Observability (deterministic systems) and realization theory

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Contents

1 Glossary

- **Observability**

  An observed (and eventually controlled) dynamical system is observable if two distinct initial conditions can be distinguished (via the observations) by choosing the control function.

- **Universal inputs**

  A universal input is a control function allowing to distinguish between all initial conditions.

- **Observer**

  An observer system is a device, given in general under the guise of a differential equation (or a differences equation in the discrete case), allowing to track asymptotically the state trajectory of the system, using only the controls and the observations.

- **Input-output map**

  An input-output map is a mapping (for fixed initial condition) which to "control functions" associates "output functions". It is in general assumed to be "causal" in some sense.

- **Realization** A realization of an input-output map is a (controlled) nonlinear system realizing the given input-output map. A realization (system) is said minimal if it is controllable and observable.

2 Definition of the subject and its importance

Observability analysis, design of nonlinear observers and realization of input-output maps are subjects of central interest in control theory and systems analysis. Related to the synthesis of observer systems is the very important question of "dynamic output stabilization": usually in practice a stabilizing feedback law is applied to the system via the estimation of the state provided by some observer device. Also, the topic is strongly connected with filtering theory, including the standard linear Kalman filter but also nonlinear filtering theory. Realization of some input-output behavior covers the practical idea of modelling systems by differential equations on the basis of input-output experiments (identification).

3 Introduction

In this article, we discuss the basic concepts and methods in observability, observation and realization theories. The area is so large that there are thousands contributions. We provide a nonexhaustive tiny list of references which is certainly far from complete, but corresponds to our taste: an entirely subjective selection. We focus on the continuous finite dimensional case, but there are very important developments for systems governed by PDE's, and for discrete time systems.

In this continuous, finite dimensional context, we chose the geometric setting, however there are other possibilities (algebraic setting, formal power series, Volterra series,..).

For more details, we provide a list of books of significant interest dealing with the topics.

After setting the general definitions, we consider very shortly linear systems for which the theory is perfectly well established for long, the pioneers being Kalman and Luenberger.

Then we state some important results from the geometric nonlinear observability theory, the most significant contributions being undoubtedly those of Hermann and Krener [10] and Sussmann [23, 24, 25]. Also, contrarily to the case of linear systems, the observability of a system depends on the control applied to it. The existence of universal controls is a very important point, clarified by Sussmann [26]. We state the main result. About observability in an analytic-geometry setting, there are also interesting and important results by Bartoziewicz.

The next part of the paper is devoted to realization theory, where mostly two problems may be considered:

1. given a nonlinear system, find a minimal realization;

2. Given some input-output mapping, find a realization of it (it will be minimal by construction).

The most important contribution in this setting is Jakubczyk's one [12, 13]. In fact, it follows a basic idea of Kalman, first for finite automata and second for linear systems. We like Jakubczyk's approach since in particular, it contains very naturally the linear case. To our knowledge, this natural approach has not been used (in the nonlinear framework) for practical identification of nonlinear systems. However it is rather clear that interesting developments are possible. Moreover, it is not so hard to show complete equivalence between this geometric approach and the formal power series approach.

The contribution of Crouch about realization of finite Volterra series is also important, original and involves a lot of geometric considerations. We just refer to the original paper.

Realizing or approximating a system by a bilinear or state linear one is an important question in view of the observer synthesis problem. We state some results on the subject. In particular, there is an important geometric representation theorem (by bilinear systems) due to Fliess and Kupka [8], that we explain.

After these theoretical considerations, we go to a more practical topic: observers. Besides the linear case, there are several contributions in nonlinear observers synthesis (sliding modes, high gain,...). Here, we focus on two natural generalizations of the linear results:

1. the output injection method (the equivalent for observability of feedback linearization) due mostly to Isidori, Krener, Respondek [17, 18];

2. The use of the deterministic version of the linear Kalman's filter: it applies to bilinear systems, that are popular also by several approximation results (Fliess and Jacob in particular [11]).

4 Preliminaries

Surprisingly in the nonlinear case controllability plays a role in the observability properties of a system. It is the reason for the title of the next section.

4.1 Nonlinear systems under consideration and controllability

We consider nonlinear systems (Σ) of the usual form:

$$(\Sigma) \left\{ \begin{array}{c} \dot{x} = f(x,u);\ u \in U, \\ y = h(x). \end{array} \right. \tag{1}$$

Here, the state space x lives either in $\mathbb{R}^n$ or more generally in some n-dimensional differentiable manifold X. The set U of values of control u is some arbitrary set (for simplicity, we assume a closed subset of $\mathbb{R}^l$, may be finite). The observation function h takes values in $\mathbb{R}^p$. To simplify, we will consider the analytic case only, i.e. f and h are real-analytic w.r.t. x. In the special cases where U has some analytic structure (i.e. $U = R^l$ for instance) we assume joint real analyticity w.r.t. (x, u).

If W is an open subset of X, we denote by $\Sigma_{|W}$ the system Σ restricted to W.

Some initial condition $x_0 \in X$ being fixed, such a system Σ defines (via Cauchy existence and uniqueness Theorem) an input-output mapping $P_\Sigma : L^\infty[U] \to AC[\mathbb{R}^p]$, $u(.) \to y(.)$, where $L^\infty[U]$ is the set of functions defined on semi-open intervals $[0, T_u[$ (depending on the control $u(.)$). Possibly $T_u = +\infty$. Here $AC[\mathbb{R}^p]$ denotes the set of absolutely continuous functions over some interval $[0, T_y[$ possibly depending on the output function $y(.)$. Moreover, $T_y = \inf(T_u,\ e(u, x_0))$, where $e(u, x_0)$ is the explosion time of the solution of (1) associated with the initial condition x_0, and the control $u(.)$.

Particular cases of systems under consideration are the usual linear systems

(L), bilinear systems (B) or state-linear systems (LX) :

$$\left\{\begin{array}{c} (L)\ \dot{x} = Ax + Bu,\ y = Cx;\ X = \mathbb{R}^n, U = \mathbb{R}^l, \\ (B)\ \dot{x} = Ax + Bx \otimes u,\ y = Cx;\ X = \mathbb{R}^n, U = \mathbb{R}^l, \\ (LX)\ \dot{x} = A(u)x,\ y = Cx,\ X = \mathbb{R}^n. \end{array}\right. \quad (2)$$

In these formulas, A, B, C are linear. Of course, in the case of a linear system (L), with initial condition $x_0 = 0$, the input-output mapping P_L is a linear mapping.

Our system Σ is said **"controllable"** if the Lie algebra $Lie(\Sigma)$ of smooth vector fields on X generated by the vector fields $f_u, u \in U$ (where $f_u(x) = f(x,u)$) has dimension n at each point of X.

Also, we say that a system Σ is **symmetric** if $\forall u \in U, \exists v \in U$ s.t. $f_v = -f_u$, and Σ is **complete** if all the vector fields f_u, $u \in U$, are complete.

The following fact is standard, for analytic systems. A system is controllable iff:

- 1. the accessibility set $A(x_0)$ of $x_0 \in X$, i.e. the set of points that can be reached from x_0 by some trajectory of Σ, **in positive time**, has open interior in X, whatever $x_0 \in X$.
- 2. The orbit $O(x_0)$ of $x_0 \in X$, i.e. the set of points that can be joined to x_0 by some continuous curve which is a concatenation of trajectories of Σ **in positive or negative time**, is equal to X, whatever $x_0 \in X$.

Moreover in 1, 2 above, it is enough to restrict to piecewise constant control functions. Also, if Σ is symmetric, $O(x_0) = A(x_0)\ \forall x_0 \in X$.

4.2 Definition and characterization of observability, minimal systems

Here, $C^\omega(X)$ denotes the vector space of real analytic functions over X. First, let $\Theta \subset C^\omega(X)$ denote the "**observation space of** Σ", i.e. the smallest vector subspace of $C^\omega(X)$ containing the p components $h_i(.)$ of the output function h and closed under Lie derivation L_{f_u} in the direction of the vector fields f_u, $u \in U$. Then, Θ is also closed under Lie derivation in the direction of vector fields in $Lie(\Sigma)$ and Θ is generated as a real vector space by the functions $(L_{f_{u_r}})^{k_r}(L_{f_{u_{r-1}}})^{k_{r-1}}......(L_{f_{u_1}})^{k_1}h_i$.

Definition 1 *The observability distribution Δ of Σ is the distribution* $\ker(d\Theta)$ *formed by the kernel of the one-forms $d\theta$, $\theta \in \Theta$. The system Σ is said rank-observable if the distribution Δ is trivial. This fact is also called the "observability rank condition".*

The important fact relating the observability and controllability properties is that the observability distribution Δ **has no singularities as soon as** Σ **is controllable**: the rank of Δ is preserved along trajectories of vector fields f_u. Moreover, it is clear that Δ is involutive, hence integrable by Frobenius's Theorem. Leaves of Δ are **levels** of Θ.

Definition 2 *(Indistinguishability and weak indistinguishability relations) Let I be the binary relation over X defined by $x_0^1 I x_0^2$ if for any (piecewise constant) control $u(.) : [0, T_u[\rightarrow U$ such that $e(u, x_0^1) = e(u, x_0^2) = T_u$, then the corresponding output functions $y_1(t)$, $y_2(t)$ from both initial conditions x_0^1, x_0^2 are equal, $t \in [0, T_u[$. The relation I is called the indistinguishability relation for Σ. If V is an open subset of X, we denote by I_V (V-indistinguishability relation) the indistinguishability relation for the restriction $\Sigma_{|V}$. The weak-indistinguishability relation, denoted by I^w is the equivalence relation associated with the foliation of X generated by Δ.*

The indistinguishability relation is an equivalence relation as soon as Σ is complete. It is not an equivalence relation in general. Hence in general, V-indistinguishability also is not equivalence over V.

Definition 3 *the system Σ is said observable if the relation I is the trivial relation. It is said weakly observable if for all $x_0 \in X$, there is a neighborhood W of x_0 such that for each neighborhood V of x_0, $V \subset W$, $I_V(x_0) = x_0$.*

Then weak observability means that **locally**, we can find inputs such that the initial conditions are distinguished by the observations, in arbitrarily short time. Observability means just that distinct initial conditions can be distinguished by observations. The system Σ being observable, analytic, this can be done in arbitrary short time.

In view of realization theory, we say that Σ is **minimal** if it is both controllable and observable. We say that it is **weakly minimal** if it is controllable and weakly observable.

Definition 4 *A universal input for Σ is an input $u(.)$, that distinguishes among any pair of distinct states in arbitrarily short time.*

4.3 Observers

For a system Σ of the form (1) (that we assume observable) an observer is a system of the form:

$$\begin{cases} \dot{z} = F(z, y, u), \\ \hat{x} = H(z, u), \end{cases} \tag{3}$$

where $z \in Z$, some manifold. The observer system is fed by $y(t)$ and $u(t)$, the output and input of Σ. The mapping $H : H \times U \rightarrow X$, and we require that, for a large set of initial condition z_0 for z, the output $\hat{x}(t)$ tracks asymptotically the state $x(t)$ of the system, i.e. at least,

$$\lim_{t \rightarrow +\infty} d(\hat{x}(t), x(t)) = 0, \tag{4}$$

where d is some (Riemannian) metric over X. In general, there are additional requirements on the rate of convergence to zero of the **estimation error** $\varepsilon(t) = d(\hat{x}(t), x(t))$ (such as exponential convergence, with arbitrary exponential rate).

Of course even without such additional requirements, this definition is very vague and not serious at all. It has to be made more precise, depending on the context. There are mostly 2 types of problems:

- This definition depends on the metric d. It may happen that $\varepsilon(t)$ goes to zero for some Riemannian metric d, although it goes to infinity for some other metric d'. Also, the state variables z or x may explode in finite time. Therefore, in general it is reasonable to require (4) only for trajectories of Σ that remain in a given compact subset of X for all positive times. In that case, the usual convergence requirements becomes independent of the Riemannian metric d.
- One cannot expect to observe unobservable systems. Then, one has to require convergence for "good" inputs only.

4.4 Abstract definition of an input-output map

We define the topological group G (resp. the topological semi group S) of extended (resp. positive time) piecewise constant controls as follows: typical elements of G and S are words of the form:

$$\check{u}(\check{t}) = (u_k, t_k)....(u_1, t_1), \tag{5}$$

where $u_i \in U$ and $t_i \in \mathbb{R}$ (resp. $\mathbb{R}_+$). The operation over G and S is the concatenation of words. We consider also the neutral element ε : $\check{u}(\check{t})\varepsilon = \varepsilon\check{u}(\check{t}) = \check{u}(\check{t})$. We define the equivalence relation $\sim$ over G and S as being generated by the relations:

$$\begin{cases} (u, 0) \sim \varepsilon, \\ (u, s)(u, \theta) \sim (u, s + \theta). \end{cases} \tag{6}$$

We consider the quotient spaces $G := G/\sim$, $S := S/\sim$. Both are embedded with the topology co-induced by the maps:

$$\check{u}(.) : \mathbb{R}^k \to G \text{ (resp. } (\mathbb{R}_+)^k \to S).$$

For $\theta \in \mathbb{R}_+$ and $\check{u}(\check{t}) \in S$, we define

$$\begin{aligned} \theta * \check{u}(\check{t}) &= (u_{r+1}, \theta - \eta_r)(u_r, t_r)...(u_1, t_1) \text{ for } \theta \in [\eta_r, \eta_{r+1}[, \\ \eta_r &= t_1 + ... + t_r, \\ \theta * \check{u}(\check{t}) &= \check{u}(\check{t}) \text{ for } \theta \geq \eta_k. \end{aligned}$$

A real mapping: $P : G \to \mathbb{R}$ (*resp.* $S \to \mathbb{R}$) with open domain D is said analytic if , for all $\check{u}(\check{t}) \in D$, the mapping $\check{t} \to P(\check{u}(\check{t}))$ is analytic at $\check{t}$ as a mapping $\mathbb{R}^k \to \mathbb{R}$.

The domain D of $P : D \subset S \to \mathbb{R}$ is said "star-shaped" if $\theta * a \in D$ for all $\theta \in \mathbb{R}_+$ and $a \in D$.

Denote $\check{B}(\check{s}) = ((\check{b}_m(\check{s}_m), ..., \check{b}_1(\check{s}_1)) \in G^m$ (resp. S^m), with

$$\check{b}_i(\check{s}_i) = (b_{in_i}, s_{in_i})....(b_{i_1}, s_{i_1}), b_{i_j} \in U,$$

and set:

$$\Psi_{\breve{u}(\breve{t})}^{\breve{B}(\breve{s})} = (\Psi_{\breve{u}(\breve{t})}^{\breve{b}_m(\breve{s}_m)}, ..., \Psi_{\breve{u}(\breve{t})}^{\breve{b}_1(\breve{s}_1)}),$$
$$\Psi_{\breve{u}(\breve{t})}^{\breve{b}_i(\breve{s}_i)} = P(\breve{b}_i(\breve{s}_i)\breve{u}(\breve{t})).$$

The rank of P is defined as

$$rank(P) = \sup_{k,\breve{B}(\breve{s}),\breve{u}(\breve{t})} rank D_{\breve{t}} \Psi_{\breve{u}(\breve{t})}^{\breve{B}(\breve{s})},$$

where $D_{\breve{t}}$ means the differential w.r.t. $\breve{t} \in \mathbb{R}^k$, and all the arguments cross the possible domain defined by the domain D of P.

Definition 5 *An (abstract) input-output mapping P is an analytic mapping, from some open and star-shaped subset $D \subset S$, with finite rank.*

An extension P^+ of an analytic mapping P is an analytic mapping such that $dom(P) \subset dom(P^+) \subset S$ and $P = P^+_{|dom(P)}$ (restriction of P^+ to $Dom(P)$).

Remark 6 *Given a pointed nonlinear system (Σ, x_0) where Σ is of the form (1) and $x_0 \in X$, it is clear that the associated input-output mapping defines an abstract input-output mapping, the rank of which is the dimension n of the state space.*

5 Linear systems

The simplest case for observability, design of observers and realization theory is the linear case.

Given a linear system (L) from (2) the following results are standard and more or less obvious:

- The observability property is independent of the control $u(.)$ applied to the system, i.e (L) is observable iff it is observable for some fixed arbitrary control $u(.)$.
- The observability distribution Δ is a field of constant planes, given by $\Delta = \cap_{i=1}^{n-1} \ker(CA^i)$. Then Σ is observable iff $rank(\Delta) = 0$. This condition is known as the **observability rank condition**.
- If (L) is observable the following device (**Luenberger observer**):

$$\begin{cases} \dot{z} = (A - \Omega C)z + \Omega y + Bu, \\ \Omega : \mathbb{R}^n \to \mathbb{R}^p, \ z \in \mathbb{R}^n, \end{cases} \quad (7)$$

  is an **arbitrary exponential rate observer**, i.e. the matrix Ω can be chosen in such a way that the the matrix $A - \Omega C$ has arbitrary spectrum, which implies:

$$||\varepsilon(t)|| = ||z(t) - x(t)|| \leq k(\alpha)e^{-\alpha t}||z_0 - x_0|| = k(\alpha)e^{-\alpha t}||\varepsilon_0||, \quad (8)$$

where $\alpha > 0$ is arbitrary, and k is some polynomial in α, independent of Ω.

- Any linear system restricts to a controllable one on some subspace, and is mapped to an observable one, by the canonical projection $\Pi : \mathbb{R}^n \to \mathbb{R}^n / I$.
- Let $Y(t)$ denote an "impulse response" ($Y(t) : \mathbb{R}^l \to \mathbb{R}^p$, $t \geq 0$). The input-output map is the causal linear mapping $P : u(.) \to y(.) = Y * u$, where $*$ denotes the convolution of (positive time) signals. Then assume that (as a formal power series) $Y(t) = \sum_{k=1}^{\infty} G_k \frac{t^k}{k!,}$ and let $\mathcal{H}$ denote the infinite block-Hankel matrix constructed from the sequence of blocks $G_1, G_2, ..., G_k, ..$

  Then, $Y(t)$ is the impulse response of a linear system (L) iff $\mathcal{H}$ has finite rank n.

6 Observability of nonlinear systems

What is clear is that if a system is rank-observable, then it is weakly observable. This is due to a Baker-campbell-Hausdorf-like formula, valid for piecewise constant controls $\check{u}(\check{t})$:

$$y(t) = \sum (L_{f_{u_k}})^{r_k} ... (Lf_{u_1})^{r_1} h(x_0) \frac{t_k^{r_k} ... t_1^{r_1}}{r_k! ... r_1!}. \tag{9}$$

Indeed by real analyticity, if $y_1(t) = y_2(t)$, all the terms $(L_{f_{u_k}})^{r_k} ... (Lf_{u_1})^{r_1} h(x_0^1)$ and $(L_{f_{u_k}})^{r_k} ... (Lf_{u_1})^{r_1} h(x_0^2)$ are equal, which contradicts the rank assumption, for x_0^1, x_0^2 close enough.

Conversely, assume that Σ is controllable and not rank-observable. Then, the observability distribution Δ is constant rank, integrable, nontrivial. Leaves of Δ are levels of Θ. By the same formula (9) points of such leaves are indistinguishable. Therefore Σ is not weakly observable. Then, the following theorem holds:

Theorem 7 *A controllable system Σ is weakly observable iff it is rank-observable.*

The other important result (Sussmann) is:

Theorem 8 *If Σ is observable, there is a universal input. Moreover, the set of universal inputs is generic.*

7 Realization theory

7.1 Minimal realizations given a realization

We are given a realization i.e. a pointed system (Σ, x_0), $x_0 \in X$. In fact, the results follow from the Sussmann's theorem on quotient manifolds: a closed equivalence relation $\mathcal{R}$ differentiably passes to the quotient (i.e. quotient is a manifold and canonical projection is submersive) if there are enough complete vector fields that respect $\mathcal{R}$. We apply this theorem to the indistinguishability relation $\mathcal{R} = I$ in the case of a complete and controllable system. Then all vector fields of $Lie(\Sigma)$ respect I. This is exactly Sussmann's requirement, so that not only there is a quotient manifold and canonical mapping is submersive, but moreover vector fields of $Lie(\Sigma)$ pass to the quotient. Also, the elements of Θ obviously pass to the quotient.

If Σ is not controllable, then, as a first step, we can use the standard Hermann-Nagano Theorem to restrict to a (controllable) leaf of the distribution $Lie(\Sigma)$. Then, we have a similar theorem to the one of the linear case.

Theorem 9 *If Σ is complete, then we can restrict to the leaf of $Lie(\Sigma)$ containing $x_0 \in X$ to get a controllable system. Passing to the quotient manifold by the indistinguishability relation I, we get a minimal realization. Moreover, two minimal realizations are unique up to a diffeomorphism of the state spaces.*

For complete systems, there is an interesting refinement of this theorem. A realization is said **weakly-minimal** if it is controllable, weakly observable. It turns out that the equivalence relation I^w associated to Δ meets also Sussmann's conditions. It follows that the system goes to quotient, and we get a weakly minimal realization $\hat{\Sigma}$ with state space $\hat{X}$. We can apply the previous theorem 9 to $\hat{\Sigma}$ to get again the (unique) minimal realization Σ_m of Σ, with state space X_m.The following theorem is almost obvious.

Theorem 10 *$\hat{X}$ is a covering space of X_m. Moreover, any covering space of X_m determines a weakly-minimal realization of Σ, by a trivial lifting procedure.*

In particular, there is (up to diffeomorphisms) a single simply-connected weakly-minimal realization.

Note that in fact the relation I^w is the same relation as: $x_0^1 I^w x_0^2$ if there is a continuous curve $\gamma : [0,1] \to X$ connecting x_0^1 to x_0^2 and for $r, s \in [0,1]$, $\gamma(r) I \gamma(s)$.

7.2 Minimal realizations given an abstract input-output map

The set of controls U being given, we consider an abstract input-output map defined over the whole group G (domain $D = G$). Note that this is the case in particular for the input-output mappings determined by a complete symmetric system.

In that case we have the following theorem, due to Jakubczyk.

Theorem 11 *An abstract input-output mapping with domain G has a unique minimal realization, which is complete.*

Remark 12 *The finite rank assumption for the input-output mapping is a generalization of the finite rank assumption of the Hankel matrix of the linear case. It is also the analog of certain finite rank assumptions appearing in the formal power series approach of Fliess, or in the Volterra-kernels approach.*

Remark 13 *There is one ugly detail in this theory: in general, we don't get a paracompact manifold as the state space X.*

The idea for the proof of the theorem is very simple: we consider the subgroup H of G defined by $H = \{a \in G | P(ca) = P(c), \forall c \in G\}$. Then , the state space will just be $X = G/H$. The finite rank condition implies that X has the structure of an Hausdorff analytic manifold. The output function h is defined by $h(gH) = P(g)$. The vector-field f_u is defined via its one parameter group: $\exp(tf_u)(gH) = \Pi((u,t)g)$, where $\Pi : G \to G/H$ is the canonical projection.

A more practical result is the following: if we assume that the set U of values of the control **is a finite set**, then the following global result (containing a local one) can be proven.

Theorem 14 *Assume U is finite, then, a necessary and sufficient condition for P to have a realization (weakly-minimal) is that P has an extension P^+ with star-shaped domain D^+. The state space X of this realization is Hausdorff, paracompact.*

In the general analytic case with infinite U, there is only existence of certain **local** realizations.

7.3 Bilinear or state-linear realization

This point will be extremely important for the problem of constructing observer systems (Section 8). A system is said control affine if the vector fields f_u form an affine family w.r.t. u. The single control case ($l = 1$) is just the case $f(x,u) = f(x) + g(x)u$ where f and g are two vector fields on X. Note that a bilinear system is just a state linear system, which is moreover affine in the controls.

A state linear realization (LX, x_0) from Formula (2) is said minimal if it is observable and controllable in the following sense: the orbit of x_0 is not contained in a strict subspace of $\mathbb{R}^n$ (the smallest such subspace would be automatically invariant under all the operators $A(u)$, $u \in U$). First, it is rather simple to show that any pointed state-linear system (LX, x_0) has a minimal state-linear realization. Of course, the additional property to be bilinear is hereditary.

Note that for state-linear systems, the observation space is a (finite-dimensional) vector space of linear forms over X. It turns out that this finite dimensionality condition is in fact a necessary and sufficient condition. This is a very important result from Fliess and Kupka:

Theorem 15 *Assume that Σ has a finite dimensional observation space Θ. Then, Σ is embeddable in a state-linear system. In other terms (Σ, x_0) has a state linear (minimal) realization.*

The proof is very easy. It is enough to take:

- $X = \Theta^*$ (dual space of Θ),
- For $\varphi \in \Theta^*$, $C_i\varphi = \varphi(h_i)$, $i = 1, ..., p$,
- $A(u) = (L_{f_u})^*$ (transpose of L_{f_u}),
- The initial state $\hat{x}_0$ meets $\hat{x}_0(\varphi) = \varphi(x_0)$ for $\varphi \in \Theta$.

Besides the fact that this result allows to solve the observer problem for such systems, "truncating" in some manner the observation space it is a way to approximate systems by state-linear ones , and to get approximate observers.

An interesting particular case where this theorem applies is the case of systems with polynomial observation h and state-linear dynamics:

$$\begin{cases} \dot{x} = A(u)x, \\ y = P(x), \end{cases}$$

where P is some polynomial mapping. It is clear that Θ is finite-dimensional. More generally, if we start with a system with state-linear dynamics, we can approximate uniformly h on compact sets by a polynomial mapping to get a state-linear realization (and later on, an approximate observer device).

7.4 State-linear skew-adjoint realization

Here, for the sake of simplicity in the exposition we limit ourselves to the single output case $p = 1$.

This section describes some particular cases and some generalizations of the results of the previous section, in view of synthesis of observers with a method presented in Section 8.4.

For some reason that will be made clear in the section 8.4 we would like to know when it is possible to embed a system (or to have a realization of a system) into a skew-symmetric, or more generally skew-adjoint, state-linear one. This means that all the matrices $A(u)$ are skew-symmetric w.r.t. the usual scalar product over the state space $\mathbb{R}^n$ of the realization. By the theorem 15, necessary conditions for the nonlinear system Σ (minimal and complete) be embeddable into a such one is that Θ be finite dimensional and hence the group of diffeomorphisms of X generated by the vector fields f_u be a Lie group G. One could think that a necessary condition is that G be a compact Lie group. This is not the case as shows the following example:

$$\begin{cases} \dot{x} = u, \ x, u \in \mathbb{R}, \\ y = \cos(x) + \cos(\alpha x), \text{ where } \alpha \text{ is irrational.} \end{cases}$$

The proper condition is given by the following theorem:

Theorem 16 *The system Σ (complete, minimal) can be embedded into a state-linear skew symmetric system iff:*

1. $\dim(\Theta) < \infty$ *(from what it follows that G is a Lie group),*

2. The observation function $h(x)$ lifts over G into $\tilde{h}$ (in a natural way), an almost periodic function over G.

Remind that an almost periodic function over G is a function that prolongs into a continuous function over the Bohr compactification $G^\flat$ of G. The two conditions of Theorem 16 are equivalent to the fact that G is a Lie group and $\tilde{h}$ is a finite linear combination of coefficients of unitary irreducible finite dimensional representations of G.

If G is "embeddable in a compact group", i.e. if G is the semi-direct product of a compact group by a finite dimensional real vector space then, any h can be approximated in some sense by an almost periodic one.

Actually, a special interesting case is the following: the system Σ is such that $X = G$, a compact Lie group, and the vector fields f_u are right invariant vector fields over G. We can take h as any continuous function $h : G \to \mathbb{R}$, and consider the abstract Fourier transform $\hat{h}$ of h. In fact, by Peter-Weyl's Theorem, h is a uniform limit over G of **finite** linear combinations of the form

$$h(g) = \sum_i \alpha_i \Phi_i(g),$$

where $\Phi_i(g)$ is a coefficient of an irreducible (hence finite dimensional) unitary representation of G. This means that h has approximations h_m that converge uniformly to h over G, such that the system

$$(\Sigma_m) \begin{cases} \dot{g} = A(u)g, \\ y = h_m(g), \end{cases}$$

has a state-linear minimal realization of the form:

$$\begin{cases} \dot{x} = A_m(u)x, \ x \in \mathbb{C}^n, \ A_m(u) \text{ is skew-adjoint}, \\ y = C_m x. \end{cases}$$

Hence **the input-output mapping of any right invariant system over a compact group can be approximated by the one of a skew-adjoint state-linear one**.

Now, let us consider again a (complete, minimal) system Σ, with finite dimensional Lie algebra, but the group G is not compact. In that case $\tilde{h}$ (a lift of h over G) can be approximated uniformly on any compact subset of G by a function h_m, which is a finite linear combination of "positive type" functions over G. This approximation result is known as the Gelfand-Raikov Theorem. As a consequence we have the theorem:

Theorem 17 *The system* $(\Sigma_n)\left\{\begin{array}{l}\dot{g}=A(u)g,\\ y=h_m(g),\end{array}\right.$ *has a (infinite dimensional) skew-adjoint state linear realization on a separable complex Hilbert space* $\mathbb{H}$, *i.e:*

$$\left\{\begin{array}{l}\dot{\Psi}=A(u)\Psi,\\ y=<\Psi,\xi>.\end{array}\right.$$

Here $\xi,\Psi\in\mathbb{H}$ and $<.,.>$ is the scalar product over $\mathbb{H}$. All the operators $A(u)$ are densely defined, essentially skew-adjoint operators, infinitesimal generators of strongly continuous one parameter groups of unitary operators over $\mathbb{H}$.

With this result, in Section 8.4, we will be able to construct reasonable approximate observers for Σ.

8 Observers

8.1 The Kalman's observer for state-linear systems

This is just the deterministic version of the linear time-dependant Kalman filter. Therefore, inputs being known, it applies to state-linear systems (LX) from (2). Contrarily to linear systems, observability for those systems is not a property independent of the inputs: for some input $u(.)$ it might be observable, for other it might be not. Clearly, if we want the observer to have some asymptotic property of convergence of the estimation error, it is reasonable to require that the input under consideration keeps a certain **minimum level of observability** when the time grows to infinity. It is natural to consider inputs living in the space $\mathcal{U}=L^{\infty}_{[0,\infty[,\mathbb{R}^p}$ of measurable U-valued bounded functions. For an input $u\in\mathcal{U}$ and for a real $a\geq 0$, set $u_a(t)=u(t+a)$. We denote by $\Phi_u(t)$ the matrix resolvent of the linear equation $\dot{\Phi}_u(t)=A(u(t))\Phi_u(t)$. Then for $T>0$, the Gramm-observability matrix:

$$G_{u,T}=\int_0^T\Phi_u(t)^*C^*C\Phi_u(t)dt, \qquad (10)$$

where * stands for adjoint operator, measures observability in the following sense: the system is observable for $u:[0,T]\rightarrow U$ iff $G_{u,T}$ is positive definite. Hence there are several type of assumptions that are possible to express that $u:[0,+\infty[\rightarrow U$ keeps a certain level of observability when the time passes. The most simple one is the following:

There are $\alpha,T,T_0>0$ such that for all $\theta\geq T_0$, $G_{u_\theta,T}\geq\alpha Id_n$, where Id_n is the identity matrix. This condition means intuitively that, from time T_0 on, the input u has minimum observability level α on all time intervals of length T. Such an input could be called a **"persistent-excitation"** for Σ.

Then, the following theorem is just a restatement of the classical results about the deterministic version of the linear time dependant Kalman's filter:

Theorem 18 *The matrices Q and R being positive definite symmetric matrices with adequate dimensions, the Riccati system:*

$$\left\{ \begin{array}{c} (1)\ \dot{S} = -A(u(t))'S(t) - S(t)A(u(t)) + C^*R^{-1}C - SQS, \\ (2)\ \dot{z} = A(u(t))z - S^{-1}C^*R^{-1}(Cz - y(t)), \end{array} \right. \tag{11}$$

is an asymptotic observer for persistent-excitations $u(.)$. Convergence of the estimation error is exponential. The matrices $S(t)$ (as soon as the same holds for the initial condition S_0) live in the open cone of positive definite symmetric matrices.

8.2 Observers for systems that are injectable in a state-linear one

Of course, the technique of the previous section applies stricto-sensu to such systems from Section (7.3).

8.3 The output-injection idea

It turns out that both the Luenberger observer (7) for linear systems and the Kalman observer (11) for state-linear systems can be applied in more general nonlinear situations.

Assume that Σ is **linear "up to output injection",** i.e.

$$(\Sigma) \left\{ \begin{array}{c} \dot{x} = Ax + \varphi(y, u) \\ y = Cx \end{array} \right\}, \tag{12}$$

or respectively that Σ is **state-linear (or bilinear) up to output injection**, i.e.

$$(\Sigma) \left\{ \begin{array}{c} \dot{x} = A(u)x + \varphi(y, u) \\ y = Cx \end{array} \right\}, \tag{13}$$

where φ (the output injection) is some nonlinear term depending on the output and input only. Then there are easy modifications of the Luenberger observer (resp. the Kalman's observer) that provide exactly the same results of convergence of the estimation error as for the corresponding systems without the output-injection term.

For case (12) we take the observer under the Luenberger-modified form:

$$\dot{z} = (A - \Omega C)z + \varphi(y, u) + \Omega(y - Cz),$$

while for case (13) we take:

$$\left\{ \begin{array}{c} \dot{S} = -A(u(t))'S(t) - S(t)A(u(t)) + C^*R^{-1}C - SQS \\ \dot{z} = A(u(t))z + \varphi(y, u) - S^{-1}C^*R^{-1}(Cz - y(t)). \end{array} \right.$$

To check the result it is enough to write the estimation error equation and to see that it is exactly the same as in the situation without output-injection.

For that reason, **it is important to characterize systems that can be put under the form of a linear or state-linear system up to output-injection**.

There is an industry around this question. It starts with works of Isidori, Krener, Respondek. The first result of this type is in the uncontrolled case. For an uncontrolled system

$$(\Sigma)\left\{\begin{array}{l} \dot{x}=f(x) \\ y=h(x) \end{array}\right. ,$$

with single output ($p=1$), consider the vector fields X_i defined by

$$\begin{array}{c} L_{X_1}(L_f)^{i-1}h=\delta_{i,n}\,,\ i=1,...,n,\ \text{where } \delta \text{ is the Kronecker symbol,} \\ X_j=-[f,X_{j-1}],\ j=2,...,n, \end{array}$$

The system Σ can be linearized up to a diffeomorphism and an output injection iff the 2 following conditions are met:

- 1. The family $\{dh, dL_f h, ..., d(L_f)^{n-1}h\}$ has full rank n at all points of X.
- 2. $[X_k, X_m]=0$ for $1\leq k,m\leq n$.

Of course, this is a "local almost everywhere result" only.

There is also a lot of results on the problem of characterizing systems that are diffeomorphic to or **embeddable in state-linear systems up to output injection**. A significant result to the problem of embedding up to output injection is the one of Jouan ([14]).

8.4 Observers for skew-adjoint state linear systems

Again, to simplify the exposition we consider the single output case $p=1$ only.

In the case we have a (minimal) state-linear realization which is also skew-adjoint, there is a construction of an observer which is **much simpler** than Kalman's one (no Riccati equation besides the prediction-correction equation (11), (2)). Moreover this construction **extends to infinite-dimensional realizations**, a fact which allows to treat any (complete minimal) system with finite dimensional Lie algebra.

To start, consider some skew-symmetric state linear system:

$$(LX)\left\{\begin{array}{c} \dot{x}=A(u)x,\ A(u) \text{ skew-symmetric } \forall u\in U, \\ y=Cx, \end{array}\right. \tag{14}$$

We consider the following candidate observer system:

$$\dot{z}=A(u)z-rC^*(Cz-y), \tag{15}$$

in which $r>0$ is a parameter. The estimation error is, with $\varepsilon=z-x$:

$$\dot{\varepsilon}=(A(u)-rC^*C)\varepsilon.$$

Then it is not so hard to show that, if $u : [0, \infty[\rightarrow U$ is a "persistent excitation" of Σ in some sense (for instance in the sense of Section 8.1), then we have:

$$\lim_{t \rightarrow +\infty} ||\varepsilon(t)|| = 0.$$

As a consequence, the systems with compact group G of diffeomorphisms (or with G semidirect product of compact by vector space), admit also approximate observers, using the results of Section 7.4.

It turns out that this method can be extended in a reasonable way to systems with (infinite dimensional) skew-adjoint state-linear realization. In particular, it is possible to construct approximate observers for all (complete minimal) systems with finite dimensional Lie algebra.

Consider a skew adjoint realization from Section 7.4:

$$\begin{cases} \dot{\Psi} = A(u)\Psi, \\ y = <\Psi, \xi> . \end{cases}$$

on the (separable) Hilbert space $\mathbb{H}$. Then, the candidate observer device is:

$$\dot{\Lambda} = A(u)\Lambda - r\xi(<\Lambda, \xi> - y(t)). \tag{16}$$

In fact, the persistency assumption cannot be of the same type as in the finite dimensional case. The reason is that the Gramm observability matrix $G_{u,T}$ is a compact operator in that case. Hence it cannot satisfy an inequality of the type $G_{u,T} \geq \alpha Id_{\mathbb{H}}$ since $\mathbb{H}$ is infinite dimensional.

Hence, the definition of a persistent excitation has to be replaced by one of the following type: there is a time $T > 0$ and a real sequence θ_n, $\theta_n \rightarrow +\infty$, with $\theta_{n+1} - \theta_n$ bounded, such that the translated inputs $u_{\theta_n} : [0, T] \rightarrow \mathbb{R}^l$ converge to u^* in the weak-* topology of $L^\infty_{[0,T],\mathbb{R}^l}$ (which topology is precompact over bounded sets) and u^* is a universal input for Σ on $[0, T]$.

This means also that a certain level of observability is preserved, on regularly spaced time intervals, while the time increases.

In that case, of course the result is weaker than in the finite dimensional case. We have only:

$$\text{weak-}\lim_{t \rightarrow +\infty} \varepsilon(t) = 0.$$

9 Future directions

For observability and synthesis of observers, besides the improvement of the current methods (including sliding modes, high gain,...) several directions have to be investigated more deeply, namely infinite dimensional systems, delay and hybrid systems.

For realization theory, and as a consequence identification theory, almost no "**practical result**" is known in the nonlinear context. However we think interesting and consistent developments are possible, even starting from the apparently abstract theory outlined there. This is clearly the challenge for future.

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