MOTION PLANNING AND FASTLY OSCILLATING CONTROLS

JEAN-PAUL GAUTHIER, BRONISLAW JAKUBCZYK, AND VLADIMIR ZAKALYUKIN

ABSTRACT. We consider the motion planning problem, when the nonholonomic constraints are given by a strong-bracket-generating distribution. Approximating a nonadmissible trajectory by an admissible one in the subriemannian setting, we prove a theorem which provides an exact asymptotic estimate of the "interpolation entropy" in the case of a free nilpotent algebra of first brackets. This theorem shows that we can approximate in an asymptotically optimal way using sinusoidal fastly oscillating controls.

In the general case we obtain similar results, however the estimates are less explicit.

1. Introduction and statement of results

The "Motion-Planning-Problem" treated below is a particular case of the very general problem, of outstanding importance in control theory: given a control system, and given a non-admissible trajectory (i.e. any parametrized curve Γ in the phase space, which is not a trajectory of the system), approximate it by an admissible one in some optimal way.

The particular case we consider here is the case of a kinematic system, defined by a linear set of nonholonomic constraints (i.e. a nonintegrable distribution). We will approximate in the "subriemannian sense".

For the main definitions and constructions we refer to the series of works of the authors [6, 7, 8, 9, 10, 11] and the series of works of F. Jean [13, 14, 15].

In particular there is a small parameter ε (we want to approximate up to ε), and certain quantities $f(\varepsilon), g(\varepsilon)$ go to $+\infty$ when ε tends to zero. We say that such quantities are equivalent ($f \simeq g$) if $\lim_{\varepsilon\to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 1$. If Δ denotes the distribution (specifying the nonholonomic constraints) then we take a Riemannian metric g over Δ, which allows to measure the length of tangent vectors to admissible curves, and therefore the length of admissible curves. Then (Δ, g) is called a subriemannian metric, which we assume to be **strong** (or one-step) bracket generating. It means that the first bracket is enough to generate the full tangent space. The resulting subriemannian distance is denoted by $d(x, y)$.

The problem is local around the given compact curve Γ with no self intersection.

Date: June 2009.

1991 *Mathematics Subject Classification.* 53C17, 49J15, 34H05.

Key words and phrases. Control Theory, robotics, motion planning, optimal control, subriemannian geometry.

[0]The second author was supported by the Ministry of Research and Higher Education, grant N201 039 32/2703, 2007-2010. The third author is partly supported by RFBR080100157, AVCP 2.1.1/5568,2904.

Hence we can always assume that the phase space is some open set in R^n. The rank of the distribution will be denoted by p, and we can always assume the existence of a global orthonormal frame $\mathcal{F}$ for the metric over Δ, $\mathcal{F} = (F_1, ...F_p)$.

The problem can be stated in "control system form":

$$\dot{x} = \sum_{i=1}^{p} F_i(x)u_i, \tag{1.1}$$

we can always choose the F_i, $i = 1, ..., p$, to be an orthonormal frame for the metric, the distribution Δ being just $span(F_1, ..., F_p)$ and the length of an admissible curve $\gamma : [0, T] \to \mathbb{R}^n$ corresponding to a control $u(t)$, $t \in [0, T]$ being just:

$$l(\gamma) = \int_0^T \sqrt{\sum_{i=1}^{p} (u_i)^2 dt}$$

Definition 1. *A motion planning problem $\mathcal{P}$ is a triple $\mathcal{P} = (\Delta, g, \Gamma)$, where (Δ, g) is a rank p strong-bracket-generating subriemannian metric over $\mathbb{R}^n$ and $\Gamma : [0, T] \to \mathbb{R}^n$ is a smooth curve, nowhere tangent to Δ. The set of motion planning problems is endowed with the C^∞ topology over compact sets.*

We are interested in **analytic or generic** (w.r.t. the topology defined above) motion planning problems only. Mainly, Γ and Δ are transversal except maybe at some isolated points, and when some singularities of Δ appear along Γ, they are always isolated along Γ. For the treatment of these isolated singularities see Remark 2, item 2, below.

Definition 2. *The interpolation entropy $E(\varepsilon)$ of $\mathcal{P}$ is $\frac{1}{\varepsilon}$ times the minimum length of an admissible curve γ_ε connecting the endpoints $\Gamma(0), \Gamma(T)$ of Γ, and ε-interpolating Γ, that is, in any segment of γ_ε of length $\geq \varepsilon$, there is a point of Γ.*

This quantity $E(\varepsilon)$ is a function of ε which tends to $+\infty$ as ε tends to zero. It is considered up to equivalence.

Definition 3. *An **asymptotic optimal synthesis** for $\mathcal{P}$ is a one-parameter family $\gamma_\varepsilon : [0, T_{\gamma_\varepsilon}] \to \mathbb{R}^n$ of admissible curves (we assume they are arclength parametrized, i.e. $\sum_{i=1}^{p} (u_i(t))^2 = 1$), which realize an equivalent of the entropy, i.e.:*

1. $\gamma_\varepsilon(0) = \Gamma(0)$, $\quad \gamma_\varepsilon(T_{\gamma_\varepsilon}) = \Gamma(1)$,

2. γ_ε is $\varepsilon(1 + \varepsilon^\alpha)$-interpolating, i.e. γ_ε connects (a finite number of) points of Γ by pieces of length less than or equal to $\varepsilon(1 + \varepsilon^\alpha)$ for a certain real $\alpha > 0$.

4. $E(\varepsilon) \simeq \frac{l(\gamma_\varepsilon)}{\varepsilon}$.

Definition 4. *Given a one parameter family of (absolutely continuous, arclength parametrized) admissible curves $\gamma_\varepsilon : [0, T_{\gamma_\varepsilon}] \to \mathbb{R}^n$, **an ε-modification of** γ_ε is another one parameter family of (absolutely continuous, arclength parametrized) admissible curves $\tilde{\gamma}_\varepsilon : [0, T_{\tilde{\gamma}_\varepsilon}] \to \mathbb{R}^n$ such that for all ε and for some $\alpha > 0$, if $[0, T_{\gamma_\varepsilon}]$ is splitted into subintervals of length ε, $[0, \varepsilon]$, $[\varepsilon, 2\varepsilon]$, $[2\varepsilon, 3\varepsilon]$, then:*

1. $[0, T_{\tilde{\gamma}_\varepsilon}]$ is splitted into corresponding intervals, $[0, \varepsilon_1]$, $[\varepsilon_1, \varepsilon_1 + \varepsilon_2]$, $[\varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_2 + \varepsilon_3]$, with $\varepsilon \leq \varepsilon_i < \varepsilon(1 + \varepsilon^\alpha)$, $i = 1, 2, ...$

2. for each couple of an interval $I_1 = [\varepsilon_i, \varepsilon_i + \varepsilon]$, and the respective interval $I_2 = [i\varepsilon, (i + 1)\varepsilon]$ $\frac{d}{dt}(\tilde{\gamma})$ and $\frac{d}{dt}(\gamma)$, coincide over I_2, i.e.:

$$\frac{d}{dt}(\tilde{\gamma})(\varepsilon_i + t) = \frac{d}{dt}(\gamma)(i\varepsilon + t), \text{ for almost all } t \in [0, \varepsilon].$$

Remark 1. *This concept of an ε-**modification** is for the following use: we will construct asymptotic optimal syntheses for certain "approximating model" of the problem (called nilpotent approximation, to be defined later). Then, the asymptotic optimal syntheses have to be slightly modified in order to realize the interpolation constraints for the original (non-modified) problem. This has to be done "slightly" for the length of paths remaining equivalent.*

To state our main result, we need the definition of the invariants of the problem. Let Ω be the affine subspace of one-forms ω vanishing on Δ and taking value 1 on $\dot{\Gamma}$. For $\omega \in \Omega$, the restriction to Δ of its exterior derivative, $d\omega_{|\Delta}$ defines a one parameter family A_θ along Γ of g-skew symmetric endomorphisms of Δ as follows:

$$\forall \theta \in [0,T],\ \forall X, Y \in \Delta_{\Gamma(\theta)},\ \ < A_\theta X, Y >_g = d\omega(X,Y).$$

For a fixed value of θ, the set $\mathcal{A}_\theta$ of these endomorphisms A_θ is an affine subspace of the space of endomorphisms of Δ_θ.

Hence, we have a one parameter family $\mathcal{A}$ of such affine spaces $\mathcal{A}_\theta$. The affine space $\mathcal{A}_\theta$ is defined by a parallel vector subspace $\mathcal{B}_\theta$ and a one parameter family Λ_θ of endomorphisms which shifts $\mathcal{B}_\theta$ to $\mathcal{A}_\theta$. Moreover, Λ_θ can be chosen uniquely in such a way that it is orthogonal to $\mathcal{B}_\theta$ with respect to the Hilbert-Schmidt scalar product over $End(\Delta_\theta)$, i.e.:

$$trace_g(\Lambda'_\theta \mathcal{B}_\theta) = 0,$$

where Λ' denotes the transpose of Λ.

Then, the decreasingly ordered sequence S_θ of the moduli of eigenvalues of Λ_θ, $S_\theta = (\lambda_\theta^1, ..., \lambda_\theta^r)$, with $r = [p/2]$, (integer part of $\frac{p}{2}$), is the main invariant of the problem.

We will say that θ^* is a **bifurcation point** of the continuous 1-parameter family $\{S_\theta\}$ if the order $\lambda_\theta^1 \geq \cdots \geq \lambda_\theta^r \geq 0$ is not constant in neigbourhood of θ^*, meaning that at least one strong inequality changes for equality, or vice versa, when θ varies in neighbourhood of θ^*. A smooth one parameter family of skew symmetric matrices is said to be **regular** if there are only finitely many bifurcation points and, on each of the subintervals obtained after removing the bifurcation points, the family can be smoothly block-diagonalized, with blocks of size 2.

Any C^ω one parameter family is regular (see [16]). In our previous paper [7], we show that the bad set $\mathcal{U}$ of smooth one-parameter families which are not regular **has infinite codimension** in the space of all one parameter families. To summarize, in the regular case, the $\lambda_\theta^1, ..., \lambda_\theta^r$ are continuous functions in a parameter θ along Γ, Γ can be splitted into a finite number of open pieces (excluding bifurcation points) on which the ordered sequence $S_\theta = (\lambda_\theta^1, ..., \lambda_\theta^r)$ is a smooth function of θ and the matrix family Λ_θ can be smoothly block-diagonalized.

Denote by $k = n - p$ the codimension of the distribution.

Definition 5. ***(Free Case)*** *If the codimension $k = \frac{p(p-1)}{2}$, i.e. the dimension k of $T\mathbb{R}^n/\Delta$ is equal to the dimension of the second homogeneous component of the free Lie algebra with p generators, we say that this case is free.*

Our first main result is the following:

Theorem 1. ***(Free case)*** *We assume that the motion planning problem* $\mathcal{P} = (\Delta, g, \Gamma)$ *is either analytic or the family* Λ_θ *is regular. Then,*

(1) the entropy $E(\varepsilon)$ *is given by the formula:*

$$E(\varepsilon) = \frac{2\pi}{\varepsilon^2} \int_0^T \frac{\sum_{j=1}^r j\lambda_\theta^j}{\sum_{j=1}^r (\lambda_\theta^j)^2} d\theta, \tag{1.2}$$

and

(2) excluding a finite number of bifurcation points, there is a splitting of Γ *into a finite number of (open) pieces, and in a neighborhood of each of these pieces there exists an orthonormal frame field* $\mathcal{F} = (F_1, ...F_p)$ *such that: an asymptotic optimal synthesis is an* ε*-modification of the one-parameter family of trajectories* $\xi_\varepsilon(t)$ *determined by applying the feedback controls:*

$$u_{2j-1}(t) = -\sqrt{\frac{j\lambda_{\theta(t)}^j}{\sum_{j=1}^r j\lambda_{\theta(t)}^j}} \sin(\frac{2\pi jt}{\varepsilon}), \tag{1.3}$$

$$u_{2j}(t) = \sqrt{\frac{j\lambda_{\theta(t)}^j}{\sum_{j=1}^r j\lambda_{\theta(t)}^j}} \cos(\frac{2\pi jt}{\varepsilon}), \quad j = 1, ..., r$$

$$u_{2r+1}(t) = 0 \textit{ if } p \textit{ is odd} .$$

In the last formula, $\theta(t) = w(\xi_\varepsilon(t))$, where w is a smooth projection of the point $\xi_\varepsilon(t)$ to Γ, $w : \mathbb{R}^n \to \Gamma$. For instance, we can chose the orthogonal projection to the w-axis in any normal coordinate system, defined below.

If the eigenvalues are distinct then the operator Λ_θ has unique 2-dimensional "eigen-spaces" in D, corresponding to different λ_i. These eigen-spaces are spanned by corresponding vector fields in the family $(F_1, ...F_p)$.

Dropping the interpolation requirement the theorem states, together with the proposition 1 (3) below, that a non-admissible trajectory Γ can be approximated by an admissible trajectory corresponding to fastly oscillating trigonometric controls, with pulsation equal to successive small multiples of the single basic pulsation $\frac{2\pi}{\varepsilon}$.

If the admissible trajectory starts from a point in Γ, after each pulse it returns close to Γ (at the distance of order $\varepsilon^{1+\alpha}$, $\alpha > 0$) and it can be "corrected" (ε-modified) to meet again Γ in additional time of order $\varepsilon^{1+\alpha}$.

In the non-free case the results are less explicit. Contrarily to the free case, the subspace $\mathcal{B}_\theta$ of the space $so(\Delta_\theta)$ of g-skew symmetric matrices over Δ_θ has codimension larger than 1. For a fixed value of θ, let us consider the family $\tilde{b}_\theta$ of all supplements $\tilde{\mathcal{B}}_\theta$ of Λ_θ, i.e. hyperspaces which do not contain $\Lambda_\theta \in so(\Delta_\theta)$, but contain the subspace $\mathcal{B}_\theta$. We get a new affine space $\tilde{\mathcal{A}}_\theta = \Lambda_\theta + \tilde{\mathcal{B}}_\theta = \tilde{\Lambda}_\theta + \tilde{\mathcal{B}}_\theta$, where $\tilde{\Lambda}_\theta$ is defined uniquely by the fact that it is Hilbert-Schmidt orthogonal to $\tilde{\mathcal{B}}_\theta$. Nonzero ordered eigenvalues of $\tilde{\Lambda}_\theta$ are denoted by $\tilde{\lambda}_\theta^1, ..., \tilde{\lambda}_\theta^r$.

Theorem 2. ***(General case)*** *Assume that the motion planning problem* $\mathcal{P} = (\Delta, g, \Gamma)$ *is* ***analytic****. Then:*

(1) the entropy is given by the formula:

$$E(\varepsilon) = \frac{2\pi}{\varepsilon^2} \int_0^T \min_{\tilde{\mathcal{B}}_\theta \in \tilde{b}_\theta} \frac{\sum_{j=1}^r j\tilde{\lambda}_\theta^j}{\sum_{j=1}^r (\tilde{\lambda}_\theta^j)^2} d\theta, \tag{1.4}$$

(2) there is a partition of $[0,T[$ *into a finite number of subintervals* $[T_i, T_{i+1}[$, $i = 1,...,s$, *and for each* i, *there is an integer number* $r_i \leq [\frac{p}{2}]$ *and an analytic orthonormal frame field* $\mathcal{F} = (F_1^i, ...F_p^i)$ *such that an asymptotic optimal synthesis on* $[T_i, T_{i+1}[$ *is an* ε*-modification of a one parameter family* $\xi_\varepsilon(t)$ *of trajectories determined by applying time dependant fastly oscillating feedback controls:*

$$u_{2j-1}^i(t) = -\sigma_j^i(\theta(t))\sin(\frac{2\pi jt}{\varepsilon}),$$
$$u_{2j}^i(t) = \sigma_j^i(\theta(t))\cos(\frac{2\pi jt}{\varepsilon}) \quad j = 1,...,r_i,$$
$$u_j^i(t) = 0, \quad \text{if } j > r_i,$$

where $\theta(t) = w(\xi_\varepsilon(t))$, *and* $w : \mathbb{R}^n \to \Gamma$ *is a smooth projection, as in Theorem 1.*

Remark 2. *1. The main difference to the free case is that it is not the family of operators* Λ_θ *which plays the crucial role in the formula for the entropy and for determining the two dimensional "eigen-subspaces" in the distribution (if the eigenvalues are distinct). The role of* Λ_θ *is taken by a family of operators* $\tilde{\Lambda}_\theta^*$ *which realize the minimum in the formula (1.4). This family does not necessarily depend continuously on* θ*. The amplitudes of the oscillating controls can be defined (like in the free case) in terms of the eigenvalues of the operator* $\tilde{\Lambda}_\theta^*$ *realizing the minimum in the formula for the entropy.*

2. The numbers r_i *may depend on* i *and they are equal to the ranks of the matrix* $\tilde{\Lambda}_\theta^*$, *constant on the subintervals mentioned in the theorem. In the section 4, we recall the fact that, for* $p = 4$, $n = 9$, *it is generic (precisely, it is open in the* C^∞ *topology) that the integers* r_i *(in Theorem 2), i.e. the number of effective multiple pulsations needed in the asymptotic optimal syntheses* ***can be equal to 1 on certain subintervals and to 2 on other subintervals.*** *In this special case the computations can provide an explicit result.*

3. If the motion planning problem is generic, but Γ *crosses transversally a codimension 1 surface* S_{ing} *where* Δ *is not one-step bracket generating, then the "Logarithmic lemma" from* [7], [10], [11] *provides an explicit formula for the entropy which depends only on the system data near crossing points: the time necessary to cross dominates.*

The present results generalize our previous works in the one-step bracket generating case. They contain our previous result in the free 4-10 case. In principle, using the formula (1.4) one should also be able to recover our previous results in the non free case (and in particular the results relative to the corank $k \leq 3$, or the results in the 4-9 case). However, deriving these results from Formula 1.4 is still a difficult problem. This is due to the nontriviality of the minimization problem in (1.4). We state this problem more explicitly.

Problem 1. *Let* $so(\mathbb{R}, p)$ *denote the vector space of skew symmetric matrices of size* $p = 2r$, *or* $p = 2r+1$, *endowed with the scalar product* $(A, B) = trace(A'B)$. *Suppose we are given an affine subspace* $\mathcal{A} \subset so(\mathbb{R}, p)$ *not containing the origin. Denote by* $\mathcal{H}(\mathcal{A})$ *the set of all hyperplanes* $\widetilde{\mathcal{A}} \subset so(\mathbb{R}, p)$ *not containing the origin and such that* $\mathcal{A} \subset \widetilde{\mathcal{A}}$ *. Find an explicit formula or a simple algorithm for the value*

$$J(\mathcal{A}) = \min_{\widetilde{\mathcal{A}} \in \mathcal{H}(\mathcal{A})} \{e(\Lambda(\widetilde{\mathcal{A}}))\}$$

where $\Lambda(\widetilde{\mathcal{A}})$ *is the element* $\Lambda \in so(\mathbb{R}, p)$ *such that* $\widetilde{\mathcal{A}} = \Lambda + \mathcal{B}$*, with* $\mathcal{B} \subset so(\mathbb{R}, p)$ *- a vector space, and* Λ *orthogonal to* $\mathcal{B}$*, and*

$$e(\Lambda) = \frac{\sum_{j=1}^{r} j\lambda_j}{\sum_{j=1}^{r}(\lambda_j)^2}$$

where $\lambda_1, ..., \lambda_r$ *denote the absolute values of the eigenvalues of* Λ*.*

Proposition 4 and the next theorem provide bounds from above and from below to the solutions of this problem. Define:

$$\psi(\theta) = \sqrt{trace(\Lambda'_\theta \Lambda_\theta)},$$

and

$$\chi(\theta) = \max_{\tilde{\mathcal{B}}_\theta \in \tilde{b}_\theta} ||\tilde{\Lambda}_\theta||.$$

Theorem 3.

$$(1.5) \qquad (1) \qquad \frac{2\pi\sqrt{2}}{\varepsilon^2}\int_0^T \frac{d\theta}{\psi(\theta)} \leq E(\varepsilon) \leq \frac{2\pi}{\varepsilon^2}\int_0^T \frac{B_r d\theta}{\psi(\theta)},$$

$$(2) \qquad \frac{2\pi}{\varepsilon^2}\int_0^T \frac{d\theta}{\chi(\theta)} \leq E(\varepsilon) \leq \frac{2\pi}{\varepsilon^2}\int_0^T \frac{C_r d\theta}{\chi(\theta)},$$

where

$$B_r = \frac{1}{2}(r+1)\sqrt{2r}, \qquad C_r = \frac{1}{4}r\sqrt{r+3} + \frac{1}{2},$$

with $r = [\frac{p}{2}]$*.*

Remark 3. *1. The estimate (1.5), item (1) is especially effective as it does not require to find any maximum.*

2. The estimate (1.5) item(2) coincides (up to factor 2, see remark 4) with the estimate (3.2) given in our previous paper [10]*, in the special case of* $n = 10$ *and* $p = 4$*. It becomes equality when* $r = 1$*.*

3.If we have an a priori knowledge that on some subinterval $I \subset [0, T]$ *the maximal* j *such that* $\lambda_\theta^j \neq 0$ *is bounded by* r^**, for* Λ_θ *realizing the minimum in (1.4), then the constants* B_r *and* C_r *can be replaced by* B_{r^*} *and* C_{r^*}*, on that subinterval. For instance, if the corank* $k \leq 3$ *we know from* [8] *that generically* $r = 1$*, and the estimate (1.5) item(2) becomes equality.*

2. Prerequisites

For a motion planning problem $\mathcal{P} = (\Delta, g, \Gamma)$, where Δ is strong bracket generating.It was proven in [8] that there are coordinates (called **normal coordinates**) (x, y, w) on an open neighborhood of Γ, $x = (x_1, ..., x_p) \in \mathbb{R}^p$, $y = (y_1, ..., y_{k-1}) \in \mathbb{R}^{k-1}$, $w \in \mathbb{R}$, such that the following properties hold.

(1) The parametrized curve Γ is just the set $\{x = 0, y = 0\}, i.e.\ \Gamma(w) = (0, 0, w)$.

(2) Denote by Σ the surface defined by the equation $x = 0$. Subriemannian distance from an arbitrary point $\xi = (x, y, w)$ to Σ is as follows: $d(\Sigma, \xi) = \sqrt{(x_1)^2 + ... + (x_p)^2}$. The geodesic minimizing the distance from an arbitrary point $\xi = (x, y, w)$ to Σ is the straight line through ξ perpendicular to Σ.

(3) The restriction $\Delta_{|\Sigma}$ of the distribution to the surface Σ consists of the horizontal planes which are common kernels of $dy_1, ..., dy_{k-1}, dw$ and the metric g along Σ is $(dx_1)^2 + ... + (dx_p)^2$.

(4) At any point $q=(0,0,\tilde{w})\in\Gamma$, the **tensor** mapping $[.,.]_{/\Delta_q}:\Delta_q\times\Delta_q\rightarrow T_q\mathbb{R}^n/\Delta_q$, $(X,Y)\rightarrow[X,Y]+\Delta_q$ has the following expression:

$$[X,Y]_{/\Delta_q}=2(y\frac{\partial}{\partial y},w\frac{\partial}{\partial w})\text{ with}\tag{2.1}$$
$$y_i=X'L_i(\tilde{w})Y,\ i=1,...,k-1,\ w=X'M(\tilde{w})Y,$$

where $L_i(w)$ and $M(w)$ are independent skew symmetric matrices, depending smoothly upon w.

In fact the surface Σ is a parametrization of the factor space $T\mathbb{R}^n/\Delta$.

Remark 4. *Notice that for the sake of simplicity of the normal form, the formula (2.1) differs by a factor 2 from the respective formulas in our previous papers. Therefore one should be aware that all the expressions for the entropy in this paper also differ by this factor.*

Notice that the affine space $\mathcal{A}$ of one-parameter-families of g-skew symmetric endomorphisms of Δ defined in Section 1, in normal coordinates, coincides with the affine one-parameter-family of pencils of skew symmetric matrices:

$$M(w)+\sum_{i=1}^{k-1}\lambda_iL_i(w).$$

Given a normal coordinate system, the **nilpotent approximation** of the problem $\mathcal{P}$ along Γ is the simplified problem $\mathcal{N}(\mathcal{P})$, with the same Γ, and with the subriemannian metric specified in normal coordinates by the following control system, with control $u\in\mathbb{R}^p$

$$\mathcal{N}(\mathcal{P})\ \begin{cases}\dot{x}=u,\\ \dot{y}_i=x'L_i(w)u,\ i=1,...,k-1,\\ \dot{w}=x'M(w)u.\end{cases}\tag{2.2}$$

and the metric consisting of minimizing $\int\sqrt{(u_1)^2+...+(u_p)^2}dt$.

In fact the nilpotent approximation "dominates" $\mathcal{P}$ in a neighborhood of Γ.

Let C_ε be the cylinder $C_\varepsilon=\{\xi,d(\Sigma,\xi)\leq\varepsilon\}$. In fact, restricting to the cylinder C_ε, the original problem $\mathcal{P}$ can be also written as a control system:

$$\begin{aligned}\dot{x}&=u+O_1(\varepsilon^2)\\ \dot{y}_i&=x'L_i(w)u+O_2(\varepsilon^2),\ i=1,...,k-1,\\ \dot{w}&=x'M(w)u+O_3(\varepsilon^2),\end{aligned}\tag{2.3}$$

where $O_i(\varepsilon^2)$ are smooth functions bounded by $C\varepsilon^2$ for some appropriate positive constant C.

Notice a very important point: these normal forms are invariant under the action of the orthogonal group over Δ (changes of coordinates in Δ have to be compensated by gauge transformations). Notice also that a change of normal coordinates of the form:

$$\tilde{w}=w+\sum_{i=1}^{k-1}\lambda_i(w)\ y_i,\tag{2.4}$$

leaves invariant the formulas (2.2, 2.3) above. Also, it doesn't change the curve Γ, neither its parametrization, The only change is that the matrix $M(w)$ becomes:

$$\tilde{M}(w) = M(w) + \sum_{i=1}^{k-1} \lambda_i(w)\ L_i(w). \tag{2.5}$$

Therefore, there is a unique (smooth) choice of the functions $\lambda_i(w)$ to get:

$$trace(L_i(w)\tilde{M}(w)') = 0, \text{ for } i = 1, ..., k-1. \tag{2.6}$$

In the following we will keep the previous notation M for $\tilde{M}$.

Proposition 1. *(1) The entropy of a motion planning problem is equivalent to that of its nilpotent approximation.*

(2) An asymptotic optimal synthesis for $\mathcal{P}$ is obtained as an ε-modification of an asymptotic optimal synthesis for the nilpotent approximation $\mathcal{N}(\mathcal{P})$,

(3) Leaving out the interpolation requirement, an asymptotic optimal synthesis γ_ε for the nilpotent approximation $\mathcal{N}(\mathcal{P})$ ε-approximates Γ. That is, applying the same controls to the original system, the resulting trajectory remains at a distance less than 2ε from Γ, in particular $\gamma_\varepsilon(0) = \Gamma(0)$, $\gamma_\varepsilon(T_\varepsilon)$ is at a distance less than 2ε from $\Gamma(T)$.

The proposition was mainly proven in [10] (Theorem8). Only item (3) requires some minor modification.

3. Proof of the results

3.1. Proof of Theorem 1. We prove the theorem for even p only. The proof for odd p requires a few obvious modifications.

At first, using the proposition 1, (1-2), we reduce the system to the nilpotent approximation. Moreover, due to the quasi-homogeneity of this nilpotent approximation, we may drop ε, and consider a system (in normal coordinates) with frozen invariants, up to certain ε-modification:

$$\begin{aligned} \dot{x} &= u, \\ \dot{y}_i &= x' L_i u, \ \ i = 1, ..., k-1, \\ \dot{w} &= x' M u. \end{aligned} \tag{3.1}$$

In fact the distance between the trajectories of the frozen system and the genuine one issued from the same point is of order ε^2, during a time interval of length ε.

To solve the problem, it is enough to find a closed curve $u(t), t \in [0,1]$ on the unit sphere $||u(t)||^2 = 1$, $u(0) = u(1)$, which provides a solution of the frozen problem (3.1) with zero initial condition: $x(0) = 0$, $y(0) = 0$, $w(0) = 0$, terminal conditions $x(1) = 0$, $y(1) = 0$, **and with maximum possible value of** $w(1)$.

Proposition 2. *(1)This maximizing solution does exist (see* [10]*) and is given by a particular solution of the linear O.D.E $\dot{u} = Su$ for some skew symmetric matrix S.*

This proposition is an immediate consequence of Pontriaguin's maximum principle.

In other words the maximizing trajectory denoted by $\gamma(t)$, is determined by some skew-symmetric matrix S and initial unit vector u_0, $||u_0|| = 1$, according to the formulas:

$$u(t) = e^{St}u_0, \tag{3.2}$$

$$x(t) = (\int_0^t e^{S\tau}d\tau)u_0. \tag{3.3}$$

The trajectory must satisfy the interpolation constraints:

$$x(1) = 0, \quad \text{equivalently } (\int_0^1 e^{S\tau}d\tau)u_0 = 0, \tag{3.4}$$

and

$$y(1) = 0, \quad \text{equivalently } \int_0^1 x'(\tau)L_j u(\tau) = 0, \tag{3.5}$$

for any $j = 1, ..., N$ and L_j forming a basis of the linear hyperspace $\mathcal{B}$.

Notice that a trajectory does not define the matrix S uniquely. In fact, the following alterations of the matrices S are possible.

Consider a basis in which S has a block-diagonal form with 2×2 blocks of the form $J_\alpha = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$. Consider a decomposition of the vector u_0 into the sum $u_0 = u_1 + ... + u_r$ of its orthogonal projections u_j to the block-coordinate planes $(e_1, e_2), (e_3, e_4), ..., (e_{2m-1}, e_{2m})$.

Assume some projection u_j vanishes. Then the respective block can be modified arbitrarily (for example changed for the zero block). Indeed, there is no influence of these blocks on the trajectories. Moreover if the matrix S has multiple eigenvalues $\pm i\alpha$, then on the eigensubspace $\mathbb{R}^{2r}$ of $\pm i\alpha$ there is a multiple choice of the orthogonal basis (up to the action of the subgroup of the orthogonal group $SO(2r, \mathbb{R})$ which commutes with the restriction $S_{|\mathbb{R}^{2r}}$). The projection of the vector u_0 to $\mathbb{R}^{2r}$ can be chosen colinear to the first basis vector in this subspace. Then the projection of the trajectory lies in the first block. Therefore if $r \geq 2$ the other blocks may be taken vanishing.

Thus we have shown the following.

Proposition 3. *For any control trajectory $u = e^{St}u_0$ there is a unique matrix $\tilde{S}$ with simple nonzero eigenvalues and maximum possible kernel subspace such that $e^{St}u_0 = e^{\tilde{S}t}u_0$.*

*We call this matrix **adapted** to the solution $\gamma(t)$.*

Lemma 1. *1. The nonzero eigenvalues of $\tilde{S}$ are distinct integer multiples of $2\pi\sqrt{-1}$.*

2. An adapted matrix $\tilde{S}$ has a block-diagonal eigenbasis which is also a block-diagonal eigenbasis for the matrix M.

Proof. Item 1 follows from the interpolation requirement (3.4) projected to the block subspace. In other words, the matrix $\int_0^1 e^{J_\alpha \tau}d\tau$ is equal to 0 iff $\alpha = 2k\pi$, $k \in \mathbb{Z} \backslash \{0\}$.

To prove the item 2, suppose that L_j in (3.5) is the matrix $e_{i,j}$ of the basic exterior two-form $\omega_{i,j} = dx_i \wedge dx_j$ where the coordinates x_i and x_j are coordinate functions in the eigenbasis, corresponding to two different diagonal blocks.

Since $I_{i,j} = \int_0^1 x'(t)e_{i,j}u(t)dt$ vanishes for these i and j, the forms $\omega_{i,j}$ belong to the hyperspace $\mathcal{B}$ of 2-forms (or skew-symmetric matrices) which provide (3.5).

Therefore, writing explicitly in the same basis $trace(M'\omega_{i,j}) = 0$, we get $M_{i,j} = 0$. This means exactly that the blocks corresponding to nonzero values of α and the kernel subspaces of $\tilde{S}$ are eigenspaces for M. □

Denote by α_k the imaginary part of nonzero eigenvalues of $\tilde{S}$. For definiteness we will chose $\alpha_k = 2\pi\beta_k$, $k = 1, ..., r$, where β_k is a strictly negative integer and $r = \frac{p}{2}$. Also M and $\tilde{S}$ are diagonal in the same basis, and we work in this basis.

Set the (moduli of the) nonzero eigenvalues of M in the decreasing order: $\lambda_1 \geq \lambda_2 \geq .. \geq \lambda_l > 0$. The other eigenvalues $\lambda_{l+1}, ..., \lambda_r$ vanish. Then, by an easy direct computation we get:

$$\chi_k = \int_0^1 x'(\tau)e_{2k-1,2k}u(\tau)d\tau = -\frac{||u_k||^2}{\alpha_k} \quad \text{if } \alpha_k \neq 0, \text{ otherwise } \chi_k = 0. \tag{3.6}$$

Indeed, if $\alpha_k = 0$, then $u_k = 0$ due to requirement (3.4). Hence we have:

$$w(1) = -\sum_{k=1}^{l} \lambda_k \chi_k, \tag{3.7}$$

Due to the interpolation requirement for the components y_i, for any choice of b_k, the condition $\sum_{k=1}^r \lambda_k b_k = 0$ implies that $\sum_{k=1}^r b_k \chi_k = 0$. Hence the χ_k are proportional to λ_k, $\chi_k = -\delta\lambda_k$ for some nonzero constant δ. It gives:

$$w(1) = -\sum_{k=1}^{l} (\lambda_k)^2 \delta,$$

Notice that if some λ_k is nonzero then χ_k is nonzero and therefore β_k is nonzero.

The remaining constraint (due to the arclength parametrization of the optimal curves) is: $||u_0|| = \sum_{k=1}^r ||u_k||^2 = 1$. It implies that $1 = \delta \sum_{k=1}^r \alpha_k \lambda_k$ and

$$w(1) = -\frac{1}{\sum_{k=1}^r \alpha_k \lambda_k} \sum_{k=1}^{l} (\lambda_k)^2.$$

Since the vectors (χ_k) and (λ_k) are proportional then the integers r and l are equal. Therefore:

$$w(1) = -\frac{1}{\sum_{k=1}^l \alpha_k \lambda_k} \sum_{k=1}^{l} (\lambda_k)^2 = -\frac{1}{\sum_{k=1}^r \alpha_k \lambda_k} \sum_{k=1}^{r} (\lambda_k)^2$$

This quantity $w(1)$ has to be maximum and the $\beta_k = \frac{\alpha_k}{2\pi}$ are nonzero different integers. Clearly, the maximum is attained for $\beta_k = -k$. Finally,

$$w(1) = \frac{1}{2\pi} \frac{\sum_{k=1}^r (\lambda_k)^2}{\sum_{k=1}^r k\lambda_k}.$$

The expression (1.3) of the controls follows, which completes the proof of Theorem 1.

3.2. **Proof of Theorem 2.** Statement (2). As earlier, using Proposition 1, (1-2), **we reduce the system to its nilpotent approximation.** As before we drop ε, and we get (in normal coordinates):

$$\left\{\begin{array}{c} \dot{x} = u, \\ \dot{y}_i = x' L_i(w)u, \; i = 1, ..., l, \\ \dot{w} = x' M(w)u. \end{array}\right.$$

Now $l < \frac{p(p-1)}{2} - 1$. The matrices $L_i(w)$ or $M(w)$ are analytic functions of w.

Further on, as in the previous section, the main term in the ε-asymptotics is given by the integration upon an interval $[0, W]$ on the w-axis of the system with frozen invariants. So we consider the following problem $\mathcal{P}_{\tilde{w}}$ where $\tilde{w}$ is a constant:

$$\text{(3.8)} \qquad \mathcal{P}_{\tilde{w}} \left\{\begin{array}{c} \dot{x} = u, \\ \dot{y}_j = x' L_j(\tilde{w})u, \; j = 1, ..., l, \\ \dot{w} = x' M(\tilde{w})u, \text{ with the additional conditions:} \\ y_j(0) = y_j(1) = 0, \; j = 1, ..., l, \qquad\qquad (1) \\ x_i(0) = x_i(1) = 0, \; i = 1, ..., p \qquad\qquad (2) \\ w(1) \text{ is maximum among the subriemannian geodesics. } (3) \end{array}\right.$$

As before the condition (3) is equivalent to the maximization of $w(1)$ among curves with $\sum_{i=1}^{p}(u_i)^2 = 1$. We limit ourselves to the case where p is even, $p = 2r$. The proof of the odd case is an obvious modification.

As suggested by the earlier proof, due to the interpolation conditions (2) and to the Maximum Principle, for fixed $\tilde{w}$ we should look for an orthogonal matrix $H(\tilde{w})$, an integer vector $k(\tilde{w}) = (k_1(\tilde{w}) \geq ... \geq k_r(\tilde{w}) \geq 0)$, and a vector $v_0(\tilde{w}) \in \mathbb{R}^p$, $||v_0(\tilde{w})|| = 1$, such that

$$\dot{u} = H(\tilde{w})S(\tilde{w})H(\tilde{w})'u, \; u(0) = H(\tilde{w})v_0(\tilde{w})$$

meeting the other conditions (1), (2), (3), with $S(\tilde{w}) = BD(2\pi k_1(\tilde{w})J, ..., 2\pi k_r(\tilde{w})J)$, $J = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)$, and $v_0(\tilde{w})_{2j-1} = v_0(\tilde{w})_{2j} = 0$ if $k_j(\tilde{w}) = 0$. Computing $u(t)$ and $x(t)$, we can find $w(1)$ in a form analogous to formula (3.7),

$$\text{(3.9)} \qquad w(1) = \sum_{i=1}^{r} \bar{\lambda}_j(H(\tilde{w}), v_0(\tilde{w}))\frac{1}{k_j(\tilde{w})},$$

where the $\bar{\lambda}_j$ are analytic, uniformly bounded functions of the arguments (not coinciding with the earlier λ_i). We will later show the following property:

$$(Q) \qquad \text{the } k_j(\tilde{w}) \text{ are bounded by } [\frac{p}{2}].$$

.

Then, we have to maximize $w(1)$ among a finite family of multi-integers $k(\tilde{w})$.

For a **fixed** multi-integer $k(\tilde{w})$, we look for $H(\tilde{w}), v_0(\tilde{w})$. The interpolation conditions (1) provide relations of the same type:

$$0 = \sum_{i=1}^{r} \mu_j(H(\tilde{w}), v_0(\tilde{w}))\frac{1}{k_j(\tilde{w})},$$

for some analytic functions μ_j.

Therefore, by standard arguments of existence of subanalytic sections, the required family $H(\tilde{w})$, $v_0(\tilde{w})$ of orthogonal matrices$H(\tilde{w})$ and of unit vectors $||v_0(\tilde{w})|| = 1$ can be chosen as piecewise-analytic functions of the (single) real parameter $\tilde{w}$.

Choosing the best among the finite family of multi-integers provides again piecewise analytic $H(\tilde{w}), v_0(\tilde{w})$.

Therefore, we may split the interval $[0, T]$ into a finite number of subintervals $I_i = [T_i, T_{i+1}[$ on which the problem $\mathcal{P}_{\tilde{w}}$ with frozen invariants, and with I_i replaced by $[0, 1]$, can be written in the form:

$$\begin{gathered} \dot{x} = u, \\ \dot{y}_j = x' L_j(\tilde{w}) u, \ j = 1, ..., l, \\ \dot{w} = x' M(\tilde{w}) u, \end{gathered}$$

with analytic $L_j(\tilde{w})$, $M(\tilde{w})$, and a **constant** multi integer $k = (k_1 \geq ... \geq k_r \geq 0)$, such that the optimum solution is specified by the relation:

$$u = H(\tilde{w}) e^{St} v_0(\tilde{w}),$$

for $H(\tilde{w})$, $v_0(\tilde{w})$ analytic and $S = BD(2\pi k_1 J, ..., 2\pi k_r J)$.

Now, provided that property (Q) holds, the theorem is already almost proven: the controls for the nilpotent approximation are already linear combinations of fastly oscillating functions $\sin(\frac{2\pi jt}{\varepsilon})$, $\cos(\frac{2\pi jt}{\varepsilon})$ with a pulsation being multiple of a basic one. We just want the first control to depend only on the first pulsation, the second control of the second one, etc...

We apply a simple idea to **transform the problem into an equivalent problem for the free case (complete to freeness)**:

Complete the set of matrices $L_i(\tilde{w})$ by adding some others, in order to get the free case. And of course impose the respective interpolation condition.

This can be done by choosing analytic $L_k(\tilde{w})$, $k = l+1, ..., N = \frac{p(p-1)}{2} - 1$, such that $L_1(\tilde{w}), ..., L_N(\tilde{w})$ are independent and the interpolation condition is met,

$$\int_0^1 x'(t) L_k(\tilde{w}) u(t) dt + \alpha_k(\tilde{w}) \int_0^1 x'(t) M(\tilde{w}) u(t) dt = 0 \tag{3.10}$$

for new matrices $\tilde{L}_k = L_k + \alpha_k M$. The last claim follows from the fact that $\int_0^1 x'(t) M(\tilde{w}) u(t) dt$, which is the maximal $w(1)$, is nonzero. Moreover the functions $\alpha_k(\tilde{w})$ can be chosen analytic (on the considered subinterval I_i).

Then, our problem at the level of the nilpotent approximation reduces to a finite set of analytic free problems to which the theorem 1 can be applied. This gives the second assertion in Theorem 2.

Now it remains only to prove (Q). But for fixed $\tilde{w}$ (frozen invariants), the argument above shows that we can "complete to freeness" and use Theorem 1. This completes the proof of statement (2).

Statement (1) follows from Theorem 1 and the above proof. Namely, the claim (1) says that the entropy is provided by the minimum of the entropies over all "completions to freeness" of the given system. Therefore, it is enough to show that there is a piecewise analytic "completion to freeness" which has the same entropy. But, this is shown in the proof of statement (2).

This ends the proof.

Remark 5. *We emphasize that,* ***even generically,*** *the integer vector k may vary along the curve Γ. It was shown in the paper* [11] *(see also Section 4 below) that for $p = 4$, $n = 9$, on certain open pieces of Γ, $k = (1, 0)$ and on other open pieces $k = (2, 1)$.*

3.3. **Proof of Theorem 3.** The theorem is an immediate consequence of Theorem 2 and the following.

Let $\Lambda : E_n \to E_n$ be a skew-symmetric linear operator in an Euclidean space $E_n = (\mathbb{R}^n, <.,.>)$ of dimension $2r$ or $2r+1$, and let $||\Lambda||_{op}$ and $||\Lambda||_{HS}$ denote its operator norm and Hilbert-Schmidt norm respectively. Consider the function

$$e(\Lambda) = e(\lambda) = \frac{\sum_{j=1}^{r} j\lambda_j}{\sum_{j=1}^{r} \lambda_j^2},$$

where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r \geq 0$ denote the absolute values of the eigenvalues of Λ. Then the following estimates hold:

Proposition 4.

$$\sqrt{2}||\Lambda||_{HS}^{-1} \leq e(\Lambda) \leq B_r||\Lambda||_{HS}^{-1}, \tag{3.11}$$

$$||\Lambda||_{op}^{-1} \leq e(\Lambda) \leq C_r||\Lambda||_{op}^{-1}, \tag{3.12}$$

Proof. We first prove (3.12). Note that $||\Lambda||_{op} = \lambda_1$. Using this fact and homogeneity of all three components in (3.12) we see that it is enough to prove it for $\lambda_1 = 1$, that is, it is enough to show that

$$1 \leq \frac{1 + 2\lambda_2 + ... + r\lambda_r}{1 + \lambda_2^2 + ... + \lambda_r^2} \leq C_r,$$

if $1 \geq \lambda_2 \geq ... \geq \lambda_r \geq 0$. The left inequality is obvious, since $\lambda_j \geq \lambda_j^2$.

In order to prove the right inequality it is enough to show that the maximal value of the estimated function $e(\lambda) = g(\lambda)/h(\lambda)$, with $\lambda = (1, \lambda_2, \ldots, \lambda_r)$ in the above defined range of $\lambda_2, ..., \lambda_r$, is equal to C_r. We first show that the maximum is taken when $\lambda_2 = ... = \lambda_r =: \alpha$. Indeed, suppose that for a maximum point $(\lambda_2^*, \ldots, \lambda_r^*)$ we have $\lambda_i^* > \lambda_{i+1}^*$, for some fixed $i \in \{2, \ldots, r-1\}$. This means that points $\lambda(s) = (1, \lambda_2, \ldots, \lambda_r)$ with $\lambda_j = \lambda_j^*$, $j \neq i$, and $\lambda_i = s$, $\lambda_{i+1}^* \leq s \leq \lambda_i^*$, are admissible in our maximization problem and $e(\lambda(s))$ takes the maximal value at the right end $s = \lambda_i^*$ of the interval $[\lambda_{i+1}^*, \lambda_i^*]$. Therefore, the derivative of the function $s \mapsto e(\lambda(s))$ is nonnegative at $s = \lambda_i^*$, i.e., $(\partial e/\partial\lambda_i)(\lambda^*) \geq 0$. The same assumption $\lambda_i^* > \lambda_{i+1}^*$ implies that the points $\bar{\lambda}(s)$, with $\bar{\lambda}_j = \lambda_j^*$, $j \neq i+1$, and $\lambda_{i+1} = s$, $\lambda_{i+1}^* \leq s \leq \lambda_i^*$ are admissible in the maximization problem and $e(\bar{\lambda}(s))$ takes the maximal value when $s = \lambda_{i+1}^*$. Thus, the derivative of this function is nonpositive, i.e., $(\partial e/\partial\lambda_{i+1})(\lambda^*) \leq 0$. Writing $e = g/h$ we compute

$$\frac{\partial e}{\partial \lambda_j} = \frac{1}{h^2}\left(h\frac{\partial g}{\partial \lambda_j} - g\frac{\partial h}{\partial \lambda_j}\right) = \frac{1}{h^2}(hj - 2g\lambda_j) = \frac{1}{h}(j - 2\lambda_j e).$$

Therefore, the earlier inequalities imply

$$\lambda_i^* \leq \frac{i}{2e(\lambda^*)} \quad \text{and} \quad \frac{i+1}{2e(\lambda^*)} \leq \lambda_{i+1}^*.$$

However, this contradicts our initial assumption that $\lambda_i^* > \lambda_{i+1}^*$.

To conclude the proof of (3.12) we find the maximum of $f(\alpha) := e(1, \alpha, \ldots, \alpha)$, with $0 \le \alpha \le 1$, where

$$f(\alpha) = \frac{1 + a\alpha}{1 + b\alpha^2} = c\frac{1+\beta}{c+\beta^2} =: \bar{f}(\beta),$$

and $a = r(r+1)/2 - 1$, $b = r - 1$, $c = a^2/b$, $\beta = a\alpha$. The maximum is taken at the only point in $[0, a]$ where the derivative $\bar{f}'(\beta)$ vanishes, which is

$$\beta_{max} = \sqrt{c+1} - 1.$$

Evaluating f at this point gives $f(a\beta_{max}) = \bar{f}(\beta_{max}) = 1/2 + \sqrt{c+1}/2$. Since $c = (r-1)(r+2)^2/4$, we easily find that $f(\alpha_{max}) = C_r$, which proves (3.12).

To show (3.11), we first note that $||\Lambda||_{HS} = \sqrt{2\sum_j \lambda_j^2}$ and by homogeneity of all three terms in (3.11), it is enough to show the inequalities under the additional assumption $||\Lambda||_{HS} = const = \sqrt{2r}$. Thus the problem is to find the extrema of the function $f(\lambda) = \sum_{j=1}^{r} j\lambda_j$ under the constraints $g_i(\lambda) = \lambda_{i+1} - \lambda_i \le 0$ and $g_{r+1}(\lambda) = \sum_j \lambda_j^2 = r$, where $\lambda_{r+1} = 0$. Clearly the minimum is attained at $\lambda = (1, 0, ..., 0)$, which gives the maximal value 1 and shows the left inequality. Finding the maximum value is a standard problem of quadratic programming replacing the last constraint by $g_{r+1}(\lambda) \le 0$. Karush-Kuhn-Tucker conditions are sufficient for the extremum. Therefore, we have to find a solution to the problem $-\bigtriangledown f(\lambda) + \sum_i \mu_i \bigtriangledown g_i(\lambda) = 0$, $\mu_i g_i(\lambda) = 0$, $\mu_i \ge 0$, $i = 1, ..., r+1$. It happens that the following values:

$$\lambda = (1, ..., 1),\ \mu_{r+1} = \frac{r+1}{4},\ \mu_i = \frac{i(r-i)}{2}, \qquad i = 1, ..., r$$

provide a solution, and the extremum is $f_{\max} = \frac{r(r+1)}{2.}$. With $\sum_i 1^2 = r$ and $||\Lambda||_{HS} = \sqrt{2r}$, we get the right hand side inequality in (3.11). □

4. The case of a 4-distribution in $\mathbb{R}^9$

As an example, we briefly recall here the case of 4-distributions in $\mathbb{R}^9$. See our paper [11] for full details. This is the very case where generically different integers r_i appear on open pieces of the curve Γ.

In this case, it is natural and useful for computations to use quaternionic notations. Set:

$$i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\imath} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \hat{\jmath} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \hat{k} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The matrices i, j, k (resp. $\hat{\imath}, \hat{\jmath}, \hat{k}$) generate the so-called pure quaternions (resp. pure skew-quaternions).

Theorem 4. *(Normal form) The nilpotent approximation along* Γ *of a generic problem* $\mathcal{P}$ *can be written as:*

$$\left\{\begin{array}{c} \dot{x} = u \\ \dot{y}_m = \frac{1}{2}x'L_m u, \quad i = 1, .., 4, \\ \dot{w} = \frac{1}{2}x'Mu, \end{array}\right.$$

with $M = \alpha(w)\hat{\imath} + \beta(w)\hat{\jmath}$, $\{L_i\} = \{i + \rho_1(w)\hat{\imath};\ j + \rho_2(w)\hat{\jmath};\ k;\ \hat{k}\}$, where $\alpha, \beta, \rho_1, \rho_2$ are smooth invariants depending on w.

Let us consider the set $\tilde{B}_t$, which is the image of the product $B_t \times B_t \subset \Delta_{\Gamma(t)} \times \Delta_{\Gamma(t)}$ of two unit balls by the bracket mapping $[.,.]$, into the quotient tangent space $T_{\Gamma(t)}\mathbb{R}^n/\Delta(\Gamma(t))$.

Definition 6. *The set* $\tilde{B}_t$ *is said strictly convex in the direction* $V_t + \Delta_{\Gamma(t)} \in T_{\Gamma(t)}\mathbb{R}^n/\Delta_{\Gamma(t)}$ *if:*

(P1) there is $x^* = \lambda V_t \in \tilde{B}_t$, $\lambda > 0$, *and* $\omega \in (T_{\Gamma(t)}\mathbb{R}^n/\Delta_{\Gamma(t)})^* \approx (\mathbb{R}^p)^*$ *(dual space of* $T_{\Gamma(t)}\Xi/\Delta_{\Gamma(t)}$*), such that for all* $y \in \tilde{B}_t$,

$$\omega(x^*) - \omega(y) \geq 0.$$

The problem $\mathcal{P}$ *is said strictly convex if* $\tilde{B}_t$ *is strictly convex in the direction* $\dot{\Gamma}(t) + \Delta_{\Gamma(t)}$.

Theorem 5. *For* $k \leq 3$ *(*$k = corank(\Delta)$*), it is generic that* $\mathcal{P}$ *is "strictly convex".*

Theorem 6. *If* $\mathcal{P}$ *is strictly convex, there is a single integer* $r_1 = 1$*. Equivalently, two controls only are nonzero, periodic trigonometric functions of same period.*

The proof of these two theorems is given in [8].

Here, for $p = 4$, $k = 5$, we have the following results, proven in [11]:

Theorem 7. *The problem* $\mathcal{P}$ *is strictly convex in the direction of* Γ *if and only if "the unit separates the squares":*

$$\frac{(\rho_1)^2 - 1}{1 - (\rho_2)^2} > 0.$$

Theorem 8. *Generically, on different open subintervals of* Γ*, the two following situations may coexist:*

1. The integer $r = 1$ *(in the strictly convex case)*
2. The integer $r = 2$ *(non strictly convex case).*

In both cases the entropy can be computed explicitly. First, set $(\alpha, \beta) = (\rho \cos(\theta), \rho \sin(\theta))$. We can reparametrize the curve Γ for $\rho(w) = 1$. In the strictly convex case, set $\chi = |\cos(\theta + \xi)|$ with $\tan(\xi) = (\frac{(\rho_1)^2 - 1}{1 - (\rho_2)^2})^{\frac{1}{2}}$.

In the non strictly convex case, set $\chi = \max(1, \frac{\cos(\theta)^2}{\rho_1} + \frac{\sin(\theta)^2}{\rho_2})$.

Theorem 9. *An estimate for the entropy is* $E(\varepsilon) \leq \frac{2\pi}{\varepsilon^2} \int_\Gamma \frac{dt}{\chi(t)}$*. It is exact in the convex case.*

References

[1] A.A. Agrachev, H.E.A. Chakir, J.P. Gauthier, Subriemannian Metrics on R^3, in Geometric Control and Nonholonomic Mechanics, Mexico City 1996, pp. 29-76, Proc. Can. Math. Soc. 25, 1998.

[2] A.A. Agrachev, J.P. Gauthier, Subriemannian Metrics and Isoperimetric Problems in the Contact Case, in honor L. Pontriaguine, 90th birthday commemoration, Contemporary Maths, Tome 64, pp. 5-48, 1999 (Russian). English version: journal of Mathematical sciences, Vol 103, N°6, pp. 639-663.

[3] G. Charlot Quasi Contact SR Metrics: Normal Form in $\mathbb{R}^{2n}$, Wave Front and Caustic in $\mathbb{R}^4$; Acta Appl. Math., 74, N°3, pp. 217-263, 2002.

[4] H.E.A. Chakir, J.P. Gauthier, I.A.K. Kupka, Small Subriemannian Balls on R^3, Journal of Dynamical and Control Systems, Vol 2, N°3, , pp. 359-421, 1996.

[5] F.H. Clarke, Optimization and nonsmooth analysis, John Wiley & Sons, 1983.

[6] J.P. Gauthier, F.Monroy-Perez, C. Romero-Melendez, On complexity and Motion Planning for Corank one SR Metrics, 2004, COCV; Vol 10, pp. 634-655.

[7] J.P. Gauthier, V. Zakalyukin, On the codimension one Motion Planning Problem, JDCS, Vol. 11, N°1, January 2005, pp.73-89.

[8] J.P. Gauthier, V. Zakalyukin, On the One-Step-Bracket-Generating Motion Planning Problem, JDCS, Vol. 11, N°2, april 2005, pp. 215-235.

[9] J.P. Gauthier, V. Zakalyukin, Robot Motion Planning, a wild case, Proceedings of the Steklov Institute of Mathematics, Vol 250, pp.56-69, 2005.

[10] J.P. Gauthier, V. Zakalyukin, On the motion planning problem, complexity, entropy, and nonholonomic interpolation, Journal of dynamical and control systems, Vol. 12, N°3, July 2006.

[11] J.P. Gauthier, V. Zakalyukin , Entropy estimations for motion planning problems in robotics, Volume In honor of Dmitry Victorovich Anosov, Proceedings of the Steklov Mathematical Institute, Vol. 256, pp. 62-79, 2007.

[12] M. Gromov, Carnot Caratheodory Spaces Seen from Within, Eds A. Bellaiche, J.J. Risler, Birkhauser, pp. 79-323, 1996.

[13] F. Jean, Complexity of Nonholonomic Motion Planning, International Journal on Control, Vol 74, N°8, pp 776-782, 2001.

[14] F. Jean, Entropy and Complexity of a Path in SR Geometry, COCV, Vol 9, pp. 485-506, 2003.

[15] F. Jean, E. Falbel, Measures and transverse paths in SR geometry, Journal d'Analyse Mathématique, Vol. 91, pp. 231-246, 2003.

[16] T. Kato, Perturbation theory for linear operators, Springer Verlag 1966, pp. 120-122.

[17] J.P. Laumond, (editor), Robot Motion Planning and Control, Lecture notes in Control and Information Sciences 229, Springer Verlag, 1998.

[18] L. Pontryagin, V. Boltyanski, R. Gamkelidze, E. Michenko, The Mathematical theory of optimal processes, Wiley, New-York, 1962.

LSIS, UMR CNRS 6168, Bat R, Université de Toulon, BP 20132, 83957, La Garde Cedex

E-mail address: gauthier@univ-tln.fr

Institute of Mathematics, Polish Academy of Sciences, ul. Sniadeckich 8, P.O. Box 21, 00-956 Warsaw, Poland

E-mail address: B.Jakubczyk@impan.gov.pl

Moscow State University, 119 993, Leninski Gori 1, Moscow, Russia

E-mail address: zakalyu@mail.ru