Hypoelliptic heat kernel over 3-step nilpotent Lie groups

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In this paper we provide explicitly the connection between the hypoelliptic heat kernel for some 3-step sub-Riemannian manifolds and the quartic oscillator. We study the left-invariant sub-Riemannian structure on two nilpotent Lie groups, namely the (2,3,4) group (called the Engel group) and the (2,3,5) group (called the Cartan group or the generalized Dido problem). Our main technique is noncommutative Fourier analysis that permits to transform the hypoelliptic heat equation into a one dimensional heat equation with a quartic potential.

1 Introduction

The study of the properties of the heat kernel in a sub-Riemannian manifold drew an increasing attention since the pioneer work of Hörmander [22]. Since then, many estimates and properties of the kernel in terms of the sub-Riemannian distance have been provided (see [5, 6, 15, 26, 32] and references therein). For some particular structures, it is moreover possible to find explicit expressions of the hypoelliptic heat kernels. In general, this computation can be performed only when the sub-Riemannian structure and the corresponding hypoelliptic heat operator present symmetry properties. For this reason, the most natural choice in this field is to consider invariant operators defined on Lie groups. Results of this kind have been first provided in [16, 23] in the case of the 3D Heisenberg group. Afterwards, other explicit expressions have been found first for 2-step nilpotent free Lie groups (again in [16]) and then for general 2-step nilpotent Lie groups (see [3, 9]). We provide in [1] the expressions of heat kernels for 2-step groups that are not nilpotent, namely $SU(2), SO(3), SL(2)$ and the group of rototranslations of the plane $SE(2)$. For other examples, see e.g. [33, 34].

In our paper we present the first results, to our knowledge, about the expression of the hypoelliptic heat kernel on the following 3-step Lie groups. The first one is the Engel group $\mathfrak{G}_4$,

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that is the nilpotent group with growth vector $(2,3,4)$. Its Lie algebra is $\mathfrak{L}_4 = \text{span}\{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}$, the generators of which satisfy

$$[\mathfrak{l}_1, \mathfrak{l}_2] = \mathfrak{l}_3,\ [\mathfrak{l}_1, \mathfrak{l}_3] = \mathfrak{l}_4,\ [\mathfrak{l}_1, \mathfrak{l}_4] = [\mathfrak{l}_2, \mathfrak{l}_3] = [\mathfrak{l}_2, \mathfrak{l}_4] = [\mathfrak{l}_3, \mathfrak{l}_4] = 0.$$

The second example is the Cartan group $\mathfrak{G}_5$, that is the free nilpotent group with growth vector $(2,3,5)$. Its Lie algebra is $\mathfrak{L}_5 = \text{span}\{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4, \mathfrak{l}_5\}$ and generators satisfy

$$\begin{aligned}&[\mathfrak{l}_1, \mathfrak{l}_2] = \mathfrak{l}_3,\ [\mathfrak{l}_1, \mathfrak{l}_3] = \mathfrak{l}_4,\ [\mathfrak{l}_2, \mathfrak{l}_3] = \mathfrak{l}_5,\\ &[\mathfrak{l}_1, \mathfrak{l}_4] = [\mathfrak{l}_1, \mathfrak{l}_5] = [\mathfrak{l}_2, \mathfrak{l}_4] = [\mathfrak{l}_2, \mathfrak{l}_5] = [\mathfrak{l}_3, \mathfrak{l}_4] = [\mathfrak{l}_3, \mathfrak{l}_5] = [\mathfrak{l}_4, \mathfrak{l}_5] = 0.\end{aligned}$$

In both cases, we consider the heat equation with the so-called intrinsic hypoelliptic Laplacian Δ_H (in the sense of [1], see also Section 2.2.1) of the sub-Riemannian structure for which $\{g\mathfrak{l}_1, g\mathfrak{l}_2\}$ (g element of the group) is an orthonormal frame. As it has been proved in [1], since $\mathfrak{G}_4$ and $\mathfrak{G}_5$ are unimodular, then the intrinsic hypoelliptic Laplacian is the sum of the square of the Lie derivative with respect to the vector fields $g\mathfrak{l}_1, g\mathfrak{l}_2$.

One interesting feature of these two sub-Riemannian problems is that they present abnormal minimizers (see [27, 28]) and it is known that in both cases Δ_H is not analytic hypoelliptic [8]. Hence, for these two examples the Trèves conjecture[2] holds. Having information about the expression of the heat kernel can help for further investigations in this direction.

Any other left-invariant sub-Riemannian structure of rank 2 on these groups is indeed isometric to the ones we study in this paper, see [27, 28]. Moreover, notice that the sub-Riemannian structures we study on $\mathfrak{G}_4$ and $\mathfrak{G}_5$ are local approximations (nilpotentizations, see [18]) of arbitrary sub-Riemannian structures at regular points with growth vector $(2,3,4)$ or $(2,3,5)$, hence, roughly speaking, the kernels on $\mathfrak{G}_4$ and $\mathfrak{G}_5$ provide approximations of the heat kernels at these points.

The goal of this paper is to transform the hypoelliptic heat equations on these Lie groups into a family of elliptic heat equations on $\mathbb{R}$, depending on one parameter. To this purpose, we apply the method developed in [1], based on the Generalized Fourier Transform (GFT for short), and hence on representation theory of these groups (see [11, p. 333–338]).

Applying the GFT to the original equation, we get an evolution equation on the Hilbert space where representations act. For both examples, this is the heat equation over $\mathbb{R}$ with quartic potential, the so-called quartic oscillator (see [10, 29]), for which no general explicit solution is known. Notice that the connection between the quartic oscillator and degenerate elliptic operators has been already noted by previously (see [17]).

It is clearly possible to use numerical approximations of the evolution equation with quartic potential (for which a huge amount of literature is available) to find numerical approximations of the hypoelliptic heat kernel. However, this analysis is outside the aims of this paper.

The organization of the paper is the following. In Section 2 we recall the main definitions from sub-Riemannian geometry, in particular for invariant structures on Lie groups. We then recall the definition of the Generalized Fourier Transform and its main properties. Finally, we

[2]We recall that Trèves conjectured in [31] that the existence of abnormal minimizers on a sub-Riemannian manifold is equivalent to the loss of analytic-hypoellipticity of the sub-Laplacian.

recall the main results of our prevuious paper [1], where we studied hypoelliptic heat equations on Lie groups.

The main part of the paper is Section 3. We first present the Lie groups $\mathfrak{G}_4$ and $\mathfrak{G}_5$, their algebras and their Euclidian and matrix presentations. We then recall results about their representations. We finally apply the method of computation of hypoelliptic heat kernels to the two groups $\mathfrak{G}_4$ and $\mathfrak{G}_5$, to find explicitly the connection between the heat kernels on these groups and the fundamental solution of the 1D heat equation with quartic potential.

2 The hypoelliptic heat equation on a a sub-Riemannian manifold

In this section we recall basic definitions from sub-Riemannian geometry, including the one of the intrinsic hypoelliptic Laplacian. Then we recall our method for computing the hypoelliptic heat kernel in the case of unimodular Lie groups, using the GFT.

2.1 Sub-Riemannian manifolds

We start by recalling the definition of sub-Riemannian manifold.

Definition 1. *A (n, m)-sub-Riemannian manifold is a triple $(M, \blacktriangle, \mathbf{g})$, where*

- *M is a connected smooth manifold of dimension n;*
- *$\blacktriangle$ is a smooth distribution of constant rank $m < n$ satisfying the* **Hörmander condition**, *i.e. $\blacktriangle$ is a smooth map that associates to $q \in M$ a m-dim subspace $\blacktriangle(q)$ of T_qM and $\forall\, q \in M$ we have*

$$\text{span}\,\{[X_1, [\ldots[X_{k-1}, X_k]\ldots]](q) \mid X_i \in \text{Vec}_H(M)\} = T_qM \tag{1}$$

  where $\text{Vec}_H(M)$ denotes the set of **horizontal smooth vector fields** *on M, i.e.*

$$\text{Vec}_H(M) = \{X \in \text{Vec}(M) \mid X(p) \in \blacktriangle(p) \quad \forall\, p \in M\}\,.$$

- *$\mathbf{g}_q$ is a Riemannian metric on $\blacktriangle(q)$, that is smooth as function of q.*

When M is an orientable manifold, we say that the sub-Riemannian manifold is orientable.

A Lipschitz continuous curve $\gamma : [0, T] \to M$ is said to be **horizontal** if $\dot\gamma(t) \in \blacktriangle(\gamma(t))$ for almost every $t \in [0, T]$. Given an horizontal curve $\gamma : [0, T] \to M$, the *length of* γ is

$$l(\gamma) = \int_0^T \sqrt{\mathbf{g}_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\; dt. \tag{2}$$

The *distance* induced by the sub-Riemannian structure on M is the function

$$d(q_0, q_1) = \inf\{l(\gamma) \mid \gamma(0) = q_0, \gamma(T) = q_1, \gamma \text{ horizontal}\}. \tag{3}$$

The hypothesis of connectedness of M and the Hörmander condition guarantee the finiteness and the continuity of $d(\cdot,\cdot)$ with respect to the topology of M (Chow's Theorem, see for instance [2]). The function $d(\cdot,\cdot)$ is called the Carnot-Charateodory distance and gives to M the structure of metric space (see [4, 18]).

Locally, the pair $(\blacktriangle, \mathbf{g})$ can be given by assigning a set of m smooth vector fields spanning $\blacktriangle$ and that are orthonormal for $\mathbf{g}$, i.e.

$$\blacktriangle(q) = \text{span}\{X_1(q), \ldots, X_m(q)\}, \quad \mathbf{g}_q(X_i(q), X_j(q)) = \delta_{ij}. \tag{4}$$

In this case, the set $\{X_1, \ldots, X_m\}$ is called a local **orthonormal frame** for the sub-Riemannian structure. When $(\blacktriangle, \mathbf{g})$ can be defined as in (4) by m vector fields defined globally, we say that the sub-Riemannian manifold is *trivializable*.

When the manifold is analytic and the orthonormal frame can be assigned through m analytic vector fields, we say that the sub-Riemannian manifold is *analytic*.

We end this section with the definition of regular sub-Riemannian manifold.

Definition 2. *Let $\blacktriangle$ be a distribution and define through the recursive formula*

$$\blacktriangle_1 := \blacktriangle, \qquad \blacktriangle_{n+1} := \blacktriangle_n + [\blacktriangle_n, \blacktriangle].$$

The small flag of $\blacktriangle$ is the sequence

$$\blacktriangle_1 \subset \blacktriangle_2 \subset \ldots \subset \blacktriangle_n \subset \ldots$$

A sub-Riemannian manifold is said to be **regular** *if for each $n = 1, 2, \ldots$ the dimension of $\blacktriangle_n(q_0)$ does not depend on the point $q_0 \in M$.*

In this paper we always deal with sub-Riemannian manifolds that are orientable, analytic, trivializable and regular.

2.2 Left-invariant sub-Riemannian manifolds

In this section we present a natural sub-Riemannian structure that can be defined on Lie groups. All along the paper, we use the notation for Lie groups of matrices. For general Lie groups, by gv with $g \in G$ and $v \in \mathfrak{L}$, we mean $(L_g)_*(v)$ where L_g is the left-translation of the group.

Definition 3. *Let G be a Lie group with Lie algebra $\mathfrak{L}$ and $\mathfrak{P} \subseteq \mathfrak{L}$ a subspace of $\mathfrak{L}$ satisfying the* **Lie bracket generating condition**

$$\text{Lie}\ \mathfrak{P} := \text{span}\{[\mathfrak{p}_1, [\mathfrak{p}_2, \ldots, [\mathfrak{p}_{n-1}, \mathfrak{p}_n]]] \mid \mathfrak{p}_i \in \mathfrak{P}\} = \mathfrak{L}.$$

Endow $\mathfrak{P}$ with a positive definite quadratic form $\langle ., . \rangle$. Define a sub-Riemannian structure on G as follows:

- *the distribution is the left-invariant distribution $\blacktriangle(g) := g\mathfrak{P}$;*
- *the quadratic form $\mathbf{g}$ on $\blacktriangle$ is given by $\mathbf{g}_g(v_1, v_2) := \langle g^{-1}v_1, g^{-1}v_2 \rangle$.*

In this case we say that $(G, \blacktriangle, \mathbf{g})$ *is a left-invariant sub-Riemannian manifold.*

Remark **4.** Observe that all left-invariant manifolds $(G, \blacktriangle, \mathbf{g})$ are regular.

In the following we define a left-invariant sub-Riemannian manifold choosing a set of m vectors $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ that are an orthonormal basis for the subspace $\mathfrak{P} \subseteq \mathfrak{L}$ with respect to the metric defined in Definition 3, i.e. $\mathfrak{P} = \mathrm{span}\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ and $\langle \mathfrak{p}_i, \mathfrak{p}_j \rangle = \delta_{ij}$. We thus have $\blacktriangle(g) = g\mathfrak{P} = \mathrm{span}\{g\mathfrak{p}_1, \ldots, g\mathfrak{p}_m\}$ and $\mathbf{g}_g(g\mathfrak{p}_i, g\mathfrak{p}_j) = \delta_{ij}$. Hence, every left-invariant sub-Riemannian manifold is trivializable.

2.2.1 The intrinsic hypoelliptic Laplacian

In this section, we recall the definition of intrinsic hypoelliptic Laplacian given in [1] and based on the Popp volume form in sub-Riemannian geometry presented in [25].

Let $(M, \blacktriangle, \mathbf{g})$ be a (n, m)-sub-Riemannian manifold and $\{X_1, \ldots X_m\}$ a local orthonormal frame. The operator obtained by the sum of squares of these vector fields is not a good definition of hypoelliptic Laplacian, since it depends on the choice of the orthonormal frame (see for instance [1]).

In sub-Riemannian geometry an invariant definition of hypoelliptic Laplacian is obtained by computing the divergence of the horizontal gradient, like the Laplace-Beltrami operator in Riemannian geometry.

Definition 5. *Let* $(M, \blacktriangle, \mathbf{g})$ *be an orientable regular sub-Riemannian manifold. We define the intrinsic hypoelliptic Laplacian as* $\Delta_H \phi := \mathrm{div}_H \mathrm{grad}_H \phi$*, where*

- *the horizontal gradient is the unique operator* grad_H *from* $\mathcal{C}^\infty(M)$ *to* $\mathrm{Vec}_H(M)$ *satisfying* $\mathbf{g}_q(\mathrm{grad}_H \phi(q), v) = d\phi_q(v) \quad \forall\, q \in M,\ v \in \blacktriangle(q)$. *(In coordinates if* $\{X_1, \ldots X_m\}$ *is a local orthonormal frame for* $(M, \blacktriangle, \mathbf{g})$*, then* $\mathrm{grad}_H \phi = \sum_{i=1}^m (L_{X_i}\phi)\, X_i$*.)*
- *the divergence of a vector field* X *is the unique function satisfying* $\mathrm{div} X \mu_H = L_X \mu_H$ *where* μ_H *is the Popp volume form.*

The construction of the Popp volume form is not totally trivial and we address the reader to [25] or [1] for details. We just recall that the Popp volume form coincide with the Lebesgue measure in a special system of coordinate related to the nilpotent approximation. In sub-Riemannian geometry one can also define other intrinsic volume forms, like the Hausdorff or the spherical Hausdorff volume. However, at the moment, the Popp volume form is the only one known to be smooth in general. However for left-invariant sub-Riemannian manifolds all these measures are proportional to the left Haar measure.

The hypoellipticity of Δ_H (i.e. given $U \subset M$ and $\phi : U \to \mathbb{R}$ such that $\Delta_H \phi \in \mathcal{C}^\infty$, then ϕ is $\mathcal{C}^\infty$) follows from the Hörmander Theorem (see [22]).

In this paper we are interested only to nilpotent Lie groups. The next proposition says that for all unimodular Lie groups, i.e. for groups such that the left and right Haar measure coincides (and in particular for real connected nilpotent groups) the intrinsic hypoelliptic Laplacian is the sum of squares.

Proposition 6. *Let* $(G, \blacktriangle, \mathbf{g})$ *be a left-invariant sub-Riemannian manifold generated by the orthonormal basis* $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\} \subset \mathfrak{l}$*. If* G *is unimodular then* $\Delta_H \phi = \sum_{i=1}^m \left(L^2_{X_i}\phi\right)$ *where* L_{X_i} *is the Lie derivative w.r.t. the field* $X_i = g\mathfrak{p}_i$*.*

2.3 Computation of the hypoelliptic heat kernel via the Generalized Fourier Transform

In this section we describe the method, developed in [1], for the computation of the hypoellitpic heat kernel for left-invariant sub-Riemannian structures on unimodular Lie groups.

The method is based upon the GFT, that permits to disintegrate a function from a Lie group G to $\mathbb{R}$ on its components on (the class of) non-equivalent unitary irreducible representations of G. For proofs and more details, see [1].

2.3.1 The Generalized Fourier Transform

Let $f \in L^1(\mathbb{R}, \mathbb{R})$: its Fourier transform is defined by the formula

$$\hat{f}(\lambda) = \int_{\mathbb{R}} f(x) e^{-ix\lambda} dx.$$

If $f \in L^1(\mathbb{R}, \mathbb{R}) \cap L^2(\mathbb{R}, \mathbb{R})$ then $\hat{f} \in L^2(\mathbb{R}, \mathbb{R})$ and one has

$$\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\hat{f}(\lambda)|^2 \frac{d\lambda}{2\pi},$$

called Parseval or Plancherel equation. By density of $L^1(\mathbb{R}, \mathbb{R}) \cap L^2(\mathbb{R}, \mathbb{R})$ in $L^2(\mathbb{R}, \mathbb{R})$, this equation expresses the fact that the Fourier transform is an isometry between $L^2(\mathbb{R}, \mathbb{R})$ and itself. Moreover, the following inversion formula holds:

$$f(x) = \int_{\mathbb{R}} \hat{f}(\lambda) e^{ix\lambda} \frac{d\lambda}{2\pi},$$

where the equality is intended in the L^2 sense. It has been known from more than 50 years that the Fourier transform generalizes to a wide class of locally compact groups (see for instance [7, 14, 20, 21, 24, 30]). Next we briefly present this generalization for groups satisfying the following hypothesis:

($\mathbf{H}_0$) G is a unimodular Lie group of Type I.

For the definition of groups of Type I see [12]. For our purposes it is sufficient to recall that all groups treated in this paper (i.e. $\mathfrak{G}_4$ and $\mathfrak{G}_5$) are of Type I. Actually, all the real connected nilpotent Lie groups are of Type I [11, 19]. In the following, the L^p spaces $L^p(G, \mathbb{C})$ are intended with respect to the Haar measure $\mu := \mu_L = \mu_R$.

Let G be a Lie group satisfying ($\mathbf{H}_0$) and $\hat{G}$ be the dual[3] of the group G, i.e. the set of all equivalence classes of unitary irreducible representations of G. Let $\lambda \in \hat{G}$: in the following we indicate by $\mathfrak{X}^\lambda$ a choice of an irreducible representation in the class λ. By definition, $\mathfrak{X}^\lambda$ is a map that to an element of G associates a unitary operator acting on a complex separable Hilbert space $\mathcal{H}^\lambda$:

[3] In this paper, by the dual of the group, we mean the support of the Plancherel measure on the set of non-equivalent unitary irreducible representations of G; we thus ignore the singular representations.

$$\mathfrak{X}^\lambda : \begin{array}{ccc} G & \to & U(\mathcal{H}^\lambda) \\ g & \mapsto & \mathfrak{X}^\lambda(g). \end{array}$$

The index λ for $\mathcal{H}^\lambda$ indicates that in general the Hilbert space can vary with λ.

Definition 7. *Let G be a Lie group satisfying* $(\mathbf{H}_0)$*, and $f \in L^1(G, \mathbb{C})$. The generalized (or noncommutative) Fourier transform (GFT) of f is the map (indicated in the following as $\hat{f}$ or $\mathcal{F}(f)$) that to each element of $\hat{G}$ associates the linear operator on $\mathcal{H}^\lambda$:*

$$\hat{f}(\lambda) := \mathcal{F}(f) := \int_G f(g)\mathfrak{X}^\lambda(g^{-1})d\mu. \tag{5}$$

Notice that since f is integrable and $\mathfrak{X}^\lambda$ unitary, then $\hat{f}(\lambda)$ is a bounded operator.

Remark **8.** $\hat{f}$ can be seen as an operator from $\int_{\hat{G}}^{\oplus} \mathcal{H}^\lambda$ to itself. We also use the notation $\hat{f} = \int_{\hat{G}}^{\oplus} \hat{f}(\lambda)$

In general $\hat{G}$ is not a group and its structure can be quite complicated. In the case in which G is abelian then $\hat{G}$ is a group; if G is nilpotent (as in our cases) then $\hat{G}$ has the structure of $\mathbb{R}^n$ for some n.

Under the hypothesis $(\mathbf{H}_0)$ one can define on $\hat{G}$ a positive measure $dP(\lambda)$ (called the Plancherel measure) such that for every $f \in L^1(G, \mathbb{C}) \cap L^2(G, \mathbb{C})$ one has

$$\int_G |f(g)|^2 \mu(g) = \int_{\hat{G}} Tr(\hat{f}(\lambda) \circ \hat{f}(\lambda)^*) dP(\lambda).$$

By density of $L^1(G, \mathbb{C}) \cap L^2(G, \mathbb{C})$ in $L^2(G, \mathbb{C})$, this formula expresses the fact that the GFT is an isometry between $L^2(G, \mathbb{C})$ and $\int_{\hat{G}}^{\oplus} \mathbf{HS}^\lambda$, the set of Hilbert-Schmidt operators with respect to the Plancherel measure. Moreover, it is obvious that:

Proposition 9. *Let G be a Lie group satisfying* $(\mathbf{H}_0)$ *and $f \in L^1(G, \mathbb{C}) \cap L^2(G, \mathbb{C})$. We have, for each $g \in G$*

$$f(g) = \int_{\hat{G}} Tr(\hat{f}(\lambda) \circ \mathfrak{X}^\lambda(g)) dP(\lambda). \tag{6}$$

where the equality is intended in the L^2 sense.

It is immediate to verify that, given two functions $f_1, f_2 \in L^1(G, \mathbb{C})$ and defining their convolution as

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g) dh, \tag{7}$$

then the GFT maps the convolution into non-commutative product:

$$\mathcal{F}(f_1 * f_2)(\lambda) = \hat{f}_2(\lambda) \hat{f}_1(\lambda). \tag{8}$$

Another important property is that if $\delta_{\mathrm{Id}}(g)$ is the Dirac function at the identity over G, then

$$\hat{\delta}_{\mathrm{Id}}(\lambda) = \mathrm{Id}_{H^\lambda}. \tag{9}$$

In the following, a key role is played by the infinitesimal version of the representation $\mathfrak{X}^\lambda$, that is the map

$$d\mathfrak{X}^\lambda : X \mapsto d\mathfrak{X}^\lambda(X) := \left.\frac{d}{dt}\right|_{t=0} \mathfrak{X}^\lambda(e^{tp}), \tag{10}$$

where $X = gp$, $(p \in \mathfrak{l}, g \in G)$ is a left-invariant vector field over G. By Stone theorem (see for instance [30, p. 6]) $d\mathfrak{X}^\lambda(X)$ is a (possibly unbounded) skew-adjoint operator on $\mathcal{H}^\lambda$. We have the following:

Proposition 10. *Let G be a Lie group satisfying $(\mathbf{H}_0)$ and X be a left-invariant vector field over G. The GFT of X, i.e. $\hat{X} = \mathcal{F} L_X \mathcal{F}^{-1}$ splits into the Hilbert sum of operators $\hat{X}^\lambda$, each one of them acting on the set $\mathbf{HS}^\lambda$ of Hilbert-Schmidt operators over $\mathcal{H}^\lambda$:*

$$\hat{X} = \int_{\hat{G}}^{\oplus} \hat{X}^\lambda.$$

Moreover,

$$\hat{X}^\lambda \Xi = d\mathfrak{X}^\lambda(X) \circ \Xi, \quad \text{for every } \Xi \in \mathbf{HS}^\lambda, \tag{11}$$

i.e. the GFT of a left-invariant vector field acts as a left-translation over $\mathbf{HS}^\lambda$.

Remark **11.** From the fact that the GFT of a left-invariant vector field acts as a left-translation, it follows that $\hat{X}^\lambda$ can be interpreted as an operator over $\mathcal{H}^\lambda$.

2.3.2 Computation of the kernel of the hypoelliptic heat equation

In this section we provide a general method to compute the kernel of the hypoelliptic heat equation on a left-invariant sub-Riemannian manifold $(G, \blacktriangle, \mathbf{g})$ such that G satisfies the assumption $(\mathbf{H}_0)$.

We begin by recalling some existence results (for the semigroup of evolution and for the corresponding kernel) in the case of the sum of squares. We recall that for all the examples treated in this paper the invariant hypoelliptic Laplacian is the sum of squares.

Let G be a unimodular Lie group and $(G, \blacktriangle, \mathbf{g})$ a left-invariant sub-Riemannian manifold generated by the orthonormal basis $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$, and consider the hypoelliptic heat equation

$$\partial_t \phi(t, g) = \Delta_H \phi(t, g). \tag{12}$$

Since G is unimodular, then $\Delta_H = L_{X_1}^2 + \ldots + L_{X_m}^2$, where L_{X_i} is the Lie derivative w.r.t. the vector field $X_i := g\mathfrak{p}_i$ $(i = 1, \ldots, m)$. Following Varopoulos [32, pp. 20-21, 106], since Δ_H is a sum of squares, then it is a symmetric operator that we identify with its Friedrichs (self-adjoint) extension, that is the infinitesimal generator of a (Markov) semigroup $e^{t\Delta_H}$. Thanks to the left-invariance of X_i (with $i = 1, \ldots, m$), $e^{t\Delta_H}$ admits a a right-convolution kernel $p_t(.)$, i.e.

$$e^{t\Delta_H}\phi_0(g) = \phi_0 * p_t(g) = \int_G \phi_0(h)p_t(h^{-1}g)\mu(h) \tag{13}$$

is the solution for $t > 0$ to (12) with initial condition $\phi(0, g) = \phi_0(g) \in L^1(G, \mathbb{R})$ with respect to the Haar measure.

Since the operator $\partial_t - \Delta_H$ is hypoelliptic, then the kernel is a $\mathcal{C}^\infty$ function of $(t, g) \in \mathbb{R}^+ \times G$. Notice that $p_t(g) = e^{t\Delta_H}\delta_{\mathrm{Id}}(g)$.

The main results of the paper are based on the following key fact.

Theorem 12. *Let G be a Lie group satisfying $(\mathbf{H}_0)$ and $(G, \blacktriangle, \mathbf{g})$ a left-invariant sub-Riemannian manifold generated by the orthonormal basis $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$. Let $\Delta_H = L^2_{X_1} + \ldots + L^2_{X_m}$ be the intrinsic hypoelliptic Laplacian where L_{X_i} is the Lie derivative w.r.t. the vector field $X_i := g\mathfrak{p}_i$.*

Let $\{\mathfrak{X}^\lambda\}_{\lambda\in\hat{G}}$ be the set of all non-equivalent classes of irreducible representations of the group G, each acting on an Hilbert space $\mathcal{H}^\lambda$, and $dP(\lambda)$ be the Plancherel measure on the dual space $\hat{G}$. We have the following:

(i) *the GFT of Δ_H splits into the Hilbert sum of operators $\hat{\Delta}^\lambda_H$, each one of which leaves $\mathcal{H}^\lambda$ invariant:*

$$\hat{\Delta}_H = \mathcal{F}\Delta_H\mathcal{F}^{-1} = \int_{\hat{G}}^{\oplus} \hat{\Delta}^\lambda_H dP(\lambda), \quad \textit{where} \quad \hat{\Delta}^\lambda_H = \sum_{i=1}^m \left(\hat{X}^\lambda_i\right)^2. \tag{14}$$

(ii) *The operator $\hat{\Delta}^\lambda_H$ is self-adjoint and it is the infinitesimal generator of a contraction semi-group $e^{t\hat{\Delta}^\lambda_H}$ over $\mathbf{HS}^\lambda$, i.e. $e^{t\hat{\Delta}^\lambda_H}\Xi^\lambda_0$ is the solution for $t > 0$ to the operator equation $\partial_t\Xi^\lambda(t) = \hat{\Delta}^\lambda_H\Xi^\lambda(t)$ in $\mathbf{HS}^\lambda$, with initial condition $\Xi^\lambda(0) = \Xi^\lambda_0$.*

(iii) *The hypoelliptic heat kernel is*

$$p_t(g) = \int_{\hat{G}} Tr\left(e^{t\hat{\Delta}^\lambda_H}\mathfrak{X}^\lambda(g)\right) dP(\lambda), \quad t > 0. \tag{15}$$

Remark **13.** As a consequence of Remark 11, it follows that $\hat{\Delta}^\lambda_H$ and $e^{t\hat{\Delta}^\lambda_H}$ can be considered as operators on $\mathcal{H}^\lambda$.

The following corollary gives a useful formula for the hypoelliptic heat kernel in the case in which for all $\lambda \in \hat{G}$ each operator $e^{t\hat{\Delta}^\lambda_H}$ admits a convolution kernel $Q^\lambda_t(.,.)$. Below by ψ^λ, we intend an element of $\mathcal{H}^\lambda$.

Corollary 14. *Under the hypotheses of Theorem 12, if for all $\lambda \in \hat{G}$ we have $\mathcal{H}^\lambda = L^2(X^\lambda, d\theta^\lambda)$ for some measure space $(X^\lambda, d\theta^\lambda)$ and*

$$\left[e^{t\hat{\Delta}^\lambda_H}\psi^\lambda\right](\theta) = \int_{X^\lambda} \psi^\lambda(\bar{\theta})Q^\lambda_t(\theta, \bar{\theta})\, d\bar{\theta},$$

then

$$p_t(g) = \int_{\hat{G}}\int_{X^\lambda} \mathfrak{X}^\lambda(g)Q^\lambda_t(\theta, \bar{\theta})\big|_{\theta=\bar{\theta}}\, d\bar{\theta}\, dP(\lambda),$$

where in the last formula $\mathfrak{X}^\lambda(g)$ acts on $Q^\lambda_t(\theta, \bar{\theta})$ as a function of θ.

3 Hypoelliptic heat kernels on $\mathfrak{G}_4$ and $\mathfrak{G}_5$

In this section we describe the groups $\mathfrak{G}_4$ and $\mathfrak{G}_5$ and we provide their matrix and Euclidean presentations. We define left-invariant sub-Riemannian structures on them and the corresponding hypoelliptic Laplacian.

We then provide representations of the groups and compute the GFT of the hypoelliptic Laplacian. We apply the method presented in Section 2.3.2 to compute the fundamental solution of the hypoelliptic heat equation.

3.1 Definitions of $\mathfrak{G}_4$ and $\mathfrak{G}_5$

In our paper we deal with two 3-step Lie groups. The first one is the nilpotent group $\mathfrak{G}_4$ with growth vector $(2,3,4)$. Its Lie algebra is $\mathfrak{L}_4 = \operatorname{span}\{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}$, whose generators satisfy

$$[\mathfrak{l}_1, \mathfrak{l}_2] = \mathfrak{l}_3,\ [\mathfrak{l}_1, \mathfrak{l}_3] = \mathfrak{l}_4,\ [\mathfrak{l}_1, \mathfrak{l}_4] = [\mathfrak{l}_2, \mathfrak{l}_3] = [\mathfrak{l}_2, \mathfrak{l}_4] = [\mathfrak{l}_3, \mathfrak{l}_4] = 0.$$

The second one is the free nilpotent group $\mathfrak{G}_5$ with growth vector $(2,3,5)$. Its Lie algebra is $\mathfrak{L}_5 = \operatorname{span}\{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4, \mathfrak{l}_5\}$, whose generators satisfy

$$\begin{aligned}&[\mathfrak{l}_1, \mathfrak{l}_2] = \mathfrak{l}_3,\ [\mathfrak{l}_1, \mathfrak{l}_3] = \mathfrak{l}_4,\ [\mathfrak{l}_2, \mathfrak{l}_3] = \mathfrak{l}_5,\\ &[\mathfrak{l}_1, \mathfrak{l}_4] = [\mathfrak{l}_1, \mathfrak{l}_5] = [\mathfrak{l}_2, \mathfrak{l}_4] = [\mathfrak{l}_2, \mathfrak{l}_5] = [\mathfrak{l}_3, \mathfrak{l}_4] = [\mathfrak{l}_3, \mathfrak{l}_5] = [\mathfrak{l}_4, \mathfrak{l}_5] = 0.\end{aligned}$$

Both $\mathfrak{G}_4$ and $\mathfrak{G}_5$ are 3-step nilpotent, as a direct consequence of the definition.

3.2 Hypoelliptic heat kernel on $\mathfrak{G}_4$

In this section we first give the matrix and Euclidean presentations of the Lie group $\mathfrak{G}_4$. We then define a sub-Riemannian structure on it. We give explicitly the representations of the group, that we use at the end to compute the hypoelliptic kernel in terms of the kernel of the quartic oscillator.

We start with the Lie algebra $\mathfrak{L}_4$, that can be presented as the follow matrix space

$$\mathfrak{L}_4 \simeq \left\{ \begin{pmatrix} 0 & -a_1 & 0 & a_4 \\ 0 & 0 & -a_1 & a_3 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a_i \in \mathbb{R} \right\}.$$

We present each $\mathfrak{l}_i$ as the matrix with $a_j = \delta_{ij}$. It is straightforward to prove that these matrices satisfy the commutation rules for $\mathfrak{L}_4$, where the bracket operation is the standard $[A,B] := BA - AB$.

A matrix presentation of the group $\mathfrak{G}_4$ is thus the matrix exponential of $\mathfrak{L}_4$:

$$\mathfrak{G}_4 \ \simeq \ \left\{ \exp\left(\begin{pmatrix} 0 & -a_1 & 0 & a_4 \\ 0 & 0 & -a_1 & a_3 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \mid a_i \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 & -x_1 & \frac{x_1^2}{2} & x_4 \\ 0 & 1 & -x_1 & x_3 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid x_i \in \mathbb{R} \right\},$$

with

$$x_1 = a_1, \qquad x_2 = a_2, \qquad x_3 = a_3 - \frac{a_1 a_2}{2}, \quad x_4 = a_4 + \frac{a_1^2 a_2}{6} - \frac{a_1 a_3}{2}.$$

We now define the isomorphism Π_4 between $\mathfrak{G}_4$ and $\mathbb{R}_4$ given by

$$\Pi_4\left(\begin{pmatrix} 1 & -x_1 & \frac{x_1^2}{2} & x_4 \\ 0 & 1 & -x_1 & x_3 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) = (x_1, x_2, x_3, x_4).$$

This isomorphism is a group isomorphism when $\mathbb{R}^4$ is endowed with the following product (see [13, p. 330]):

$$(x_1, x_2, x_3, x_4) \cdot (y_1, y_2, y_3, y_4) := \left(x_1 + y_1, x_2 + y_2, x_3 + y_3 - x_1 y_2, x_4 + y_4 + \frac{1}{2} x_1^2 y_2 - x_1 y_3\right)$$

The isomorphism Π_4 induces an isomorphism of tangent spaces $T_g\mathfrak{g} \simeq T_{\Pi_4(g)}\mathbb{R}^4$, that is explicitly $g\mathfrak{l}_i \simeq X_i$, with X_i given by

$$\begin{aligned} X_1(x) &= \frac{\partial}{\partial x_1}, \qquad X_2(x) = \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} + \frac{x_1^2}{2}\frac{\partial}{\partial x_4}, \\ X_3(x) &= \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_4}, \qquad X_4(x) = \frac{\partial}{\partial x_4}, \end{aligned} \tag{16}$$

where $x = (x_1, x_2, x_3, x_4)$.

3.2.1 Left-invariant sub-Riemannian structure on $\mathfrak{G}_4$

We endow $\mathfrak{G}_4$ with a left-invariant sub-Riemannian structure as presented in Section 2.2. We define the sub-Riemannian manifold $(\mathfrak{G}_4, \blacktriangle, \mathbf{g})$ where $\blacktriangle(g) = g\mathfrak{p}$ with $\mathfrak{p} = \operatorname{span}\{\mathfrak{l}_1, \mathfrak{l}_2\}$ and $\mathbf{g}_g(g\mathfrak{l}_i, g\mathfrak{l}_j) = \delta_{ij}$ with $i, j = 1$ or 2.

Since $\mathfrak{G}_4$ is nilpotent, then it is unimodular, thus the intrinsic hypoelliptic Laplacian Δ_H is the sum of squares (see [1, Proposition 17]). In terms of the Euclidean presentation of $\mathfrak{G}_4$, the hypoelliptic Laplacian is thus $\Delta_H = X_1^2 + X_2^2$, with the X_i given by (16).

We thus want to find the fundamental solution for the following heat equation:

$$\partial_t \phi(t, x) = \Delta_H \phi(t, x). \tag{17}$$

3.2.2 Representations of $\mathfrak{G}_4$

We now recall the representations of the group $\mathfrak{G}_4$, as computed by Dixmier in [13, p. 333]. As stated before, we may consider only representations on the support of the Plancherel measure.

Proposition 15. *The dual space of* $\mathfrak{G}_4$ *is* $\hat{G} = \left\{\mathfrak{X}^{\lambda,\mu} \mid \lambda \neq 0, \mu \in \mathbb{R}\right\}$, *where*

$$\mathfrak{X}^{\lambda,\mu}(x_1,x_2,x_3,x_4) : \begin{array}{ccc} \mathcal{H} & \to & \mathcal{H} \\ \psi(\theta) & \mapsto & \exp\left(i\left(-\frac{\mu}{2\lambda}x_2+\lambda x_4-\lambda x_3\theta+\frac{\lambda}{2}x_2\theta^2\right)\right)\psi(\theta+x_1) \end{array}$$

whose domain is $\mathcal{H}=L^2(\mathbb{R},\mathbb{C})$, *endowed with the standard product* $<\psi_1,\psi_2>:=\int_{\mathbb{R}}\psi_1(\theta)\overline{\psi_2}(\theta)\,d\theta$ *where* $d\theta$ *is the Lebesgue measure.*

The Plancherel measure on $\hat{G}$ *is* $dP(\lambda,\mu)=d\lambda d\mu$, *i.e. the Lebesgue measure on* $\mathbb{R}^2$.

Remark **16.** Notice that in this case the domain $\mathcal{H}$ of the representation $\mathfrak{X}^{\lambda,\mu}$ does not depend on λ,μ.

3.2.3 The kernel of the hypoelliptic heat equation

Consider the representation $\mathfrak{X}^{\lambda,\mu}$ of $\mathfrak{G}_4$ and let $d\mathfrak{X}_i^{\lambda,\mu}$ be the corresponding representations of the differential operators L_{X_i} with $i=1,2$. Recall that $d\mathfrak{X}_i^{\lambda,\mu}$ are operators on $\mathcal{H}$. Again from [13, p. 333], or by explicit computation, we have

$$\left[d\mathfrak{X}_1^{\lambda,\mu}\psi\right](\theta)=\frac{d}{d\theta}\psi(\theta),\quad [d\mathfrak{X}_2^{\lambda,\mu}\psi](\theta)=\left(-\frac{i}{2}\frac{\mu}{\lambda}+\frac{i}{2}\lambda\theta^2\right)\psi(\theta),$$

thus

$$\left[\hat{\Delta}_H^{\lambda,\mu}\psi\right](\theta)=\left(\frac{d^2}{d\theta^2}-\frac{1}{4}\left(\lambda\theta^2-\frac{\mu}{\lambda}\right)^2\right)\psi(\theta).$$

The GFT of the hypoelliptic heat equation is thus

$$\partial_t\psi=\left(\frac{d^2}{d\theta^2}-\frac{1}{4}\left(\lambda\theta^2-\frac{\mu}{\lambda}\right)^2\right)\psi(\theta). \tag{18}$$

We rewrite it as

$$\partial_t\psi=\left(\frac{d^2}{d\theta^2}-\left(\alpha\theta^2+\beta\right)^2\right)\psi(\theta), \tag{19}$$

with $\alpha=\frac{\lambda}{2}$, $\beta=-\frac{\mu}{2\lambda}$.

The operator $\frac{d^2}{d\theta^2}-(\alpha\theta^2+\beta)^2$ is the Laplacian with quartic potential, see e.g. [29]. As already stated, no general explicit solutions are known for this equation. We call

$$\Psi_t\left(\theta,\bar{\theta};\alpha,\beta\right)$$

the solution of

$$\begin{cases}\partial_t\psi(t,\theta)=\left(\frac{d^2}{d\theta^2}-(\alpha\theta^2+\beta)^2\right)\psi(t,\theta),\\ \psi(0,\theta)=\delta_{\bar{\theta}},\end{cases}$$

i.e. the solution of (19) evaluated in θ at time t, with initial data $\delta_{\bar{\theta}}$ and parameters α,β.

Applying Corollary 14 and after straightforward computations, one gets the kernel of the hypoelliptic heat equation on the group $\mathfrak{G}_4$:

$$p_t(x_1,x_2,x_3,x_4)=\int_{\mathbb{R}\setminus\{0\}}d\lambda\int_{\mathbb{R}}d\mu\int_{\mathbb{R}}d\theta\,e^{i\left(-\frac{\mu}{2\lambda}x_2+\lambda x_4-\lambda x_3\theta+\frac{\lambda}{2}x_2\theta^2\right)}\Psi_t\left(\theta+x_1,\theta;\frac{\lambda}{2},-\frac{\mu}{2\lambda}\right). \tag{20}$$

3.3 Hypoelliptic heat kernel on $\mathfrak{G}_5$

The Lie algebra $\mathfrak{L}_5$ of the group $\mathfrak{G}_5$ can be presented as the following matrix space

$$\mathfrak{L}_5 \simeq \left\{ \begin{pmatrix} \mathbf{M_1}(a_1,a_2,a_3,a_4) & \mathbf{0}_{4\times4} \\ \mathbf{0}_{4\times4} & \mathbf{M_2}(a_1,a_2,a_3,a_5) \end{pmatrix} \mid a_i \in \mathbb{R} \right\},$$

where

$$\mathbf{M_1}(a_1,a_2,a_3,a_4) = \begin{pmatrix} 0 & -a_1 & 0 & a_4 \\ 0 & 0 & -a_1 & a_3 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{M_2}(a_1,a_2,a_3,a_5) = \begin{pmatrix} 0 & a_2 & 0 & a_5 \\ 0 & 0 & a_2 & -a_3 \\ 0 & 0 & 0 & -a_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We present each $\mathfrak{l}_i$ as the matrix with $a_j = \delta_{ij}$. It is straightforward to prove that these matrices satisfy the commutation rules for $\mathfrak{L}_5$, where the bracket operation is the standard $[A,B] := BA - AB$.

A matrix presentation of the group $\mathfrak{G}_5$ is thus the matrix exponential of $\mathfrak{L}_5$:

$$\begin{aligned} \mathfrak{G}_5 &\simeq \left\{ \exp\left(\begin{pmatrix} \mathbf{M_1}(a_1,a_2,a_3,a_4) & \mathbf{0}_{4\times4} \\ \mathbf{0}_{4\times4} & \mathbf{M_2}(a_1,a_2,a_3,a_5) \end{pmatrix} \right) \mid a_i \in \mathbb{R} \right\} = \\ &= \left\{ \begin{pmatrix} \exp\left(\mathbf{M_1}(a_1,a_2,a_3,a_4)\right) & \mathbf{0}_{4\times4} \\ \mathbf{0}_{4\times4} & \exp\left(\mathbf{M_2}(a_1,a_2,a_3,a_5)\right) \end{pmatrix} \mid a_i \in \mathbb{R} \right\} = \\ &= \left\{ \begin{pmatrix} \mathbf{N_1}(x_1,x_2,x_3,x_4) & \mathbf{0}_{4\times4} \\ \mathbf{0}_{4\times4} & \mathbf{N_2}(x_1,x_2,x_3,x_5) \end{pmatrix} \mid x_i \in \mathbb{R} \right\} \end{aligned}$$

with

$$\mathbf{N_1}(x_1,x_2,x_3,x_4) = \begin{pmatrix} 1 & -x_1 & \frac{x_1^2}{2} & x_4 \\ 0 & 1 & -x_1 & x_3 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{N_2}(x_1,x_2,x_3,x_5) = \begin{pmatrix} 1 & x_2 & \frac{x_2^2}{2} & x_5 - \frac{x_1 x_2^2}{2} \\ 0 & 1 & x_2 & -x_3 - x_1 x_2 \\ 0 & 0 & 1 & -x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$x_1 = a_1, \qquad x_2 = a_2, \qquad x_3 = a_3 - \frac{a_1 a_2}{2}, \quad x_4 = a_4 + \frac{a_1^2 a_2}{6} - \frac{a_1 a_3}{2}, \quad x_5 = a_5 + \frac{a_1 a_2^2}{6} - \frac{a_2 a_3}{2}.$$

We now define the isomorphism Π_5 between $\mathfrak{G}_5$ and $\mathbb{R}_5$ given by

$$\Pi_4\left(\begin{pmatrix} \mathbf{N_1}(x_1,x_2,x_3,x_4) & \mathbf{0}_{4\times4} \\ \mathbf{0}_{4\times4} & \mathbf{N_2}(x_1,x_2,x_3,x_5) \end{pmatrix} \right) = (x_1,x_2,x_3,x_4,x_5).$$

This isomorphism is a group isomorphism when $\mathbb{R}^5$ is endowed with the following product (see [13, p. 331]):

$$(x_1,x_2,x_3,x_4,x_5)\cdot(y_1,y_2,y_3,y_4,y_5) \quad :=$$

$$\left(x_1+y_1, x_2+y_2, x_3+y_3-x_1y_2, x_4+y_4+\frac{1}{2}x_1^2y_2 - x_1y_3, x_5+y_5+\frac{1}{2}x_1y_2^2 - x_2y_3 + x_1x_2y_2 \right).$$

The isomorphism Π_5 induces an isomorphism of tangent spaces $T_g\mathfrak{g} \simeq T_{\Pi_5(g)}\mathbb{R}^5$, that is explicitly $g\mathfrak{l}_i \simeq X_i$, with X_i given by

$$\begin{aligned} X_1(x) &= \frac{\partial}{\partial x_1}, \qquad X_2(x) = \frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_3} + \frac{x_1^2}{2}\frac{\partial}{\partial x_4} + x_1x_2\frac{\partial}{\partial x_5}, \\ X_3(x) &= \frac{\partial}{\partial x_3} - x_1\frac{\partial}{\partial x_4} - x_2\frac{\partial}{\partial x_5}, \qquad X_4(x) = \frac{\partial}{\partial x_4}, \qquad X_5(x) = \frac{\partial}{\partial x_5}, \end{aligned} \tag{21}$$

where $x = (x_1, x_2, x_3, x_4, x_5)$.

We endow $\mathfrak{G}_5$ with a left-invariant sub-Riemannian structure as presented in Section 2.2, where $\mathfrak{p} = \operatorname{span}\{\mathfrak{l}_1, \mathfrak{l}_2\}$ and $\mathbf{g}_g(g\mathfrak{l}_i, g\mathfrak{l}_j) = \delta_{ij}$ with $i, j = 1$ or 2.

We thus want to find the fundamental solution for the following heat equation:

$$\partial_t\phi(t,x) = \Delta_H\phi(t,x), \tag{22}$$

with $\Delta_H = X_1^2 + X_2^2$.

3.3.1 Representations of $\mathfrak{G}_5$

We now recall the representations of the group $\mathfrak{G}_5$, as computed by Dixmier in [13, p. 338]. As stated before, we may consider only representations on the support of the Plancherel measure.

Proposition 17. *The dual space of* $\mathfrak{G}_5$ *is* $\hat{G} = \left\{\mathfrak{X}^{\lambda,\mu,\nu} \mid \lambda^2 + \mu^2 \neq 0, \nu \in \mathbb{R}\right\}$, *where*

$$\mathfrak{X}^{\lambda,\mu,\nu}(x_1,x_2,x_3,x_4,x_5) : \begin{array}{ccc} \mathcal{H} & \to & \mathcal{H} \\ \psi(\theta) & \mapsto & \exp\left(iK^{\lambda,\mu,\nu}_{x_1,x_2,x_3,x_4,x_5}(\theta)\right)\psi\left(\theta + \dfrac{\lambda x_1 + \mu x_2}{\lambda^2+\mu^2}\right) \end{array}$$

with

$$\begin{aligned} K^{\lambda,\mu,\nu}_{x_1,x_2,x_3,x_4,x_5}(\theta) &= -\frac{1}{2}\frac{\nu}{\lambda^2+\mu^2}(\mu x_1 - \lambda x_2) + \lambda x_4 + \mu x_5 + \\ &- \frac{1}{6}\frac{\mu}{\lambda^2+\mu^2}\left(\lambda^2x_1^3 + 3\lambda\mu x_1^2x_2 + 3\mu^2x_1x_2^2 - \lambda\mu x_2^3\right) + \mu^2x_1x_2\theta + \lambda\mu(x_1^2 - x_2^2)\theta + \\ &+ \frac{1}{2}\left(\lambda^2+\mu^2\right)(\mu x_1 - \lambda x_2)\,\theta^2. \end{aligned}$$

The domain of $\mathfrak{X}^{\lambda,\mu,\nu}(x_1,x_2,x_3,x_4,x_5)$ *is* $\mathcal{H} = L^2(\mathbb{R},\mathbb{C})$, *endowed with the standard product* $<\psi_1,\psi_2> := \int_{\mathbb{R}}\psi_1(\theta)\overline{\psi_2}(\theta)\,d\theta$ *where* $d\theta$ *is the Lebesgue measure.*

The Plancherel measure on $\hat{G}$ *is* $dP(\lambda,\mu,\nu) = d\lambda d\mu d\nu$, *i.e. the Lebesgue measure on* $\mathbb{R}^3$.

Remark **18.** Notice that in this case the domain $\mathcal{H}$ of the representation $\mathfrak{X}^{\lambda,\mu,\nu}$ does not depend on λ, μ, ν.

3.3.2 The kernel of the hypoelliptic heat equation

Consider the representation $\mathfrak{X}^{\lambda,\mu,\nu}$ of $\mathfrak{G}_5$ and let $d\mathfrak{X}_i^{\lambda,\mu,\nu}$ be the corresponding representations of the differential operators L_{X_i} with $i=1,2$. Recall that $d\mathfrak{X}_i^{\lambda,\mu,\nu}$ are operators on $\mathcal{H}$. Again from [13, p. 338], or by explicit computation, we have

$$\begin{aligned} \left[d\mathfrak{X}_1^{\lambda,\mu,\nu}\psi\right](\theta) &= \left(-\frac{i}{2}\frac{\mu\eta}{\lambda^2+\mu^2}+\frac{\lambda}{\lambda^2+\mu^2}\frac{d}{d\theta}-\frac{i}{2}\mu\left(\lambda^2+\mu^2\right)\theta^2\right)\psi(\theta)\\ \left[d\mathfrak{X}_2^{\lambda,\mu,\nu}\psi\right](\theta) &= \left(\frac{i}{2}\frac{\lambda\mu}{\lambda^2+\mu^2}+\frac{\mu}{\lambda^2+\mu^2}\frac{d}{d\theta}+\frac{i}{2}\lambda\left(\lambda^2+\mu^2\right)\theta^2\right)\psi(\theta), \end{aligned}$$

thus

$$\left[\hat{\Delta}_H^{\lambda,\mu,\nu}\psi\right](\theta)=\frac{1}{\lambda^2+\mu^2}\frac{d^2\psi(\theta)}{d\theta^2}-\frac{\left(\nu+(\lambda^2+\mu^2)^2\theta^2\right)^2}{4\left(\lambda^2+\mu^2\right)}.$$

The GFT of the hypoelliptic heat equation is thus

$$\partial_t\psi=\frac{1}{\lambda^2+\mu^2}\frac{d^2\psi(\theta)}{d\theta^2}-\frac{\left(\nu+(\lambda^2+\mu^2)^2\theta^2\right)^2}{4\left(\lambda^2+\mu^2\right)}\psi(\theta). \tag{23}$$

We rewrite it as

$$\partial_\tau\psi=\left(\frac{d^2}{d\theta^2}-\left(\alpha\theta^2+\beta\right)^2\right)\psi(\theta), \tag{24}$$

with $\tau=\frac{t}{(\lambda^2+\mu^2)}$, $\alpha=\frac{\lambda^2+\mu^2}{2}$, $\beta=-\frac{\nu}{2}$.

The operator $\frac{d^2}{d\theta^2}-(\alpha\theta^2+\beta)^2$ is the Laplacian with quartic potential, see e.g. [29]. As already stated, no general explicit solutions are known for this equation. We call

$$\Psi_\tau\left(\theta,\bar{\theta};\alpha,\beta\right)$$

the solution of

$$\begin{cases}\partial_\tau\psi(\tau,\theta)=\left(\frac{d^2}{d\theta^2}-\left(\alpha\theta^2+\beta\right)^2\right)\psi(\tau,\theta),\\ \psi(0,\theta)=\delta_{\bar{\theta}},\end{cases}$$

i.e. the solution of (24) evaluated in θ at time τ, with initial data $\delta_{\bar{\theta}}$ and parameters α,β.

Applying Corollary 14 and after straightforward computations, one gets the kernel of the hypoelliptic heat equation on the group $\mathfrak{G}_5$:

$$p_t(x_1,x_2,x_3,x_4,x_5)=$$

$$=\int_{\lambda^2+\mu^2\neq 0}d\lambda d\mu d\nu\int_{\mathbb{R}}d\theta\exp\left(iK^{\lambda,\mu,\nu}_{x_1,x_2,x_3,x_4,x_5}(\theta)\right)\Psi_{\frac{t}{\lambda^2+\mu^2}}\left(\theta+\frac{\lambda x_1+\mu x_2}{\lambda^2+\mu^2},\theta;\frac{\lambda^2+\mu^2}{2},-\frac{\nu}{2}\right). \tag{25}$$

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