ON THE SPHERICAL HAUSDORF MEASURE IN SUBRIEMANNIAN GEOMETRY

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Dedicated to A. Agrachev for his 60th birthday

Abstract. In this paper, we consider generic corank-2 subriemannian structures, and we show that the Spherical Hausdorf measure is always a $\mathcal{C}^1$-smooth volume, which is in fact generically $\mathcal{C}^2$-smooth out of a stratified subset of codimension 7. In particular, for rank 4, it is generically $\mathcal{C}^2$.

This is a continuation of previous works of the auhors.

1. Introduction

In this paper, we consider subriemannian structures $s = (\Delta, g)$ over a n-dimensional manifold M. The distribution Δ has rank p and corank $k = n - p$, and g is a riemannian metric over Δ. In most of the paper, $k = 2$. Moreover, the distribution is assumed to be 2-step bracket generating.

The set $\mathcal{S}$ of such (corank 2, 2-step bracket generating) subriemannian structures over M is embedded withe the $\mathcal{C}^\infty$ Whitney topology.

As will be recalled in the next section, there is a natural smooth measure associated with the structure s, called the Popp measure (see [8]). It has been shown in [1] that the Radon-Nykodim derivative $R_a(\xi)$ of the spherical Haussdorf measure with respect to the Popp measure is just given by the Popp-volume of the unit ball of the nilpotent approximation of s at the point ξ. Moreover, in the same paper, when $k = 1$, it is shown that $R_a(\xi)$ is a $\mathcal{C}^3$ function, which is not $\mathcal{C}^5$ in general.

The nonsmoothness is due to certain generic singularity of the structure. The degree 3 of differentiability is due to the fact that, in the corank one case, the conjugate locus of the Nilpotent approximation coincides with the cut locus. This is no more true for higher corank.

In particular, this is shown in [4], in the corank 2 case, and an explicit characterization of the cut-locus is given. In the same paper, it has been shown that $R_a(\xi)$ is generically $\mathcal{C}^1$ for $n = 4, k = 2$.

The purpose of this paper is to show that in fact, due to to this explicit simple characterization of the cut-locus, the following holds:

Theorem 1. *(corank $k = 2$) 1. The Radon-Nykodim derivative $R_a(\xi)$ is always $\mathcal{C}^1$*

2. The Radon-Nykodim derivative $R_a(\xi)$ is generically (residual) $\mathcal{C}^2$, out of a stratified set of codimension 7. In the particular case $n = 4$, *there is an open-dense subset of $\mathcal{S}$ for which $R_a(\xi)$ is $\mathcal{C}^2$-smooth.*

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The paper is organized as follows: in the next section 2, we recall the definition of the Popp measure, and of the nilpotent approximation of $s = (\Delta, g)$. We restate the main result of [4], which is our key point.

In Section 3, we give the proof of Theorem 1. For this, we need an adaptation of a very old result of Arnold [3], to the case of versal defomations of real skew-symmetric matrices. This adaptation is presented in our appendix, together with certain basic facts about quaternions, that are a very convenient tool in this study.

2. Prerequisites

2.1. **Nilpotent approximation.** We define the nilpotent approximation in the

two-step baracket generating case only. The tensor mapping:

$$[.,.] : \Delta_\xi \times \Delta_\xi \to T_\xi M/\Delta_\xi, \tag{2.1a}$$

is skew symmetric. Then, for any $Z^* \in (T_\xi M/\Delta_\xi)^*$ we have:

$$Z^*([X,Y] + \Delta_\xi) =< A_{Z^*}(X), Y >_g$$

for some g-skew-symmetric endomorphism A_{Z^*} of Δ_ξ. The mapping $Z^* \to A_{Z^*}$ is linear, and its image is denoted by $\mathcal{L}_\xi$.

The space $L_\xi = \Delta_\xi \oplus T_\xi M/\Delta_\xi$ is endowed with the structure of a 2-step nilpotent Lie algebra with the bracket:

$$[(V_1, W_1), (V_2, W_2)] = (0, [V_1, V_2] + \Delta_q).$$

The associated simply connected nilpotent Lie group is denoted by G_ξ, and the exponential mapping $E_{xp} : L_\xi \to G_\xi$ is one-to-one and onto. By translation, the metric g_ξ over Δ_ξ allows to define a left-invariant subriemannian structure over G_ξ, called the nilpotent approximation of (Δ, g) at ξ.

Any k-dimensional vector subspace $\mathcal{V}_\xi$ of $T_\xi M$, transversal to Δ_ξ, allows to identify L_ξ and G_ξ to $T_\xi M \simeq \Delta_\xi \oplus T_\xi M/\Delta_\xi$. If we fix $\xi_0 \in M$,we can chose linear coordinates x in Δ_{ξ_0} such that the metric g_{ξ_0} is the standard Euclidean metric, and for any linear coordinate system y in $\mathcal{V}_\xi$, there are skew-symmetric matrices $L_1, ..., L_k \in so(p, \mathbb{R})$ such that te mapping 2.1a writes:

$$[X,Y] + \Delta_\xi = \begin{pmatrix} X' L_1 Y \\ . \\ . \\ X' L_k Y \end{pmatrix},$$

where X' denotes the transpose of the vector X.

This construction works for any Δ, but Δ is one-step bracket generating iff the endomorphisms of Δ_ξ, Li, $i = 1, ..., k$ (respectively the matices L_i if coordinates y in $\mathcal{V}_\xi$ are chosen) are linearly independant.

2.2. **Popp Measure.** In the 2-step bracket generating case, the linear coordinates y in $T_\xi M/\Delta_\xi$ can be chosen in such a way that the endomorphisms Li, $i = 1, ..., k$ are orthonormal with respect to the Hilbert-Schmidt scalar product $< L_i, L_j >= \frac{1}{p} Trace_g(L_i' L_j)$. This choice defines a canonical euclidean structure over $T_\xi M/\Delta_\xi$ and a corresponding volume in this space. Then using the euclidean structure over Δ_ξ, we get a canonical eucildean structure over $\Delta_\xi \oplus T_\xi M/\Delta_\xi$. The choice of the subspace $\mathcal{V}_\xi$ induces an euclidean structure on $T_\xi M$ that depends on the choice of $\mathcal{V}_\xi$, but the associated volume over $T_\xi M$ is independant of this choice.

Definition 1. *This volume form on M is called the Popp measure.*

The Popp measure is a smooth volume form.

Let us recall a main result from [1].

Theorem 2. *The value $R_a(\xi)$ at $\xi \in M$ of the Radon-Nykodim derivative of the spherical Hausdorf measure with respect to the Popp measure is equal to the Popp volume of the unit ball of the nilpotent approximation at ξ.*

2.3. **Geodesics and Cut-locus.** We restrict to the corank 2 case. Here, we consider geodesics of the nilpotent approximation of $s = (\Delta, g)$ in $T_{\xi_0}M \simeq \mathbb{R}^n$, issued from the origin. A transversal subspace $\mathcal{V}_{\xi_0}$ is chosen, together with the linear Hilbert-Schmidt-orthonormal coordinates y in $\mathcal{V}_{\xi_0}$, and euclidean coordinates x in Δ_{ξ_0}. The geodesics are projections on $\mathbb{R}^n$ of trajectories of the hamiltonian H on $T^*\mathbb{R}^n$:

$$H(p, q, x, y) = \sup_{u \in \mathbb{R}^p} (-||u||^2 + \sum_{i=1}^{p} p_i u_i + q_1 x' L_1 u + q_2 x' L_2 u). \tag{2.2}$$

where p, q are the coordinates dual to x, y.

Geodesics are arclength-parametrized as soon as the initial covector p_0 verifies $||p_0|| = 1$.

The following result is shown in [4], and is crucial for the proof of our result.

Theorem 3. *The cut time t_{cut} of the arclength-parametrized geodesic corresponding to p, q_1, q_2 is given by:*

$$tcut = \frac{2\pi}{\max(\sigma(q_1 L_1 + q_2 L_2)},$$

where $\max(\sigma(A))$ denotes the maximum modulus of the eigenvalues of the skew symmetric matrix A. Ingeneral, the conjugate time is not equal to the cut time. If $q_1 L_1 + q_2 L_2$ has a double maximum eigenvalue or if $[L1, L2] = 0$, then the cut time is also conjugate.

It turns out that the singularities of the Hausdorf measure will appear at collision points of the spectrum of $q_1 L_1 + q_2 L_2$. The set of skew-symmetric matrices that have a double eigenvalue is a codimension 3 algebraic subset of $so(p, \mathbb{R})$. Then, from the tranversality theorems ([2]), for generic (open, dense) subriemannian structures, the set $\overline{\Sigma_2}$ of points of M such that $q_1 L_1 + q_2 L_2$ has a double (at least) eigenvalue for some q_1, q_2 has codimension 2 in M. The problems of smoothness of the Hausdorf measure will occur on $\overline{\Sigma_2}$ only.

Note that q_1, q_2 are constant along geodesics, since the hamiltonian (2.2) does not depend on the y-coordinates. Along the paper we set, for the geodesic under consideration:

$$q_1 = r\cos(\theta),\ q_2 = r\sin(\theta),\ A_\xi(\theta, r) = \frac{2\pi}{\max(\sigma(q_1 L_1 + q_2 L_2)},\ \text{where } \xi = (x, y) \in M.$$

It is known ([6, 7, 9]) that $A_\xi(\theta, r)$ is a Lipschitz function of all parameters ξ, θ, r. We write also $A_\xi(\theta) = A_\xi(\theta, 1)$.

3. Proof of the Theorem

For a fixed point $\xi_0 = (x_0, y_0) \in M$, let us consider the exponential mapping $\mathcal{E}$ associated with the nilpotent approximation at ξ_0, where x, y are coordinates as in Section 2.2:

$$\mathcal{E}_t(p_0, q_0) = \pi(e^{t\vec{H}}(\xi_0, p_0, q_0)),$$

where $\pi : T^*M \to M$ is the canonical projection, and $\vec{H}$ is the hamiltonian vector field associated with the (smooth) hamiltonian (2.2). We have $p_0 = u_0$, and as above $q(t) = q_0 = (q1, q2) = (r\cos(\theta), r\sin(\theta))$. Also, by quasihomogeneity, $\mathcal{E}_t(p, q_0) = \mathcal{E}_1(t\ u_0, t\ q_0)$.

It follows that the volume V_ξ at a point $\xi \in M$ of the unit ball of the nilpotent approximation is given by the formula:

$$V_\xi = \int_0^{2\pi} \int_0^{A_\xi(\theta)} \int_B J_\varepsilon(u, \theta, r, \xi) du\ dr\ d\theta \tag{3.1}$$

where B is the unit ball in the euclidean p-dimensional u-space, and $J_\varepsilon(u, \theta, r, \xi)$ is the jacobian determinant of $\mathcal{E}_1(u, r\cos(\theta), r\sin(\theta))$.

We set $f_\xi(\theta, r) = \int_B J_\varepsilon(u, \theta, r, \xi)du$, and $W_\xi(\theta) = \int_0^{A_\xi(\theta)} f_\xi(\theta, r)dr$. If we show that $W_\xi(\theta)$ is $\mathcal{C}^1$ or $\mathcal{C}^2$ w.r.t (θ, ξ), it will imply that V_ξ is $\mathcal{C}^1$ or $\mathcal{C}^2$ w.r.t ξ. But, in a neighborhood of a fixed $(\xi_0, \theta_0) \in M \times S_1$,

$$\begin{aligned} W_\xi(\theta) &= \int_0^{A_\xi(\theta)} f_\xi(\theta, r)dr \\ &= \int_0^{A_{\xi_0}(\theta_0)} f_\xi(\theta, r)dr \ + \int_{A_{\xi_0}(\theta_0)}^{A_\xi(\theta)} f_\xi(\theta, r)dr \\ &= (I) + (II). \end{aligned} \tag{3.2}$$

The term (I) is smooth. Then we are concerned with eximining the smoothness of $II(\xi, \theta)$ only.

3.1. Proof of the fact that $W_\xi(\theta)$ is always $\mathcal{C}^1$. The tangent mapping to $II(\xi, \theta)$, at (θ_0, ξ_0) is , setting $z = (\theta, \xi)$, and $f(z, r) = f_\xi(\theta, r)$, $A(z) = A_\xi(\theta)$:

$$D\ II(z_0)(h) = \sum_{i=1}^{n+1} f(z_0, A(z_0)) \frac{\partial A}{\partial z_i}(z_0) h_i. \tag{3.3}$$

This last expression makes sense, and is continuous w.r.t z_0 for the following reasons: first as we said, $A(z)$ is Lipschitz-continuous, then the derivatives are bounded. And, precisely, at points z_0 such that A is not differentiable, $f(z_0, A(z_0))$ vanishes. This last poin follows from the fact that for a multiple eigenvalue $A(z_0)$, the conjugate time is equal to the cut time, which makes the jacobian determinant $J_\varepsilon(u, \theta_0, A(\theta_0, \xi_0), \xi_0)$ vanish for all u. This comes from the section II.3 1 in the paper [1].

Remark 1. *In fact, it follows from the same paper that, if $A(z_0)$ is a multiple eigenvalue, the rank of $J_\varepsilon(u, \theta_0, A(\theta_0\xi_0), \xi_0)$ drops by 2 at least, independantly of u. This point will be very important in the next section.*

This ends the proof.

3.2. **Proof of the $\mathcal{C}^2$ result.** It follows from the transversality theorems ([2, 5]) and from Lemma 1, that we can start from an open dense subsetof subriemannian metrics, still denoted by $\mathcal{S}$, such that all elements s of $\mathcal{S}$ meet: the set $U_s \subset M \times S_1$ of (θ, ξ) such that $A_\xi(\theta)$ corresponds to a triple (at least) eigenvalue is a locally finite union of manifolds, regularly embedded, of codimension 8 in $M \times S_1$, and the set $\tilde{U}_s \subset M \times S_1$ of (θ, ξ) such that $A(\theta, \xi)$ corresponds to a double (and not triple) eigenvalue is a locally finite union of manifolds, of codimension 3.

To show this, we can work locally, in a neighborhood $\mathcal{N}$ of $(s_0, \xi_0, \theta_0) \subset \mathcal{S} \times M \times S_1$, in coordinates $\xi = (x, y)$ in M, relative to s_0, from section 2.2. Then, in coordinates, the (Nilpotentized at ξ) subriemannian metric s is specified by two skew-symmetric matrices $L_1(\xi), L_2(\xi)$. It is easy to see that the mapping:

$$(\xi, \theta, L_1(\xi), L_2(\xi) \to L_1(\xi)\cos(\theta) + L_2(\xi)\sin(\theta)$$

is a submersion. Then, we consider the stratification by the multiplicity of eigenvalues of the space of p-skew-symmetric matices. Generically (open-dense), s can be put transversal to this (closed) Whitney stratification.

We want to show the following property (P), for a generic (residual in the Whitney topology) set $\mathcal{S}_0$ of subriemannian metrics over M:

(P) the partial derivatives $D_i(z) = f(z, A(z))\frac{\partial A}{\partial z_i}(z)$ from 3.3 are $\mathcal{C}^1$ in a neighborhood of all points z_0 such that $A(z_0)$ corresponds to a double (and not triple) eigenvalue.

To do this, we fix s_0 and $z_0 \in \tilde{U}_{s_0}$ and we consider a (mini)versal deformation of $L(\xi_0, \theta 0) = L_1(\xi_0)\cos(\theta_0) + L_2(\xi_0)\sin(\theta_0) = L(z_0)$, as introduced in section 4.1. It follows that:

$$L(\xi, \theta) = L(z) = g(z)^{-1} N(\tilde{\varphi}(z)) g(z)$$

where $N(\tilde{\varphi}(z))$ is the block-diagonal $N(\tilde{\varphi}(z)) = Bd(\lambda(z)\hat{q} + q(z), \Delta(z))$, and $g(z)$ belongs to the orthogonal group.

The functions $g(z), \lambda(z), q(z), \Delta(z)$ are smooth functions. We will temporarily assume the following property (R):

(R): the map $z \to q(z)$, $M \times S_1 \to \mathbb{R}^3$, has rank 3 at z_0.

Then, assume that (R) holds at all points z of $\tilde{U}_{s_0}$.

It means that we can change the coordinates in $M \times S_1$ for the three components of $q(z)$ around $z_0 \in \tilde{U}_{s_0}$, are the three first coordinates, z_1, z_2, z_3. Note that these 3 coordinates vanish at z_0.

Locally, the codimension 3 manifold $\tilde{U}_{s_0}$ is determined by the equations $z_1 = z_2 = z_3 = 0$.

As we said in Remark 1, the rank of $J_\varepsilon(u, z, A(z))$ drops by 2 at least, independantly of u, at each point $z \in \tilde{U}_{s_0}$, and for all u. Formula 4.1 in the appendix tell us that $A(z) = \frac{2\pi}{\lambda(z) + \sqrt{z_1^2 + z_2^2 + z_3^2}}$ where $\lambda(z)$ is smooth and nonzero. We set $\hat{z}_4 = (z_4, ..., z_{n+1})$ and $\hat{z}_1 = (z_1, z_2, z_3)$. Then, J_ε and $f(z, r)$ vanish with all their first order partial derivatives w.r.t. z, r, at points $\hat{z}_1 = 0$, and $r = \frac{2\pi}{\lambda(z)}$. Hence,

$$f(z, r) = \tilde{Q}_{z,r}(\hat{z}_1, r - \frac{2\pi}{\lambda(z)}). \tag{3.4}$$

Here, $\tilde{Q}_{z,r}(.)$ is a quadratic form depending smoothly on z, r.

We go bach to the second partial derivatives of $W_\xi(\theta) = \int_0^{A(\theta_0,\xi_0)} f_\xi(\theta, r)dr + \int_{A(\theta_0,\xi_0)}^{A(\theta,\xi)} f_\xi(\theta, r)dr$, or with the new notations, $W(z) = \int_0^{A(z_0)} f(z, r)dr + \int_{A(z_0)}^{A(z)} f(z, r)dr$.

The first partial derivatives, at any point z_0 were:

$$\begin{aligned}\frac{\partial W(z_0)}{\partial z_i} &= \int_0^{A(z_0)} \frac{\partial}{\partial z_i} f(z, r)dr + f(z_0, A(z_0))\frac{\partial A}{\partial z_i}(z_0), \\ &= III(z_0) + IV(z_0)\end{aligned}$$

To show that $\frac{\partial III(z)}{\partial zj}$ exists and is continuous, we proceed exactly as in Section 3.1, using the fact that $\frac{\partial}{\partial z_j} f(z, r)$ also vanishes at $(\hat{z}_1 = 0,\ r = \frac{2\pi}{\lambda(z)})$.

The more difficult point is to show that $\frac{\partial IV(z)}{\partial zj}$ exists and is continuous.

$$\frac{\partial IV(z)}{\partial zj} = \frac{\partial}{\partial z_j}(f(z, A(z))\frac{\partial A(z)}{\partial z_i}).$$

We get:

$$\begin{aligned}\frac{\partial IV}{\partial zj}(z) &= \frac{\partial f}{\partial z_j}(z, A(z))\frac{\partial A(z)}{\partial z_i}) + \frac{\partial f}{\partial r}(z, A(z))\frac{\partial A(z)}{\partial z_i}\frac{\partial A(z)}{\partial z_j} + f(z, A(z))\frac{\partial^2 A(z)}{\partial z_i \partial z_j}. \\ &= V(z) + VI(z) + VII(z).\end{aligned}$$

The cases of $V(z), VI(z)$ are obvious, since again $\frac{\partial A(z)}{\partial z_i}$ is bounded, and the functions $\frac{\partial f}{\partial z_j}(z, A(z)), \frac{\partial f}{\partial r}(z, A(z))$ are continuous and go to zero when $\hat{z}_1$ tends to zero. The only difficulty is the case of $VII(z)$.

Remind that $A(z) = \frac{2\pi}{\lambda(z)+||\hat{z}_1||.}$ where $\lambda(z)$ is nonzero, smooth. Then the only problem may occur for $i = 1, 2, 3$.

Let us consider only 2 cases: (1) $i = 1, j = 2$, (2) $i = 1, j = 4$, the other cases being similar.

Case (2): $\frac{\partial A(z)}{\partial z_1} = \frac{-2\pi}{(\lambda(z)+||\hat{z}_1||)^2}(\frac{\partial\lambda}{\partial z_1} + \frac{z_1}{||z_1||})$, and $\frac{\partial^2 A(z)}{\partial z_1 \partial z_4}$ is bounded. It is multiplied by $f(z, A(z))$, which tends to zero when $\hat{z}_1$ tends to zero. Then it is zero at points $\hat{z}_1 = 0$, and it is continuous.

Case(1): $\frac{\partial A(z)}{\partial z_1} = \frac{-2\pi}{(\lambda(z)+||\hat{z}_1||)^2}(\frac{\partial\lambda}{\partial z_1} + \frac{z_1}{||z_1||})$, and $\frac{\partial^2 A(z)}{\partial z_1 \partial z_4} = C(z) + D(z)\frac{z_1 z_2}{||\hat{z}_1||^{\frac{3}{2}}}$, where $C(z)$ is bounded, $D(z)$ is continuous. Then, the question is the continuity of $\varphi(z) = \frac{f(z,A(z))}{||\hat{z}_1||}$, in a neighborhood of the set $E = \{\hat{z}_1 = 0\}$. Let us use Formula 3.4. It gives $f(z, A(z)) = \tilde{Q}_{z,r}(\hat{z}_1, A(z) - \frac{2\pi}{\lambda(z)})$. But $A(z) = \frac{2\pi}{\lambda(z)+||\hat{z}_1||}$, then, $A(z) - \frac{2\pi}{\lambda(z)} = \psi(z)||\hat{z}_1||$, where $\psi(z)$ is continuous. It follows that $\varphi(z)$ tends to zero when $\hat{z}_1$ tends to zero. The subriemannian volume is $\mathcal{C}^2$ in a neighborhood of $\tilde{U}_{s_0}$.

If we show that generically, property (R) holds, it follows that $R_a(\xi)$ is $\mathcal{C}^2$ except on a set of codimension 8 in $M \times S_1$, the theorem is proved. It is shown in the appendix 2 that property (R) holds for a residual in $\mathcal{S}$. In the case $n = 6$, the bad set is generically empty in $M \times S_1$ and property (R) is open dense in $\mathcal{S}$.

4. Appendix

4.0.1. *Pure Quaternions in $so(4)$.* In $so(4)$, it is natural and useful for computations to use quaternionic notations. Set:

$$i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\imath} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \hat{\jmath} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \hat{k} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The matrices i, j, k (resp. $\hat{\imath}, \hat{\jmath}, \hat{k}$) generate the so-called pure quaternions (resp. pure skew-quaternions), the space of which is denoted by Q (resp. $\hat{Q}$). The Lie algebra $so(4) = Q \oplus \hat{Q}$, and quaternions commute with skew-quaternions: $[Q, \hat{Q}] = 0$.

We endow $so(4)$ with the Hilbert-Schmidt scal product: $< L_1, L_2 >= trace(L_1' L_2)$.

Then, $i, j, k, \hat{\imath}, \hat{\jmath}, \hat{k}$ form an orthonormal basis. The eigenvalues ω_1, ω_2 of $A = q + \hat{q}$ meet:

$$-(\omega_{1,2})^2 = (||q|| \pm ||\hat{q}||)^2. \tag{4.1}$$

As a consequence, an element $A \in so(4)$ has a double eigenvalue iff $A \in Q \cup \hat{Q}$.

4.1. **Versal deformation of skew-symmetric matrices.**

The results of Arnold in [3] can be easily extended to the real smooth case ($\mathcal{C}^\infty$), for skew-symmetric matrices, under the action of the orthogonal group:

Theorem 4. [3] *Let $N(p)$ be a family of $n \times n$ matrices smoothly depending on p at $(\mathbb{R}^l, 0)$. Let O_M be the orbit of $N = N(0)$ under the action of $Gl(n, \mathbb{R})$ by conjugation. Let $T(\mu)$ be a smooth family of matrices, depending on the parameter $\mu \in \mathbb{R}^k$, such that the mapping $\varphi : \mu \to T(\mu)$* ***transversally*** *intersects O_N at some $\tilde{N} = g^{-1} N g$. Then, there is a family of (smoothly depending on p) matrices $g(p)$ and a smooth mapping $\tilde{\varphi} : p \to \mu(p)$, such that $N(p) = g(p)^{-1} A(\tilde{\varphi}(p)) g(p)$. Moreover, for the transversal $T(\mu)$, one can chose the centralizer of N in $gl(n, \mathbb{R})$.*

We rephrase the result in the case of a skew-symmetric matrix N that has a double (but not triple) eigenvalue. Then, by section 4.0.1, we can assume that N is (conjugate to) a block-diagonal $Bd(\alpha\hat{q}\ , \delta)$, where $\hat{q}$ is a unit skew-quaternion and δ is a block-diagonal skew symmetric matrix with 2×2 blocks and non multiple eigenvalues. The centralizer of $\hat{q}$ in $so(4, \mathbb{R})$ is the vector space of matrices of the form $\lambda\hat{q} + q$, where q varies over pure quaternions. Then, the centralizer of N in $so(n, \mathbb{R})$ is the space of block diagonal matrices $Bd(\lambda\hat{q} + q, \Delta)$, where q varies over pure quaternions and Δ varies over 2×2 skew-symmetric block diagonal matrices.

Hence, we can find a smooth $g(p) \in SO(n, \mathbb{R})$, and a smooth $\tilde{\varphi}(p)$ such that:

$$\begin{aligned} N(p) &= g(p)^{-1} T(\tilde{\varphi}(p)) g(p), \quad \text{with} \\ T(\mu) &= Bd(\lambda(\mu)\hat{q} + q(\mu), \Delta(\mu)). \end{aligned} \tag{4.2}$$

The versal deformation $T(\mu)$ is not universal (which means that $\tilde{\varphi}$ is not uniquely determined by $N(p)$), however, the nondiagonal eigenvalues of $T(\mu)$ are given by the formula 4.1. It follows that q is determined modulo conjugation by a unit quaternion. On the other hand, the functions $\lambda(\mu), \Delta(\mu)$ are smooth and $\lambda(\mu)$ is nonzero.

4.2. **Lemma.**

Lemma 1. *1.The set of skew symmetric matrices with a double eigenvalue is an algebraic subset of codimension 3 in skew symmetric matrices.*

2. The set of skew symmetric matrices with a triple eigenvalue is an algebraic subset of codimension 8 in skew symmetric matrices.

The proof can be obtained as in the appendix of [10].

4.3. **Genericity of (R).** We consider the set $\mathcal{S}$ of corank-2 subriemannian metrics on a fixed manifold M, equipped with the Whitney topology. The result being essentially local, we work in a neighborhood of a point, in coordinates.

Lemma 2. *Property (R) is resisual in* $\mathcal{S}$.

Note that the function $q(\mu)$ in 4.2 is defined modulo conjugation by a unit quaternion $q_1(\mu)$, depending smoothly on μ.

Let us show that the property (R) does not depend on this choice: actually, if we set $\tilde{q}(\mu) = q_1(\mu)q(\mu)q_1(\mu)^{-1}$, then the differential $Dq(\mu)$ is changed for:

$$D\tilde{q}(\mu)(v) = [Dq1(\mu)(v), q(\mu)] + q_1(\mu)Dq(\mu)(v)q_1(\mu)^{-1}.$$

But $q(0) = 0$, therefore the image of $D\tilde{q}(0)$ has the same dimension as the one of $Dq(0)$.

The codimension d_0 of the algebraic set of $3\times(n+1)$ matrices that have corank 1 at least is $d_0 = (n-1)$ [product of coranks] in the $3\times(n+1)$ matrices. By Lemma 1, the set of skew-symmetric matrices that have double maximum eigenvalue is $d_1 = 3$. Consider the map $\mathcal{M} : (z, L_1(z), L_2(z)) \to Dq(z)$, where q is the map considered in property (R). This map $\mathcal{M}$ is a submersion. Therefore, there is a residual set $\mathcal{S}$ in subriemannian metrics, for which the codimension of the set of (θ, ξ) in $S_1 \times M$ where $A(z)$ corresponds to a double eigenvalue, an property (R) holds at (θ, ξ) is a stratified set of codimension $d_0 + d_1 = n + 2$.

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