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Generalized Fourier Descriptors with Applications to Objects Recognition in SVM Context

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Abstract This paper is about generalized Fourier descriptors, and their application to the research of invariants under group actions. A general methodology is developed, crucially related to Pontryagin's, Tannaka's, Chu's and Tatsuuma's dualities, from abstract harmonic analysis. Application to motion groups provides a general methodology for pattern recognition. This methodology generalizes the classical basic method of Fourier-invariants of contours of objects. In the paper, we use the results of this theory, inside a Support-Vector-Machine context, for 3D objects-recognition. As usual in practice, we classify 3D objects starting from 2D information. However our method is rather general and could be applied directly to 3D data, in other contexts.

Our applications and comparisons with other methods are about human-face recognition, but also we provide tests and comparisons based upon standard data-bases such as the COIL data-base. Our methodology looks extremely efficient, and effective computations are rather simple and low cost.

The paper is divided in two parts: first, the part relative to applications and computations, in a SVM environment. The second part is devoted to the development of the general theory of generalized Fourier-descriptors, with several new results, about their completeness in particular. These results lead to simple formulas for motion-invariants of images, that are "complete" in a certain sense, and that are used in the first part of the paper. The computation of these invariants requires only standard FFT estimations, and one dimensional integration.

Keywords Harmonic analysis · Invariants theory · SVM · Fourier descriptors · Pattern recognition

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1 Introduction

1.1 The Purposes and Contents of the Paper

Our contributions in this paper are at several different levels, from the point of view of both the theory and applications:

A. From the theoretical point of view, we develop a theory of Fourier-based descriptors for functions spaces on a group or a homogeneous space of a group. Typical application is the case where the functions space is the space of 2D or 3D **images** and the group is the group of motions, or the group of motions plus dilations. The purpose is to construct a "complete set of invariants" under this action. Completeness means that this set of invariants will allow to discriminate between all possible images, up to the effect of motions, or motions plus dilations.

The method we develop is inspired from the classical method of Fourier-Descriptors for contours of 2D objects, the basic ideas of which are rather simple. If one wants to discriminate among two contours of objects, (which is a simple way to provide motion invariants of the objects under consideration), one may consider the two following quantities: the contour being denoted by $\rho(\theta)$, and its Fourier series by $\hat{\rho}_n$, the **"spectral densities"** $|\hat{\rho}_n|^2$, and the **"shifts of phases"**[1] $\frac{\varphi(\hat{\rho}_n)}{n} - \frac{\varphi(\hat{\rho}_m)}{m}$ form a complete set of invariants under the action $\alpha \to \rho(\theta+\alpha)$ of translations of the contour. It means that all the information about the contour is contained in these invariants, and the contour can be reconstructed, modulo translation, from these quantities. Notice that this method is just based upon the classical covariance properties of the usual Fourier series (of the contours) under the effect of translations.

In fact, there is a deep abstract reason behind this, that will allow to generalize these Fourier-descriptors to much more general situations: the Pontriaguin's duality theory. Pontriaguin's duality theory (available for Abelian groups) can be generalized in several ways. First generalization is Tannaka duality, available for compact groups. As we shall see, using Tannaka duality, we will be able to generalize perfectly the classical Fourier-descriptors to the case of function spaces over an arbitrary compact group.

A second generalization (not very popular), is Chu-duality, valid for the so-called Moore groups. The group M_2 of motions of the plane is not a Moore group, however, it is in some sense a "limit of Moore groups": the motions of the plane (translations plus rotations) that have a rotation part which is a multiple of a certain elementary angle θ, form a Moore-group, that we denote by $M_{2,N}$ when $\theta = \frac{2\pi}{N}$. Clearly, the group of Motions M_2 is the limit when $N \to +\infty$ of the groups $M_{2,N}$ and at least in practice for objects recognition, it is enough to consider motions that belong to $M_{2,N}$ for N large enough. It turns out that the Fourier descriptors generalize to these Moore groups, and that again, **we are able to prove certain "completeness results"**, but surprisingly **in the case where N is odd only**.

General Motion groups such as the group of motions on the plane M_2 belong to another class (namely the Tatsuuma class), for which a duality theory is also available. However, this class is less tractable, and we are not able to show that the invariants generalizing the Fourier descriptors are complete in the case of M_2, even in a weak sense. However, they are very simple and have several good qualities in practice. For plane images, on which the 2D motions act, we get the following two simple expressions, for the quantities generalizing the classical Fourier descriptors:

"Spectral densities"-type invariants:

$$I_1^r(f) = \int_0^{2\pi} |\tilde{f}(r,\theta)|^2 d\theta, \tag{1.1}$$

[1] Here $\varphi(z)$ denotes the phase of the complex number z.

"Shift of phases"-type invariants:

$$I^{\xi_1,\xi_2}(f) = \int_0^{2\pi} \tilde{f}(R_\theta(\xi_1+\xi_2)\overline{\tilde{f}}(R_\theta(\xi_1))\overline{\tilde{f}}(R_\theta(\xi_2))d\theta, \tag{1.2}$$

where $\tilde{f}(r,\theta)$ is the Fourier transform expressed in polar coordinates in the frequency-plane, of the image $f(x,y)$, and the variables ξ_1, ξ_2 live in the frequency plane. Here R_θ is the two dimensional rotation operator with angle θ.

Remark 1

1. All along the paper, we focus on applications to objects recognition and discrimination. But our method is very general, and can be applied to many other problems in different contexts.
2. It is reasonable to weaken the notion of completeness (for a set of invariants of functions on a group G—or a homogeneous space of a group G—, under the action of G): there is no hope to get a general completeness result in a so abstract setting. We will be happy with "weak completeness", i.e. discrimination among a **"big"** subset of the full set of functions or images. By **"big"**, we mean **residual**, i.e. countable intersection of open-dense sets. For more details about this question of completeness in the case of Abelian groups, see the thesis [18] and the paper [19].

Notation 1 In the paper, we will use indistinctly two terminologies: "Fourier descriptors" and "Motion descriptors". The second terminology will be used when we want to focus on the case of motion groups.

B. To go from 2D information to 3D discrimination, we have chosen the following strategy. A 3D object is represented by a number of 2D "model" pictures from several points of view. We use the fact that the motion descriptors are continuous functions of both their parameters r, ξ_1, ξ_2 (that are homogeneous to frequencies) and the images f (with the $\mathbb{L}^2$ topology of the energy of signals). This continuity expresses **robustness with respect to small deformations**.

An object will be given under the guise of a number of "model" pictures (from several points of view). **Due to the invariance under motions, this number can be small**. Depending on the class of problems, a certain range of the parameters r, ξ_1, ξ_2 is selected, and the corresponding Fourier descriptors are computed. This set of values of the descriptors is the data characterizing an object. If the range of values of the parameters is properly selected, this data determines a cloud of points in the space of parameters, and this cloud of points is characteristic of the object modulo 3D motions: the object is recognized when we decide that its set of motion descriptors belongs to the corresponding cloud, i.e. the picture is close to one (at least) of the "model" pictures, up to motion, but not close to the others.

Then, to discriminate between two objects modulo motions (and possibly modulo rescaling, after renormalization of the descriptors), one has just to decide whether or not a measurement (the set of descriptors of an object, which forms a point in the parameters space) belongs to a certain cloud in the parameters space (and not to the others). To do this, we use this characteristic data inside a **nonlinear classifier**. We have chosen a classifier of SVM (Support-Vector-Machine) type, in order to introduce a **"learning step"**, during which some "separation criteria" of the different clouds are computed, or actuated when adding new data.

Remark 2 The range of values of the parameters r, ξ_1, ξ_2 is selected depending on the problem. This is done in practice just by trials. But, the parameters being homogeneous to frequencies, it is easily understood that if discrimination is due to "texture", high frequencies will be chosen. If discrimination is due to "shape" properties, low frequencies will discriminate.

Remark 3

1. The formulas (1.1), (1.2) for descriptors (and other formulas in the paper) show that the values of the descriptors can be easily computed: they are just usual Fourier transforms (evaluated by FFT) plus integration over circles. Hence, **numerical part of this work is more or less obvious**.
2. These formulas have another important nice property (which is not true for the weakly-complete set of invariants we exhibit at the end of the paper). The final (weakly complete) set of invariants requires a preliminary estimation of the centroid of the image, all the other computations depending on this preliminary result. Accuracy of the computation of this centroid reflects on accuracy of all the other invariants. It is not the case in formulas (1.1), (1.2), that are, in some sense, more intrinsically related to the group of motions. Determining first a centroid corresponds to eliminating the effect of translations, and restricting to the action of the rotations group. Intuitively, this two-step way of thinking is not robust.
3. There are other motion-invariant formulas that are usually applied in the area of objects recognition. We have found a substantial improvement of our results by coupling our motion descriptors with other classical invariants, the Zernike moments namely.

To finish with this introductory presentation, let us provide the following self-justification of our work.

There is actually no need of the "heavy" theory we develop in this paper to perform our applications: after exhausting the formulas (1.1), (1.2), (and other formulas in the paper), one could just observe that these formulas are motion-invariant, rescaling-covariant, and go directly to the applications.

In fact the justification of our theoretical contribution is the following:

1. Our theory here provides a very general methodology applicable to a lot of practical problems, that are concerned with the action of small groups on large spaces (although the method is based upon the basic idea behind the classical Fourier descriptors for contours).
2. The theory is very interesting, even from a purely mathematical point of view.
3. The question of completeness of our "generalized Fourier descriptors" was still open.
4. The invariants obtained are simple, easily computed and physically make sense, since they are homogeneous to spectral densities.

1.2 History and Related Works

There are 3 key related directions in which other recent works have been developed:

1. A lot has been done around applications of group theory and abstract harmonic analysis in signal and image processing. A nice review in the area of "image understanding" is provided in [27]. In this approach, several "modeling assumptions" are made to perform the transition from 2D to 3D. See also [28] for instance. In our paper, we completely ignore this problematic of "understanding", since we have no assumption and no information other than 2D images from several points of view.
2. From the computational point of view, harmonic analysis leads unavoidably to evaluation of "abstract" Fourier transforms, that very often reduce to usual Fourier transforms or Fourier series, computed in practice by FFT. A lot of work is around generalization of this FFT. The basic paper in this direction is [2] and many applied contributions start from this theoretical contribution. A recent reference is [46]. Related recent work is also [8, 31].
3. One of the reasons for which abstract harmonic analysis is interesting in applications for shape discrimination is the covariance of the Fourier transform with respect to the group action (the effect of the group action reflects on the Fourier transform by multiplication by some unitary operator). This covariance property is of course a key-point that we use in our theory. In this direction, besides our papers, a lot of other contributions are important, and we don't claim to be exhaustive at all. Recent contributions are [33, 52].

In this paper, we follow the initial approach of one of the authors: considering the group of motions of the plane, Gauthier et al. [15, 19] introduced the motion descriptors.

H. Fonga [13] improved on this work. Several other nice contributions in the same area are [17, 30]. In many contributions the completeness question is not considered and even "shift of phases" invariants are omitted.

Other tools may be used to exploit the idea of invariance/covariance w.r.t. group actions. They are mostly of two types:

a. computing some invariant moments, the most popular being the Zernike moments [29, 38, 39], that we use also in this paper.
b. making group-invariant multiscale analysis: wavelets adapted to certain group actions. This is certainly a promising direction. See [33, 41, 48].

Also, except in very special contexts, our descriptors will not be applied directly to images: one has for instance to "isolate" the pertinent piece of the image and apply to it our methodology or similar other. In this perspective, several local approaches have been developed recently [26, 34, 36, 37, 50].

In this paper, we forget about these local questions. Our assumption is: images of isolated objects, subject to different motions, or visible under different points of view. It is why the data bases we use in the paper (COIL and others) are certainly not those that are used nowadays for these (different) local problematics. (Typical recent benchmark databases may be found on [44].)

1.3 Organization of the Paper

As claimed at the end the above Sect. 1.1, our purposes in the paper are twofold: we want to develop some theory of "generalized Fourier descriptors", and we want to apply it to invariant object recognition, in SVM environment. For that reason, our paper is divided in two parts. In order not to bother the reader interested only with applications, we have decided to put the applicative part first.

The second part of the paper develops the theory, exhausts our final formulas and final completeness results. But we claim that this part is very important, in the sense that it provides a general methodology, applicable to a lot of other areas than invariant objects recognition.

We have rejected all the complicated proofs in a long appendix, together with the computational methods for our motion descriptors. As we said, the computation is very easy, since it reduces to two steps: 1. FFT computations, 2. Integrating over circles. Therefore the appendix is organized as follows:

In Appendix 1, we give some trivial technical details about the classical motion descriptors for contours, that are useful for the understanding of our methodology.

In Appendix 2, we justify the notion of the "cyclic lift" of an image $f(x, y)$ (a function on the plane) to a certain function $f(x, y, \theta)$ on the group of motions, necessary for our study. It turns out that the natural "trivial lift" $f(x, y) \to f(x, y)$ is not enough for our purposes, since it doesn't produce complete invariants.

Appendix 3 states a simple transversality fact, necessary to prove that our final invariants are weakly complete (i.e. complete over a residual subset of the set of images). With this elementary result, we can apply standard transversality theorems.

Appendix 4 is devoted to the practical computation of the motion descriptors.

Appendix 5 proves a convergence result that is crucially needed to apply Tannaka-Chu duality theories, which are our main technical tools.

Finally, Appendix 6 contains the proofs of several very technical lemmas that we need in our developments. Most of them are stated in Sect. 3.

A standard reader can easily understand our methodology in its full generality without reading a single line of these appendices, if he believes that the lemmas we state in the text are true.

In the first part of the paper, besides our aim to demonstrate empirically the ability of such descriptors to be used successfully in color objects recognition, we also want to show how they can be combined with another well known set of invariant descriptors: the Zernike Moments. We present results obtained by testing our method with standard data-bases in the objects recognition community: the COIL data-base [11, 40] which contains images from 100 objects, the A R face data-base [35] (126 people), the ORL data-base [43], a self-made cellular phones data-base (20 phones) and a self-made data-base of few objects under different lighting conditions.

Sections 2.1 and 2.2 of are devoted to the review of Motion Descriptors and Zernike Moments. Then in Sect. 2.3 the basic theory of Support-Vector-Machines is briefly recalled for the sake of completeness. Our experimental and numerical results are illustrated in Sect. 2.4.

The second part of the paper is divided in several sections:

First, Sect. 3.1 provides more details and comments about the classical Fourier descriptors for contours. Section 3.1.1 recalls the definition of the abstract Fourier transform in the group-theoretic context, together with its main properties, including the crucial "covariance property" w.r.t. translations. We include the explicit computation of the Fourier transform in the case of the group M_2 of motions of the plane.

Then, in Sect. 3.1.2, we show how one can formally copy the classical Fourier-descriptors for contours to get an abstract expression of the Fourier descriptors in a general context. Invariance under the group action, is shown. Section 3.2 treats the special case of M_2. In Sect. 3.3, we treat the case of compact groups, and we introduce Tannaka and Chu dualities that are the main ingredients of our work. Based upon Tannaka duality, **we prove (weak) completeness of the Fourier descriptors in the general case of compact groups**. This is one of our most beautiful result (known for long, but still unpublished).

The last part of the paper, Sect. 3.4 contains our new results, and is specially devoted to the group $M_{2,N}$ of motions with discrete rotations. Chu duality allows (overcoming several hard technical details) to generalize the result over compact groups. More or less, we construct a (weakly) complete set of invariants, containing the "spectral densities" and "shift of phases" motion descriptors. This is obtained by following exactly the same strategy as in the previous cases. The strange fact is that **it works only if N is an odd number**.

Finally our short conclusions are stated in Sect. 4.

2 The Objects Recognition Process

Feature extraction and objects recognition are subject to large research in the field of image processing. To classify objects from images two steps are usually required: first, extracting some features from the images, second, use these features in a classification tool. Feature extraction needs to consider the effectiveness on both data representation and class separability [14]. We are interested in the problem of recognition of individual objects. We describe three methods for objects recognition and their applications for classifying objects.

2.1 Review of Motion Descriptors

2.1.1 Definition of First-Type-Motion-Descriptors (Spectral Densities Type)

First-Type-Motion-Descriptors (1^{st}MD) are defined as follows. Let f be a square summable function on the plane, and $\tilde{f}$ its Fourier transform[2]:

$$\tilde{f}(\xi) = \int_{\mathbb{R}^2} f(x) e^{-i\langle x,\xi\rangle_{\mathbb{R}^2}} dx. \tag{2.1}$$

If (λ, θ) are polar coordinates of the point ξ, we shall denote again by $\tilde{f}(\lambda, \theta)$ the Fourier transform of f at the point (λ, θ). We define [15, 19] the mapping:

$$I_1^r(f) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+,$$

$$r \longrightarrow I_1^r(f),$$

by

$$I_1^r(f) = \int_0^{2\pi} |\tilde{f}(r,\theta)|^2 d\theta. \tag{2.2}$$

Here I_1^r is the feature vector which describes each image f and will be used as an input of our first supervised classification method.

2.1.2 Properties

Fourier descriptors I_1^r calculated according to equation (2.2), have several elementary properties crucial for invariant object recognition [15]:

Motion-Descriptors are motion and reflection-invariant:

- If M is a "Motion" such as $g = f \circ M$,

$$I_1^r(f) = I_1^r(g), \quad \forall r \in \mathbb{R}^+. \tag{2.3}$$

- If there exists a reflexion $\Re$ such that $g = f \circ \Re$,

$$I_1^r(f) = I_1^r(g), \quad \forall r \in \mathbb{R}^+. \tag{2.4}$$

- Motion descriptors are scaling-covariant:

If k is a real constant such as $g(x) = f(kx)$ for all $x \in \mathbb{R}^2$,

$$I_1^r(g) = \frac{1}{k^4} I_1^{\frac{r}{k}}(f), \quad \forall r \in \mathbb{R}^+. \tag{2.5}$$

The proof is obvious and left to the reader.

2.1.3 Definition of Second-Type-Motion-Descriptors (Shift of Phases Type)

Second-Type-Motion-Descriptors (2^{nd} MD) are a second family of invariants (containing the first one) which is "closer to completeness" and very natural as explained in the second part of this paper. Originally they were defined in [13, 18, 19]. They are denoted by I^{ξ_1,ξ_2} and they are defined by:

$$I^{\xi_1,\xi_2}(f) = \int_{S_1} \tilde{f}(R_\theta(\xi_1+\xi_2)) \overline{\tilde{f}}(R_\theta(\xi_1)) \overline{\tilde{f}}(R_\theta(\xi_2)) d\theta,$$

$$\xi_1, \xi_2 \in \mathbb{R}^2. \tag{2.6}$$

Here $R_\theta(\xi)$ denotes the rotation of angle θ of the vector $\xi \in \mathbb{R}^2$, i.e. $R_\theta(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$.

Remark 4

1. It is clear that I^{ξ_1,ξ_2} is invariant with respect to motions.
2. It is also clear that the set of invariants I^{ξ_1,ξ_2} is completely determined by the smaller set obtained by taking ξ_1 of the form $(0, r_1)$, $r_1 \in \mathbb{R}^+$.

[2] All along the paper, we omit the important detail that certain formulas make sense in fact on $\mathbb{L}^1 \cap \mathbb{L}^2$ spaces only, but prolong in a unique way to $\mathbb{L}^2$ spaces. It is the case here.

Hence an alternative definition of I^{ξ_1,ξ_2} is given by:

$$\begin{aligned} I_f^{\omega}(\lambda_1,\lambda_2) = \int_{S_1} & [\tilde{f}(-\lambda_1 \sin(\theta+\omega) - \lambda_2 \sin\theta, \\ & \lambda_1 \cos(\theta+\omega) + \lambda_2 \cos\theta) \\ & \times \overline{\tilde{f}}(-\lambda_1 \sin(\theta+\omega), \lambda_1 \cos(\theta+\omega)) \\ & \times \overline{\tilde{f}}(-\lambda_2 \sin\theta, \lambda_2 \cos(\theta))] d\theta, \end{aligned} \quad (2.7)$$

where $\lambda_1, \lambda_2 \in \mathbb{R}^+$ and $\omega \in [0, 2\pi[$.

2.1.4 Properties

The following properties are elementary and left to the reader to check:

- For a real-valued f, $I^{\xi_1,\xi_2}(f)$ is a real number.
- The quantity $I^{\xi_1,\xi_2}(f)$ is symmetric in ξ_1, ξ_2, i.e.:

$$I_f^{\omega}(\lambda_1,\lambda_2) = I_f^{-\omega}(\lambda_2,\lambda_1). \quad (2.8)$$

2.1.5 Important Remark

In Sects. 3.4.4, 3.4.5 appears a third-type of Generalized-Motion-Descriptors. We don't use them in practice. There are two reasons for this:

– First, using the first and second Type-Descriptors we obtain already extremely good results as the reader shall see.
– Second the computation of each of the Third Type Motion-Descriptors requires a preliminary estimation of the centroid of the image (although the computation of the first and second type descriptors do not require any such estimation). This estimation can be sensitive to noise and affects the sensitivity of **all** third-type Motion-Descriptors. Notice that the same problem appears for any invariant system requiring the preliminary estimation of this centroid. However this third type class is not far from being complete as is shown in Sect. 3.4.5.

2.2 Zernike Moments

The Zernike Moments (ZM) are computed from the set of orthogonal Zernike polynomials defined over the polar coordinates (r,θ) space inside a unit circle. The two dimensional Zernike Moments ζ_{pq} of an image intensity function f are defined as in [6]:

$$\zeta_{pq} = \frac{p+1}{\pi} \int_0^1 \int_{-\pi}^{\pi} f(r,\theta) \overline{V_{pq}}(r,\theta) r dr d\theta, \quad |r| \le 1 \quad (2.9)$$

where the Zernike polynomials are defined as:

$$V_{pq} = R_{pq}(r) e^{-jq\theta}. \quad (2.10)$$

The real-valued radial polynomials R_{pq} are:

$$R_{pq}(r) = \sum_{s=0}^{p-|q|} (-1)^q \frac{(p-s)!}{s!(\frac{p-2s+|q|}{2})!(\frac{p-2s-|q|}{2})!} r^{p-2s}. \quad (2.11)$$

That is the Zernike moments are just the scalar product of f with the V_{pq}.

Moduli of the Zernike moments are rotation-invariant: image rotation in the spatial domain just implies a phase shift of the Zernike moments.

Mukandan et al. [38], and Khotanzad [29] have shown that translation- invariance of Zernike moments can be achieved using some image normalization method. In [6] Chee-Way Chong, presents a mathematical framework for the derivation of translation invariance of radial moments defined in polar form.

2.3 Review of SVM Based Classification

Most of the methods in objects recognition include a classification step. Here we have chosen the famous and efficient SVM approach.

SVM is a universal learning machine (developed in particular by Vladimir Vapnik [5, 53]). A review of the basic principles follows, considering a 2-class-problem (whatever the number of classes, it can be reduced to a 2-class-problem by a "one-against-others" method).

The SVM method maps the input vectors (the motion-invariants of the objects from several points of view), or the "initial feature space" R^d into a higher dimensional "feature space" Q. The mapping is determined by a kernel function K. The "separation" properties of this kernel mapping are theoretically based upon the well known Mercer's Theorem.

After this embedding in higher dimension, a decision rule in the feature space Q is chosen, under the form of a separating hyperplane maximizing the separation margin. This optimization problem (of maximizing the margin) can be expressed as a standard quadratic-programming problem, i.e. maximize $W(\alpha)$:

$$W(\alpha) = \sum_{i=1}^n \alpha_i - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j K(x_i, x_j), \quad (2.12)$$

under the constraints:

$$\sum_{i=1}^n \alpha_i y_i = 0, \quad (2.13)$$

and $0 \leq \alpha_i \leq T$ for $i = 1, 2, \ldots, n$ where $\alpha_i \in \mathbb{R}^{\mathrm{d}}$ are the training sample set vectors, and $y_i \in \{-1, 1\}$ the corresponding class labels. T is a constant needed in the case of non-separable classes. The kernel $K(u, v)$ determines an inner product in the feature space Q. The condition required by Mercer's Theorem is that the kernel K be a symmetric positive definite function, i.e. for $g \neq^{\mathbb{L}_2} 0$:

$$\int_{\Omega}\int_{\Omega} K(u, v)g(u)g(v)dudv > 0, \tag{2.14}$$

on a certain compact set Ω.

The choice of the kernel $K(u, v)$ determines the structure of the feature space Q. A kernel that satisfies the positive-definiteness assumption (2.14) may be presented under the form:

$$K(u, v) = \sum_k a_k \Phi_k(u)\Phi_k(v), \tag{2.15}$$

where a_k are positive scalars and the functions Φ_k form a basis of the feature space Q. Vapnik considered mostly three types of SVM kernels [5]:

- Polynomial SVM:

$$K(x, y) = (xy + 1)^p. \tag{2.16}$$

- Radial Basis Function SVM (RBF):

$$K(x, y) = e^{\left(\frac{-\|x-y\|^2}{2\sigma^2}\right)}. \tag{2.17}$$

- Two-layer neural network SVM:

$$K(x, y) = \tanh(kxy - \theta). \tag{2.18}$$

The kernel is chosen a-priori (depending on the problem). Other parameters of the decision rule are obtained from (2.12), i.e. the set of numerical parameters $\{\alpha_i\}$ which determine the support vectors and the scalar b, defined just below.

The separating plane is constructed from those input vectors, for which $\alpha_i \neq 0$. These vectors are called support-vectors and lie on the boundary margin. The number N_s of support-vectors determines the accuracy and the speed of the SVM procedure. Mapping the separating plane back to the input space $\mathbb{R}^{\mathrm{d}}$, gives a separating surface leading to the following nonlinear decision rules:

$$C(x) = Sgn\left(\sum_{i=1}^{Ns} y_i\alpha_i.K(\delta_i, x) + b\right). \tag{2.19}$$

Where δ_i belongs to the set N_s of support vectors defined at the training step.

A SVM based classifier contracts all the informations contained in the training set relevant for classification, into the support vectors. This procedure reduces the size of the training set by identifying its most important points.

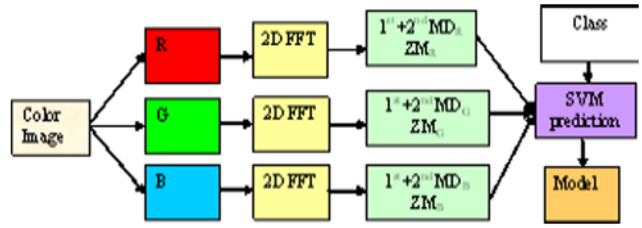

Fig. 1 Training process

Moreover, if the feature space is already a high dimensional space (which is our case here in) then SVM is a quite natural procedure [54].

In this paper, we used LIBSVM [9]. It consists of an integrated software for support vector classification, regression, and distribution estimation. It supports multi-class classification.

2.4 Experimental Results

2.4.1 Test Protocol

In order to test our approach, we performed a cross validation using:

– Three public data-bases: the COIL-100 [11, 29], the ORL face data-base [43] and the A R face color data-base [35].
– Two self made data-bases: One consisting of similar objects (cellular phones) and the second consisting of 15 different objects subject to two different lightings.

- **Training Step:**

During the training step (Fig. 1), the data flow is as follows:

The input image is resampled to 128×128 pixels, and a standard FFT is computed for each color channel (Red, Green, and Blue). The three corresponding first and second type Motion-Decriptors are computed from the FFT values and the Zernike moments are also computed from the 3 color channels.

Hence the final size of the feature-vector used for SVM training is $d = 63 \times 3 = 189$ for first-type Motion-Descriptors, $d = 63 \times 3 = 189$ for second-type Motion-Descriptors and $d = 14 \times 3 = 42$ for Zernike Moments. The result of the training step consists of the (Model) set of support vectors determined by the SVM based method.

- **Decision Step:**

During the decision step, the Motion Descriptors or Zernike Moments are computed in the same way, and the model determined during the training step is used to perform the SVM decision. The output is the image class (Fig. 2).

The classification error rate was evaluated using cross-validation.

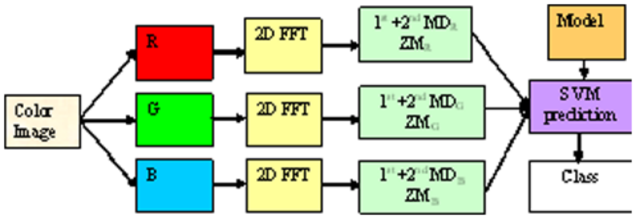

Fig. 2 Decision process

For each database, we evaluated separately the classification error obtained using the First-Type Motion-Descriptors, the Second-Type Motion Descriptors, the Zernike Moments, and the combination of all three feature vectors. In this last case, the dimension of the feature space is $d = 2 \times 189 + 42 = 420$.

Since we used the RBF kernel in the SVM classification process, we have to tune the kernel size, i.e. the value of σ in (2.17). This has been done empirically for each database, choosing the kernel σ_{opt} value providing minimum error rate.

For certain bases we also studied the influence of the number of training images (number of points of view for each object) on the performance of classification, in order to minimize the duration of the training step for a-priori given performances.

As already stated in the introduction we have also tested the robustness with respect to lighting changes. As expected, reasonable robustness with respect to lighting is obtained under the condition of a contour-pretreatment only.

2.4.2 Experiments

A. The COIL-100 database

The Columbia Object Image Library (COIL-100, Fig. 3) [11] is a database of color images of 100 different objects, where 72 images of each object were taken at pose intervals of 5°. The images were pre-processed in such a way that each of them fits the size of 128×128 pixels.

• **Classification performance**

Table 1 and Fig. 4 present results obtained testing our object recognition method with the COIL-100. Tests have been performed using 5-fold cross validation (58 images used for training, 14 images used for testing, for each validation step). Optimum error values are depicted in ▲ (Fig. 4). In this case, first-type Motion-Descriptors outperform Zernike Moments, and the combination of both descriptors improve significantly the global performances of the system. For this data-base second-type Motion-Descriptors do not improve the results.

Other methods testing the COIL-100 database, in the literature, provide error rates varying from 12.5% to 0.1%. See for instance [42].

Fig. 3 Sample objects of COIL-100 database

Table 1 Cross validated error rate on COIL-100 data-base

	ZM	1^{st}MD	1^{st}MD + ZM	2^{st}MD
$\sigma_{opt} = 0.1$	0.22%	0.09%	0.01%	0.09%

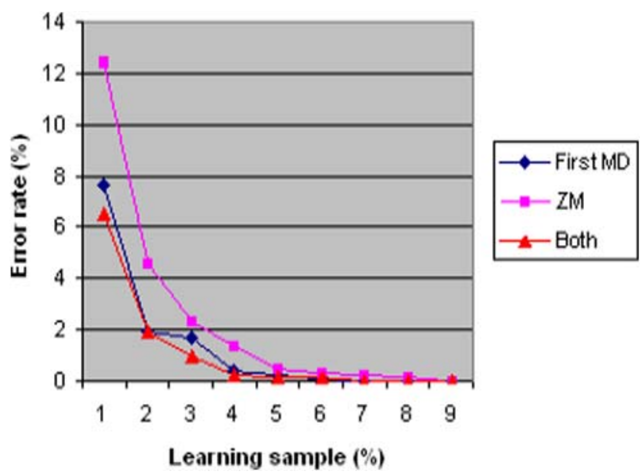

Fig. 4 Influence of number of training samples for COIL

In our global approach, we reach the error-rate $e = 0.01\%$, which corresponds to one faulty image over 7200 only.

We studied the influence of the number of image samples used during the training step. Results are depicted in Fig. 4. The faster convergence is obtained for the combination of first type Motion-Descriptors together with Zernike moments. Using only 20% of images (∼14 images per object) at the training step, we get $e = 2\%$.

• **Robustness against noise**

In order to study the robustness with respect to noise of the Zernike moments and Generalized-Motion-Descriptors, we have used a noisy data-base. This database has been cre-

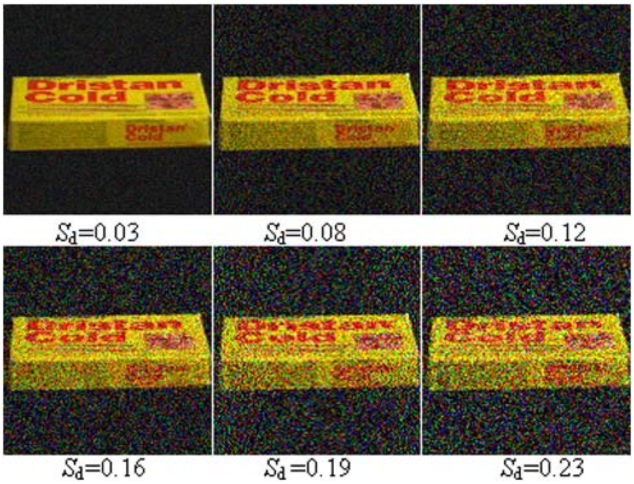

Fig. 5 Sample of COIL noisy object

Table 2 Error rate on COIL-100 noisy database

S_d	ZM	1^{st} MD	1^{st}MD and ZM	2^{nd}MD
0.03	0.40%	0.29%	0.4%	0.02%
0.08	0.29%	0.36%	0.54%	0.02%
0.12	0.27%	0.38%	0.51%	0.02%
0.16	0.34%	0.40%	0.42%	0.04%
0.19	0.26%	0.47%	0.48%	0.05%
0.23	0.43%	0.38%	0.61%	0.06%

ated by adding some Gaussian noises to the COIL images. In order to test several noise levels, we created data-bases with different standard deviation S_d ($0.08 < S_d < 0.23$). Some examples of noisy images are depicted in Fig. 5.

Table 2 shows our results with noisy data-bases. Tests have been done using 9-fold cross validation and the best set of SVM parameters obtained in Sect. 2.3. Results show that noise has little influence on classification performances when we use either Zernike moments or first type Motion-descriptors or both. However second type Motion-Descriptors seem to be much more robust to additive noise.

B. The ORL database

Face detection is a difficult problem for which a lot of methods have been studied [4, 23, 24, 32, 51].

The ORL database used in this paper (Fig. 6) is composed of 400 grey level images of size 112×92. There are 40 persons with ten images per each. The images are taken at different time occurrences with varying lighting conditions, facial expressions (open/closed eyes, smiling/no-smiling), and facial details (glasses/no glasses). All the subjects are in upright, frontal position (with tolerance for some pose variation).

Published results in the literature show a range of error rate varying from 7.5% to 0% [20, 35]. The protocol for testing is different from one paper to another.

Fig. 6 Face samples from the ORL database

Table 3 Error rate on ORL data-base

SVM Kernel RBF	ZM	1^{st}MD	2^{nd}MD	1^{st}MD and ZM	2^{nd}MD and ZM
$\sigma = 0.1$	25%	9.5%	3.25%	4.75%	2.25%

In [23], Hjelmas reported a classification error rate $e = 15\%$ using the ORL data-base with feature vector consisting of Gabor-wavelet coefficients.

In [24], the PCA based method (from [51]), the LDA-based method (from [4]) and a nearest-neighbor-based method (NN) where tested for comparisons. With 10 images of each subject for the training step the error rate is 6.25% with LDA-based method and the best performance is an error of 2.1% with NN-based method.

In [47], a hidden Markov model (HMM) based approach is used, and the best model resulted in a 13% error rate.

Lawrence et al. [32] take the convolutional neural network approach for the classification of ORL database, and the best error is 3.83%.

We performed experiments on the ORL data-base using the Zernike moments, first-type Motion-Descriptors and second-type Motion-Descriptors. The results are shown in Table 3. The second-type Motion-Descriptors applied to the ORL data-base clearly improved on the result. The best result are obtained with combination of Zernike moments and second-type Motion-Descriptors.

C. The A R face data-base

The second face-data-base we used to validate our approach (Fig. 7) was created by Martinez [35]. It contains over 4.000 color images corresponding to 126 people's faces (70 men and 56 women).

This data-base consists of frontal view faces with different facial expressions, illumination conditions, and occlusions (sun glasses and scarf). Each image in the data-base is a 786×576 pixels array and each pixel is represented by 24 bits of RGB color.

Fig. 7 Face samples from the A R data-base

Table 4 Error rate on A R face data-base

SVM RBF Kernel	Z M	1^{st}MD	2^{nd}MD	1^{st}MD and ZM	2^{nd}MD and ZM
$\sigma = 0.1$	10.61%	2.31%	0.92%	0.46%	0.46%

For our experiments, reported in Table 4, the images were normalized to a final 512 × 512 pixel size array.

The performance obtained are:

- $e = 2.31\%$, using a 10-fold cross validation and first-type Motion-Descriptors.
- $e = 0.92\%$ with second-type Motion-Descriptors.
- The addition of Zernike Moment to the first type Motion-Descriptors provides the best performance: $e = 0.46\%$.

It should be noticed that our approach gives much better results than in [35]. The errors obtained there vary from 15 to 5%. However other problems are dealt with, such as detecting occlusions.

D. The Cellular-phones data-base

This cellular-phones (Fig. 8) data-base has been created in our laboratory in order to illustrate the ability of Motion-Descriptors and Zernike Moments to discriminate between very similar objects. The data-base contains 20 objects (phones) and 300 images by object. The acquisition protocol is similar to the COIL acquisition, each object being put on a turntable in order to perform an acquisition each 1.2 degree.

Applied to this cellular-phone data-base, first type Motion-Descriptors and Zernike Moments (and combination) give both a null error using cross validation. It is the reason why we did not test the second type Motion-Descriptors (that are more complicated to compute and will not improve anything since they contain the first type Motion-Descriptors which already give zero error).

We also studied the influence of the number of samples used during the learning step. The results are reported in

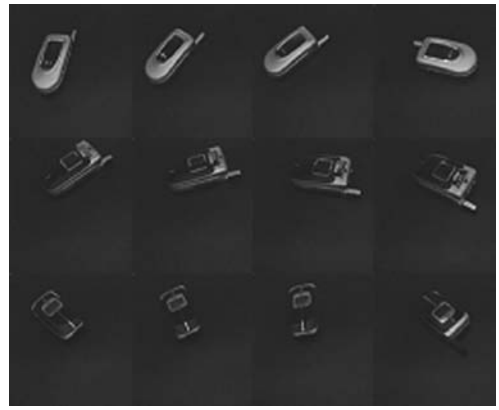

Fig. 8 Sample objects of the cellular phone database

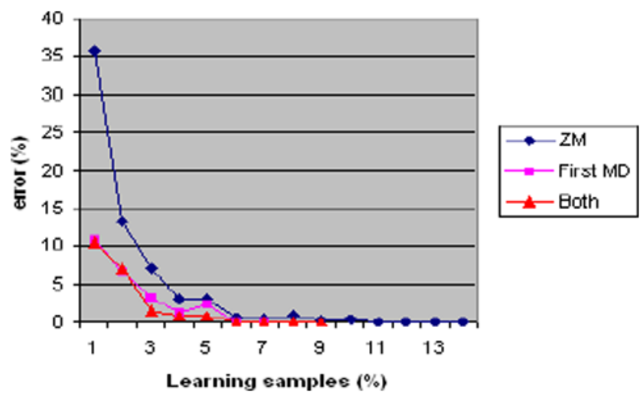

Fig. 9 Influence of the number of training samples for the cellular-phone data-base

Fig. 9. First-type Motion-Descriptors are globally more efficient than Zernike Moments and one can note that as in the COIL case, the combination of both provides a faster convergence: $e < 2\%$ is obtained when only 3% of the available samples are used during the training step.

Robustness study with respect to lighting. The purpose is to test the robustness of the methods with respected to illumination changes.

A data-base of 15 objects has been created. We provide images corresponding to two lighting conditions (Fig. 10). The study is illustrated with these two experiments.

In the first, we train the system with images taken under lighting 1 and we test the data set corresponding to the second lighting condition.

As we already said it is very intuitively reasonable in such conditions to perform a pre-treatment consisting of contour extraction. For this purpose we preprocess the images through a Sobel edge filter.

The results are depicted on Fig. 11. The horizontal axis represents the learning sample percentage and the vertical axis represents the error rate. In this experiment the feature vector is just the first type Motion-Descriptors. We observe that the contour extraction improves on the results, as ex-

Fig. 10 Different lightings

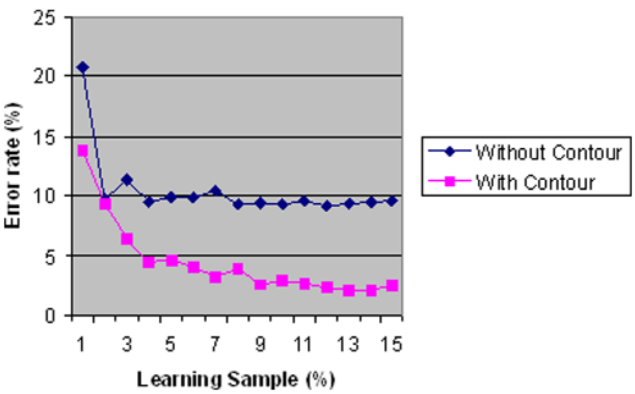

Fig. 11 Influence of contour-extraction on the number of training samples with first motion-descriptors

pected, since the error $e < 5\%$ is obtained when only 4% of samples are used during the training step, while without contour-extraction the error is $e \approx 10\%$.

We also observed that the use of Second-type Motion-Descriptors does not improve on these robustness results, except for the number of training samples that can be reduced to 3% to get approximately the same robustness.

3 Theory of Generalized Fourier Descriptors

Let us start with a few preliminaries about:

- The classical Fourier descriptors for contours.
- The main facts about the abstract Fourier transform from group harmonic analysis. The example of the group M_2 of motions of the plane is treated explicitly.
- The generalization of Fourier Descriptors for contours to Fourier Descriptors in the large.

3.1 Preliminaries

The Fourier-Descriptors method is a very old method used for pattern analysis from the old days on. The oldest reference we were able to find is [45]. Recent ones are [30, 52]. One of the authors and his co-workers have several contributions in the area [13, 15, 18, 19]. Basically, the method uses the good properties of standard Fourier series with respect to translations. For the sake of completeness, let us recall this basic idea, that has been used successfully several times for pattern recognition. For details, see for instance [45].

The method applies to the problem of discrimination of 2D-patterns by their **exterior** contour. Let the exterior contour be well defined, and regular enough (piecewise smooth, say). Assume that it is represented as a closed curve, arclength parametrized and denoted by $s(\theta)$. The variable θ is the arclength, from some arbitrary reference point θ_0 on the contour, and $s(\theta)$ denotes the value of the angle between the tangent to the contour at θ and some privileged direction (the x-axis, say). By construction, the function $s(\theta)$ is obviously invariant under 2D translation of the pattern. Let now $\hat{s}_n$ denote the Fourier series of the periodic function $s(\theta)$. The only arbitrary object that makes the function s non-invariant under motions (translations plus rotations) of the pattern, is the choice of the initial point θ_0. As it is well known, a translation of θ_0 by a, $\theta_0 := a + \theta_0$, changes $\hat{s}_n$ for $e^{ian}\hat{s}_n$, where $i = \sqrt{-1}$. (Here, the total arclength is normalized to 2π.) Set $\hat{s}_n = \rho_n e^{i\varphi_n}$. Let us define the "shifts of phases" $R_{n,m} = \frac{\varphi_n}{n} - \frac{\varphi_m}{m}$. Then, it is easy to check that the "discrete power spectral densities" $Pn = |\hat{s}_n|^2$ and the "shifts of phases" $R_{n,m}$ **form a complete set of invariants** of exterior contours, under motions of the plane. They are also homotetic-invariants as soon as the total arclength is normalized.

This result is extremely efficient for shape discrimination, it has been used an incredible number of times in many areas. It is very robust and physically interesting for several reasons (in particular the fact that the P_n are just discrete "power spectral densities", and that both P_n and $R_{n,m}$ can be computed very quickly using FFT algorithms). Also, the extraction of the "exterior-contour" is more or less a standard procedure in image processing.

The main default of the method is that it doesn't take any account of the "texture" of the pattern: two objects with similar exterior contours have similar "Fourier-Descriptors" P_n and $R_{n,m}$.

This apparently naive method is in fact conceptually very important: as soon as one knows a bit about abstract harmonic analysis, one immediately thinks about possible abstract generalizations of this method. The first paper that we know in which this idea of "abstract generalization" of the method appears is the paper [7]. One of the authors here in worked on the subject, with several co-workers [13, 15, 18, 19]. In particular, there is a lot of very interesting results in the theses [18] and [13]. A recent reference is [52]. Unfortunately, our results being very incomplete, they were never completely published. We would like here to give a series of more or less final result, not yet completely satisfactory, but very interesting and convincing.

They lead to the **"Generalized-Fourier-Descriptors"** that are used in the first part of this paper, and that look extremely efficient for objects discrimination, in addition to a standard Support-Vector-Machine technique. Moreover, at the end, they are computed in practice with standard Fourier integrals, then with FFT algorithms, and hence the algorithms are "fast".

3.1.1 First Preliminary: The Fourier Transform on Locally Compact Unimodular Groups

Classical Fourier descriptors for exterior contours will just correspond to the case of the "circle" group as the reader can check, i.e. the group of rotations $e^{i\theta}$ of the complex plane.

By a famous theorem of Weil, a locally compact group possesses a (almost unique) Haar-measure [57], i.e. a measure which is invariant under (left or right) translations. For instance the Haar measure of the circle group is $d\theta$ since $d(\theta + a) = d\theta$. A group is said unimodular if its left and right Haar measures can be taken equal (that is, the Haar measure associated with left or right translations). An Abelian group is obviously automatically unimodular. A less obvious result is that a compact group is automatically unimodular.

The most pertinent examples for pattern recognition are of course the following:

1. The circle group C.
2. The group of motions of the plane M_2. It is the group of rotations and translations (θ, x, y) of the plane. As one can check, the product law on M_2 is

$$\begin{aligned}(\theta_1, x_1, y_1).(\theta_2, x_2, y_2) \\ = (\theta_1 + \theta_2, \cos(\theta_1)x_2 - \sin(\theta_1)y_2 + x_1, \\ \sin(\theta_1)x_2 + \cos(\theta_1)y_2 + y_1).\end{aligned} \tag{3.1}$$

   It represents the geometric composition of two motions. The main difference with the circle group is that it is not Abelian (commutative). This expresses the fact that rotations and translations of the plane do not commute. However, it is unimodular: the measure $d\theta dx dy$ is simultaneously left and right invariant.
3. The group of y-homotheties and x-translations of the upper two dimensional half plane: $(y_1, x_1).(y_2, x_2) = (y_1 y_2, x_1 + x_2)$. Here, the y_i's are positive real numbers. Left and right Haar measure is $dx\frac{dy}{y}$ since $dx\frac{dy}{y} = d(x+a)\frac{d(by)}{by}$.

   This Abelian group is related to the classical Fourier-Mellin transform. A similar group of interest is the (Abelian) group of θ-rotations and λ homotheties of the complex or two dimensional plane: $(\theta_1, \lambda_1).(\theta_2, \lambda_2) = (\theta_1 + \theta_2, \lambda_1\lambda_2)$. Here again, the λ_i's are positive real numbers but the θ_i's belong to the circle group. Of course, if one takes an image centered around it's gravity center, then, the effect of translations is eliminated, and it remains only the action of rotations and homotheties. Applying the theory developed in the second part of this paper to the case of this group leads to complete invariants with respect to motions and homotheties. This is related with the nice work of [17].

   Unfortunately, in this case, the computation of all the invariants is based upon a preliminary estimation of the gravity center of the image. Hence, the invariants are simultaneously very sensitive to this preliminary estimation.
4. The group of translations, rotations and homotheties of the 2D plane itself (we don't write the multiplication but it is obvious) is unfortunately not unimodular. Hence the theory in this section does not apply. It is why one has to go back to the previous group.
5. The group SO_3 of rotations of $\mathbb{R}^3$. It is related to the human mechanisms of vision (see the paper [7]).
6. Certain rather **unusual groups** play a fundamental role in our theory below: the groups $M_{2,N}$ of motions, the rotation component of which is an integer multiple of $\frac{2\pi}{N}$. They are subgroups of M_2, and if N is large, $M_{2,N}$ could be reasonably called the "group of translations and sufficiently small rotations". In some precise mathematical sense, M_2 is the limit when N tends to infinity of the groups $M_{2,N}$.

For standard Fourier series and Fourier transforms, there are several general ingredients. Fourier series correspond to the circle group, Fourier transforms to the $\mathbb{R}$ (or more generally $\mathbb{R}^p$) group. In both cases, we have the formulas:

$$\begin{aligned}\hat{s}_n &= \int_G s(\theta)e^{-in\theta}\, d\theta, \\ \hat{f}(\lambda) &= \int_G f(x)e^{-i\lambda x}\, dx.\end{aligned} \tag{3.2}$$

Formally, in these two formulas appear an integration over the group G with respect to the Haar measure (respectively $d\theta, dx$) of the function (respectively s, f) times (the inverse of) the "mysterious" term $e^{in\theta}$ (resp. $e^{i\lambda x}$).This term is the "character" term. It has to be interpreted as follows: For each n (resp. λ), the map $\mathbb{C} \to \mathbb{C}$, $z \to e^{in\theta}z$ (resp. the map $z \to e^{i\lambda x}z$) is a unitary map (i.e. preserving the norm over $\mathbb{C}$), and the map $\theta \to e^{in\theta}$ (resp. $x \to e^{i\lambda x}$) is a continuous[3] group-homomorphism to the group of unitary linear transformations of $\mathbb{C}$. For a general topological group G, such a mapping is called a "character" of G.

The main basic result is the Pontryagin's duality theorem [21], that claims the following:

Theorem 1 (Pontryagin's duality Theorem) *The set of characters of an Abelian locally-compact group G is a locally-compact Abelian group* (*under natural multiplication of characters*), *denoted by $G^\wedge$, and called the dual group of G.The dual group $(G^\wedge)^\wedge$ of $G^\wedge$ is isomorphic to G.*

Then, the Fourier transform over G is defined like that: it is a mapping from $\mathbb{L}^2(G, dg)$ (space of square integrable functions over G, with respect to the Haar measure dg), to the space $\mathbb{L}^2(G^\wedge, d\hat{g})$, where $d\hat{g}$ is the Haar measure over $G^\wedge$:

$$\begin{aligned} &f \to \hat{f}, \\ &\hat{f}(\hat{g}) = \int_G f(g)\chi_{\hat{g}}(g^{-1})dg. \end{aligned} \tag{3.3}$$

Here, $\hat{g} \in G^\wedge$ and $\chi_{\hat{g}}(g)$ is the value of the character $\chi_{\hat{g}}$ on the element $g \in G$.

As soon as one knows that the dual group of $\mathbb{R}$ is $\mathbb{R}$ itself, and the dual group of the circle group is the discrete additive group $\mathbb{Z}$ of integer numbers, it is clear that formulas (3.2) are particular cases of formula (3.3).

It happens that there is a generalization of the usual **Plancherel's Theorem**: The Fourier Transform[4] is an isometry from $\mathbb{L}^2(G, dg)$ to $\mathbb{L}^2(G^\wedge, d\hat{g})$. The general form of the inversion formula follows:

$$f(g) = \int_G \hat{f}(\hat{g})\chi_{\hat{g}}(g)d\hat{g}. \tag{3.4}$$

In our cases ($\mathbb{R}$, C), this gives of course the usual formulas.

In the case of non-Abelian groups, the generalization starts to be less straightforward. To define a reasonable Fourier transform, one cannot consider only characters (this is not enough for a good theory, leading to Plancherel's Theorem). One has to consider more general objects than characters, namely, unitary irreducible representations of G. A (continuous) unitary representation of G consists of replacing $\mathbb{C}$ by a general complex Hilbert[5] space H, and the characters $\chi_{\hat{g}}$ by unitary linear operators $T_{\hat{g}}(g) : H \to H$, such that the mapping $g \to T_{\hat{g}}(g)$ is a continuous[6] homomorphism. Irreducible means that there is no nontrivial closed subspace of H which is invariant under all the operators $T_{\hat{g}}(g)$, $g \in G$. Clearly, characters are very special cases of continuous unitary irreducible representations. The main fact is that, for locally compact non-Abelian groups, to get Plancherel's formula, it is enough to replace characters by these representations.

Definition 1 Two representations T_1, T_2 of G, with respective underlaying Hilbert spaces H_1, H_2 are said equivalent if there is a linear invertible operator $A : H_1 \to H_2$, such that, for all $g \in G$:

$$T_2(g) \circ A = A \circ T_1(g). \tag{3.5}$$

More generally, a linear operator A, eventually noninvertible, meeting condition (3.5), is called an **intertwining operator** between the representations T_1, T_2.

The set of equivalence classes of unitary irreducible representations of G is called the dual set of G, and is denoted by $G^\wedge$.

One of the main differences with the Abelian case is that $G^\wedge$ has in general no group structure. However, in this very general setting, Plancherel's Theorem holds:

Theorem 2 *Let G be a locally compact unimodular group with Haar measure dg. Let $G^\wedge$ be the dual of G. There is a measure over $G^\wedge$* (*called the Plancherel's measure, and denoted by $d\hat{g}$*), *such that, if we define the Fourier transform over G as the mapping*:

$$\begin{aligned} &\mathbb{L}^2(G, dg) \to \mathbb{L}^2(G^\wedge, d\hat{g}), \\ &f \to \hat{f}, \\ &\hat{f}(\hat{g}) = \int_G f(g)T_{\hat{g}}(g^{-1})dg, \end{aligned} \tag{3.6}$$

then, $\hat{f}(\hat{g})$ is a Hilbert-Schmidt operator over the underlaying space $H_{\hat{g}}$, and the Fourier transform is an isometry.

As a consequence, the following inverse formula holds:

$$f(x) = \int_{G^\wedge} \mathrm{Trace}[\hat{f}(\hat{g})T_{\hat{g}}(g)]d\hat{g}. \tag{3.7}$$

More generally, if T is a unitary representation of G—**not necessarily irreducible**—one can define the Fourier transform $\hat{f}(T)$ by the same formula (3.6).

All this could look rather complicated. In fact, it is not at all, and we shall immediately make it explicit in the case of main interest for our applications to pattern recognition, namely the group of motions M_2.

[3] Along the paper, the topology over unitary operators on a Hilbert or Euclidean space is not the normic, but the strong topology.

[4] Precisely, Haar measures can be normalized so that Fourier transform is isometric.

[5] In the paper, all Hilbert spaces are assumed separable.

[6] For the strong topology of the unitary group $U(H)$.

In the following, for the group M_2, (and later on $M_{2,N}$), we take up the notations below:

Notation 2 Elements of the group are denoted indifferently by $g = (\theta, x, y) = (\theta, X)$, where $X = (x, y) \in \mathbb{R}^2$. The usual scalar product over $\mathbb{R}^2$ is denoted by $\langle ., . \rangle_{\mathbb{R}^2,}$ or simply $\langle ., . \rangle$ if no confusion is possible. Then, the product over M_2 (resp. $M_{2,N}$) writes $(\theta, X).(\alpha, Y) = (\theta + \alpha, R_\theta Y + X)$, where R_θ is the rotation operator of angle θ.

Example 1 Group M_2 of motions of the plane [55]. In that case, the unitary irreducible representations fall in two classes: 1. characters (one dimensional Hilbert space of the representation), 2. The other irreducible representations have infinite dimensional underlaying Hilbert space $H = \mathbb{L}^2(C, d\theta)$ where C is the circle group $\mathbb{R}/2\pi\mathbb{Z}$, and $d\theta$ is the Lebesgue measure over C. These representations are parametrized by any ray R from the origin in $\mathbb{R}^2$, $R = \{\alpha V$, V some fixed nonzero vector in $\mathbb{R}^2$, α a real number, $\alpha > 0\}$. For $r \in R$ (the ray), the representation T_r expresses as follows, for $\varphi(.) \in H$:

$$[T_r(\theta, X).\varphi](u) = e^{i\langle r, R_u X\rangle}\varphi(u + \theta). \tag{3.8}$$

The Plancherel's measure has support the second class of representations, i.e. characters play no role in that case.

It is easily computed that the Fourier transform of $f \in \mathbb{L}^2(M_2, \text{Haar})$ writes, with $X = (x, y)$:

$$[\hat{f}(r).\varphi](u) = \iiint_{M_2} f(\theta, x, y)e^{-i\langle r, R_{u-\theta} X\rangle} \times \varphi(u - \theta)d\theta dx dy. \tag{3.9}$$

The main property of the general Fourier-transform that we will use in the paper concerns obviously its behavior with respect to translations of the group. Let $f \in \mathbb{L}^2(G, dg)$ and set $f_a(g) = f(ag)$. Due to the invariance of the Haar measure w.r.t. translations of G, we get the **crucial** generalization of a well known formula:

$$\hat{f}(\hat{g}) \circ T_{\hat{g}}(a) = \widehat{f_a}(\hat{g}). \tag{3.10}$$

3.1.2 *Second Preliminary: General Definition of the Generalized Fourier Descriptors, from Those Over the Circle Group*

In the case of exterior contours of 2D patterns, the group under consideration is the circle group C. The set of invariants P_n, $R_{m,n}$ has first to be replaced by the (**almost equivalent**) set of invariants, P_n, $\tilde{R}_{m,n}$, where the new "phase invariants" $\tilde{R}_{m,n}$ are defined by:

$$\tilde{R}_{m,n} = \hat{s}_n \hat{s}_m \overline{\hat{s}_{n+m}}. \tag{3.11}$$

The first 3 Lemmas 9, 10, 11 of Appendix 1 justify this definition: at least on a residual subset of $\mathbb{L}^2(C)$, these sets of invariants are equivalent. This is enough for our practical purposes.

Remark 5

1. There is a counterexample in [19] showing that the second set of invariants is weaker (does not discriminate among all functions).

   But in practice, discriminating over a very big dense subset of functions is enough. Moreover, it is unexpected to be able to do more, in general.
2. Nevertheless, in the case of the additive groups $\mathbb{R}^n$, these second invariants discriminate completely. This is shown in [18].
3. For complete invariants over $\mathbb{L}^2(G)$ in **the general Abelian case**, generalizing those, see [13, 18, 19].

Now, an important fact has to be pointed out. There is a natural interpretation and generalization of the "phase-invariants" $\tilde{R}_{m,n}$ in terms of representations.

We are given an arbitrary unimodular group G, with Haar measure dg. We define the Fourier transform $\hat{f}$ of f, as the map from the set of (equivalence classes of) unitary irreducible representations of G, given by formula (3.6).

Let us state now a crucial definition, and a crucial theorem.

Definition 2 The following sets $I_1, I_{2,}$ are called respectively the first and second-Fourier-Descriptors (or Motion-Descriptors) of a map $f \in \mathbb{L}^2(G)$. For $\hat{g}, \hat{g}_1, \hat{g}_2 \in G^\wedge$,

$$\begin{aligned} &I_1^{\hat{g}}(f) = \hat{f}(\hat{g}) \circ \hat{f}(\hat{g})^*, \\ &I_2^{\hat{g}_1,\hat{g}_2}(f) = \hat{f}(\hat{g}_1)\hat{\otimes}\hat{f}(\hat{g}_2) \circ \hat{f}(\hat{g}_1\hat{\otimes}\hat{g}_2)^*, \end{aligned} \tag{3.12}$$

where $\hat{f}(\hat{g})^*$ denotes the adjoint of $\hat{f}(\hat{g})$, and where $\hat{g}_1\hat{\otimes}\hat{g}_2$ denotes the (equivalence class of) (Kronecker) Hilbert tensor product of the representations $\hat{g}_1$ and $\hat{g}_2$, and $\hat{f}(\hat{g}_1)\hat{\otimes}\hat{f}(\hat{g}_2)$ is the Hilbert tensor product of the Hilbert-Schmidt operators $\hat{f}(\hat{g}_1)$ and $\hat{f}(\hat{g}_2)$.[7]

Then, clearly, in the particular case of the circle group, **these formulas coincide** with those defining P_n, $\tilde{R}_{m,n}$.

Let us temporarily say that a (grey-level) image f on G is a compactly supported real nonzero function over G, with positive values (the grey levels).

Theorem 3 *The quantity $I_1(f)$ is determined by $I_2(f)$ (by abuse, we write $I_1(f) \subset I_2(f)$) and $I_1(f)$, $I_2(f)$ are invariant under translations of f by elements of G.*

[7] Later, it will be easier to compute the adjoint operator I_2^*, better than I_2.

Proof That $I_1(f)$ is determined by $I_2(f)$ comes from the fact that, f being an image, taking for $\hat{g}_2$ the trivial character c_0 of G, we get that $I_2^{\hat{g}_1,\hat{g}_2}(f)=av(f)I_1^{\hat{g}_1}(f)$, where the "mean value" of $f, av(f)=\int_G f(g)dg>0$, $av(f)=(I_2^{c_0,c_0})^{1/3}$. That $I_1^{\hat{g}}(f_a)=I_1^{\hat{g}}(f)$ (where $f_a(g)=f(ag)$, the translate of f by a) comes from the classical property (3.10) of Fourier transforms. That $I_2^{\hat{g}_1,\hat{g}_2}(f)=I_2^{\hat{g}_1,\hat{g}_2}(f_a)$, comes from the other trivial fact, just a consequence of the definition,

$$\hat{f}_a(\hat{g}_1\hat{\otimes}\hat{g}_2)=\hat{f}(\hat{g}_1\hat{\otimes}\hat{g}_2)\circ(T_{\hat{g}_1}(a)\hat{\otimes}T_{\hat{g}_2}(a)),$$

and from the unitarity of the representations. □

Our purpose in the remaining of the paper is to compute these invariants and to investigate about their completeness (at least on a big subset of $\mathbb{L}^2(G)$) and their pertinence. We will mostly consider either an Abelian or compact group G, or one of our motion groups M_2 and $M_{2,N}$.

3.2 The Generalized Fourier Descriptors for the Motion Group M_2

Here, using the results stated in Example 1, let us compute the generalized Fourier Descriptors from the Definition 2 and observe that **these invariants coincide with the invariants** (1.1), (1.2) **under consideration from the beginning of this paper**.

Remark 6 We consider functions f on the group of motions that are functions of $X=(x,y)$ only (they do not depend on θ, i.e. they are the "trivial" lifts on the group M_2 of functions on the plane $\mathbb{R}^2$).

Straightforward computations from formulas (3.8), (3.9) of Example 1 give:

$$[I_1^r(f)\varphi](u)=\int_C|\tilde{f}(R_\theta r)|^2d\theta\langle\varphi,1\rangle_{\mathbb{L}^2(\mathrm{C})},$$

$$\begin{aligned}&[I_2^{r_1,r_2}(f)\varphi](u_1,u_2)\\&\quad=\int_C\tilde{f}(R_\theta(\hat{r}_1+\hat{r}_2))\overline{\tilde{f}}(R_\theta\hat{r}_1)\\&\qquad\times\overline{\tilde{f}}(R_\theta\hat{r}_2)d\theta\iint_{C\times C}\varphi(a,b)dadb,\end{aligned}\tag{3.13}$$

with $\hat{r}_i=R_{-u_i}r_i,\ i=1,2$.

Clearly, these invariants are completely determined by those used in the first part of the paper:

$$I_1^r(f)=\int_C|\tilde{f}(R_\theta r)|^2d\theta,\quad r\in R,$$

$$I_2^{\xi_1,\xi_2}(f)=\int_C\tilde{f}(R_\theta(\xi_1+\xi_2))\overline{\tilde{f}}(R_\theta\xi_1)\overline{\tilde{f}}(R_\theta\xi_2)d\theta,\tag{3.14}$$

for $\xi_1,\xi_2\in\mathbb{R}^2$.

Remark 7 The Generalized-Fourier-Descriptors are real quantities (this is not an obvious fact for the second type invariants, but it is easily checked).

Completeness of these invariants is still an open question. However in the remaining of the paper we will prove certain completeness results in other very close cases.

3.3 The Case of Compact (Non-Abelian) Groups

This is the most beautiful part of the theory, showing in a very convincing way that the formulas (3.12) are really pertinent: in the compact case, (including the classical Abelian case of exterior contours), the Generalized Fourier Descriptors are weakly complete. This is due to the Tannaka-Krein duality theory. (See [22, 25].)

3.3.1 Chu and Tannaka Categories, Chu and Tannaka Dualities

Tannaka Theory is the generalization to compact groups of Pontriaguin's duality theory.

The following facts are standard: The dual of a compact group is a **discrete set**, and all its unitary irreducible representations are **finite dimensional**.

The main lines of Tannaka theory is like that: we start with a compact group G.

1. There is the notion of a Tannaka category $\mathcal{T}_G$, that describes the structure of the finite dimensional unitary representations of G;
2. There is the notion of a quasi representation $\mathcal{Q}$ of a Tannaka category $\mathcal{T}_G$;
3. The set $rep(G)^\wedge$ of quasi representations of the Tannaka category $\mathcal{T}_G$ has the structure of a topological group;
4. The groups $rep(G)^\wedge$ and G are naturally isomorphic. (Tannaka duality).

This scheme completely generalizes the scheme of Pontryagin's duality to the case of compact groups.

In fact, Tannaka duality theory is just a particular case of Chu duality, which will be **the crucial form of duality needed** for our purposes. Hence, let us introduce precisely Chu duality [10, 22], and Tannaka duality will just be **the particular case of compact groups**.

Let temporarily G be an arbitrary topological group.

For all $n\in\mathbb{N}$ the set $rep_n(G)$ denotes the set of continuous unitary representations of G over $\mathbb{C}^n$. $rep_n(G)$ is endowed with the following topology: a basis of open neighborhoods of $T\in rep_n(G)$ is given by the sets $W(K,T,\varepsilon)$,

$\varepsilon > 0$, and $K \subset G$, a compact subset,

$$W(K, T, \varepsilon) = \{\tau \in rep_n(G) \mid \|T(g) - \tau(g)\| < \varepsilon, \forall g \in K,$$

where the norm of operators is the usual Hilbert-Schmidt norm. If G is locally compact, so is $rep_n(G)$.

Definition 3 The Chu-Category of G is the category $\pi(G)$, the objects of which are the finite dimensional unitary representations of G, and the morphisms are the intertwining operators.

Definition 4 A quasi-representation of the category $\pi(G)$ is a function Q over $ob(\pi(G))$ such that $Q(T)$ belongs to $U(H_T)$, the unitary group over the underlaying space H_T of the representation T, with the following properties:

0. Q commutes with Hilbert direct-sum: $Q(T_1 \dot{\oplus} T_2) = Q(T_1) \dot{\oplus} Q(T_2)$,
1. Q commutes with the Hilbert tensor product: $Q(T_1 \hat{\otimes} T_2) = Q(T_1) \hat{\otimes} Q(T_2)$,
2. Q commutes with the equivalence operators: for an equivalence A between T_1 and T_2, $A \circ Q(T_1) = Q(T_2) \circ A$,
3. the mappings, $rep_n(G) \to U(\mathbb{C}^n)$, $T \to Q(T)$ are continuous.

The set of quasi-representations of the category $\pi(G)$ is denoted by $rep(G)^\wedge$.

There are "natural" quasi representations of G: for each $g \in G$, the mapping $\Omega_g(T) = T(g)$ defines a quasi-representation of $\pi(G)$.

Remark 8 $rep(G)^\wedge$ is a group with the multiplication $Q_1.Q_2(T) = Q_1(T).Q_2(T)$.

The neutral element is E, with $E(T) = \Omega_e(T) = T(e)$, for e the neutral of G.

There is a topology over $rep(G)^\wedge$ such that it becomes a topological group. A fundamental system of neighborhoods of E is given by the sets $W(K^\wedge_{n_1}, \ldots, K^\wedge_{np}, \varepsilon)$, $\varepsilon > 0$ and $K^\wedge_{n_i}$ is compact in $rep_{n_i}(G)$, with $W(K^\wedge_{n_1}, \ldots, K^\wedge_{np}, \varepsilon) = \{Q \in rep(G)^\wedge \mid \|Q(T) - E(T)\| < \varepsilon, \forall T \in \cup K^\wedge_{n_i}\}$.

The first main result is that, as soon as G is locally compact, the mapping $\Omega : G \to rep(G)^\wedge$, $g \to \Omega_g$ is a continuous homomorphism.

Definition 5 A locally compact G has the **duality property if** Ω is a topological group isomorphism.

The main result is:

Theorem 4 *If G is locally compact, Abelian, then G has the duality property. (This is no more than Pontryagin's duality.)*

If G is compact, G has the duality property. (This is Tannaka-Krein theory.)

In the last section of the paper, for the purpose of pattern recognition, we will use crucially the fact that **another class of groups, namely the Moore groups, have also the duality property**.

3.3.2 Generalized Fourier Descriptors over Compact Groups

Our result in this section is based upon Tannaka theory, and shows that the **weak-completeness**—i.e. completeness over a residual subset of $\mathbb{L}^2(G, dg)$—of the Generalized-Fourier-Descriptors (which holds on the circle group, and which is crucial for pattern recognition of "exterior contours") **generalizes to compact separable groups**.

If G is compact separable, then, we have the following crucial but obvious lemma:

Lemma 1 *The subset R of functions $f \in \mathbb{L}^2(G, dg)$ such that $\hat{f}(\hat{g})$ is invertible for all $T = \hat{g} \in G^\wedge$ is residual in $\mathbb{L}^2(G, dg)$.*

Proof It follows from [12] that if G is compact separable, then $G^\wedge$ is countable. For a fixed $\hat{g}$, the set of f such that $\hat{f}(\hat{g})$ is not invertible is clearly open, dense. Hence, R is a countable intersection of open-dense sets, in a Hilbert space. □

The main theorem is:

Theorem 5 *Let G be a compact separable group. Let R be the subset of elements of $\mathbb{L}^2(G, dg)$ on which the Fourier transform takes values in invertible operators. Then R is residual in $\mathbb{L}^2(G, dg)$, and the Generalized Fourier Descriptors discriminate over R.*

Proof Let us take two functions $f, h \in R$, such that the associated Generalized-Fourier-Descriptors are equal. The equality of the first-type Fourier-Descriptors gives $\hat{f}(\hat{g}) \circ \hat{f}(\hat{g})^* = \hat{h}(\hat{g}) \circ \hat{h}(\hat{g})^*$, for all $\hat{g} \in G^\wedge$. Since $\hat{f}(\hat{g})$ is invertible, we deduce that there is $u(\hat{g}) \in U(H_{\hat{g}})$, such that $\hat{f}(\hat{g}) = \hat{h}(\hat{g})\, u(\hat{g})$.

If T is a reducible unitary representation, it is a finite direct sum of irreducible representations, and therefore, the equality $\hat{f}(T) \circ \hat{f}(T)^* = \hat{h}(T) \circ \hat{h}(T)^*$, for all $\hat{g}_i \in G^\wedge$ also defines an invertible $u(T) = \hat{h}(T)^{-1}\, \hat{f}(T)$. (By the finite sum decomposition, $\hat{h}(T) = \dot{\oplus} \hat{h}(\hat{g}_i)$, hence $\hat{h}(T)$ is invertible.) Moreover it is obvious that the mappings $rep_n(G) \to M(n, \mathbb{C})$, $T \to \hat{f}(T)$ are continuous therefore the mapping $T \to u(T) = \hat{h}(T)^{-1}\, \hat{f}(T)$ is also continuous.

Also, the equality of the (second) Fourier-Descriptors for the irreducible representations, [due to the finite decomposition of any representation in a direct sum of irreducible ones, plus the usual properties of Hilbert tensor product] implies the equality of Fourier-Descriptors for arbitrary (non-irreducible) unitary finite-dimensional representations, i.e., if T, T' are such unitary representations, non-necessarily irreducible, we have also:

$$\hat{f}(T)\hat{\otimes}\hat{f}(T') \circ \hat{f}(T\hat{\otimes}T')^* = \hat{h}(T)\hat{\otimes}\hat{h}(T') \circ \hat{h}(T\hat{\otimes}T')^*. \tag{3.15}$$

Replacing in this last equality $\hat{f}(T) = \hat{h}(T)\ u(T)$, and taking into account the fact that all the $\hat{f}(T), \hat{h}(T)$ are invertible, we get that:

$$u(T\hat{\otimes}T') = u(T)\hat{\otimes}u(T'), \tag{3.16}$$

for all finite dimensional unitary representations T, T' of G.

Now, for such T, T', and for A intertwining T and T', we have also $A\hat{f}(T) = \int_G f(g)AT(g^{-1})dg = \int_G f(g)T'(g^{-1})Adg = \hat{f}(T')A$. It follows that $A\hat{h}(T)u(T) = \hat{h}(T')u(T')A$, hence, $\hat{h}(T')Au(T) = \hat{h}(T')u(T')A$, in which $\hat{h}(T')$ is invertible. Therefore, $Au(T) = u(T')A$. By Definition 4, u is a quasi-representation of the category $\pi(G)$. By Theorem 4, G has the duality property, and for all $\hat{g} \in G^\wedge$, $u(\hat{g}) = T_{\hat{g}}(g_0)$ for some $g_0 \in G$. Then:

$$\hat{f}(\hat{g}) = \hat{h}(\hat{g})T_{\hat{g}}(g_0),$$

and, by the main property (3.10) of Fourier transforms, $\hat{f} = \widehat{h_a}$, $f = h_a$ for some $a \in G$. □

3.4 The Case of the Group of Motions with Small Rotations $M_{2,N}$

This section contains our final results. We will consider the action on the plane of the group $M_{2,N}$ of translations and small rotations. In the case where N is an odd number, we will be able to achieve a full theory and to get a weak-completion result. To focus on main ideas, the proof of several crucial technical lemmas is postponed to Appendix 6.

3.4.1 Moore Groups and Duality for Moore Groups

For details, we refer to [22]. We already know that compact groups have all their unitary irreducible representations of finite dimension. But they are not the only ones.

Definition 6 A Moore group is a locally-compact group, such that all its unitary irreducible representations have finite-dimensional underlaying Hilbert space.

Theorem 6 *The groups $M_{2,N}$ are Moore groups.*

Proof These groups are semidirect products of the type $G_0 \ltimes \mathbb{R}^2$, where G_0 is a (Abelian) finite group. Then we can use Mackey's Imprimitivity Theorem to compute their dual (see [56] for instance). By this theorem, their unitary irreducible representations are parametrized by the (contragredient) action of the action of G_0 on $\mathbb{R}^2$, and their underlaying Hilbert spaces are the spaces of square summable functions on these orbits, with respect to the corresponding quasi-invariant measures. These orbits are finite. Hence, their $\mathbb{L}^2$-space is isomorphic to $\mathbb{C}^N$. □

Theorem 7 (Chu duality) [22] *Moore groups (separable) have the duality property.*

Then, we will try to copy what has been done for compact groups to our Moore groups. There are several difficulties, due to the fact that the functions under consideration (the images) are very special functions over the group. In fact, they are functions over the homogeneous space $\mathbb{R}^2$ of $M_{2,N}$.

3.4.2 Representations, Fourier Transform and Generalized Fourier Descriptors over $M_{2,N}$

In fact, considering "images", we will be interested only with functions on $M_{2,N}$ that are also functions on the plane $\mathbb{R}^2$. One of the main problems, as we shall see, is that there are several possible "lifts" of the functions of $\mathbb{L}^2(\mathbb{R}^2)$ on $\mathbb{L}^2(M_{2,N})$, and that the "trivial" lift is bad for our purposes.

Typical elements of $M_{2,N}$ are still denoted by $g = (\theta, x, y) = (\theta, X)$, $X = (x, y) \in \mathbb{R}^2$, but now, $\theta \in \check{N} = \{0, \dots, N-1\}$. Each such θ represents a rotation of angle $\frac{2\theta\pi}{N}$, that we still denote by R_θ.

The Haar measure is the tensor product of the uniform measure over $\check{N}$ and the Lebesgue measure over $\mathbb{R}^2$. The dual space $\hat{G}$ is the union of the finite set $\mathbb{Z}/N\mathbb{Z} = \check{N}$ (characters) with the "Slice of Cake" $\mathcal{S}$, corresponding to nonzero values of $r \in \mathbb{R}^2$ of angle $\alpha(r)$, $0 \le \alpha(r) < \frac{2\pi}{N}$. The support of the Plancherel Measure is $\mathcal{S}$ (characters are of no use).

Here $\varphi \in \mathbb{C}^N$, i.e. $\varphi : \check{N} \to \mathbb{C}$. We have exactly the same formula as for M_2:

$$[T_r(\theta, X).\varphi](u) = e^{i\langle r, R_u X\rangle}\varphi(u + \theta), \tag{3.17}$$

but $r \in \mathcal{S}$ and the map $l^2(\check{N}) \to l^2(\check{N})$, $\varphi(u) \to \varphi(u + \theta)$, is just the $\theta-$shift operator over $\mathbb{C}^{\mathrm{N}}$.

The Fourier transform has a similar expression to formula (3.9):

$$[\hat{f}(r).\varphi](u) = \sum_{\check{N}} \left(\iint_{\mathbb{R}^2} f(\theta, x, y) e^{-i\langle r, R_{u-\theta}X\rangle} \times \varphi(u - \theta) dx dy \right). \tag{3.18}$$

Similar computations to those of Sect. 3.2 lead to the final formula for the Fourier descriptors relative to the trivial lift of functions f over $\mathbb{R}^2$ into functions over $M_{2,N}$ (not depending on θ):

$$I_1^r(f) = \sum_{\theta \in \check{N}} |\tilde{f}(R_\theta r)|^2 d\theta, \quad r \in R, \tag{3.19}$$

$$I_2^{\xi_1,\xi_2}(f) = \sum_{\theta \in \check{N}} \tilde{f}(R_\theta(\xi_1+\xi_2))\overline{\tilde{f}}(R_\theta \xi_1)\overline{\tilde{f}}(R_\theta \xi_2) d\theta,$$

for $\xi_1, \xi_2 \in \mathbb{R}^2$.

By Theorem 3, these **Generalized Fourier Descriptors are invariant under the action of** $M_{2,N}$ **on** $\mathbb{L}^2(\mathbb{R}^2)$. Let us explain the main problem that appears when we try to generalize Theorem 5 of Sect. 3.3.2.

For this, we have to consider the special expression of the Fourier transform of the "trivial lift" of a function on the plane. Similarly to the case of M_2, we get:

$$\begin{aligned}[\hat{f}(r)\varphi](u) &= \sum_{\check{N}} \tilde{f}(R_{\theta-u} r)\varphi(u-\theta) \\ &= \langle \varphi(\theta), \overline{\tilde{f}}(R_{-\theta} r)\rangle_{l^2(\check{N})}.\end{aligned} \tag{3.20}$$

The **crucial** point in the proof of the main theorem 5 is that the operators $\hat{f}(r)$ are all invertible. But, here, it is not at all the case since the operators defined by the formula above are far from invertible: **they always have rank 1**.

To overcome this difficulty, **we have to chose another lift of functions on the plane to functions on** $M_{2,N}$**, the trivial lift being too rough**. This is what we do in the next section.

3.4.3 *The Cyclic-Lift from* $\mathbb{L}^2(\mathbb{R}^2)$ *to* $\mathbb{L}^2$ $(M_{2,N})$

From now on, we consider functions on $\mathbb{R}^2$, that are square-summable, and that have their support contained in a translated of a given compact set K (the "screen").

Given a compactly supported function in $\mathbb{L}^2(\mathbb{R}^2, \mathbb{R})$, we can define its average and its (weighted) centroid, as follows:

$$av(f) = \int_K f(x,y)dxdy,$$

$$\begin{aligned}centr(f) &= (x_f, y_f) = X_f \\ &= \left(\int_K xf(x,y)dxdy, \int_K yf(x,y)dxdy\right).\end{aligned}$$

Definition 7 The cyclic-lift of a compactly supported $f \in \mathbb{L}^2(\mathbb{R}^2, \mathbb{R})$, with nonzero average, onto $\mathbb{L}^2(M_{2,N})$ is the function $f^c(\theta, x, y) = f(R_\theta X + \frac{centr(f)}{av(f)})$.

Note that $\frac{centr(f)}{av(f)}$ is the "geometric center" of the image f and that $f^c(0, X)$ is the "centered image".

The set of K-supported real valued functions is a closed subspace $\mathcal{H} = \mathbb{L}^2(K)$ of $\mathbb{L}^2(\mathbb{R}^2)$. The set $\mathcal{I}$ of elements of $\mathcal{H}$ with nonzero average is an open subset of $\mathcal{H}$, therefore it has the structure of a Hilbert manifold. This is important since we shall apply to this space the parametric transversality theorem of [1].

Definition 8 From now on, a (grey level, or one-color) "image" f is an element of $\mathcal{I}$.

Notice that moreover, usual images have positive value (grey or color levels vary between zero and 1). This will be of no importance here in.

By Lemma 12 in Appendix 2, we know that f and g differ from a motion angle $\frac{4k\pi}{N}$ if and only if f^c and g^c differ from a motion with angle equal to $\frac{2k\pi}{N}$.

In this way, we reduce the problem of equivalence with rotation of certain multiples of a small angle, to **the problem of equivalence of the cyclic lifts over** $M_{2,N}$.

This problem will be treated now, with the same method as in Sect. 3.3 (case of compact groups). For crucial reasons that will appear clearly below, we will consider only the case of an **odd** $N = 2n+1$. Note that if N is odd, when k varies in $\check{N}$, $2k \bmod N$ also varies in $\check{N}$.

3.4.4 *Fourier-Transform, Generalized-Fourier-Descriptors of Cyclic-Lifts over* $M_{2,2n+1}$

Using the expression (3.17) of the unitary irreducible representations over $M_{2,N}$, easy computations give the following results:

For $r_1, r_2 \in \mathcal{S}$,

$$\begin{aligned}&[T_{r_1 \hat{\otimes} r_2}(\theta, V)\varphi](u1, u2) \\ &\quad = e^{i\langle R_{-u_2} r_1 + R_{-u_1} r_2, V\rangle}\varphi(u_1+\theta, u_2+\theta).\end{aligned} \tag{3.21}$$

As a consequence:

$$\begin{aligned}&[T_{r_1 \hat{\otimes} r_2}(\theta, X)^*\varphi](u_1, u_2) \\ &\quad = e^{-i\langle R_{\theta-u_2} r_1 + R_{\theta-u_1} r_2, X\rangle}\varphi(u_1-\theta, u_2-\theta).\end{aligned} \tag{3.22}$$

For the Fourier transform of a cyclic lift f^c, we get:

$$\begin{aligned}&[\widehat{f^c}(r)\Psi](u) \\ &\quad = \sum_{\alpha} \tilde{f}(R_{2\alpha+u} r) e^{i\langle R_{2\alpha+u} r, \frac{1}{av(f)} Xf\rangle}\Psi(-\alpha), \\ &\quad = \sum_{\alpha \in \check{N}} \tilde{f}(R_{u-2\alpha} r) e^{i\langle R_{u-2\alpha} r, \frac{1}{av(f)} Xf\rangle}\Psi(\alpha).\end{aligned} \tag{3.23}$$

Here, as above, $\tilde{f}(V)$ denotes the usual 2-D Fourier transform of f at V. We get also:

$$[\widehat{f^c}(r)^*\Psi](u) = \sum_{\alpha\in\check{N}} \overline{\tilde{f}}(R_{\alpha-2u}r)e^{-i\langle R_{\alpha-2u}r, \frac{1}{av(f)}Xf\rangle}\Psi(\alpha). \tag{3.24}$$

The last expression we need is:

$$\begin{aligned}
&[\widehat{f^c}(r_1\hat{\otimes}r_2)\varphi](u_1,u_2) \\
&\quad = \sum_{\alpha\in\check{N}} \tilde{f}(R_{2\alpha-u_2}r_1 + R_{2\alpha-u_1}r_2) \\
&\qquad \times e^{i\langle R_{2\alpha-u_2}r_1+R_{2\alpha-u_1}r_2, \frac{1}{av(f)}Xf\rangle}\varphi(u_1-\alpha, u_2-\alpha).
\end{aligned} \tag{3.25}$$

Formula (3.24) leads to:

$$\begin{aligned}
&[\widehat{f^c}(r_1)^*\hat{\otimes}\widehat{f^c}(r_2)^*\varphi(u_1,u_2) \\
&\quad = \sum_{(\alpha_1,\alpha_2)\in\check{N}\times\check{N}} \overline{\tilde{f}}(R_{\alpha_2-2u_2}r_1)\overline{\tilde{f}}(R_{\alpha_1-2u_1}r_2) \\
&\qquad \times e^{-i\langle R_{\alpha_2-2u_2}r_1+R_{\alpha_1-2u_1}r_2, \frac{1}{av(f)}Xf\rangle}\varphi(\alpha_1,\alpha_2).
\end{aligned} \tag{3.26}$$

Now, we can perform the computation of the Generalized Fourier Descriptors. After computations based upon the formulas just established, we get for the self adjoint matrix $I_1^r(f) = \hat{f}(r)\circ\hat{f}(r)^*$:

$$\begin{aligned}
I_1^r(f)_{l,k} &= \sum_{j\in\check{N}} \tilde{f}(R_{l-2j}r)\overline{\tilde{f}}(R_{k-2j}r) \\
&\qquad \times e^{i\langle (R_l-R_k)R_{-2j}r, \frac{1}{av(f)}Xf\rangle},
\end{aligned}$$

and for the phase invariants $I_2^{r_1,r_2}(f)$:

$$\begin{aligned}
&[I_2^{r_1,r_2}(f)\Psi](u_1,u_2) \\
&\quad = \sum_{j\in\check{N}}\sum_{(\omega_1,\omega_2)\in\check{N}} \tilde{f}(R_{2j-u_2}r_1 + R_{2j-u_1}r_2) \\
&\qquad \times \overline{\tilde{f}}(R_{\omega_2-2u_2+2j}r_1)\overline{\tilde{f}}(R_{\omega_1-2u_1+2j}r_2) \\
&\qquad \times e^{i\langle (I-R_{\omega 2}-u_2)R_{2j-u_2}r_1+(I-R_{\omega 1}-u_1)R_{2j-u_1}r_2, \frac{1}{av(f)}Xf\rangle} \\
&\qquad \times \Psi(u_1,u_2).
\end{aligned}$$

Since N is odd, setting $m=2j$, we get:

$$\begin{aligned}
I_1^r(f)_{l,k} &= \sum_{m\in\check{N}} \tilde{f}(R_{l-m}r)\overline{\tilde{f}}(R_{k-m}r) \\
&\qquad \times e^{i\langle (R_l-R_k)R_{-m}r, \frac{1}{av(f)}Xf\rangle},
\end{aligned} \tag{3.27}$$

and also, we see easily that $I_2^{r_1,r_2}(f)$ is completely determined by the quantities:

$$\begin{aligned}
&\widetilde{I_2^{r_1,r_2}}(f)(u_1,u_2,\omega_1,\omega_2) \\
&\quad = \sum_{m\in\check{N}} \tilde{f}(R_{m-u_2}r_1 + R_{m-u_1}r_2)\overline{\tilde{f}}(R_{\omega_2-2u_2+m}r_1) \\
&\qquad \times \overline{\tilde{f}}(R_{\omega_1-2u_1+m}r_2) \\
&\qquad \times e^{i\langle (I-R_{\omega 2}-u_2)R_{m-u_2}r_1+(I-R_{\omega 1}-u_1)R_{m-u_1}r_2, \frac{1}{av(f)}Xf\rangle}.
\end{aligned} \tag{3.28}$$

Setting $u_2=-l_2, \omega_2-2u_2=k2, u_1=-l_1, \omega_1-2u_1=k_1$, we get:

$$\begin{aligned}
&\widetilde{I_2^{r_1,r_2}}(f)(l_1,l_2,k_1,k_2) \\
&\quad = \sum_{m\in\check{N}} \tilde{f}(R_{m+l_2}r_1 + R_{m+l_1}r_2)\overline{\tilde{f}}(R_{k_2+m}r_1) \\
&\qquad \times \overline{\tilde{f}}(R_{k_1+m}r_2)e^{i\langle (R_{l_2}-R_{k_2})R_m r_1+(R_{l_1}-R_{k_1})R_m r_2, \frac{1}{av(f)}Xf\rangle}.
\end{aligned} \tag{3.29}$$

Remark 9 Consider the particular case $l_2=k_2, l_1=k_1$, and set $\xi_1 = R_{k_2}r_1, \xi_2 = R_{k_1}r_2$, then, we get:

$$\widetilde{I_2^{\xi_1,\xi_2}}(f)(l_1,l_2) = \sum_{m\in\check{N}} \tilde{f}(R_m(\xi_1+\xi_2))\overline{\tilde{f}}(R_m\xi_1)\overline{\tilde{f}}(R_m\xi_2). \tag{3.30}$$

Note that this is just the **discrete version of the (continuous) invariants of type 2**, in formula (3.14). Note also that, making the change of variables $\xi_1 = R_{k_2}r_1, \xi_2 = R_{k_1}r_2$, $\xi_3 = R_{l_2}r_1 + R_{l_1}r_2$, we get:

$$\begin{aligned}
\widetilde{I_3^{\xi_1,\xi_2,\xi_3}}(f) &= \sum_{m\in\check{N}} \tilde{f}(R_m\xi_3)\overline{\tilde{f}}(R_m\xi_1)\overline{\tilde{f}}(R_m\xi_2) \\
&\qquad \times e^{i\langle R_m(\xi_3-\xi_1-\xi_2), \frac{1}{av(f)}Xf\rangle}
\end{aligned}$$

which is the final (discrete form of our invariants).

Therefore, at the end, we have 3 sets of Generalized-Fourier-Descriptors (type-1, type-2, type-3):

$$\begin{aligned}
&I_1^r(f)_{l,k}, \widetilde{I_2^{\xi_1,\xi_2}}(f) \subset \widetilde{I_3^{\xi_1,\xi_2,\xi_3}}(f), \\
&I_1^r(f)_{l,k} = \sum_{m\in\check{N}} \tilde{f}(R_{l-m}r)\overline{\tilde{f}}(R_{k-m}r) \\
&\qquad \times e^{i\langle (R_l-R_k)R_{-m}r, \frac{1}{av(f)}Xf\rangle},
\end{aligned}$$

$$\widetilde{I_2^{\xi_1,\xi_2}}(f) = \sum_{m\in\check{N}} \tilde{f}(R_m(\xi_1+\xi_2)) \times \overline{\tilde{f}}(R_m\xi_1)\overline{\tilde{f}}(R_m\xi_2),$$

$$\widetilde{I_3^{\xi_1,\xi_2,\xi_3}}(f) = \sum_{m\in\check{N}} \tilde{f}(R_m\xi_3)\overline{\tilde{f}}(R_m\xi_1)\overline{\tilde{f}}(R_m\xi_2) \times e^{i\langle R_m(\xi_3-\xi_1-\xi_2), \frac{1}{av(f)}Xf\rangle}.$$

As we shall see these descriptors are weakly complete (i.e. they discriminate over a residual subset of the set of images under the action of motions of angle $\frac{4k\pi}{2n+1}$, i.e. $\frac{2k'\pi}{2n+1}$).

3.4.5 Completeness of the Discrete Generalized Fourier Descriptors

This is a rather hard work. We try to follow the scheme of the proof of Theorem 5, and at several points, there are technical difficulties.

Here, as above, a compact $K\subset\mathbb{R}^2$ is fixed, containing a neighborhood of the origin (K is the "screen"), and an image is an element of $\mathcal{I}$, from Definition 8.

Let us consider the subset $\mathcal{G}\subset\mathcal{I}$ of "generic images", defined as follows. For $f\in\mathcal{I}$, $\tilde{f}^t$ denotes the ordinary 2-D Fourier transform of $f^c(0,X)$ as an element of $\mathbb{L}^2(\mathbb{R}^2)$. Set as above $X=(x,y)\in\mathbb{R}^2$ (but here X should be understood as a point of the frequency plane). The function $\tilde{f}^t(X)$ is a complex-valued function of X, analytic in X (Paley-Wiener). For $r\in\mathbb{R}^2$, denote by $\omega_r\in\mathbb{C}^{\mathrm{N}}$ the vector $\omega_r=(\tilde{f}^t(R_0r),\ldots,\tilde{f}^t(R_{\theta_i}r),\ldots,\tilde{f}^t(R_{\theta_{N-1}}r))$.

Denote also by Ω_r the circulant matrix associated to ω_r.If F_N denotes the usual DFT matrix of order N (i.e. the $N\times N$ unitary matrix representing the Fourier transform over the Abelian group $\mathbb{Z}/N\mathbb{Z}$), then the vector of eigenvalues δ_r of Ω_r meets $\delta_r=F_N\omega_r$.

Definition 9 The generic set $\mathcal{G}$ is the subset of $\mathcal{I}$ of elements such that Ω_r is an invertible matrix for all $r\in\mathbb{R}^2$, $r\neq 0$, except for a (may be countable) set of isolated values of r, for which Ω_r has a zero eigenvalue with simple multiplicity.

The next Lemma shows that if N is an odd integer number, then $\mathcal{G}$ is very big.

Lemma 2 *Assume that N is odd. Then, $\mathcal{G}$ is residual.*

Proof We consider the following mappings $\varrho_k:\mathcal{I}\times\mathbb{R}^2\backslash\{0\}\to\mathbb{R}^2$, $k\in\check{N}$, $\varrho_k(f,r)$ is the (real and imaginary part of the) k^{th} eigenvalue of Ω_r (it makes sense to talk about the k^{th} eigenvalue since all circulant matrices are simultaneously diagonalized by the DFT F_N). Lemma 13 from the appendix shows that, applying Abraham's parametric transversality Theorem [1] to ϱ_k, we find a residual subset $\mathcal{G}_{\mathrm{k}}\subset\mathcal{I}$, such that $\varrho_k(f)$ is transversal to zero, for all $f\in\mathcal{G}_{\mathrm{k}}$. Here, $\varrho_k(f)(x)$ means $\varrho_k(f,x)$. Set $\mathcal{G}=\bigcap_{\mathrm{k}\in\check{N}}\mathcal{G}_{\mathrm{k}}$. Clearly, $\mathcal{G}$ is residual, and for $f\in\mathcal{G}$ (for dimension 2 and codimension 2 reasons) Ω_r can have a zero eigenvalue at isolated points of $\mathbb{R}^2\backslash\{0\}$ only. A similar transversality argument shows that at these special points the zero eigenvalue is simple. □

Remark 10 Notice that here, once more, the fact that N is odd is crucial.

Now let us take $f,g\in\mathcal{G}$, and assume that their discrete Generalized Fourier descriptors from Sect. 3.4.4 are equal.

We can apply the reasoning of Sect. 3.3.2 to construct a quasi-representation of the category $\pi(M_{2,N})$ at points where $\Omega_r(f)$ and $\Omega_r(g)$ are invertible only.

Recall the formula (3.23) for our Fourier Transform in the case of $M_{2,N}$:

$$\begin{aligned}[\widehat{f^c}(r)\Psi](u) &= \sum_{\alpha\in\check{N}} \tilde{f}(R_{u-2\alpha}r)e^{i\langle R_{u-2\alpha}r,\frac{1}{av(f)}Xf\rangle}\Psi(\alpha)\\ &= \sum_{\alpha\in\check{N}} \tilde{f}^t(R_{u-2\alpha}r)\Psi(\alpha),\end{aligned}$$

$$\text{with } f^t(x) = f\left(x+\frac{X_f}{av(f)}\right) = f^c(0,x),$$

by the basic property of the usual 2D Fourier transform with respect to translations.

Since N is odd (a crucial point again), it is also equal to:

$$[\hat{f}^c(r)\Psi](u) = \sum_{\alpha\in\check{N}} \tilde{f}^t(R_{u-\alpha}r)(C\Psi)(\alpha), \tag{3.31}$$

where C is a certain universal unitary operator (permutation).

This formula can also be read, in a matrix setting, as:

$$\hat{f}^c(r) = \Omega_r(f)C. \tag{3.32}$$

Also, by the equality of the invariants, the points where $\Omega_r(f)$ and $\Omega_r(g)$ are non-invertible are the same.

Out of these isolated points, we can apply the same reasoning as in the compact case, Sect. 3.3.2. Hence, the equality of the first invariants gives:

$$\begin{aligned}\hat{f}^c(r)\hat{f}^c(r)^* &= \Omega_r(f)\Omega_r(f)^*\\ &= \hat{g}^c(r)\hat{g}^c(r)^* = \Omega_r(g)\Omega_r(g)^*.\end{aligned}$$

Since at nonsingular points $\Omega_r(f)$ and $\Omega_r(g)$ are invertible, this implies that there is a unitary matrix $U(r)$ such that $\hat{g}^c(r)=\hat{f}^c(r)U(r)$.

Let $I=\{r_i|\Omega_{r_i}$ is singular$\}$. Out of I, $U(r)$ is an analytic function of r, since $U(r)=[\hat{f}^c(r)]^{-1}\hat{g}^c(r)$.

Now, we will need some results about unitary representations, namely:

R1. Two finite dimensional unitary representations that are equivalent are unitarily equivalent, and the more difficult one, that we state in our special case only, and which is a consequence of the "Induction-reduction" theorem of Barut [3] (however, once one knows the result, he can easily check it by direct computations in the special case).

R2. For $r_1, r_2 \in \mathbb{R}^2$, the representation $T_{r_1 \hat{\otimes} r_2}$ is equivalent (hence unitarily equivalent by **R1**) to the direct Hilbert sum of representations $\dot{\bigoplus}_{k\in\check{N}} T_{r_1+R_k r_2}$.

This means that, if we take r_1, r_2 out of I, but $r_1 + R_{k_0} r_2 \in I$, and $r_1 + R_k r_2 \notin I$ for $k \neq k_0$ (which is clearly possible), and if A denotes the unitary equivalence between $T_{r_1 \hat{\otimes} r_2}$ and $\dot{\bigoplus}_{k\in\check{N}} T_{r_1+R_k r_2}$, setting $\xi_k = r_1 + R_k r_2$, we can write that the block diagonal matrix $\Delta_f = \operatorname{diag}(\hat{f}^c(\xi_0), \ldots, \hat{f}^c(\xi_{N-1}))$ satisfies:

$$\Delta_f = \Delta_g A U(r_1)^* \hat{\otimes} U(r_2)^* A^{-1}. \tag{3.33}$$

Indeed, this comes from the equality of the second-type descriptors:

$$\begin{aligned}&\hat{f}^c(T_{r_1}) \hat{\otimes} \hat{f}^c(T_{r_2}) \circ \hat{f}^c(T_{r_1} \hat{\otimes} T_{r_2})^* \\ &\quad = \hat{g}^c(T_{r_1}) \hat{\otimes} \hat{g}^c((T_{r_2}) \circ \hat{g}^c(T_{r_1} \hat{\otimes} T_{r_2})^*,\end{aligned} \tag{3.34}$$

and since $\hat{g}^c(\chi_{r_1}) \hat{\otimes} \hat{g}^c((T_{r_2}) = \hat{f}^c(T_{r_1}) \hat{\otimes} \hat{f}^c(T_{r_2}) \circ U(r_1) \hat{\otimes} U(r_2)$ and both are invertible operators, then, replacing in (3.34), we get:

$$\begin{aligned}&\hat{f}^c(T_{r_1} \hat{\otimes} T_{r_2}) \circ \hat{f}^c(T_{r_1})^* \hat{\otimes} \hat{f}^c(T_{r_2})^* \\ &\quad = \hat{g}^c(T_{r_1} \hat{\otimes} T_{r_2}) \circ U(r_1)^* \hat{\otimes} U(r_2)^* \circ \hat{f}^c(T_{r_1})^* \hat{\otimes} \hat{f}^c(T_{r_2})^*,\end{aligned}$$

which implies,

$$\hat{f}^c(T_{r_1} \hat{\otimes} T_{r_2}) = \hat{g}^c(T_{r_1} \hat{\otimes} T_{r_2}) \circ U(r_1)^* \hat{\otimes} U(r_2)^*.$$

Using the equivalence A, we get:

$$\begin{aligned}&A \hat{f}^c(T_{r_1} \hat{\otimes} T_{r_2}) A^{-1} \\ &\quad = A \hat{g}^c(T_{r_1} \hat{\otimes} T_{r_2}) A^{-1} A \circ U(r_1)^* \hat{\otimes} U(r_2)^* A^{-1}.\end{aligned}$$

This last equality is exactly (3.33).

Remark 11 The following fact is important: the matrix A is a constant. This comes again from the "Induction-Reduction" Theorem of [3] (or from direct computation): the equivalence $A : \mathbb{L}^2(\check{N}) \hat{\otimes} \mathbb{L}^2(\check{N}) \approx \mathbb{L}^2(\check{N} \times \check{N}) \to \dot{\bigoplus}_{k\in\check{N}} \mathbb{L}^2(\check{N})$, is given by $A\varphi = \dot{\bigoplus}_{k\in\check{N}} \varphi_k$, with $\varphi_k(l) = \varphi(l, l-k)$. Hence, its matrix is independent of r_1, r_2.

Let us rewrite (3.33) as $\Delta_f = \Delta_g H$, for some unitary matrix H. Since $N-1$ corresponding blocks in Δ_f and Δ_g are invertible, it follows that H is also block diagonal. Since it is unitary, all diagonal blocks are unitary. In particular, the k_0^{th} block is unitary. Also, $H = A \circ U(r_1)^* \hat{\otimes} U(r_2)^* A^{-1}$ is an analytic function of r_1, r_2. Moving r_1, r_2 in a neighborhood moves $r_1 + R_{k_0} r_2$ in a neighborhood. If we read the k_0^{th} line of the equality $\Delta_f = \Delta_g H$, we get $\Delta_f(T_{r_1+R_{k_0} r_2}) = \Delta g(T_{r_1+R_{k_0} r_2}) H_{k_0}(r_1, r_2)$, where $H_{k_0}(r_1, r_2)$ is unitary, and analytic in r_1, r_2. It follows that, by analyticity outside I, that $U(r)$ prolongs analytically to all of $\mathbb{R}^2 \backslash \{0\}$, in a unique way. The equality $\hat{g}^c(r) = \hat{f}^c(r) U(r)$ holds over $\mathbb{R}^2 \backslash \{0\}$.

Now, for the characters $\hat{K}_n$, $n \in \mathbb{Z}/N\mathbb{Z}$, it is easily computed that $\hat{f}^c(\hat{K}_n) = av(f) \sum_k e^{2\pi i n k/N}$. In particular $\hat{f}^c(0) = N av(f)$.

The equality of the second type invariants imply that $av(f) = av(g)$. Moreover, if $\hat{f}^c(\hat{K}_0) \neq 0$, $\hat{g}^c(\hat{K}_0) = \hat{f}^c(\hat{K}_0)$. This implies the choice $U(\hat{K}_0) = 1$.

For $n \neq 0$, note that $\hat{f}^c(\hat{K}_n)$ and $\hat{g}^c(\hat{K}_n)$ are zero. Hence we cannot define $U(\hat{K}_n)$ in the same way. In fact, we will consider the representations $T_{n,r} \approx T_r$,

$$T_{n,r} = \hat{K}_n \otimes T_r. \tag{3.35}$$

The representation $T_{n,r}$ is equivalent to T_r, the equivalence being A_n,

$$A_n(u) = e^{\frac{2\pi i}{N} un} = \varepsilon^{un}. \tag{3.36}$$

Also, we set

$$U(T_{n,r}) = U(n,r) = [\hat{f}^c(T_{n,r})]^{-1} \hat{g}^c(T_{n,r}) = A_{-n} U(r) A_n. \tag{3.37}$$

It follows that, wherever $U(r)$ is defined, $U(n,r)$ is also defined. We set also:

$$U(\hat{K}_n) Id = U(r)^* A_{-n} U(r) A_n = U(r)^* U(n,r). \tag{3.38}$$

A-priori, $U(\hat{K}_n)$ is ill defined, for several reasons. The crucial Lemma 3 below shows not only that it is actually well defined but also:

$$U(\hat{K}_n) = e^{in\theta_0}, \quad \text{for some } \theta_0 = \frac{2\pi k_0}{N}. \tag{3.39}$$

Remark 12 At this point, we could already conclude from (3.39) directly (but not so easily) our result, i.e. $\widetilde{h}^t(R_\theta r)$ and $\widetilde{f}^t(R_\theta r)$ differ from a rotation R_{θ_0}. However it is rather easy to see that this is in fact just "Chu-duality".

Note that, to conclude (3.39), we need Lemma 3, which is the most complicated among the series of lemmas just below.

Let us define $U(T)$ for any p-dimensional representation T (p arbitrary).

As a unitary representation T is unitarily equivalent to $\dot{\bigoplus}_{i=1}^{p} T_{r_i} \dot{\bigoplus}_{i=1}^{k} \hat{K}_{n_i} = \dot{\bigoplus} T_i$, $r_i \in \mathcal{S}$, i.e. $T = A \Delta T_i A^*$, where A is some unitary matrix, and ΔT_i is a block diagonal of irreducible representations T_i.

We define $U(T) = A \Delta U(T_i) A^*$.

The proof of the following Lemmas 3, 7, 8 are given in Appendix 6. The proof of Lemma 7 requires the crucial Lemma 14, Appendix 5, characterizing the convergence in $rep_n(M_{2,N})$.

Lemma 3 *U is well defined.*

Lemma 4 *At a point* $T = A(T_{r_1} \dot{\bigoplus} \cdots \dot{\bigoplus} T_{r_p} \dot{\bigoplus} \hat{K}_{k1} \dot{\bigoplus} \cdots \dot{\bigoplus} \hat{K}_{kl}) A^* = A(T_r \dot{\bigoplus} T_{\hat{K}}) A^*$, *where* $r_1, \ldots, r_p \notin I$, *we have*: $U(T) = A(\cdots \hat{f}^c(T_r)^{-1} \hat{g}^c(T_r) \dot{\bigoplus} \cdots e^{ik\theta_0} \dot{\bigoplus} \cdots) A^*$.

Proof By definition of I at such points $r_1, \ldots, r_p$, $\hat{f}^c(T_r)$ and $\hat{g}^c(T_r)$ are invertible. Also, by equality of the first descriptors, $\hat{f}^c(T_{rj})\ \hat{f}^{c^*}(T_{r_j}) = \hat{g}^c(T_{rj})\ \hat{g}^{c^*}(T_{r_j})$, we have $\hat{g}^c(T_{rj}) = \hat{f}^c(T_{rj}) U(T_{rj})$. Also, by definition, $U(\hat{K}_j) = e^{ij\theta_0}$. This shows the result. □

Lemma 5 $U(T \dot{\bigoplus} T') = U(T) \dot{\bigoplus} U(T')$.

Lemma 6 *If* $AT = T'A$, *A unitary, then*: $AU(T) = U(T')A$.

The Lemmas 5, 6 are just trivial consequences of the definition of $U(T)$.

Lemma 7 *U is continuous.*

Lemma 8 $U(T \hat{\otimes} T') = U(T) \hat{\otimes} U(T')$.

Lemmas 3, 4, 5, 6, 7, show that U is a quasi-representation of the category $\pi(M_{2,N})$.

Since $M_{2,N}$ has the duality property, $U(T) = T(g_0)$ for some $g_0 \in M_{2,N}$.

Also, we have:

$\hat{g}^c(T_r) = \hat{f}^c(T_r) U(T_r) = \hat{f}^c(T_r) T_r(g_0) = \hat{f}^c_{g_0}(T_r)$, by the fundamental property of the Fourier transform.

The support of the Plancherel's measure being given by the (non-character) unitary irreducible representations T_r, by the inverse Fourier transform, we get $g^c = f^c_{g_0}$, for some $g_0 \in M_{2,N}$, which is what we needed to prove. By Lemma 12 we have shown our final result.

Theorem 8 *If the (Three types of) Discrete Generalized Fourier Descriptors of two images* $f, g \in \mathcal{G}$ *are the same, and if N is odd, then the two images differ from a motion, the rotation of which has angle* $\frac{4k\pi}{N}$ *(i.e.* $\frac{2k'\pi}{N}$ *since N is odd) for some k. Remind that* $\mathcal{G}$ *is a residual subset of the set of images of size K.*

4 Conclusion

In this paper, we have developed a rather general theory of "Motion Descriptors", based upon the basic duality concepts of abstract harmonic analysis.

We have applied this theory to several motion groups, and to the general case of compact groups, completing previous results.

This theory leads to rather general families of invariants under group actions operating on functions (images). We have proved weak completeness -i.e. completeness over a large (residual) subset of the set of images- in the case of several special groups, including motion groups "with small basic rotation". These invariants are at most cubic expressions of the functions (images).

A number of interesting theoretical questions remain open (such as completeness for the usual group of motions M_2).

In the first part of the paper, we have applied our practical theory to four cases, namely the COIL data-base, The AR and ORL data bases for human faces, and to a personal database of cellular phones. We have also made several tests of robustness with respect to lighting, using another special data-base.

In our methodology, we have used the "Motion Descriptors" provided by our theory in the context of a standard SVM method (that we have recalled briefly). We have also compared, in this context, our Descriptors to other classical families of invariants, such as the Zernike moments.

About the theoretical results, let us point out the following facts:

1. There is a final form of duality Theory, which is given by "Tatsuuma Duality", see [22, 49]. This is a generalization of Chu duality, to general locally compact (type 1) groups. In particular, it works for M_2. Unfortunately, huge difficulties appear when trying to use it in our context. However this is a challenging subject.
2. Computation of the Generalized Fourier Descriptors reduces to usual FFT evaluations.
3. The first and second-type Descriptors, that arise via the trivial or the cyclic lift have a very interesting practical feature: they don't depend on an estimation of the centroid of the image. This is a strong point in practice.
4. Otherwise, the variables that appear in the Generalized-Fourier-descriptors have clear frequency interpretation. Hence, depending on the problem (a high or low frequency texture), one can chose the actual values of these frequency variables in certain adequate ranges.

We leave the reader to conclude that our results are at least extremely promising.

Appendix 1: A Few Technical Facts about Standard Fourier Descriptors for Contours

We start with the statement of 3 very elementary lemmas, the proof of which is easy and left to the reader.

Lemma 9 *Let* $\{a_n\}, \{b_n\}, n \in \mathbb{Z}$, *be two sequences in* $\mathbb{R}/2\pi\mathbb{Z}$ *with* $a_{-n} = -a_n$, $b_{-n} = -b_n$, *and for all* m, n,

$$a_n + a_m - a_{m+n} = b_m + b_n - b_{n+m}, \tag{5.1}$$

then:

$$a_0 = b_0 = 0, \qquad \frac{a_n}{n} - \frac{a_m}{m} = \frac{b_n}{n} - \frac{b_m}{m}, \quad \text{for all } m, n \neq 0. \tag{5.2}$$

Conversely, (5.2) *implies* $a_n + a_m - a_{m+n} = b_m + b_n - b_{n+m}$.

Lemma 10 *Let* f, g *be real* $\mathbb{L}^2$ *functions on the circle. Let* $\{f_n\}, \{g_n\}$ *be their respective Fourier series.*

Assume that: (a) $|f_n| = |g_n| \neq 0, \forall n \in \mathbb{Z}$, (b) $f_n f_m \bar{f}_{n+m} = g_n g_m \bar{g}_{n+m}$ $\forall n, m$. *Then* g *is a translate of* f.

Lemma 11 *The set of real* $\mathbb{L}^2$ *functions* f *on the circle, such that* $f_n \neq 0$ *for all* $n \in \mathbb{Z}$ (*where* f_n *is the Fourier series of* f) *is residual in* $\mathbb{L}^2$.

Appendix 2: Justification of the Concept of the Cyclic-Lift

The lemma below justifies the use of the "cyclic lift" of a function f over the plane to a function f^c over one of our motion groups M_2 or $M_{2,N}$.

Lemma 12 *Two functions* $f, g \in L^2(\mathbb{R}^2)$ *with nonzero average differ from a motion* $(\theta, a, b) = (\theta, A)$ *iff their cyclic lifts differ from a motion, the rotation component of which has angle* $\frac{\theta}{2}$, *and the translation is zero.*

Proof Set $g(X) = f(R_\omega X + A)$ for $(\omega, A) \in G = M_2$ or $M_{2,N}$.

Then, $av(g) = \int_{\mathbb{R}^2} f(R_\omega X + A)dX = \int_{\mathbb{R}^2} f(R_\omega X + A)d(R_\omega X) = \int_{\mathbb{R}^2} f(Y)d(Y) = av(f)$.

Also, $centr(g) = X_g = \int_{\mathbb{R}^2} Xf(R_\omega X + A)dX = R_{-\omega} \int_{\mathbb{R}^2} R_\omega X f(R_\omega X + A)d(R_\omega X) = R_{-\omega} \int_{\mathbb{R}^2} (R_\omega X + A) f(R_\omega X + A)d(R_\omega X + A)) - R_{-\omega} A \int_{\mathbb{R}^2} f(R_\omega X + A) \times d(R_\omega X + A))$

$= R_{-\omega} X_f - R_{-\omega} A av(f)$. Hence we get two first conclusions:

For $g(X) = f(R_\omega X + A)$,

1. $av(g) = av(f)$, (6.1)
2. $X_g = R_{-\omega}(X_f - Aav(f))$.

Now, consider the cyclic lifts f^c, g^c of f and g:

$$\begin{aligned}
f^c(\alpha, X) &= f\left(R_\alpha X + \frac{1}{av(f)} X_f\right), \\
g^c(\alpha, X) &= g\left(R_\alpha X + \frac{1}{av(g)} X_g\right), \\
&= f\left(R_\omega\left(R_\alpha X + \frac{1}{av(f)} R_{-\omega}(X_f - Aav(f))\right) + A\right), \\
&= f\left(R_{\omega+\alpha} X + A + \frac{1}{av(f)}(X_f - Aav(f))\right), \\
&= f\left(R_{\omega+\alpha} X + \frac{1}{av(f)} X_f\right).
\end{aligned}$$

Otherwise $(\lambda, B) f^c(\alpha, X) = f(R_{\alpha+\lambda}(R_\lambda X + B) + \frac{1}{av(f)} X_f)$
$= f(R_{\alpha+2\lambda} X + R_{\alpha+\lambda} B + \frac{1}{av(f)} X_f)$. Therefore, choosing $\lambda = \frac{\omega}{2}$ and $B = 0$ we get:

$(\lambda, B) f^c(\alpha, X) = f(R_{\alpha+\omega} X + \frac{1}{av(f)} X_f) = g^c(\alpha, X)$.

Conversely, we assume that $(\lambda, 0) f^c(\alpha, X) = g^c(\alpha, X)$. This means that $f^c(\alpha + \lambda, R_\lambda X) = g^c(\alpha, X)$ which is equivalent to:

$$f\left(R_{\alpha+\lambda} R_\lambda X + \frac{1}{av(f)} X_f\right) = g\left(R_\alpha X + \frac{1}{av(g)} X_g\right).$$

This is true for all α, X. Let us take the particular case where $\alpha = -2\lambda$. It gives:

$$f\left(X + \frac{1}{av(f)} X_f\right) = g\left(R_{-2\lambda} X + \frac{1}{av(g)} X_g\right).$$

This is true for all X. Let us set $Y = X + \frac{1}{av(f)} X_f$. Then $X = Y - \frac{1}{av(f)} X_f$, and for all Y, we have:

$$f(Y) = g\left(R_{-2\lambda} Y + \frac{1}{av(g)} X_g - \frac{1}{av(f)} R_{-2\lambda} X_f\right).$$

$$f(Y) = g(R_{-2\lambda} Y + H),$$

for a certain H. This shows that f and g differ from a motion, with rotation angle 2λ. □

Appendix 3: A Crucial Transversality Result

The following lemma is a more or less obvious technical result we need in Sect. 3.4.5. A compact $K \subset \mathbb{R}^2$ is fixed, containing a neighborhood of the origin. The set

$\mathcal{H} = \mathbb{L}^2(K, \mathbb{R})$ is a closed subspace in the Hilbert space $\mathbb{L}^2(\mathbb{R}^2, \mathbb{R})$, hence it is a Hilbert subspace. The set $\mathcal{I}$ of images (of size K) is the open subset of $\mathcal{H}$ formed by the functions f with nonzero average. Let $N \in \mathbb{N}$ and $r \in \mathbb{R}^2$ be fixed, $r \neq 0$. Consider the map $\mathcal{M} : \mathcal{I} \to \mathbb{C}^N$, $f \to \omega_r = (\tilde{f}^t(R_0 r), \dots, \tilde{f}^t(R_{\theta_i} r), \dots, \tilde{f}^t((R_{\theta_{N-1}} r))$, where $\tilde{f}^t$ is the usual 2D Fourier transform of f^t as an element of $\mathbb{L}^2(\mathbb{R}^2, \mathbb{R})$.

Lemma 13 *$\mathcal{M}$ is a linear submersion if and only if N is odd.*

The proof is easy and left to the reader. A very simple idea for the proof is to show that, for suitably chosen $X_m \in K$, the distributions that are linear real combinations $f = \sum_m \alpha_m \delta_{X_m}$, where δ_{X_m} is the Dirac function at X_m, provide $\mathcal{M}(f)$ which span the realification of $\mathbb{C}^N$. Although, if N is even, this is clearly not true.

Appendix 4: Computation of First-Type and Second-Type Generalized-Motion-Descriptors

There are two computational-steps for estimation of First-Type and second-Type Motion-Descriptors:

– First, computation of the Fourier transform $\tilde{f}$ of the image f.
– Second, computation of some integral expressions of $\tilde{f}$ over circles in the frequency-plane.

• **Estimation of $\tilde{f}$.**
The Fourier transform $\tilde{f}$ is computed from FFT estimation over the grid formed by pixels on the screen. We assume the values of the grey levels (or color levels) to be constant over each pixel. Hence we have to compute the Fourier transform of a piecewise-constant function over the regular pixel-grid.

FFT algorithm do not produce the exact value of the Fourier transform. In particular for high frequencies there is a large deviation. In usual situations in signal or image processing it is a nonsense to consider this deviation: due to respect of Shannon sampling rule, the deviation will be negligible.

In our problem the situation might be very different: For instance for a human face data-base, it is reasonable to work after contour-extraction.[8] In that case, after contours extraction, of course the sampling will not respect the Shannon rule. However, we have to compute the exact Fourier transform $\tilde{f}$ of the image. Here the deviation will be significant and we have to correct it.

To perform this correction we use the following remark which is probably very naive and well known by signal processing engineers.

For a function f constant over the cells of a regular grid it is easily computed that the **exact** correction term from the values of the FFT to values of the usual Fourier transform $\tilde{f}$ at the points of the grid, is given by:

$$\tilde{f}\left(\frac{r-1}{\sqrt{N}}, \frac{s-1}{\sqrt{N}}\right) = \left(\frac{e^{2\pi i \frac{(r-\frac{N}{2})}{N}} - 1}{2\pi i \frac{r-\frac{N}{2}}{\sqrt{N}}}\right)\left(\frac{e^{2\pi i \frac{(s-\frac{N}{2})}{N}} - 1}{2\pi i \frac{s-\frac{N}{2}}{\sqrt{N}}}\right) \times FFT_{r,s}. \quad (8.1)$$

Notice the very important point that this correction term preserves the $N \log_2 N$ complexity of the FFT algorithm.

• **Estimation of integrals over circles**
We explain only the computation of the second Type Motion-Descriptors. Computation of the integrals corresponding to the first type are easier and based upon the same principle.

We have to evaluate formula (2.6) for a function f constant over each cell of the grid. The only approximation we make is to consider the values of $\tilde{f}$ constant over the dual grid of the frequency plane, and given by formula (8.1). Therefore the value of $I^{\xi_1,\xi_2}(f)$ will be equal to the sum of the values of $\tilde{f}(R_\theta(V_1+V_2)\overline{\tilde{f}}(R_\theta(V_1))\overline{\tilde{f}}(R_\theta(V_2))$ weighted by the length of the arcs encountered.

V_1 and V_2 are two fixed vectors in the Fourier space and $V_3 = V_1 + V_2$ (see Fig. 12), then on some elementary arcs (the length of which can be pre-computed once for all as soon as the values of V_1, V_2 are given) the value of $\tilde{f}(R_\theta(V_1+V_2)\overline{\tilde{f}}(R_\theta(V_1))\overline{\tilde{f}}(R_\theta(V_2))$ is constant by our approximation. The contribution of this arc to $I^{\xi_1,\xi_2}(f)$ will be equal to this value times the length L of the arc. A trivial undergraduate computation shows that:

$$L = R \times \theta,$$

with R the radius of the smallest among the three arcs and:

$$\theta = \arcsin\left[\sqrt{1 - \frac{b^2}{R^2}}\right] - \arcsin\left[\frac{a}{R}\right].$$

Here a (resp. b) is the x (resp y) coordinate of the first (resp. second) endpoint of the arc (see Fig. 13).

Appendix 5: Convergence of Representations of $M_{2,N}$

Now, we state and prove a lemma characterizing the convergence of sequences on $rep_n(M_{2,N})$. This lemma is crucial to

[8] By "contour" we mean here the result obtained after applying a standard contour filter. This "contour" contains information about the "texture" of the image. This contour notion has to be distinguished from the (natural) notion of an "exterior-contour" also used here. If well defined, the "exterior-contour" is a connected component of the "contour".

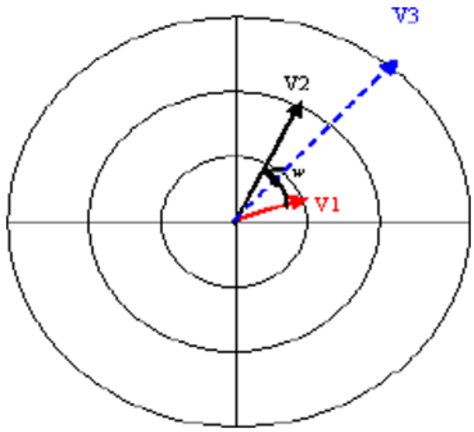

Fig. 12 The vectors positions in the Fourier-space

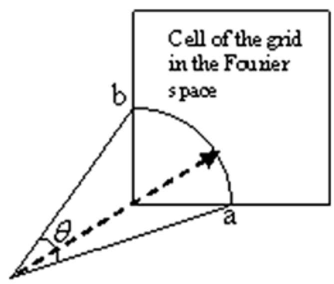

Fig. 13 Estimation of integrals in the discrete-Fourier-space

prove the continuity of the quasi-representation of $\pi(M_{2,N})$ that we construct in Sect. 3.4.5.

Let T^p be a sequence of finite dimensional representations of $M_{2,N}$ of the same dimension n. Assume that:

$$T = T_1 \otimes I_{k_1} \dot{\oplus} \cdots \oplus T_l \otimes I_{k_l}, \tag{9.1}$$

where T_j is either a character $T_j = \hat{K}_{n_j}(\alpha, X) = e^{in_j\alpha}$, or an irreducible representation of the form $T_{r_j}, r_j \in \mathcal{S}$, and $\begin{cases} r_i \neq r_j \\ n_i \neq n_j \end{cases}$ for $i \neq j$.

Let $\mathcal{S}_\varepsilon$ denote the "modified slice of cake", i.e. $\mathcal{S}_\varepsilon = \{(\lambda \cos\alpha, \lambda \sin\alpha), \lambda > 0, -\varepsilon \leq \alpha < \frac{2\pi}{N} - \varepsilon\}$. We can assume that ε is small enough for $r_j \in \mathcal{S}_\varepsilon$ for all j.

Lemma 14 $T^p \longrightarrow T$ *if and only if there exists* A_p, *a unitary matrix, and* ϱ_i^p , n_i^p *such that*:

1. $\begin{cases} \varrho_i^p \longrightarrow r_i \in \mathcal{S}_\varepsilon, \\ \hat{K}_i^p \longrightarrow \hat{K}_{ni}. \end{cases}$
2. $T^p = A_p(\oplus_i T_{\varrho_i^p} \oplus_{i'} \hat{K}_{n_{i'}}^p) A_p^*$.
3. *For all convergent subsequence* $A_p \longrightarrow A$, $A = I_{k_1} \hat{\otimes} \Lambda_1 \oplus \cdots \oplus I_{k_l} \hat{\otimes} \Lambda_l$, *for certain unitary matrices* $\Lambda_1, \ldots, \Lambda_l$.

Proof T^p is *completely* reducible. Then:

$$T^p = A_p \Delta T_p A_p^*,$$

where ΔT_p is a block-diagonal of irreducible representations (either $T_{r_j^p}$ or $\hat{K}_{n_j^p}$). First, when $p \longrightarrow +\infty$, all the r_j^p remain bounded: it would contradict the equicontinuity on any compact $K \subset M_{2,N}$ of the sequence $T_{|K}^p$ (T^p restricted to K). Second, consider any convergent subsequence (still denoted by A_p) and the corresponding subsequences (r^p), (n^p). Note that the vectors (r^p) and (n^p) may have different dimensions depending on p.

In the following we shall consider extracted subsequences such that (r_j^p), $(\hat{K}_j^p)$ both converge. We shall show that all of them converge to the same required limit. Hence the whole extracted sequence A_p will converge to a limit with the required form.

Since $(\hat{K}_j^p)$ converges, and since $(\hat{K}_j^p)$ is bounded among characters, $\hat{K}_j^p$ is constant, after a certain rank, $\hat{K}_j^p = \hat{K}_j^*$, and also, $\varrho_j^p \longrightarrow \varrho_j^*$.

The corresponding diagonal matrix we denote by $\Delta T'$. We have $T_p - T = (A_p - A)\Delta T_p A_p^* + A\Delta T_p(A_p^* - A^*) + A\Delta T_p A^* - T$.

This shows that $A\Delta T_p A^* - T \longrightarrow 0$ (since $(A_p - A) \longrightarrow 0$ and since all other terms remain bounded in restriction to any compact $K \subset M_{2,N}$). Now, $\Delta T_p \longrightarrow \Delta T'$. Hence $A\Delta T_p A^* - T = A(\Delta T_p - \Delta T')A^* + A\Delta T' A^* - T$. It follows that $A\Delta T' A^* - T = 0$ (ΔT_p converges uniformly to $\Delta T'$ on any compact $K \subset M_{2,N}$).

$$A\Delta T' = TA. \tag{9.2}$$

The representations $\Delta T'$ and T are unitarily equivalent. This shows that $\hat{K}_j^* = \hat{K}_j$, $\varrho_j^* = r_j$, with adequate multiplicity.

Then, up to some relabelling, $A\Delta T = TA$ and $A = A_1 \dot{\oplus} \cdots \dot{\oplus} A_l$.

Let us consider a non-character-block of this decomposition, the first block A_1 say.

The relation $A\Delta T = TA$ gives (considering the block decomposition of A_1 in $N \times N$ dimensional blocks) $A_1 = (A_{1i,j})$:

$$A_{1i,j} T_{r_1} = T_{r_1} A_{1i,j}. \tag{9.3}$$

By Shur's Lemma, $A_{1i,j}$ is a scalar multiple of the identity.

$A_{1i,j} = \lambda_{ij} Id$. This can be rewritten as:
$A_1 = I_{k_1} \hat{\otimes} \Lambda_1$, $A_1(T_{r_1} \hat{\otimes} I_{k_1}) = (T_{r_1} \hat{\otimes} I_{k_1}) A_1$.

It follows since A_1 is unitary that Λ_1 is also unitary. This ends the proof, since the converse statement is easily checked. □

Appendix 6: Proofs of Technical Lemmas

Proof of Lemma 3 **The constructed quasi-representation U is well defined.**

First, we will show that $U(\hat{K}_n)$ is well defined. To do this, we set $U_0(n,r) = U(r)^* A_{-n} U(r) A_n = U(r)^* U(n,r)$. By Lemma 15, $U_0(n,r)$ and $U(r)$ are circulant. In particular, they commute.

By the end of the proof of Lemma 8, $U(r_1) \otimes U(r_2) = U(T_{r_1}) \otimes U(T_{r_2}) = U(T_{r_1} \otimes T_{r_2})$ (when both are defined). It follows that $U(n_1, r_1) \otimes U(n_2, r_2) = U(T_{n_1,r_1} \otimes T_{n_2,r_2})$, where $U(n,r) = U(\hat{k}_n \otimes T_r) = U_0(n,r)U(r) = U(r) \times U_0(n,r)$. This implies:

$$U_0(n_1,r_1)U(r_1) \otimes U_0(n_2,r_2)U(r_2)$$
$$= U(\hat{K}_{n_1+n_2} \otimes T_{r_1} \otimes T_{r_2}),$$

or:

$$U_0(n_1,r_1) \otimes U_0(n_2,r_2) U(r_1) \otimes U(r_2)$$
$$= A^* \dot{\oplus}_k U(\hat{K}_{n_1+n_2} \otimes T_{r_1+R_k r_2}) A,$$

where A is an equivalence. Hence:

$$U_0(n_1,r_1) \otimes U_0(n_2,r_2) A^* \dot{\oplus}_k U(T_{r_1+R_k r_2}) A$$
$$= A^* \dot{\oplus}_k U_0(n_1+n_2, r_1+R_k r_2) U(r_1+R_k r_2) A,$$

and:

$$A U_0(n_1,r_1) \otimes U_0(n_2,r_2) = \dot{\oplus}_k U_0(n_1+n_2, r_1+R_k r_2) A.$$

This can be rewritten as: $A_k U_0(n_1,r_1) \otimes U_0(n_2,r_2) = U_0(n_1+n_2, r_1+R_k r_2) A$, and the equivalence between $T_{r_1} \otimes T_{r_2}$ and $\dot{\oplus}_k T_{r_1+R_k r_2}$ is given by $(A_k \varphi)(l) = \varphi(l, k-l)$. Taking $\varphi(k,l) = \delta_{i,k}\delta_{j,l}$ where δ is the Kronecker symbol, we get:

$$U_0(n_1,r_1)(u,i) U_0(n_2,r_2)(u-k,j)$$
$$= \begin{cases} 0 & \text{for } k \neq i-j, \\ U_0(n_1+n_2, r_1+R_k r_2)(u,i) & \text{for } k = i-j. \end{cases} \tag{10.1}$$

We know that the diagonal of U_0 is a constant (U_0 being circulant). Assume that $U_0(n_2,r_2)(u-k,u-k) = 0$ (identically in r_2, as an analytic function of r_2 out of isolated points), then, (10.1) implies that $U_0(n_1+n_2, r_1+R_k r_2)(u,i) = 0$ for all r_1, n_1, i, $u \neq i$. Then, $U_0(n_1+n_2, r_1+R_k r_2)$ is zero, which is impossible since it should be unitary. Hence $U_0(n,r)(u,u) \neq 0$ whatever U.

By (10.1) again, $U_0(n,r)(u-k,j) = 0$ for $k \neq u-j$, or $u-k \neq j$. Hence, U_0 is diagonal, circulant. $U_0(n,r) = e^{i\theta_0(n,r)} Id$. By (10.1) once more,

$$\theta_0(n_1,r_1) + \theta_0(n_2,r_2) = \theta_0(n_1+n_2, r_1+R_k r_2).$$

Therefore $\theta_0(n,r) = n\theta_0$, and finally $U_0(n,r) = e^{in\theta_0} Id$. Also, we get that $\theta_0 = \frac{2\pi k_0}{N}$ for some k_0.

Second we have to show that two equalities (with A and B unitary): $T = A \Delta T_i A^* = B \Delta T_i B^*$, don't lead to contrary definitions of $U(T)$.

Then, $B^* A \Delta T_i = \Delta T_i B^* A$. Consider a primary-labeling of ΔT_i:

$$\Delta T_i = T_1 \hat{\otimes} I_{d_{k_1}} \dot{\oplus} \cdots \dot{\oplus} T_l \hat{\otimes} I_{d_{k_l}},$$

where $T_i \neq T_j$ for all $i \neq j$.

With an argument similar to the one at the end of the proof of Lemma 14 (from formula (9.3) on), we get that:

$$B^* A = (Id_{k_1} \hat{\otimes} \Lambda_1) \dot{\oplus} \cdots \dot{\oplus} (Id_{k_p} \hat{\otimes} \Lambda_l),$$

where $\Lambda_1, \ldots, \Lambda_l$ are certain unitary matrices.

Then we have to show that $B^* A \Delta U(T_i) = \Delta U(T_i) B^* A$, or equivalently:

$$B^* A \Delta U(T_i) A^* B = \Delta U(T_i). \tag{10.2}$$

This is true as soon as:

$$\Delta U(Tj) \hat{\otimes} Id_{k_j}$$
$$= (Id_{k_j} \hat{\otimes} \Lambda_j)(\Delta U(Tj) \hat{\otimes} Id_{k_j})(Id_{k_j} \hat{\otimes} \Lambda_j)^*, \tag{10.3}$$

for all j.

But $(Id_{k_j} \hat{\otimes} \Lambda_j)(\Delta U(Tj) \hat{\otimes} Id_{k_j})(Id_{k_j} \hat{\otimes} \Lambda_j)^* = (Id_{k_j} \hat{\otimes} \Lambda_j)(\Delta U(Tj) \hat{\otimes} Id_{k_j})(Id_{k_j} \hat{\otimes} \Lambda_j^*) = (\Delta U(T_j) \hat{\otimes} \Lambda_j)(Id_{k_j} \hat{\otimes} \Lambda_j^*) = \Delta U(T_j) \hat{\otimes} \Lambda_j \Lambda_j^* = \Delta U(T_j) \hat{\otimes} Id_{k_j}$, since Λ_j is unitary. This ends the proof. □

Proof of Lemma 7 **Continuity of** U. Assume that $T^p \in \mathrm{Re}\, p_n(G)^\wedge$, $T^p \longrightarrow T'$ set $T' = B(T_1 \hat{\otimes} I_{k_1} \dot{\oplus} \cdots \dot{\oplus} T_l \hat{\otimes} I_{k_l}) B^* = BTB^*$ with $T_i \neq T_j$ for $i \neq j$.

Then, we apply to $B^* T^p B$ the result of Lemma (14). $B^* T^p B$ tends to T iff $B^* T^p B$ meets the statements 1, 2, 3 of Lemma 14.

Using the notations of Lemma (14), it follows that $B^* T^p B = A_p(\dot{\oplus}_i T_{\varrho_i^p} \dot{\oplus}_{i'} \hat{K}_{n_i^p}) A_p^*$, with properties 1.2.3.

Set $\varepsilon = e^{\frac{2\pi i}{N}}$, then $U(\hat{K}_n) = \varepsilon^{n\theta_0}$, θ_0 from (3.39). By definition of U,

$$U(B^* T^p B) = A_p(\dot{\oplus}_i U(T_{\varrho_i^p}) \dot{\oplus} U(\hat{K}_{n_i^p})) A_p^*$$
$$= A_p(\dot{\oplus}_i U(T_{\varrho_i^p}) \dot{\oplus}_k \varepsilon^{n_k} \cdots) A_p^*,$$

and, for any convergent subsequence A_p,

$A_p \longrightarrow A = (I_{k_1} \hat{\otimes} \Lambda_1 \dot{\oplus} \cdots \dot{\oplus} I_{k_p} \hat{\otimes} \Lambda_p)$ and using Lemma 6,

$$U(B^* T^p B) \longrightarrow A((U(T_1) \hat{\otimes} I_{k_1}) \dot{\oplus} \cdots \dot{\oplus} (U(T_l) \hat{\otimes} I_{k_l})) A^*$$
$$B^* U(T^p) B \longrightarrow \dot{\oplus}_j (I_{kj} \hat{\otimes} \Lambda_j)(U(T_j) \hat{\otimes} I_{k_j})(I_{kj} \hat{\otimes} \Lambda_j)^*.$$

Then,

$$B^*U(T^p)B \longrightarrow \dot{\oplus}_j(U(T_j)\hat{\otimes}\Lambda_j)(I_{kj}\hat{\otimes}\Lambda_j^*),$$
$$\longrightarrow \dot{\oplus}_j(U(T_j)\hat{\otimes}\Lambda_j\Lambda_j^*),$$
$$\longrightarrow \dot{\oplus}_j(U(T_j)\hat{\otimes}Id_{kj}).$$

Therefore,

$$U(T^p) \longrightarrow B(\dot{\oplus}_j U(T_{r_j})\hat{\otimes}Id_{kj})B^*$$
$$= BU(T)B^*.$$

Hence by Lemma 6,

$$U(T^p) \longrightarrow U(BTB^*)$$
$$= U(T').$$

Exhausting all convergent subsequences $A^p \longrightarrow A$ (not the same, may be) it remains only a finite number of terms and for each corresponding subsequence $U(T^p) \longrightarrow U(T')$.

Therefore the whole sequence $U(T^p)$ meets:

$U(T^p) \longrightarrow U(T')$ and U is sequentially continuous hence continuous. □

Proof of Lemma 8 **Commutation of U (the constructed quasi-representation) with tensor product**.

$$T = A(T_1\dot{\oplus}\cdots\dot{\oplus}T_l)A^* = A\Delta T A^*,$$

$$T' = B(T_1'\dot{\oplus}\cdots\dot{\oplus}T_p)B^* = B\Delta T' B^*,$$

$$T\hat{\otimes}T' = A(T_1\dot{\oplus}\cdots\dot{\oplus}T_l)A^*\hat{\otimes}B(T_1'\dot{\oplus}\cdots\dot{\oplus}T_p')B^*$$
$$= (A\hat{\otimes}B)(T_1\dot{\oplus}\cdots\dot{\oplus}T_l)\hat{\otimes}A^*(T_1'\dot{\oplus}\cdots\dot{\oplus}T_p')B^*$$
$$= (A\hat{\otimes}B)(T_1\dot{\oplus}\cdots\dot{\oplus}T_l)\hat{\otimes}(T_1'\dot{\oplus}\cdots\dot{\oplus}T_p')(A^*\hat{\otimes}B^*)$$
$$= (A\hat{\otimes}B)(T_1\dot{\oplus}\cdots\dot{\oplus}T_l)\hat{\otimes}(T_1'\dot{\oplus}\cdots\dot{\oplus}T_p')(A\hat{\otimes}B)^*$$
$$= (A\hat{\otimes}B)(\dot{\oplus}_{i,j}T_i\hat{\otimes}T_j')(A\hat{\otimes}B)^*.$$

$$U(T\hat{\otimes}T') = (A\hat{\otimes}B)\dot{\oplus}_{i,j}U(T_i\hat{\otimes}T_j')(A\hat{\otimes}B)^*$$

(by Lemmas 5, 6).

Assume that:

$$U(T_l\hat{\otimes}T_j') = U(T_l)\hat{\otimes}U(T_j'). \tag{10.4}$$

Then,

$$U(T\hat{\otimes}T') = (A\hat{\otimes}B)\dot{\oplus}_{i,j}U(T_i)\hat{\otimes}U(T_j')(A\hat{\otimes}B)^*$$
$$= (A\hat{\otimes}B)U(\Delta T)\hat{\otimes}U(\Delta T')(A\hat{\otimes}B)^*$$
$$= (AU(\Delta T)\hat{\otimes}BU(\Delta T'))(A^*\hat{\otimes}B^*)$$
$$= AU(\Delta T)A^*\hat{\otimes}BU(\Delta T')B^*$$
$$= U(T)\hat{\otimes}U(T'),$$

by Lemma 6.

It remains to prove (10.4).

If T_l and T_j are both characters, then (10.4) can be rewritten as $e^{il\theta_0}e^{ij\theta_0} = e^{i(l+j)\theta_0}$.

If T_l is not character and T_j is, (10.4) can be rewritten as:

$$U(T_r\hat{\otimes}\hat{K}_n) = U(T_r)\hat{\otimes}U(\hat{K}_n), \tag{10.5}$$

which results from the definition of $U(\hat{K}_n)$.

The last case is to show: $U(T_{r_1}\hat{\otimes}T_{r_2}) = U(T_{r_1})\hat{\otimes}U(T_{r_2})$.

Actually, this is true if r_1, r_2 and $r_1 + R_k r_2 \notin I$ for all $k \in \breve{N}$: By the equality of the second Descriptors, $\hat{g}^c(T_{r_1}\hat{\otimes}T_{r_2}) = \hat{f}^c(T_{r_1}\hat{\otimes}T_{r_2})U(T_{r_1}\hat{\otimes}T_{r_2})$, $\hat{g}^c(T_{r_1})\hat{\otimes}\hat{g}^c(T_{r_2}) = \hat{f}^c(T_{r_1})\hat{\otimes}\hat{f}^c(T_{r_2})U(T_{r_1})\hat{\otimes}U(T_{r_2})$.

Then,

$$\hat{g}^c(T_{r_1}\hat{\otimes}T_{r_2})\hat{g}^c(T_{r_1})^*\hat{\otimes}\hat{g}^c(T_{r_2})^*$$
$$= \hat{f}^c(T_{r_1}\hat{\otimes}T_{r_2})U(T_{r_1}\hat{\otimes}T_{r_2})U(T_{r_1})^*\hat{\otimes}U(T_{r_2})^*$$
$$\circ \hat{f}^c(T_{r_1})^*\hat{\otimes}\hat{f}^c(T_{r_2})^*$$
$$= \hat{f}^c(T_{r_1}\hat{\otimes}T_{r_2})\hat{f}^c(T_{r_1})^*\hat{\otimes}\hat{f}^c(T_{r_2})^*.$$

But, since $r_1, r_2, r_1 + R_k r_2 \notin I$, $\hat{f}^c(T_{r_1}\hat{\otimes}T_{r_2})$ is invertible (remind that $T_{r_1}\hat{\otimes}T_{r_2} \approx \dot{\oplus}_k T_{(r_1+R_k r_2)}$).

Therefore, $U^*(T_{r_1}\hat{\otimes}T_{r_2})U(T_{r_1})\hat{\otimes}U(T_{r_2}) = Id$,

$U(T_{r_1}\hat{\otimes}T_{r_2}) = U(T_{r_1})\hat{\otimes}U(T_{r_2})$.

But, the set of $(r_1, r_2) \in \mathbb{R}^2\backslash\{0\} \times \mathbb{R}^2\backslash\{0\}$ such that this holds is open, dense.

Otherwise, the mapping $(T, T') \longrightarrow T\hat{\otimes}T'$ is clearly continuous, and U is continuous by the Lemma 7. Also, the mapping $(r, \alpha, X) \longrightarrow T_r(\alpha, X)$ is continuous (it is analytic in (r, α, X)). Hence, on any compact $K \subset M_{2,N}$, the mapping $r \longrightarrow T_{r|K}$ is continuous. Therefore, in the diagram,

$$\begin{array}{ccc} (r_1, r_2) & \rightarrow T_{r_1}\hat{\otimes}T_{r_2} \rightarrow & U(T_{r_1}\hat{\otimes}T_{r_2}) \\ \downarrow & & \downarrow \\ U(T_{r_1})\hat{\otimes}U(T_{r_2}) & \rightarrow & \begin{array}{c} U^*(T_{r_1}\hat{\otimes}T_{r_2})\circ \\ U(T_{r_1})\hat{\otimes}U(T_{r_2}) \end{array} \end{array}$$

all arrows are continuous maps.

It follows that $U(T_{r_1}\hat{\otimes}T_{r_2}) = U(T_{r_1})\hat{\otimes}U(T_{r_2})$, since it is true on a dense subset of $\mathbb{R}^2\backslash\{0\} \times \mathbb{R}^2\backslash\{0\}$. □

Lemma 15 *The matrices $U_0(n, r)$ and $U(r)$ are unitary, circulant.*

Proof We need some classical facts about circulant matrices. A perfect reference is [16]. We set $U_0(n,r) = U(r)^* A_{-n} U(r) A_n$.

First, $U(r)$ is circulant: $U(r) = \hat{f}^c(r)^{-1}\hat{h}(r) = C^*\Omega_r(f)^{-1}\Omega_r(h)C$. Here C is the permutation matrix defined in (3.31). But $\Omega_r(f)C = \tilde{C}\tilde{\Omega}_r(f)$ where $\tilde{C}$ is another permutation, and $\tilde{\Omega}_r(f)$ is another circulant. This last point follows from the following observation:

$$[\widehat{f^c}(r)\Psi](u) = \sum_{\alpha\in\check{N}} \tilde{f}^t(R_{u-2\alpha}r)\Psi(\alpha) = [\Omega_r(f)C\Psi](u),$$

then, setting $u = 2v \bmod N$, we get:

$$[\widehat{f^c}(r)\Psi](v) = \sum_{\alpha\in\check{N}} \tilde{f}^t(R_{2(v-\alpha)}r)\Psi(\alpha) = [\tilde{C}\tilde{\Omega}_r(f)\Psi](v).$$

Therefore, $U(r) = \hat{f}^c(r)^{-1}\hat{h}(r) = \tilde{\Omega}_r(f)^{-1}\tilde{\Omega}_r(h)$, which is circulant.

Hence $U(r) = F_N \Delta F_N^*$, where Δ is diagonal, unitary, and F_N is the usual N-DFT matrix. We have:

$$U_0(n,r) = F_N\Delta^* F_N^* A_{-n} F_N \Delta F_N^* A_n F_N F_N^*.$$

But $F_N^* A_{-n} F_N = R_{-n}$ and $F_N^* A_n F_N = R_n$, where R_n is the n-shift matrix. Therefore, $U_0(n,r) = F_N\Delta^* R_{-n}\Delta R_n \times F_N^*$. But $R_{-n}\Delta R_n$ is diagonal, and $\Delta^* R_{-n}\Delta R_n$ is another diagonal. It follows that $U_0(n,r)$ is circulant. □

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