Sensor fault reconstruction and observability for unknown inputs, with an application to WasteWater Treatment Plants

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Abstract

In this paper, we propose a general methodology for identifying and reconstructing sensor faults on dynamical processes. This methodology is issued from the general identification theory developed in the previous papers (Busvelle and Gauthier, 2003, 2004, 2005): In fact, this identification theory provides also a general framework for the problem of "observability with unknown inputs". Indeed, many problems of fault detection can be formulated as such observability problems, the (eventually additive) faults being just considered as unknown inputs. Our application to "sensor fault detection" for WasteWater Treatment Plants (WWTP) constitutes an ideal academic context to apply the theory: First, in this 3-5 case (3 sensors, 5 states), the theory applies generically and, second, any system is naturally under the "observability canonical form" required to apply the basic high-gain observer from Gauthier and Kupka (1994). A simulation study on the Bleesbrük WWTP is proposed to show the effectiveness of this approach.

Keywords:
Sensor fault detection, High-gain observers, Fault reconstruction, observers with unknown inputs, Wastewater treatment plants

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1. Introduction

State estimation and Fault Detection and Isolation (FDI) constitute the purpose of this paper. The main purpose of a FDI scheme is to detect the fault when it occurs, by generating an alarm, but also by identifying the nature and the location of the fault. A fault is a malfunction of actuators or sensors, or more generally of internal state variables of the system. These malfunctions occur due to certain abnormal circumstance. If unchecked, such an unallowable deviation of at least one characteristic property or variable from its acceptable range may be devastating (Isermann and Ball, 1996; Isermann, 2011; Palade and Bocaniala, 2010). Various FDI approaches have been proposed (Frank, 1990, 1996; Patton and Chen, 1993). Others methods based on computational Intelligence techniques can be found in Palade and Bocaniala (2010). In Isermann (2011), several model-based methods are defined and developed: Fault detection with parameter estimation, with parity equations, with state observers and state estimation.
In general the FDI methods do not always afford the shape, the magnitude of the time-dependent fault.

Among these approaches, observer based FDI attract a great deal of attention from the research community (Chen and Patton, 1999; Frank and Ding, 1997; Yang and Saif, 1995; De Persis and Isidori, 2000, 2001). In this model-based subcategory, residuals are constructed as the difference between the actual process behavior and the expected one described by its mathematical model. Using these residuals, a decision is easily achievable whether there is a fault or not. One difficulty is to make a robust observer w.r.t. disturbances which are not faults (De Persis and Isidori, 2000, 2001; Besançon, 2003).

In this paper, where continuous (smooth) nonlinear systems in state-space representation are considered, we propose a systematic methodology dedicated to fault reconstruction with an application to the field of wastewater treatment systems. Via this method, it is possible to detect sensor drift faults and incipient faults, which are not readily detected using other methods. Along the paper, we make the reasonable assumption that several faults do not occur simultaneously, i.e. we deal with the problem of observability with a single unknown input function.

In the context of observer-based methods, sliding mode observers are applied to reconstruct the faults by an appropriate processing of the so-called *"equivalent output error injection"* concept. Readers may refer to Tan and Edwards (2002, 2003). In other papers, (Edwards, 2004), unknown-input observers are used in order to reconstruct the fault.
Here, we develop a general theory of observability for unknown inputs, in order to reconstruct simultaneously the states and the graph of the fault. This theory is a by-product of the identification theory developed in Busvelle and Gauthier (2003, 2004, 2005), and it leads naturally to the use of high-gain observers.

The structure of the paper is as follows: First (Section 2), we state the main lines of the theory of "observability for unknown inputs". In Section 3, we briefly recall the structure of the basic high-gain observer that comes naturally to the rescue. In Section 4, the proposed method is illustrated by an application to the Bleesbrük WWTP. Finally, the short section 5 is devoted to a comparison to another popular method, with a similar geometric flavor (De Persis and Isidori, 2000, 2001).

2. Observability for unknown inputs, versus identification

2.1. generalities

It turns out that the concept of "observability for unknown inputs" (or "unknown-observability") can be seen just as a rephrasing of the concept of identifiability in the sense of Busvelle and Gauthier (2003, 2004, 2005). These three papers contain a complete theory for the case of a single unknown input (or a single function of the state to be identified). In the context of FDI, a single unknown input corresponds to a single fault. If several faults may occur simultaneously, one should consider several unknown inputs (the additive faults that could appear simultaneously on different sensors for instance).

The theory is parallel to the "deterministic observation theory" of Gauthier and Kupka (1994, 1996, 2001). It requires the same mathematical tools and methods to be understood. In this Section, we state the main results of the theory. Although these results can be stated in a clear intrinsic way, we limit ourselves to the characterizations in terms of "normal forms". Moreover we ignore certain classical difficulties (such as finite escape-time, analyticity versus smoothness, global-Lipschitzness...). For more details, the reader should refer to Busvelle and Gauthier (2003, 2004, 2005). The concept of genericity under consideration in the paper is the usual one from differential topology, i.e. it is genericity w.r.t. the Whitney topology. Since in most cases the problems are located on a compact subset of the state space, it is enough in practice to consider the metric C^∞ topology: A function is close to zero if its values together with the values of all its derivatives are small enough.

A main idea that the reader should keep in mind is the following: The observability property (resp. Identifiability, observability for unknown inputs) is the property of injectivity of a certain mapping. Therefore it is a very unstable property: For instance, the function $f(x) = x^3$ is injective, but it does not remain injective under perturbation by a very small function with very small derivatives. Due to this unstability, it is impossible to expect interesting general results. However, the injectivity becomes stable if we require the additional property of "infinitesimal injectivity", i.e. injectivity of the linearizations (Note that the function $f(x) = x^3$ is not infinitesimally injective at $x = 0$).

These considerations are the reasons why it is not realistic to avoid considering the concept of "infinitesimal observability" (resp: Identifiability, unknown-observability).

2.2. definitions and systems under consideration

Systems under consideration are smooth (C^ω or C^∞) systems of the form:

$$\Sigma \begin{cases} \frac{dx}{dt} = f(x, \varphi(t)) \\ y = h(x, \varphi(t)) \end{cases} \tag{1}$$

Where the state $x = x(t)$ lies in an n-dimensional manifold X, $x(0) = x_0$. The observation y is $\mathbb{R}^{d_y}$- valued and f, h are respectively a smooth (parameterized) vector field and a smooth function. The function φ (the unknown input) is a function of time (in the context of identifiability, it is an unknown function of the state). To simplify, each trajectory is assumed to be defined on some interval $[0, T_{x_0,\varphi}[$ depending on both the initial condition and the unknown function φ, but containing a fixed time interval $I = [0, i]$.

The goal is to estimate both the state variable x and the unknown function $\varphi \colon \mathbb{R}^+ \to \mathbb{R}$. In the applied part of the paper (Section 4), the unknown

φ will be denoted by d (for "disturbance").

Let $\Omega = X \times L^\infty[I]$, where $L^\infty[I]$ is the set of $\mathbb{R}$-valued measurable bounded functions defined over I, and by $L^\infty[\mathbb{R}^{d_y}]$ the set of measurable bounded functions from I to $\mathbb{R}^{d_y}$.

Then we can define the input/output mapping P_Σ, mapping the initial state x_0 and the input function $\widehat{\varphi}$ to the output function y :

$$P_\Sigma : \begin{array}{c} \Omega \rightarrow L^\infty[\mathbb{R}^{d_y}] \\ (x_0, \widehat{\varphi}(.)) \rightarrow y(.) \end{array} \tag{2}$$

Definition 1. *Σ is said to be "unknown-observable" if P_Σ is injective.*

The infinitesimal version of unknown-observability is defined as follows. Let us consider the first variation of the system (1), where T_x denotes the tangent mapping w.r.t. x, and d_φ denotes the differential w.r.t. :

$$T\Sigma_{x_0,\widehat{\varphi},\xi_0,\eta} \begin{cases} \dfrac{dx}{dt} = f(x, \widehat{\varphi}) \\ \dfrac{d\xi}{dt} = T_x f(x, \widehat{\varphi})\xi + d_\varphi f(x, \widehat{\varphi})\eta \\ \widehat{y} = d_x h(x, \widehat{\varphi})\xi + d_\varphi h(x, \widehat{\varphi})\eta \end{cases} \tag{3}$$

and the input/output mapping of $T\Sigma$ is:

$$P_{T\Sigma,x_0,\widehat{\varphi}} : \begin{array}{r} T_{x_0}X \times L^\infty[\mathbb{R}] \rightarrow L^\infty[\mathbb{R}^{d_y}] \\ (\xi_0, \eta(.)) \rightarrow \widehat{y}(.) \end{array} \tag{4}$$

Definition 2. *Σ is said to be infinitesimally unknown-observable if $P_{T\Sigma,x_0,\widehat{\varphi}}$ is injective for any $(x_0, \widehat{\varphi}(.)) \in \Omega$ i.e. ker $(P_{T\Sigma,x_0,\widehat{\varphi}}) = \{0\}$ for any $(x_0, \widehat{\varphi}(.))$.*

In other terms, the linearizations along any trajectory of the system are observable linear time-dependent systems.

Remark: Both identifiability and infinitesimal identifiability mean injectivity of certain mapping. Clearly injectivity depends on the domain (restricting the domain provides a weaker property). Therefore, it could seem that these notions are not well defined, since they depend on the regularity assumed for the inputs (the domain for $\widehat{\varphi}$). In fact it is not the case: Indeed, if an analytic system Σ is not (infinitesimally) unknown-observable for certain L^{∞} input function, then there exists another analytic function which makes the system not (infinitesimally) unknown-observable.

2.3. Main results stated in terms of "canonical forms"

The theory is parallel to the observability theory from Gauthier and Kupka (2001): Every unknown-observable system may be put (up to a change of coordinates) into one of the canonical forms presented in the theorems 1, 2 and 3 below.
In order to achieve default reconstruction, it is enough to develop a observer for unknown inputs adapted to each of these canonical forms.

In the previous papers (Busvelle and Gauthier, 2003, 2004, 2005), the following results are established:

- Systems are generically unknown-observable if and only if the number of observations is three or more. Generic systems can be put under the canonical form of theorem 3 below.
- Contrarily, unknown-observability is not at all generic when the number of observations is only one or two. In this case, infinitesimally unknown-observable systems are exhausted by certain geometric properties that are equivalent to the normal forms presented in theorems 1 and 2 below.

Theorem 1. ($d_y = 1$) *if Σ is infinitesimally unknown-observable, then, there is a subanalytic closed subset Z of X, of codimension 1 at least, such that for any ($x_0 \in X \setminus Z$, there is a coordinate neighborhood $(x_1, \cdots, x_n, V_{x_0})$, $V_{x_0} \subset X \setminus Z$ in which Σ (restricted to V_{x_0}) can be written:*

$$\Sigma_1 \begin{cases} \dot{x_1} = x_2 \\ \quad\vdots \\ \dot{x}_{n-1} = x_n & \textit{and} \quad \dfrac{\partial \psi(x,\varphi)}{\partial \varphi} \neq 0 \\ \dot{x}_n = \psi(x,\varphi) \\ y = x_1 \end{cases} \tag{5}$$

Theorem 2. $(d_y = 2)$ *if Σ is infinitesimally unknown-observable, then, there is an open-dense subanalytic subset $\widetilde{U}$ of $X \times \mathbb{R}$, such that each point (x_0, φ_0) of $\widetilde{U}$, has a neighborhood $V_{x_0} \times I_{\varphi_0}$, and coordinates x on V_{x_0} such that the system Σ restricted to $V_{x_0} \times I_{\varphi_0}$, denoted by $\Sigma_{|V_{x_0} \times I_{\varphi_0}}$, has one of the three following normal forms:*

- **Type 1 normal form:**

$$\Sigma_{2,1} \left\{ \begin{array}{rclcrcl} y_1 &=& x_1 & & y_2 &=& x_2 \\ \dot{x}_1 &=& x_3 & & \dot{x}_2 &=& x_4 \\ & \vdots & & & & \vdots & \\ \dot{x}_{2k-3} &=& x_{2k-1} & & \dot{x}_{2k-2} &=& x_{2k} \\ \dot{x}_{2k-1} &=& f_{2k-1}(x_1, \ldots, x_{2k+1}) & & & & \\ \dot{x}_{2k} &=& x_{2k+1} & & & & \\ & \vdots & & & & & \\ \dot{x}_{n-1} &=& x_n & & & & \\ \dot{x}_n &=& f_n(x, \varphi) & & & & \end{array} \right. \tag{6}$$

with $(\frac{\partial f_n}{\partial \varphi} \neq 0)$

- **Type 2 normal form:**

$$\Sigma_{2,2} \left\{ \begin{array}{rclcrcl} y_1 &=& x_1 & & y_2 &=& x_2 \\ \dot{x}_1 &=& x_3 & & \dot{x}_2 &=& x_4 \\ & \vdots & & & & \vdots & \\ \dot{x}_{2r-3} &=& x_{2r-1} & & \dot{x}_{2r-2} &=& x_{2r} \\ \dot{x}_{2r-1} &=& \psi(x, \varphi) & & \dot{x}_{2r} &=& F_{2r}(x_1, \ldots, x_{2r+1}, \psi(x, \varphi)) \\ & & & & \dot{x}_{2r+1} &=& F_{2r+1}(x_1, \ldots, x_{2r+2}, \psi(x, \varphi)) \\ & & & & & \vdots & \\ & & & & \dot{x}_{n-1} &=& F_{n-1}(x, \psi(x, \varphi)) \\ & & & & \dot{x}_n &=& F_n(x, \psi(x, \varphi)) \end{array} \right. \tag{7}$$

with $\frac{\partial \psi}{\partial \varphi} \neq 0, \frac{\partial F_{2r}}{\partial x_{2r+1}} \neq 0, \ldots, \frac{\partial F_{n-1}}{\partial x_n} \neq 0$

- **Type 3 normal form:**

$$\Sigma_{2,3} \left\{ \begin{array}{rclrcl} y_1 & = & x_1 & y_2 & = & x_2 \\ \dot{x}_1 & = & x_3 & \dot{x}_2 & = & x_4 \\ & \vdots & & & \vdots & \\ \dot{x}_{n-3} & = & x_{n-1} & \dot{x}_{n-2} & = & x_n \\ \dot{x}_{n-1} & = & f_{n-1}(x,\varphi) & \dot{x}_n & = & f_n(x,\varphi) \end{array} \right. \tag{8}$$

with $\frac{\partial (f_{n-1}, f_n)}{\partial \varphi} \neq 0$

Here is the result for the generic case:

Theorem 3. $(d_y = 3)$ *if Σ is an infinitesimally unknown-observable generic system, then there is a connected open dense subset Z of X such that for any $x_0 \in Z$, there exist a smooth C^∞ function F and a $(\breve{y}, \breve{y}', \ldots, \breve{y}^{(2n)})$-dependent embedding $\Phi_{\breve{y},\ldots,\breve{y}^{(2n)}}(x)$ such that on Z, trajectories of $\Sigma_{x_0,\varphi}$ are mapped via $\Phi_{\breve{y},\ldots,\breve{y}^{2n}}$ into trajectories of the following system.*

$$\Sigma_{3+} \left\{ \begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{2n} &= z_{2n+1} \\ \dot{z}_{2n+1} &= F(z_1, \ldots, z_{2n+1}, \breve{y}, \ldots, \breve{y}^{(2n+1)}) \\ \bar{y} &= z_1 \end{aligned} \right. \tag{9}$$

where z_i, $i = 1, \ldots, 2n+1$ has dimension $p-1$, and with

$$\left\{ \begin{aligned} x &= \Phi^{-1}_{\breve{y},\ldots,\breve{y}^{(2n)}}(z) \\ \varphi &= \psi(x, \breve{y}) \end{aligned} \right. \tag{10}$$

for a certain smooth function ψ.

Here $\check{y}$ is a certain selected output among the outputs $y_i,\ y_1$ for instance, and $\bar{y}$ consists of the remaining outputs y_2, y_3.
The proof of this theorem, with detailed results in the generic case, can be found in Busvelle and Gauthier (2004). This is the crucial result for our application.

2.4. The generic 3-5 case

The 3-outputs 5-states case is the most simple generic case. It has the additional good property that it is naturally under a useful canonical form, as soon as the outputs are components of the state, which is often the case.

We start with a system of the form:

$Y = (y_1, y_2, y_3) = (x_1, x_2, x_3),\ x = (x_1, \ldots, x_5),\ \dot{x}(t) = f(x)$

We would like to realize Fault Reconstruction for an additive default $d(t)$ on the first output, i.e. in fact, $y_1(t) = x_1(t) + d(t)$. Setting $z_1(t) = x_1(t) + d(t),\ z_2(t) = x_2(t), \ldots,\ z_5(t) = x_5(t)$, the system can be rewritten as:

$$
\begin{aligned}
y_1(t) &= z_1(t), y_2(t) = z_2(t), y_3(t) = z_3(t), \\
\dot{z}_1(t) &= f_1(z_1(t) - d(t), z_2(t), \ldots, z_5(t)) + \dot{d} \\
\dot{z}_i(t) &= f_i(z_1(t) - d(t), z_2(t), \ldots, z_5(t)), i = 2, \cdots 5.
\end{aligned}
\tag{11}
$$

or:

$$\dot{z} = g(z, d, \dot{d}) \tag{12}$$

2.4.1. The most naive strategy

A simple way to proceed is to assume that $\dot{d} = 0$. We get a 6-state equation of the form:

$$
\begin{aligned}
\dot{z}(t) &= g(z_1(t), z_2(t), \ldots, z_5(t), d) \\
\dot{d} &= 0
\end{aligned}
\tag{13}
$$

or, setting $Z = (z, d)$,

$$\begin{aligned} \dot{Z} &= G(Z) \\ y &= (Z_1, Z_2, Z_3) \end{aligned} \tag{14}$$

Then, a step change on d corresponds exactly to a (maybe large) jump of the state Z in the model (14).
In that case, a high-gain observer will do the reconstruction job: It has precisely the property to recover arbitrarily fast large changes in the initial conditions.

System (14) is a rather general 6-state 3-output system, but the form (14) is all ready enough for our purposes.

Indeed, in general (for a generic system), the 3×3 matrix formed by the lines:

$$(\frac{\partial G_i}{\partial z_4}, \frac{\partial G_i}{\partial z_5}, \frac{\partial G_i}{\partial d}), i = 1, \ldots, 3 \tag{15}$$

is invertible, which means by the implicit function theorem that, freezing the variables z_1, z_2, z_3.
The mapping $\tilde{G} = (G_1(z_4, z_5, d), G_2(z_4, z_5, d), G_3(z_4, z_5, d))$ has an inverse $\tilde{G}_1$.

It is then clear that the system is unknown-observable: Knowing the output $Y(t) = (z_1(t), z_2(t), z_3(t))$ and differentiating, we get $(\dot{z}_1(t), \dot{z}_2(t), \dot{z}_3(t)) = \tilde{G}(z_4(t), z_5(t), d(t))$, which we can invert for each value of $z_1(t), z_2(t), z_3(t)$, and we get the knowledge of $z_4(t), z_5(t), d(t)$.

This shows that actually the system is unknown-observable (which we know), but also provides a practical way to observe, by using approximate derivators.

2.4.2. The general strategy

A more general strategy is to use as in Busvelle and Gauthier (2003, 2004, 2005) a **local model** for the fault $d(t)$. For example, a simple local model is $d^{(k)} = 0$. The question is not that this polynomial models the function d globally as a function of t, but only locally, on reasonable time intervals (reasonable w.r.t. the performances required for input-state reconstruction).

Now, we are in the general situation of a 6+k-state, 3-output system. The fact that the original system is infinitesimally unknown-observable implies that the extended 6+k-system can be put under certain appropriate observability normal form.
Again, for this normal form, the use of approximate derivators would allow state reconstruction.

2.5. Observers for unknown inputs

It is a remarkable fact that, for all the normal forms described above, such a polynomial local model allows the use of the high-gain observers from Gauthier and Kupka (2001).

We leave the details to the reader and we just explain below (section 3) what happens in the 3-5 case (our application), when we make the naive assumption $\dot{d} = 0$ of Section 2.4.1 above.

2.6. The necessity of the theoretical analysis

One could ask: Why is it necessary to perform such a heavy theoretical analysis to get the trivial conclusion that "high gain observers must be used"?
In fact, the preliminary analysis of the unknown-input observability property is absolutely necessary, as shows the following example. It shows also that "parametric identification" may be very dangerous without careful analysis. The example is even linear, therefore it leads more simply to the use of a standard Luenberger observer (not high gain). One can imagine that in the nonlinear case, more catastrophic phenomena may appear.

Consider the linear system on $\mathbb{R}^2$

$$\Sigma_e \begin{cases} \dot{x_1} = x_2 - u \\ \dot{x}_2 = u \\ \quad y = x_1 \end{cases} \tag{16}$$

This system is not unknown-observable: Actually, setting $X = (x_{10}, x_{20})$, the mapping $(u(.), X) \to y(.)$ is linear, and it is easily seen that it is not injective: Its Kernel K is the set of couples of the form $(u = e^t x_{20}, X = (0, x_{20}))$.

However chosing, without observability analysis, a local model of the form $u^{(k)} = 0$, one obtains the extended linear system:

$$\Sigma_{e,1} \begin{cases} \dot{x}_1 = x_2 - u \\ \dot{x}_2 = u \\ \dot{u} = u_1 \\ \vdots \\ \dot{u}_{k-1} = 0 \\ y = x_1 \end{cases} \tag{17}$$

Note that $(\Sigma_{e,1})$ is an observable linear system, and that a standard Luenberger observer will provide "some result", with arbitrary exponential decay.

However, this result may be a nonsense. In fact, the system Σ_e, although non unknown-observable, is unknown-observable inside the class of polynomial unknown-inputs.

3. Our choice of the high gain observer in the 3-5 case

3.1. Preliminary

Let us go back to the system (14), and consider the 3×3 matrix J defined in formula (15) above, $J_{ij} = \frac{\partial G_i}{\partial Z}, i = 1, \ldots, 3, j = 4, \ldots, 6.$

The invertibility of this Jacobian matrix characterizes the infinitesimal observability in the sense of Gauthier and Kupka (2001), as was observed above. In this particular 3×5 case, it provides a generalization of the basic single-output observability normal form from Gauthier and Kupka (1994) (See also theorem 2.1, p.22 in Gauthier and Kupka (2001)).

Actually, in the 2-dimensional single output case considered in Gauthier and Kupka (2001),we would have the corresponding normal form:

$$\begin{cases} y = x_1 \\ \dot{x}_1 = f_1(x_1, x_2, u) \\ \dot{x}_2 = f_2(x_1, x_2, u) \quad with \ \dfrac{\partial f_1}{\partial x_2} \neq 0 \end{cases} \tag{18}$$

The condition $\frac{\partial f_1}{\partial x_2} \neq 0$ for infinitesimal observability is the analog of our condition that J is invertible.

At this step, we could use (up to a certain additional simple change of coordinates) a High-gain extended Kalman filter. In fact, here, there is a simpler solution. Due to Hammouri and Farza (2003) a multi-output generalization of the results in Gauthier and Kupka (1994) shows that we can directly apply the basic version of the (constant gain) high-gain Luenberger observer.

3.2. The multi-output high-gain Luenberger observer

We forget about the usual difficulty in high-gain observers of any kind, that consists of smoothly prolongating the system out of a compact set (the "physical" space), in order that it meets certain global-Lipschitz assumptions. In the case of our application, this is more or less trivial.

Physical space will be

$$\mathrm{Ps} = \{M_1 \geq z_1 - d \geq \varepsilon_1, M_i \geq z_i \geq \varepsilon_i > 0\}\,, \tag{19}$$

for $i = 2,\ldots,5$.

For this 3-5 case, it is easily seen that the condition from Hammouri and Farza (2003), that allows the use of a constant gain high-gain observer reduces to the following property (P):

(P) There is a constant 3×3 matrix S such that all (which means for all possible values of the variables in the physical domain) the 3×3 matrices J satisfy: $S^{\mathsf{T}}J + J^{\mathsf{T}}S \leq -aId$, for a certain $a > 0$.

This will be the case in our application, reconstruction of sensor fault for the Bleesbrük WWTP, presented in Section 5.

Let us point out again that, when property (P) holds, it is possible to construct a constant gain, high-gain Luenberger observer that guarantees arbitrarily fast state reconstruction (or fault reconstruction in our case).

4. Application

4.1. Activated Sludge Process

Due to its efficiency, the Activated Sludge Process (ASP) is the most frequent device for wastewater treatment. An ASP is a chemical-biological process, where a mixed community of microorganisms (called Activated Sludge), is used to remove pollutant. A basic ASP layout is composed of an aerated tank and a settler (Fig. 1).

Insert figure 1 about here

Wastewater is treated first in the tank, where the level of substrate is degraded by microorganisms. Next, sedimentation takes place in the settler, in order to separate the clean water and the settled solid. A portion of the sludge is recycled with the aim to maintain an appropriate biomass concentration. The remaining amount of sludge is purged.

Insert Nomenclature about here

Several mathematical models are proposed for the WWTP. The most popular model is the Activated Sludge Model No.1 (ASM1). However, this nonlinear model is rather complex: 11 state variables and 19 constant parameters. Different kinds of reduced models for the Activated sludge plant have been proposed (Jeppsson and Olsson, 1993; Mulas et al., 2007; Smets et al., 2003; Steffens et al., 1997). Here we consider the reduced 5-dimensional dynamical model that was developed by Chachuat et al. (2003).

The following simplifications were applied:

- *Dynamic simplification*: When applying a homotopy method, heterophic ($X_{B,H}$), autrophic ($X_{B,A}$) biomass and inert particulate organic compounds (X_I) were detected as the slowest state dynamics. Thus, these variables can be assumed constant over a few days. Eliminating these three states with the concentration of soluble inert organic compound (S_I), a 7-dimensional dynamic model was obtained.

- *Organic compounds simplification*: Based on more heuristic considerations, soluble (S_S) and particulate (X_S) concentrations are glued into a single organic compound (denoted by X_{DCO}).

- *Nitrogenized compounds simplification*: Due to a simplification of the mathematical expression that describes the organic nitrogen hydrolysis, the dynamics with respect to soluble and particulate organic nitrogen becomes a separated independent system that we do not consider.

Also the following standard assumptions are considered:

- The reactor is well mixed
- The settler is perfect: No reaction occurs there and the separation between solid and liquid is ideal.

These simplifications lead to the following set of equations:

$$\begin{aligned} \dot{S_{NO}} &= D^{in}\,(S_{NO}^{in} - S_{NO}) - \alpha_1 \tfrac{X_{DCO}}{K_{DCO}+X_{DCO}} \\ &\cdot \tfrac{K_{O,H}}{K_{O,H}+S_O} \tfrac{S_{NO}}{K_{NO}+S_{NO}} + \alpha_2 \tfrac{S_{NH}}{K_{NH,A}+S_{NH}} \tfrac{S_O}{K_{O,A}+S_O} \end{aligned} \tag{20}$$

$$\begin{aligned} \dot{S_{NH}} &= D^{in}\,(S_{NH}^{in} - S_{NH}) - \alpha_3 \tfrac{X_{DCO}}{K_{DCO}+X_{DCO}} \\ &\cdot \tfrac{S_O}{K_{O,H}+S_O} + \alpha_4 \tfrac{X_{DCO}}{K_{DCO}+X_{DCO}} \tfrac{K_{O,H}}{K_{O,H}+S_O} \tfrac{S_{NO}}{K_{NO}+S_{NO}} \\ &-\alpha_2 \tfrac{S_{NH}}{K_{NH,A}+S_{NH}} \tfrac{S_O}{K_{O,A}+S_O} + \alpha_5 S_{ND} \end{aligned} \tag{21}$$

$$\begin{aligned} \dot{S_O} &= -D^{in} S_O - \alpha_6 \tfrac{X_{DCO}}{K_{DCO}+X_{DCO}} \tfrac{S_O}{K_{O,H}+S_O} \\ &-\alpha_7 \tfrac{S_{NH}}{K_{NH,A}+S_{NH}} \tfrac{S_O}{K_{O,A}+S_O} + k_L a\,(S_O^{sat} - S_O) \end{aligned} \tag{22}$$

$$\begin{aligned} \dot{X_{DCO}} &= D^{in}\,(X_{DCO}^{in} - \alpha_8 X_{DCO}) - \alpha_9 \tfrac{X_{DCO}}{K_{DCO}+X_{DCO}} \tfrac{S_O}{K_{O,H}+S_O} \\ &+\alpha_{10} \tfrac{X_{DCO}}{K_{DCO}+X_{DCO}} \tfrac{K_{O,H}}{K_{O,H}+S_O} \tfrac{S_{NO}}{K_{NO}+S_{NO}} + \alpha_{11} \end{aligned} \tag{23}$$

$$\begin{aligned} \dot{S_{ND}} &= D^{in}\,(S_{ND}^{in} - S_{ND}) - \alpha_5 S_{ND} + \alpha_{12} \tfrac{X_{DCO}}{K_{ND}+X_{DCO}} \tfrac{S_O}{K_{O,H}+S_O} \\ &+\alpha_{13} \tfrac{X_{DCO}}{K_{ND}+X_{DCO}} \tfrac{K_{O,H}}{K_{O,H}+S_O} \tfrac{S_{NO}}{K_{NO}+S_{NO}} \end{aligned} \tag{24}$$

With: $\alpha_1 = \mu_H.X_{B,H}.\eta_{NO,g}.\frac{1-Y_H}{2.86Y_H}$, $\alpha_2 = \frac{\mu_A}{Y_A}.X_{B,A}$, $\alpha_3 = \mu_H.X_{B,H}.i_{NBM}$, $\alpha_4 = \mu_H.X_{B,H}.i_{NBM}.\eta_{NO,g}$, $\alpha_5 = k_a.X_{B,H}$, $\alpha_6 = \mu_H.X_{B,H}.\frac{1-Y_H}{Y_H}$, $\alpha_7 = 4.57.\frac{\mu_A}{Y_A}.X_{B,A}$, $\alpha_8 = \frac{K_S}{K_{DCO}}$, $\alpha_9 = \frac{\mu_H.X_{B,H}}{Y_H}$, $\alpha_{10} = \frac{\mu_H.X_{B,H}}{.}Y_H.\eta_{NO,g}$, $\alpha_{11} = (1 - f_{r_{XI}}).(b_H.X_{B,H} + b_A.X_{B,A})$, $\alpha_{12} = k_h.\frac{X_{ND}}{X_S}.X_{B,H}$, $\alpha_{13} = k_h.\frac{X_{ND}}{X_S}.X_{B,H}.\eta_{NO,h}$.

In this paper, we work in simulation using certain data generated by the team of modeling and simulation of LTI-CRP Henri Tudor in Luxembourg, by using the ASM1 model and SIMBA software (see http://www.enic.impl-nancy-fr/COSTWWTP/Benchmark).
In fact dry, rain and storm data files are generated from a benchmark simulation of the results for the Bleesbrük wastewater plant (in Luxembourg).

The measured concentrations of this station are: The dissolved oxygen (S_O), nitrate (S_{NO}) and ammonia (S_{NH}).

4.2. The Luenberger high-gain observer

The purpose of the study is the reconstruction of the sensor faults. A sensor fault is an unknown function that will be identified on-line. Consider the reduced ASP system described by equations 20, 21, 22, 23, 24. The unknown function d will represent the fault signal applied to the S_{NO} sensor. It is assumed to be an additive fault. As explained above, in order to reconstruct the function d, the state vector is extended by making d a state variable, and we just model the fault as a jump of initial conditions: $\dot{d} = 0$.

The vector G is as follows:

$$G(Z) = \begin{pmatrix} D^{in}(S_{NO}^{in} - (z_1 - d)) + \alpha_2 \frac{z_2}{K_{NH,A}+z_2}\frac{z_3}{K_{O,A}+z_3} \\ -\alpha_1 \frac{K_{O,H}}{K_{O,H}+z_3}\frac{(z_1-d)}{K_{NO}+(z_1-d)} z_4, \\ D^{in}(S_{NH}^{in} - z_2) - \alpha_2 \frac{z_2}{K_{NH,A}+z_2}\frac{z_3}{K_{O,A}+z_3} - (\alpha_3\frac{z_3}{K_{O,H}+z_3} \\ +\alpha_4 \frac{K_{O,H}}{K_{O,H}+z_3}\frac{(z_1-d)}{K_{NO}+(z_1-d)})z_4 + \alpha_5 z_5, \\ -D^{in}z_3 - \alpha_7 \frac{z_2}{K_{NH,A}+z_2}\frac{z_3}{K_{O,A}+z_3} \\ +k_l a(S_O^{sat} - z_3) - \alpha_6\frac{z_3}{K_{O,H}+z_3}z_4, \\ \frac{(D^{in}XDCO^{in}+\alpha_{11})}{K_{DCO}}(1-z_4)^2 - D^{in}\alpha_8(1-z_4)z_4 \\ -\frac{1}{K_{DCO}}(\alpha_9\frac{z_3}{K_{O,H}+z_3} + \alpha_{10}\frac{K_{O,H}}{K_{O,H}+z_3}\frac{(z_1-d)}{K_{NO}+(z_1-d)})(1-z_4)^2 z_4, \\ D^{in}(S_{ND}^{in} - z_5) - \alpha_5 z_5 \\ +(\alpha_{12}\frac{z_3}{K_{O,H}+z_3} + \alpha_{13}\frac{K_{O,H}}{K_{O,H}+z_3}\frac{(z_1-d)}{K_{NO}+(z_1-d)})\frac{z_4 K_{DCO}}{K_{ND}+z_4(K_{DCO}-K_{ND})} \\ 0 \end{pmatrix}$$

Here, of course, $z_6 = d$.

Remark: Here, for simplicity in the expressions, we have made the extra change of variables $z_4 = \frac{X_{dco}}{K_{dco}+X_{dco}}$. But this is not absolutely necessary.

The equation of the standard high gain Luenberger observer is:

$$\dot{\hat{X}}(t) = G(\hat{X}) - K_\theta(C\hat{X} - y) \tag{25}$$

Where $K_\theta = \Delta_\theta K$ for $\theta > 1$, large enough and:

- Δ_θ is the block diagonal matrix $\Delta_\theta = BD(\theta I_3, \theta^2 I_3)$, where I_3 is the 3-dimensional identity matrix,

- K is a certain constant gain, such that: $(\tilde{G}^*(\hat{X}) - KC)'L - L(\tilde{G}^*(\hat{X}) - KC) \leq -aId$, $a > 0$, L constant symmetric positive definite.

Here $\tilde{G}^*(X)$ denotes the Jacobian matrix of $\tilde{G}(X)$ w.r.t. X ($\tilde{G}(X)$ defined in Section 2.4.1).

In the single output case, the existence of such a K comes from Gauthier and Kupka (1994).
The multi-output case is much more complicated and has been studied in Hammouri and Farza (2003). The existence of K is guaranteed by the property (P) of Section 3 above.
To check that property (P) holds in our case, it is enough to observe that the Jacobian matrix J has the following form on the "physical space" Ps (from (19)):

$$J = \begin{pmatrix} -a & 0 & f \\ -b & \alpha & e \\ -c & 0 & 0 \end{pmatrix},$$

where all the functions a, b, c, f, e, α are strictly positive. The technical lemma in our appendix provides property (P).

Here, we did not use the explicit construction of the constant gain K provided by (Hammouri and Farza, 2003), but a heuristic one that works quite well. We have chosen $K = L^{-1}C'$, where L is the solution of the following Riccati equation:

$$-G'L - LG + C'C - LQL = 0 \tag{26}$$

With $Q = diag(10^{-3}, 10^{-3}, 10^{-3}, 10^{-3}, 10^{-3}, 10^{-1})$,

and

$$G = \begin{pmatrix} 0 & 0 & 0 & \frac{\partial g_1(x)}{\partial x_4} & \frac{\partial g_1(x)}{\partial x_5} & \frac{\partial g_1(x)}{\partial d} \\ 0 & 0 & 0 & \frac{\partial g_2(x)}{\partial x_4} & \frac{\partial g_2(x)}{\partial x_5} & \frac{\partial g_2(x)}{\partial d} \\ 0 & 0 & 0 & \frac{\partial g_3(x)}{\partial x_4} & \frac{\partial g_3(x)}{\partial x_5} & \frac{\partial g_3(x)}{\partial d} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ at a typical (or average) point } x.$$

We obtain the constant Luenberger gain:

$$K = \begin{pmatrix} 3.73 & 1.43 \times 10^{-2} & 5 \times 10^{-3} \\ 1.43 \times 10^{-2} & 5.69 \times 10^{-1} & 4.255 \times 10^{-1} \\ 5 \times 10^{-3} & 4.255 \times 10^{-1} & 2.9189 \\ 0 & -2 \times 10^{-4} & -10^{-3} \\ 0 & 10^{-3} & -2 \times 10^{-4} \\ 10^{-1} & 3 \times 10^{-4} & -5 \times 10^{-4} \end{pmatrix}$$

4.3. Numerical simulations

The three outputs are corrupted by an additive colored noise. In a standard way, we have chosen an Ornstein-Uhlenbeck process X_t, simulating the following stochastic equation (Uhlenbeck and Ornstein, 1930):

$$dX_t = -aX_t dt + \delta\sqrt{2a} dW_t, \tag{27}$$

where W_t is a standard Wiener process.

The coefficients a ,δ have been chosen in order to get the realistic noise level shown in the results below.

The kinetic and stoichiometric parameter values considered are those defined for the ASM1 model (Smets et al., 2003) (see table 1). The complete others parameters values can be found in table 2.

Insert table 1 about here

Insert table 2 about here

4.3.1. Step fault

At the second day, a step fault is applied to the S_{NO} sensor (Fig. 2). The amplitude equal to 2 mg/l (compared to an average value of 6 mg/l). The 3 state variables S_{NO}, S_{NH} and S_O are measured.

Insert figure 2 about here

Simulations, displayed on Fig. 3, 4 and 5, show the observer outputs : d, X_{DCO}, S_{ND}. They demonstrate the effectiveness of the proposed method to estimate states and simultaneously reconstruct the sensor faults even for systems subject to noisy measurements.

Insert figure 3 about here

Insert figure 4 about here

Insert figure 5 about here

Although we reconstruct simultaneously the unknown state variables X_{DCO}, S_{ND}, the main purpose of these simulations is to detect and reconstruct the additive sensor fault d. One readily checks on Fig. 3, that the observer's output d is close to zero when there is no fault (before day 2), while it reaches quickly the value 2 mg/l when the fault occurs.

4.3.2. Slow drift and intermittent fault

In order to validate completely the method, it is interesting to consider, besides the step, the most classical types of malfunctions: Slow drift and intermittent fault. The corresponding simulation results are shown respectively on Fig. 6 and 7.

Insert figure 6 about here

Insert figure 7 about here

On these two figures, one can see that the method preserves the shape and amplitude of the fault with high fidelity, despite the noisy measurements.

5. Comparison with other methods

Our method lies in the framework of geometric control theory. Another popular method of this type (referred to as the DPIM) has been developed by De Persis and Isidori (2000, 2001). Let us analyze what is different in our approach.

The DPIM is rather closely related to ours, however the basic problem is different: One wants (1) to detect the occurrence of the fault and simultaneously (2) to reject perturbations. What we do here is in a sense weaker since

we do not ask rejection of any perturbation.
In this case where there is no perturbation, however, the DPIM makes sense, and we feel that our method is stronger, from two points of view:
a) we do not only detect the occurrence of the fault, but we reconstruct the fault.
b) we do not limit ourselves to control affine systems (w.r.t. the fault in particular), but we consider general nonlinearities.

This last point (b) has to be developed: Assume for instance an additive sensor fault on the output, of the form $y(t) = h(x) + d(t)$ for simplicity. Without loss of generality, we may assume that $y(x) = x_1 + d$. Then, setting $x_1 + d = \tilde{x_1}$, we get $y = \tilde{x_1}$, and the equations for the dynamics are already fully nonlinear w.r.t. $d(t)$, even starting from a system affine w.r.t. $d(t)$. The DPIM simply does not apply.
It is the case in our application. Now, let us have a look to the example in De Persis and Isidori (2001), where the DPIM not only works, but allows to reconstruct fully the fault (we cite: "In this particular example, it is even possible to identify the value of m.").

It turns out that, in their case, l = number of controls = 3, k = number of "unknown faults" = 1. Assuming the l (= 3) controls as known constants, we are in the generic situation of $m = 3$ outputs, $k = 1$ unknown input: The generic case.
Actually, it is easily seen that our theorem 3 applies, and that the change of variables chosen in De Persis and Isidori (2001) leads exactly to our normal form Σ_{3+} of Theorem 3.
Considering now the controls as nonconstant, it is easy to see that we obtain the normal form Σ_{3+}, but with its linear part becoming time-dependent through the 3 controls. Hence, our high-gain observer still applies, and this is more or less what is suggested in De Persis and Isidori (2001) at the end of the paper.

Other related works in the same spirit are:
a) Hou and Patton, but in the linear case (Hou and Patton, 1998),
b) Kabore and Wang (Kabore and Wang, 2001), where conditions are given for observability (detectability) for unknown inputs. This work has not really a geometric flavor, and moreover, it applies to control affine problems only.

6. Conclusion

An approach for sensor fault identification and reconstruction for a class of nonlinear systems has been proposed based on a theory of observability for unknown inputs. The sensor fault is considered as the unknown input. Our theory naturally leads to the use of a Luenberger-type high gain observer. The Bleesbrük ASP with ASM^1 model provides an ideal case study. Simulations with ASP have shown the effectiveness of our strategy for fault reconstruction, in the presence of noisy measurements. The Luenberger high gain observer used for this application is specially simple.

There are several open questions after this work: First, from theoretical point of view, it seems to us that it is now necessary to complete our theory (to the case of simultaneous faults, for instance). Although it is rather clear how to proceed, the task is not technically so obvious. From the point of view of the application, we are starting to apply the method to a real waste-water system. As usual, this is presumably the beginning of a long story.

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A. Appendix: Technical lemma

Let C be a compact subset contained in the set of matrices of the form:

$$A = \begin{pmatrix} -a & 0 & d \\ -b & \alpha & e \\ -c & 0 & 0 \end{pmatrix}$$

with $a, b, c, d, e, \alpha > 0$.
Let N be of the form

$$N = \begin{pmatrix} 0 & 0 & -r \\ 0 & -1 & 0 \\ rs & 0 & 0 \end{pmatrix}$$

Then, for $s, r > 0$ large enough

$$N'A + A'N < -\beta Id,\ \ \beta > 0\ \forall A \in C$$

Proof.

$$N'A = \begin{pmatrix} -crs & 0 & 0 \\ b & -\alpha & -e \\ ar & 0 & -dr \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$X^{'}N^{'}AX = -crsx^2 - \alpha y^2 - drz^2 + bxy - eyz + arxz$$

$$-\alpha y^2 + bxy - eyz = -(\alpha y^2 - 2\sqrt{\alpha}y\tfrac{bx-ez}{2\sqrt{\alpha}} + (\tfrac{bx-ez}{2\sqrt{\alpha}})^2) + \tfrac{(bx-ez)^2}{4\alpha^2}$$

$$-\alpha y^2 + bxy - eyz = -(\sqrt{\alpha}y - \tfrac{bx-ez}{2\sqrt{\alpha}})^2 + \tfrac{b^2x^2}{4\alpha^2} + \tfrac{e^2z^2}{4\alpha^2} - \tfrac{bexz}{2\alpha^2}$$

$$X^{'}N^{'}AX = -(\sqrt{\alpha}y - \tfrac{bx-ez}{2\sqrt{\alpha}})^2 + ((\tfrac{b^2}{4\alpha^2} - crs)x^2 + (\tfrac{e^2}{4\alpha^2} - dr)z^2 + (ar - \tfrac{be}{2\alpha^2})xz)$$

The result follows QED.

Nomenclature

S_I	Soluble inert organic matter concentration (mgl^{-1})	i_{NBM}	Mass of nitrogen in the biomass (gNg_{COD}^{-1})
S_S	Readily biodegradable substrate concentration (mgl^{-1})	i_{NXI}	Mass of nitrogen in the inert particulate organic matter (gNg_{COD}^{-1})
S_O	Dissolved oxygen concentration (mgl^{-1})	K_l a	Coefficient of oxygen rate (d^{-1})
S_O^{sat}	Dissolved oxygen saturation concentration (mgl^{-1})	$K_{NH,A}$	Half-saturation coefficient of ammonia for autotrophs ($gNHm^{-3}$)
S_{NO}	Nitrate and nitrite nitrogen concentration (mgl^{-1})	K_{N0}	Half-saturation coefficient of nitrate for denitrifying heterotrophs ($gNOm^{-3}$)
S_{NH}	Ammonia nitrogen concentration (mgl^{-1})	$K_{0,A}$	Half-saturation coefficient of oxygen autotrophs (gO_2m^{-3})
S_{ND}	Soluble biodegradable organic nitrogen concentration (mgl^{-1})	$K_{0,H}$	Half-saturation coefficient of oxygen heterotrophs (gO_2m^{-3})
X_I	Particulate inert organic matter concentration (mgl^{-1})	K_S	Half-saturation coefficient for heterotrophic organisms ($gDCOm^{-3}$)
X_S	Slowly biodegradable substrate concentration (mgl^{-1})	K_X	Half-saturation coefficient for hydrolysis of slowly biodegradable substrate $gDCOg_{DCO}^{-1}$
$X_{B,H}$	Active heterotrophic biomass concentration (mgl^{-1})	Y_A	Yield coefficient for autotrophic organisms ($-$)
$X_{B,A}$	Active autotrophic biomass concentration (mgl^{-1})	Y_H	Yield coefficient for heterotrophic organisms ($-$)
X_{ND}	Particulate biodegradable organic nitrogen concentration (mgl^{-1})	μ_A	Maximum specific growth rate for autotrophic organisms (d^{-1})
b_A	Autotrophic organisms decay rate coefficient (d^{-1})	μ_H	Maximum specific growth rate for heterotrophic organisms (d^{-1})
b_H	Heterotrophic organisms decay rate coefficient (d^{-1})	^{in}	influent (d^{-1})
fr_{XI}	Fraction of biomass generating the particulate products ($-$)	D^{in}	Influent flow rate (m^3d^{-1})

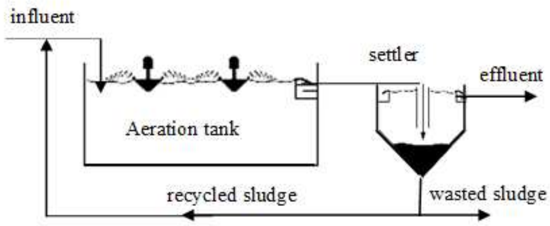

Figure 1: Typical small-size activated sludge treatment plant.

Parameter	Value	Range of variation
Y_H	0.67	0.38 - 0.75
i_{NBM}	0.08	-
K_S	10	5-225
$K_{0,H}$	0.2	0.01-0.20
K_{NO}	0.5	0.01-0.50
$K_{NH,A}$	1.0	-
$K_{0,A}$	0.40	0.40-2.0
$\eta_{NO,g}$	0.8	0.6-13.2
$\eta_{NO,h}$	0.8	-
Y_A	0.24	0.07-0.28
fr_{XI}	0.08	-
μ_H	4.0	0.60-13.2
μ_A	0.5	0.20-1.0
k_a	0.05	-
k_h	3.0	-
f_{SS}	0.79	-
$D^{in}(d^{-1})$	69.2 (mean)	62.85-79.52

Table 1: ASM1 Kinetic and stoichiometric parameters.

Parameter	Value
α_1	3923
α_2	283
α_3	796
α_4	637
α_5	124
α_6	3904
α_7	1293
α_8	0.045
α_9	14860
α_{10}	11888
α_{11}	693
α_{12}	480
α_{13}	384
K_{DCO}	220
K_{ND}	258
$X_{B,A}$	136 $gDCOm^{-3}$
$X_{B,H}$	2489 $gDCOm^{-3}$
X_{ND}	$6gNm^{-3}$
$k_l a$	240 d^{-3}
V_O	$1333m^3$

Table 2: Different parameter values.

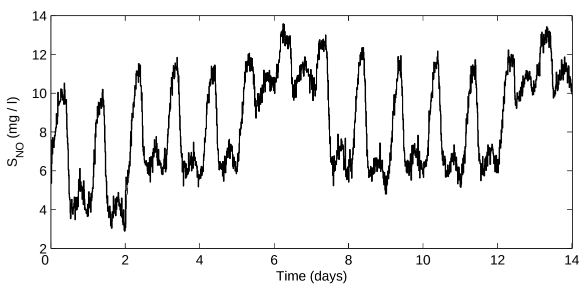

Figure 2: The faulty S_{NO} sensor.

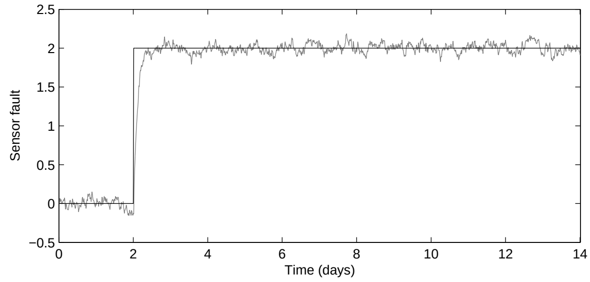

Figure 3: The difference between the applied and the reconstructed step sensor fault.

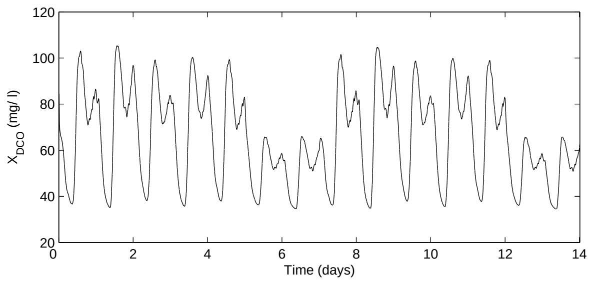

Figure 4: The difference between estimated and real X_{DCO} (unmeasured state) - No visible difference.

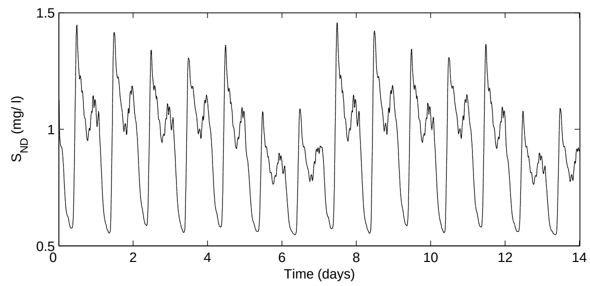

Figure 5: The difference between estimated and real S_{ND} (unmeasured state) - No visible difference.

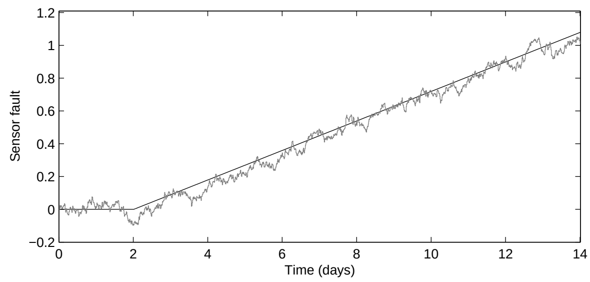

Figure 6: The difference between the applied and the reconstructed slow drift sensor fault.

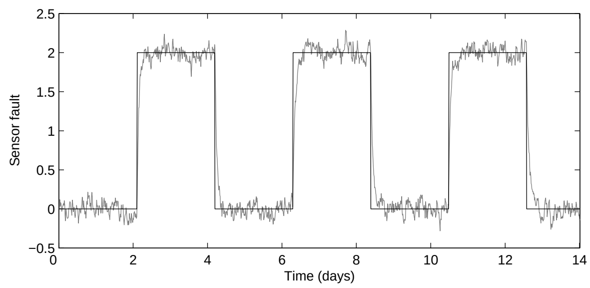

Figure 7: The difference between the applied and the reconstructed intermittent sensor fault .