WITH A TRAILER

We start by using Theorem 19, to reduce to the nilpotent approximation along  $\Gamma$ :

$$\begin{aligned} (\hat{\mathcal{P}}_{6,2}) \quad \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{y} &= \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \\ \dot{z}_1 &= x_2\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \\ \dot{z}_2 &= x_1\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \\ \dot{w} &= Q_w(x_1, x_2)\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right). \end{aligned} \tag{19}$$

By Lemma 16, we can consider that

$$Q_w(x_1, x_2) = \delta(w) \left( (x_1)^2 + (x_2)^2 \right) \tag{20}$$

where  $\delta(w)$  is **the main invariant**. In fact, it is the only invariant for the nilpotent approximation along  $\Gamma$ . Moreover, if we reparametrize  $\Gamma$  by setting  $dw := \frac{dw}{4\delta(w)}$ , we can consider that  $\delta(w) = 1/4$ .

Then, we want to maximize  $\int \dot{w} dt$  in fixed time  $\varepsilon$ , with the interpolation conditions:  $x(0) = 0$ ,  $y(0) = 0$ ,  $z(0) = 0$ ,  $w(0) = 0$ ,  $x(\varepsilon) = 0$ ,  $y(\varepsilon) = 0$ ,  $z(\varepsilon) = 0$ .

From Lemma 28 in the appendix, we know that the optimal trajectory is smooth and periodic, (of period  $\varepsilon$ ).

Clearly, the optimal trajectory has also to be a length minimizer, then we have to consider the usual Hamiltonian for length:  $H = \frac{1}{2}((PF_1)^2 + (PF_2)^2)$ , in which  $P = (p_1, \dots, p_6)$  is the adjoint vector. It is easy to see that the abnormal extremals do not come into the picture (cannot be optimal with our additional interpolation conditions), and in fact, we will show that **the Hamiltonian system corresponding to the Hamiltonian  $H$  is integrable.**

*Remark 25:* This integrability property is no more true in the general 6-2 case. **It holds only for the ball with a trailer.**

As usual, we work in Poincaré coordinates, i.e. we consider level  $\frac{1}{2}$  of the Hamiltonian  $H$ , and we set:

$$u_1 = PF = \sin(\varphi), \quad u_2 = PG = \cos(\varphi).$$

Differentiating twice, we get

$$\dot{\varphi} = P[F, G], \quad \ddot{\varphi} = -PFFG.PF - PGFG.PG,$$

where  $FFG = [F, [F, G]]$  and  $GFG = [G, [F, G]]$ . We set  $\lambda = -PFFG$ ,  $\mu = -PGFG$ , and we get the equation:

$$\ddot{\varphi} = \lambda \sin(\varphi) + \mu \cos(\varphi). \quad (21)$$

Now, we compute  $\dot{\lambda}$  and  $\dot{\mu}$ . We get, with similar notations as above for the brackets (we bracket from the left):

$$\begin{aligned} \dot{\lambda} &= PFFFG.PF + PGFFG.PG, \\ \dot{\mu} &= PFGFG.PF + PGGFG.PG, \end{aligned}$$

and computing the brackets, we see that  $GFFG = FGFG = 0$ . Also, since the Hamiltonian does not depend on  $y, z, w$ , we get that  $p_3, p_4, p_5$ , and  $p_6$  are constants. Computing the brackets  $FFG$  and  $GFG$ , we get that

$$\lambda = \frac{3}{2}p_5 + p_6x_1, \quad \mu = \frac{3}{2}p_4 + p_6x_2,$$

and then,  $\dot{\lambda} = p_6 \sin(\varphi)$  and  $\dot{\mu} = p_6 \cos(\varphi)$ . Then, by (21),  $\ddot{\varphi} = \frac{\lambda \dot{\lambda}}{p_6} + \frac{\mu \dot{\mu}}{p_6}$ , and finally:

$$\begin{aligned} \dot{x}_1 &= \sin(\varphi), \\ \dot{x}_2 &= \cos(\varphi), \\ \dot{\varphi} &= K + \frac{1}{2p_6}(\lambda^2 + \mu^2), \\ \dot{\lambda} &= p_6 \sin(\varphi), \\ \dot{\mu} &= p_6 \cos(\varphi). \end{aligned} \quad (22)$$

Setting  $\omega = \frac{\lambda}{p_6}, \delta = \frac{\mu}{p_6}$ , we obtain:

$$\begin{aligned} \dot{\omega} &= \sin(\varphi), \\ \dot{\delta} &= \cos(\varphi), \\ \dot{\varphi} &= K + \frac{p_6}{2}(\omega^2 + \delta^2). \end{aligned}$$

It means that the plane curve  $(\omega(t), \delta(t))$  has a curvature which is a quadratic function of the distance to the origin. Then, the optimal curve  $(x_1(t), x_2(t))$  projected to the horizontal plane of the normal coordinates has a curvature which is a quadratic function of the distance to some point.

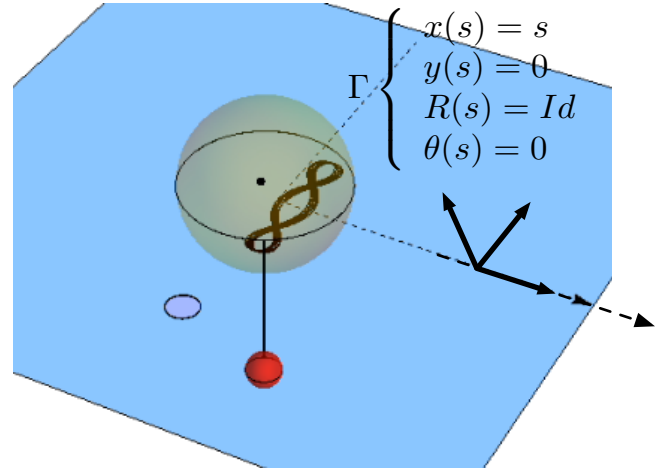


Fig. 6. Parking the ball with a trailer. See also the simulation available on the website [38]

Following Lemma 23 in the appendix, this system of equations is integrable.

Summarizing all the results, we get the following theorem.

**Theorem 26: (asymptotic optimal synthesis for the ball with a trailer)** The asymptotic optimal synthesis is an  $\varepsilon$ -modification of the one of the nilpotent approximation. The latter has the following properties in normal coordinates, in projection to the horizontal plane  $(x_1, x_2)$ :

- 1) it is a closed smooth periodic curve, whose curvature is a function of the square distance to some point,
- 2) the area and the  $2^{nd}$  order moments  $\int_{\Gamma} x_1(x_2 dx_1 - x_1 dx_2)$  and  $\int_{\Gamma} x_2(x_2 dx_1 - x_1 dx_2)$  are zero,
- 3) the entropy is given by the formula:  $E(\varepsilon) = \frac{\sigma}{\varepsilon^4} \int_{\Gamma} \frac{dw}{\delta(w)}$ , where  $\delta(w)$  is the main invariant from (20), and  $\sigma$  is a universal constant.

In fact we can go a little bit further to integrate explicitly the system (22). Set  $\bar{\lambda} = \cos(\varphi)\lambda - \sin(\varphi)\mu$ ,  $\bar{\mu} = \sin(\varphi)\lambda + \cos(\varphi)\mu$ . We obtain:

$$\begin{aligned} \frac{d\bar{\lambda}}{dt} &= -\bar{\mu}(K + \frac{1}{2p_6}(\bar{\lambda}^2 + \bar{\mu}^2)), \\ \frac{d\bar{\mu}}{dt} &= p_6 + \bar{\lambda}(K + \frac{1}{2p_6}(\bar{\lambda}^2 + \bar{\mu}^2)). \end{aligned}$$

This is a 2 dimensional (integrable) Hamiltonian system. The Hamiltonian is:

$$H_1 = -p_6 \bar{\lambda} - \frac{p_6}{2} (K + \frac{1}{2p_6}(\bar{\lambda}^2 + \bar{\mu}^2))^2.$$

This Hamiltonian system is therefore integrable, and solutions can be expressed in terms of hyperelliptic functions. A little numerics allows to show, on Fig. 7.C, the optimal  $x$ -trajectory in the horizontal plane of the normal coordinates.

On Fig. 6, we show the motion of the ball with a trailer on the plane (motion of the contact point between the ball and the plane). Here, the problem is to move along the  $x$ -axis, in the physical coordinates, while keeping both the frame attached to the ball and the angle of the trailer constant.

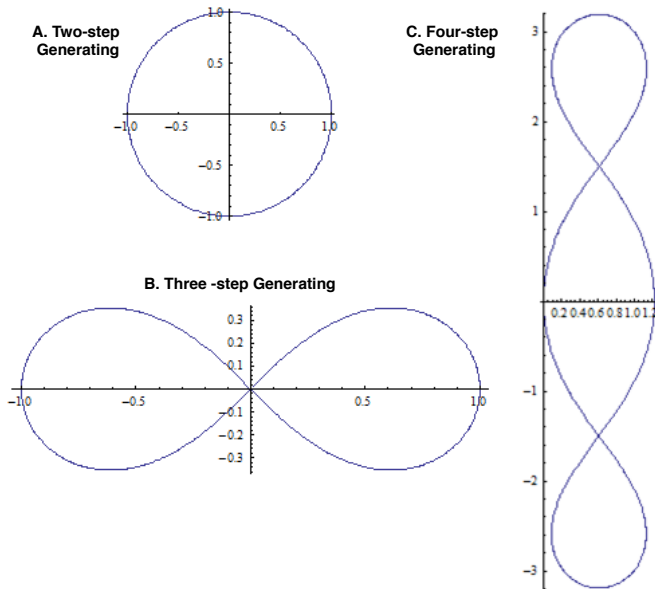


Fig. 7. The dances of minimal entropy in the 2-control case

## V. EXPECTATIONS AND CONCLUSIONS

### A. Universality of Some Pictures in Normal Coordinates

Our first conclusion is the following: there are certain universal pictures for the motion planning problem, in corank less or equal to 3, and in rank 2, with 4 brackets at most (could be 5 brackets at a singularity, with the logarithmic lemma).

These figures are, in the two-step bracket generating case: a circle, for the third bracket: the periodic elastica, for the 4<sup>th</sup> bracket: the plane curve of Fig. 7.C.

They are periodic plane curves whose curvature is respectively: a constant, a linear function of the position, a quadratic function of the position. This is, as shown on Fig. 7, the clear beginning of a series.

In the case of more than two inputs the question is: what is the equivalent of this series? In fact, according to our results in Sections III-B, III-C we just know the first term (first bracket) of the series: up to corank 3 it is still a circle (Section III-C) and for higher corank, it is still a trigonometric curve but with several periods that are successive multiples of a basic one, see Formula (18).

### B. Robustness

As one can see, in many cases (2 controls, or corank  $k \leq 3$ ), our strategy is extremely robust in the following sense: the asymptotic optimal syntheses do not depend, from the qualitative point of view, of the metric chosen. They depend only on the number of brackets needed to generate the space.

### C. The Practical Importance of Normal Coordinates

The main practical problem of implementation of our strategy comes with the  $\varepsilon$ -modifications. How to compute them? How to implement them? In fact, the  $\varepsilon$ -modifications count at higher order in the entropy. If not applied, they may cause deviations that are not negligible. The high order w.r.t.  $\varepsilon$  in

the estimates of the error between the original system and its nilpotent approximation (Formulas 10, 12, 13, 16) make these deviations very small. It is why the use of our concept of a nilpotent approximation along  $\Gamma$ , based upon normal coordinates is very efficient in practice.

On the other hand, when a correction appears to be needed (after a non-negligible deviation), it corresponds to brackets of lower order. For example, in the case of the ball with a trailer (4<sup>th</sup> bracket), the  $\varepsilon$ -modification corresponds to brackets of order 2 or 3. The optimal pictures corresponding to these orders can still be used to perform the  $\varepsilon$ -modifications.

### D. Final Conclusion

This approach, to approximate optimally non-admissible paths of nonholomic systems, looks very efficient, and in a sense, universal. Of course, the theory is not complete, but the cases under consideration (first, 2-step bracket-generating, and second, two controls) correspond to many practical situations. Nonetheless, there is still a lot of work to do in order to cover all interesting cases. However, the methodology to go ahead is rather clear.

## APPENDIX A

### NORMAL FORM IN THE 6-2 CASE

We start from the general normal form (11) in normal coordinates:

$$\begin{aligned}\dot{x}_1 &= (1 + (x_2)^2\beta)u_1 - x_1x_2\beta u_2, \\ \dot{x}_2 &= (1 + (x_1)^2\beta)u_2 - x_1x_2\beta u_1, \\ \dot{y}_i &= \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right)\gamma_i(x, y, w), \\ \dot{w} &= \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right)\delta(x, y, w).\end{aligned}$$

We perform a succession of changes of parametrization of the surface  $\mathcal{S}$  (w.r.t. which normal coordinates were constructed). These coordinate changes will always preserve the fact that  $\Gamma(t)$  is the point  $x = 0, y = 0, w = t$ .

Remind that  $\beta$  vanishes on  $\Gamma$ , and since  $x$  has order 1, we can already write on  $T_\varepsilon$ :  $\dot{x} = u + O(\varepsilon^3)$ . One of the  $\gamma_i$ 's (say  $\gamma_1$ ) has to be nonzero for  $\Gamma$  not to be tangent to  $\Delta'$ . Then,  $y_1$  has order 2 on  $T_\varepsilon$ . Set  $\tilde{y}_i = y_i - \frac{\gamma_i}{\gamma_1}y_1$ , for  $i > 1$ . The differentiation gives  $\frac{d\tilde{y}_i}{dt} = \dot{y}_i - \frac{\gamma_i}{\gamma_1}\dot{y}_1 + O(\varepsilon^2)$ . We set  $z_1 = \tilde{y}_2$ ,  $z_2 = \tilde{y}_3$  since they have order 3. We also set  $w := w - \frac{\delta}{\gamma_1}y_1$ . We are at the following point:

$$\begin{aligned}\dot{x} &= u + O(\varepsilon^3), \\ \dot{y} &= \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right)\gamma_1(w) + O(\varepsilon^2), \\ \dot{z}_i &= \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right)L_i(w).x + O(\varepsilon^3), \\ \dot{w} &= \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right)\delta(w).x + O(\varepsilon^3),\end{aligned}$$

where  $L_i(w).x$ , and  $\delta(w).x$  are linear in  $x$ . The function  $\gamma_1(w)$  can be put to 1 in the same way by setting  $y := \frac{y}{\gamma_1(w)}$ .

Now let  $T(w)$  be an invertible  $2 \times 2$  matrix. Set  $\tilde{z} = T(w)z$ . It is easy to see that we can choose  $T(w)$  such that:

$$\begin{aligned}\dot{x} &= u + O(\varepsilon^3), \\ \dot{y} &= \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right) + O(\varepsilon^2), \\ \dot{z}_i &= \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right)x_i + O(\varepsilon^3), \\ \dot{w} &= \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right)\delta(w).x + O(\varepsilon^3).\end{aligned}$$

Next, a change of the form:  $w := w + L(w).z$ , where  $L(w).z$  is linear in  $z$  kills  $\delta(w)$  and yields  $\dot{w} = \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right)O(\varepsilon^2)$ .

This  $O(\varepsilon^2)$  has to be of the form  $Q_w(x) + h(w)y + O(\varepsilon^3)$ , where  $Q_w(x)$  is quadratic in  $x$ . If we kill  $h(w)$ , we get the expected result. This is done with a change of coordinates of the form:  $w := w + \varphi(w)\frac{y^2}{2}$ .

## APPENDIX B

### PLANE CURVES WHOSE CURVATURE IS A FUNCTION OF THE DISTANCE TO THE ORIGIN

Although this result is already known [35], we provide here a very simple proof.

Consider a plane curve  $(x(t), y(t))$ , whose curvature is a function of the distance from the origin, that is:

$$\dot{x} = \cos(\varphi), \quad \dot{y} = \sin(\varphi), \quad \dot{\varphi} = k(x^2 + y^2), \quad (23)$$

for a certain smooth function  $k(\cdot)$ .

*Lemma 27:* Equation (23) is integrable.

*Proof:* Set  $\bar{x} = x \cos(\varphi) + y \sin(\varphi)$ ,  $\bar{y} = -x \sin(\varphi) + y \cos(\varphi)$ . Then  $k(\bar{x}^2 + \bar{y}^2) = k(x^2 + y^2)$ . The derivatives are:

$$\frac{d\bar{x}}{dt} = 1 + \bar{y}k(\bar{x}^2 + \bar{y}^2), \quad \frac{d\bar{y}}{dt} = -\bar{x}k(\bar{x}^2 + \bar{y}^2). \quad (24)$$

We only need to show that (24) is a Hamiltonian system. Indeed, since it is a two dimensional problem, it is always Liouville-integrable. Therefore, we are looking for solutions of the system of PDE's:

$$\frac{\partial H}{\partial \bar{x}} = 1 + \bar{y}k(\bar{x}^2 + \bar{y}^2), \quad \frac{\partial H}{\partial \bar{y}} = -\bar{x}k(\bar{x}^2 + \bar{y}^2).$$

They always do exist since the Schwartz integrability conditions are satisfied:  $\frac{\partial^2 H}{\partial \bar{x} \partial \bar{y}} = \frac{\partial^2 H}{\partial \bar{y} \partial \bar{x}} = 2\bar{x}\bar{y}k'$ . ■

## APPENDIX C

### PERIODICITY OF THE OPTIMAL CURVES IN THE 6-2 CASE

We consider the nilpotent approximation  $\hat{\mathcal{P}}_{6,2}$  given in formula (15):

$$\begin{aligned}(\hat{\mathcal{P}}_{6,2}) \quad \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{y} &= \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \\ \dot{z}_1 &= x_2\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \\ \dot{z}_2 &= x_1\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \\ \dot{w} &= Q_w(x_1, x_2)\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right).\end{aligned} \quad (25)$$

We consider the particular case of the ball with a trailer. Then, according to Lemma 16, the ratio  $r(\xi) = 1$ .

It follows that the last equation can be rewritten  $\dot{w} = \delta(w)((x_1)^2 + (x_2)^2)\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right)$  for some never vanishing function  $\delta(w)$  (otherwise it contradicts the full rank of  $\Delta^{(3)}$ ). We can change the coordinate  $w$  for  $\tilde{w}$  such that  $d\tilde{w} = \frac{dw}{\delta(w)}$ , which leads to:

$$\begin{aligned}(\hat{\mathcal{P}}_{6,2}) \quad \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{y} &= \left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \\ \dot{z}_1 &= x_2\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \\ \dot{z}_2 &= x_1\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right), \\ \dot{w} &= ((x_1)^2 + (x_2)^2)\left(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2\right).\end{aligned} \quad (26)$$

This is a right invariant system on  $\mathbb{R}^6$  with coordinates  $\xi = (\varsigma, w) = (x, y, z, w)$ , for a certain nilpotent Lie group structure over  $\mathbb{R}^6$  (denoted by  $G$ ). The group law is of the form  $(\varsigma_2, w_2)(\varsigma_1, w_1) = (\varsigma_1 * \varsigma_2, w_1 + w_2 + \Phi(\varsigma_1, \varsigma_2))$ , for a certain function  $\Phi$  and where  $*$  is the multiplication of another Lie group structure on  $\mathbb{R}^5$ , with coordinates  $\varsigma$  (denoted by  $G_0$ ). In order to establish the formula for the group law, although it is possible, we need not to compute it completely. In fact,  $G$  is a central extension of  $\mathbb{R}$  by  $G_0$ , and for this type of central extension, the result is a consequence of the following facts only:

- 1)  $\frac{\partial}{\partial w}$  is a right invariant vector field,
- 2) associativity of the law,
- 3)  $\{e, w\}$  is the center of  $G$  where  $e$  denotes the unit of  $G_0$ .

*Lemma 28:* The trajectories of (26) that maximize  $\int \dot{w} dt$  in fixed time  $\varepsilon$ , with interpolating conditions  $\varsigma(0) = \varsigma(\varepsilon) = 0$ , have a periodic projection on  $\varsigma$  (i.e.  $\varsigma(t)$  is smooth and periodic of period  $\varepsilon$ ).

- Remark 29:*
- 1) Due to the invariance of (26) with respect to  $w$ , it is equivalent to consider the problem with the more restrictive terminal conditions  $\varsigma(0) = \varsigma(\varepsilon) = 0$ ,  $w(0) = 0$ .
  - 2) The scheme of this proof works also to show periodicity in the case of distributions with flag 2-3-4 and 2-3-5.

The idea for this proof was given to us by A. Agrachev.

*Proof:* Let  $(\varsigma, w_1), (\varsigma, w_2)$  be initial and terminal points of an optimal solution of our problem with relaxed boundary conditions  $\varsigma(0) = \varsigma(\varepsilon)$  only. By right translation by  $(\varsigma^{-1}, 0)$ , this trajectory is mapped into another trajectory of the system, with initial and terminal points  $(0, w_1 + \Phi(\varsigma, \varsigma^{-1}))$  and  $(0, w_2 + \Phi(\varsigma, \varsigma^{-1}))$ . Hence, the cost  $\int \dot{w}(t) dt$  for this new trajectory has the same value. As one can see, the optimal cost is in fact independent of the  $\varsigma$ -coordinate of the initial and terminal conditions.

Therefore, our problem is the same as maximizing  $\int \dot{w}(t) dt$  but with the (larger) endpoint condition  $\varsigma(0) = \varsigma(\varepsilon)$  (free).

We can now apply the general transversality conditions of Theorem 12.15, page 188 of [5]. It says that the initial and terminal covectors  $(p_\zeta^1, p_w^1)$  and  $(p_\zeta^2, p_w^2)$  are such that  $p_\zeta^1 = p_\zeta^2$ . This is enough to show periodicity. ■

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