ENTROPY ESTIMATIONS FOR MOTION PLANNING PROBLEMS IN ROBOTICS

JEAN-PAUL GAUTHIER, VLADIMIR ZAKALYUKIN

To Dmitry Victorovich Anosov with great respect

Abstract. This is the concluding work of our series devoted to evaluations of the complexity and entropy of a motion planning problem for a sub-Riemannian distribution. We consider some new cases of dimensions and codimensions of the distribution, in particular $(2, 3)$, $(3, 4)$ and some which are one-step-bracket-generating. We summarize all known estimations for low dimensional generic systems. They include all generic systems of corank less than 4, and other cases going up to corank 10.

1. Introduction

The motivations to study motion planning problems come from robotics [12]. The kinematic constraints of the robot are specified by a non-integrable distribution Δ of dimension k and codimension p in the phase space $\mathbb{R}^n$.

A Riemannian metric g on Δ (called sub-Riemannian metric) allows to measure the length (practical cost) of the curves that may actually be realized by the robot. These curves (called admissible) are absolutely continuous and are almost everywhere tangent to the distribution.

The problem is to approximate a segment of smoothly embedded non-admissible program curve $\Gamma : [0, 1] \to \mathbb{R}^n$, by an admissible one. In practice, this is a part of the problem to choose the trajectory avoiding the obstacles for robot motion. We assume there are plenty of obstacles, so we are interested in *approximating*, or *interpolating* the curve Γ by admissible curves, ε-close (in the sub-Riemannian sense), and we want to analyze what happens when ε tends to zero.

We define the *entropy* $E(\varepsilon)$ of the problem as the asymptotics for $\varepsilon \to 0$ of the the minimal number of admissible arcs of length $\leq \varepsilon$ which interpolate the program curve Γ. Notice that in all cases considered $E(\varepsilon)$ can be shown to be double the minimal number of ε-sub-Riemannian balls centered on Γ and covering Γ. Another, less trivial consequence of our results, is that in all the cases treated $E(\frac{\varepsilon}{2}) = E_K(\varepsilon)$, where $E_K(\varepsilon)$ is the Kolmogorov's entropy, which is defined as the minimal number of ε-sub-Riemannian balls covering Γ (and not necessarily centered on it). We leave the detailed comparison of these definitions beyond the paper.

The complexity $C(\varepsilon)$ is another similar characteristic which is the asymptotic (for $\varepsilon \to 0$) of the minimal length divided by ε of an admissible curve which belongs to the ε- neighbourhood of the curve Γ and joins its end-points.

Date: June, 2006.

1991 *Mathematics Subject Classification.* 53C17, 49J15, 34H05.

Key words and phrases. Robotics, Subriemannian geometry.

the work of the second author was partially supported by RFBI050100458 grant.

In a series of our previous papers [1, 2, 3, 4] on complexity and entropy estimations we considered generic configurations of Δ, g, Γ for a range of k and p values and described an explicit procedure (synthesis) to get the optimal solution.

As usual, by generic systems we mean an arbitrary motion planning problem from an open dense subset (with respect to the C^∞-topology) of the space of all problems.

In general, the estimations are either given by the integral along Γ of certain positive numerical invariant of a germ of the system along Γ or by a sum of values of the derived invariant upon the set of degenerate points along Γ.

The present paper is in a sense a conclusion of [1, 2, 3, 4] describing the range of k and p where the estimations for generic motion planning problem are known, including several new cases. We aim to to cover the most physically interesting cases of low dimensions. Actually, up to corank 3, our classifcation covers all generic systems.

The simplest results arise in the one-step-bracket-generating case that is when $\Delta' = [\Delta, \Delta]$ span the ambient space $T_m\mathbb{R}^n$ at each point $m \in \Gamma$. In this case, of course, $p \leq \frac{k(k-1)}{2}$.

In the paper [2] for any point m of Γ we have defined a principal invariant $\chi(m)$ provided that the sub-Riemanian distribution is 1-step-bracket generating. Given an orthonormal frame in Δ_m at a point m on Γ the system determines an affine subspace

$$\mathcal{A} = \left\{ M + \sum_{i=1}^{s} \lambda_i L_i \right\}$$

of dimension $s = p-1$ in the space of Ω of all skew-symmetric $k \times k$ matrices, which represent the external differentials of 1-forms which vanish on the distribution and take unit value on the velocity vector $\dot{\Gamma}$ of the curve Γ (the curve is always assumed to be transversal to the distribution). In particular, the one-forms corresponding to L_i vanish on $\dot{\Gamma}$. The invariant χ defined as

$$\chi = \min_{\lambda} \|M + \sum_i \lambda_i L_i\|. \tag{1.1}$$

plays a key role. It provides an estimation from below of the system entropy and in the strictly convex case it provides the precise estimation and simple optimal strategy.

The problem is *strictly convex* at the point $m \in \Gamma$ if there are two unit vectors from the distribution space at m whose Lie bracket at m (defined modulo the distribution) coincides with the velocity $\dot{\Gamma}/\Delta$ at m. The complexity C of an everywhere strictly convex problem is given by the simple basic formula

$$C = \frac{2}{\varepsilon^2} \int_{\gamma} \frac{dt}{\chi(t)} \tag{1.2}$$

The entropy in this case is proportional to the complexity with the factor 2π.

Generic one - step - bracket - generating systems with the codimension of the distribution less than 4 are everywhere strictly convex [2].

The example of a different behaviour of the entropy was described in [4]. This is the generic 4 dimensional distribution in 10 dimensional ambient space. The first brackets form a free nilpotent algebra at each point. There is another invariant

$\rho \in [0,1]$ of the affine subspace $\mathcal{A}$ at a point $m \in \Gamma$. The entropy is given by the formula

$$E(\varepsilon) = \frac{2\pi}{\varepsilon^2} \int\limits_{\gamma} \frac{(3 - |\rho(t)|)dt}{\chi(t)}. \tag{1.3}$$

In the present paper we show that the last formula provides estimations of the entropy for a rather wide range of systems. Generic systems with dimension 4 of the distribution and codimension $p = 4$ or 5 are one-step-bracket generating. We present normal forms of the families of matrices $\mathcal{A}$. They depend on several invariants. For $p = 5$ we calculate the χ value in terms of these invariants. However, the precise formula for the entropy requires long computations which can be replaced by a rough but easy inequality using a projection lemma. It relates the value of entropy for the initial system and a system with higher codimension and free 1-bracket algebra (treated in [4]).

It happens that the one-step-bracket-generating systems with 5 dimensional distribution are similar to that of 4-dimensional.

When the codimension $p = \frac{1}{2}k(k-1)$, the first-bracket extension $\Delta' = [\Delta, \Delta]$ of a generic system can fail to span the ambient space at some isolated points of Γ. We call these points *singular*. We prove that, generically, if singular points exist, then the leading term of the of the entire entropy asymptotics depends only on the neighbourhoods of singular points. It is equivalent to the sum over the set of singular points of terms proportional to $\frac{\ln \varepsilon}{\varepsilon^2}$.

The lowest dimensional case $k = 2,\ p = 2$ when second brackets are needed to span the ambient space was considered in [4]. The optimal interpolation happened to be provided by inflectional periodic elliptic curves (Euler "elastica").

Here we extend there results for the following $(2,3)$ and $(3,4)$ generic systems, including possible degenerate isolated points.

In the next section we list down all the known results mentioning the reference for complete proof and the description of the optimal synthesis.

Throughout the paper we use the normal coordinates and normal forms introduced in [1] to [9]. Those are useful generalizations to the sub-Riemannian case of geodesic coordinates in Riemannian geometry. We refer to these papers for complete settings and proofs.

In the papers [1] to [5] it was also shown that the existence of the following two types of singularities along Γ has no influence on the estimates (that is why we omit the discussion of these singularities in the paper):

1. Some invariants used in our estimates tend to infinity when Γ becomes tangent to the initial distribution or to some bracket extension. For instance, in the 3-space the points of tangency of a generic curve with a 2-distribution are unavoidable. In fact, the estimates in the vicinity of these points are negligible.

2. At some isolated points the invariants fail to be smooth, but our formulas still hold (see [1] to [5]). Therefore, almost everywhere in the paper we ignore this possible non-smoothness.

2. Preliminaries. List of known estimates.

Two functions $f_1(\varepsilon)$, $f_2(\varepsilon)$, tending to $+\infty$ when ε tends to zero, are called *weakly equivalent* ($f_1 \simeq f_2$) if $k_1 f_1(\varepsilon) \leq f_2(\varepsilon) \leq k_2 f_1(\varepsilon)$, for certain strictly positive constants k_1, k_2 and sufficiently small ε.

They are called it strongly equivalent ($f_1 \asymp f_2$) if $\lim_{\varepsilon\to 0} \frac{f_1(\varepsilon)}{f_2(\varepsilon)} = 1$.

We will write also $f_1 \succeq f_2$ if $\liminf_{\varepsilon\to 0} \frac{f_1(\varepsilon)}{f_2(\varepsilon)} \geq 1$.

Consider the space of all motion planning problems with given dimension k and codimension p of distribution, and assume that the system is generic (generic distribution along generic curve).

The dimensions of the flag of successive bracket extensions $\Delta \subset \Delta' \subset \dots$ at a generic point are determined uniquely by k and p. In particular, the following holds.

Prorosition 1. *Generically, the first brackets extension Δ'_m/Δ_m has maximal possible dimension $k_1 = \frac{1}{2}k(k-1)$, at each point $m \in \Gamma$, if $p > \frac{1}{2}k(k-1)$. For $p < \frac{1}{2}k(k-1)$, generically $\Delta'_m = \mathbb{R}^n$ at any m, and for $p = \frac{1}{2}k(k-1)$ $\Delta' m = \mathbb{R}^n$ for all $m \in \Gamma$ except may be for some isolated singular points of* Γ, *where* $\dim(\Delta''/\Delta) = 1$, *and* $\dim(\Delta'/\Delta) = \mathrm{p} - 1$.

Proof. The space Δ'/Δ is spanned by the brackets $[X_i, X_j]$ at m of basic vector fields from Δ. Each bracket determines a p-vector in $\mathbb{R}^n/\Delta$. So the rank of the $p \times k_1$ matrix B of these vectors is maximal outside the subvariety Σ of singular points of codimension c which is the product of coranks $c = 1 \times |p - k_1 - 1|$. Only for $p = k_1$ the codimension $c = 1$ and generic Γ intersects Σ. Clearly the intersection is transversal, the singular point is a regular point of the hypersurface Σ and after an appropriate linear transformation of the space $T_m\mathbb{R}^n/\Delta$ the matrix B takes the form

$$B = (t\mathbf{b}_1(t), \mathbf{b}_2(t), \dots, \mathbf{b}_{k_1-1}(t)) + O(t^2),$$

provided that

$$B_* = (\mathbf{b}_1(0), \mathbf{b}_2(0), \dots, \mathbf{b}_{k_1-1}(0))$$

is not degenerate. Again, by the genericity assumption the vector $\mathbf{b}_1(0) \in \Delta''$.

For similar reasons, degenerations of the dimensions of the higher flag components $\Delta \dots \subset \Delta^s \subset \dots$ can happen at isolated points of Γ only when p equals the sum of maximal possible dimensions of successive extensions Δ^s/Δ^{s-1}, $s = 1, \dots, r$. At these points the dimension of Δ^r/Δ^{r-1} drops by 1.

For example, the lowest case of this kind (that is a degeneratioin for non one-step-bracket generating), which is the only one considered in this paper, is $k = 2,\ p = 3$. Here $\dim(\Delta'/\Delta) = 1$, and $\dim(\Delta''/\Delta') = 2$.

In the section 4 below we show that generically the invariants of the systems used in the estimates are differentiable at singular points. The main lemma 3 of the section implies all the estimates involving singular points.

2.1. **Known entropy estimations**
. **1.** If $k=2,\ p=1$ we have generically a contact distribution, with maybe several singular Martinet points. This starting case was treated in [1, 10]. If there are no Martinet points on Γ, then the entropy estimation is

$$E(\varepsilon) \asymp \frac{4\pi}{\varepsilon^2}\int_\Gamma \frac{dt}{\chi(t)}. \tag{2.1}$$

In this case $\chi(t)$ is the modulus of non-zero eigenvalue of the skew-symmetric matrix M. In the presence of Martinet points $w_1,\dots,w_s$ the estimation has to be replaced by

$$E(\varepsilon) \asymp -\sum_{i=1}^{s} \frac{8\pi \ln\varepsilon}{|\chi'(w_i)|\varepsilon^2}. \tag{2.2}$$

2. If $k=2,\ p=2$ then generically no singular points arise, Δ' is three dimensional, Δ'' spans the ambient space, and

$$E(\varepsilon) \asymp \frac{2}{3\ \sigma\ \varepsilon^3}\int_\Gamma \frac{dt}{\delta(t)}.$$

where σ denotes a certain universal constant, $\sigma \approx 0.00580305$, and the invariant $\delta(t)$ was defined in [4] as follows: A canonical 3-frame which consists of the normalized abnormal vector field in the distribution, its orthonormal and their commutator determines a metric g' on Δ'. The one-form γ vanishing on Δ' and taking value 1 on $\dot\Gamma$ corresponds to the following skew-symmetric endomorphisms of $\Delta'(\Gamma(t))$:

$$< \hat{A}(t)X, Y >_{g'} = d\gamma(X,Y) = \gamma([X,Y]), \quad \forall X, Y \in \Delta'(\Gamma(t)).$$

We set:

$$\delta(t) = ||\hat{A}(t)||_{g'},\ \forall t \in [0,1].$$

See [4] for the proof and description of optimal synthesis.

3. If $k=2,\ p=3$ and there are no singular points on Γ then

$$E(\varepsilon) \asymp \frac{2}{3\ \sigma\ \varepsilon^3}\int_\Gamma \frac{dt}{\gamma_1(t)}.$$

The invariant $\gamma_1(t)$ is defined in the section 3 below, where the proofs and the description of optimal synthesis, reducing the problem to certain $2,2$-problem are also given.

If still $k=2, p=3$, but there are singular points $w_1,\dots,w_r$ on Γ then

$$E \asymp \sum_{i=1}^{r} -\frac{4\pi}{3\sigma\gamma_1'(w_i)}\frac{\ln\varepsilon}{\varepsilon^3}$$

4. If $k=3$, and p is either 1 or 2, the generic distribution is one-step-bracket generating and has no singular points. The formula (2.1) is valid.

5. For $k=3$, and $p=3$, then isolated singular points are possible. Without them the generic distribution is one-step-bracket generating and the formula (2.1) is still valid. Respectively in the presence of singular points $w_1,\dots,w_r$ the estimation is given by the logarithm formula (2.2).

6. If $k = 3$, and $p = 4$, then for a generic distribution Δ, the distribution Δ'' spans the ambient space (no singular points are possible). The following inequality holds

$$\frac{2}{3\ \sigma\ \varepsilon^3}\int_\Gamma \frac{dt}{\delta_a(t)} \preceq E \preceq \frac{2}{\sigma\ \varepsilon^3}\int_\Gamma \frac{dt}{\delta_a(t)},$$

where δ_a is an invariant of an associated $(2,2)$ problem defined in the section 8.

7. If $k \geq 4$, and $p \leq 3$ the generic problem is always strictly convex one-step bracket generating (see [2]). The formula (2.1) is valid.

8. If $k = 4$, and p is either 4, or 5, the generic problem is one-step bracket generating, but not always strictly convex. The inequality

$$\frac{4\pi}{\varepsilon^2}\int_\Gamma \frac{dt}{\chi(t)} \preceq E(\varepsilon) \preceq \frac{6\pi}{\varepsilon^2}\int_\Gamma \frac{dt}{\chi(t)} \tag{2.3}$$

is proven in section 5. Moreover, for $p = 5$ this section contains an explicit formula for χ in terms of the normal form of the affine family $\mathcal{A}$ of skew-symmetric matrices.

9. The case $k = 4$, and $p = 6$ without singular points was considered in [4], and the invariant $\rho \in [0,1]$ of the normal form of the family $\mathcal{A}$ was introduced. Then the asymptotics takes the form

$$E(\varepsilon) \asymp -\frac{2\pi}{\varepsilon^2}\int_\Gamma \frac{(3-|\rho(t)|)dt}{\chi(t)}.$$

Consequently, the inequality 2.3 also holds. According to logarithm lemma 3, the estimation is replaced by

$$E(\varepsilon) \asymp -\sum_{i=1}^{r} \frac{4\pi\ln\varepsilon}{\varepsilon^2}\frac{(3-|\rho(w_i)|)}{\chi'(w_i)},$$

if singular points $w_1, \ldots, w_r$ arise on the program curve.

10. For $k = 5$ and $4 \leq p \leq 10$ as well as for $k > 5$ and $4 \leq p \leq 8$ generic systems are one-step-bracket generating and, as it is shown in sections 6,7 the inequality 2.3 still holds. Even for $k = 5,\ p = 10$ when the singular points $w_1, \ldots, w_r$ appear similar inequality holds

$$-\sum_{i=1}^{r} \frac{8\pi\ln\varepsilon}{\varepsilon^2|\chi'(w_i)|} \preceq E(\varepsilon) \preceq -\sum_{i=1}^{s} \frac{12\pi\ln\varepsilon}{\varepsilon^2|\chi'(w_i)|}$$

Finally, notice that the inequality

$$\frac{4\pi}{\varepsilon^2}\int_\Gamma \frac{dt}{\chi(t)} \preceq E(\varepsilon)$$

always holds for an one-step-bracket generating system [1].

3. Normal form, invariants and entropy in the (2,3) case

Consider a 2-distribution $\Delta = \{F, G\}$ in $\mathbb{R}^5$ determined by an orthonormal frame of two vector fields $F,\ G$. Assume that $F,\ G,\ [F,G]$ has rank 3 everywhere along the generic curve Γ (according to "product of coranks" theorem the set where this does not hold has generically codimension 3).

Assume also $\Delta'' = [\Delta, \Delta']$ has rank 5 at all points of Γ. Generically it can vanish at some isolated points of Γ. Then, as it can be easily seen, the Logarithm lemma from the previous section can be applied.

Lemma 1. *Through each point of* Γ *there is a unique (up to reversing the orientation) arclength parametrized abnormal curve meeting the transversality conditions of the Pontryagin Maximum Principle with respect to* Γ.

Proof. Denote the adjoint vector of the system by P. The conditions for abnormalily and transversality to Γ are

(3.1)
$$P_0F = P_0G = 0,\ P_0[F,G] = 0,\ P_0\dot{\Gamma} = 0,\ P_0[F,[F,G]]u_0 + P_0[G,[F,G]]v_0 = 0,$$

$$u_0^2 + v_0^2 = 1.$$

This implies, since P_0 is non-zero,

$$\det\{\dot{\Gamma}, F, G, [F,G], [F,[F,G]]\}u_0 + \det\{\dot{\Gamma}, F, G, [F,G], [G,[F,G]]\}v_0 = 0, \tag{3.2}$$

By the genericity assumptions there is a unique (up to a sign) solution and a unique $p_0 \in \mathbb{R}P^4$ meeting (3.1).

Now for small arclength s the equations

$$P[F,[F,G]]u + P[G,[F,G]]v = 0, \quad u^2 + v^2 = 1$$

have a unique feedback solution $u(P,x),\ v(P,x)$.

Then the Cauchy problem:

$$\begin{cases} \frac{dx}{ds} &= F(x)u(p,x) + G(x)v(p,x), \\ \frac{dp}{ds} &= -p\frac{dF(x)}{dx}u(p,x) - p\frac{dG(x)}{dx}v(p,x), \\ & x(0) = x_0 \in \Gamma(t), \end{cases}$$

where p_0 is as above, has a unique smooth solution for s small enough. Along this solution $PF = 0,\ PG = 0, P[F,G] = 0$ by construction. The curve Γ being compact, we can take a uniform bound s_0 valid for all points of Γ. This ends the proof.

Choose any smooth vector field F from the distribution having the abnormals of lemma 1 as trajectories, and take its orthonormal G in Δ. Clearly, G is defined by F up to a sign. Flows of vector fields $H = [F,G],\ I = [F[F,G]],\ J = [G[F,G]]$ determine the 3 dimensional parametrized surface

$$S(y,z,w) = \exp(zJ) \circ \exp(yI)(\Gamma(w)),$$

which is transversal to the distribution.

Choose the normal coordinates with respect to S (in the sense of the papers [1] to [9]) meeting the extra conditions along $\Gamma \subset S$:

$$F(w) = \frac{\partial}{\partial x_1}, \quad G(w) = \frac{\partial}{\partial x_2}.$$

Straightforward computations (see also, [4]) show now that inside the ε tube around Γ the system takes the following form in these coordinates:

$$(3.3)\qquad \begin{cases} \dot x_1 &= u + O(\varepsilon^2) \\ \dot x_2 &= v + O(\varepsilon^2) \\ \dot y &= \frac{x_2}{2}u - \frac{x_1}{2}v + O(\varepsilon^2) \\ \dot z &= (\beta_1 x_1 + \beta_2 x_2)\left(\frac{x_2}{2}u - \frac{x_1}{2}v\right) + O(\varepsilon^3) \\ \dot w &= (\gamma_1 x_1 + \gamma_2 x_2)\left(\frac{x_2}{2}u - \frac{x_1}{2}v\right) + O(\varepsilon^3). \end{cases}$$

Calculating the leading terms of the commutators, we get $I = -\frac{3}{2}\beta_1\frac{\partial}{\partial z} - \frac{3}{2}\gamma_1\frac{\partial}{\partial w}$, $J = -\frac{3}{2}\beta_2\frac{\partial}{\partial z} - \frac{3}{2}\gamma_2\frac{\partial}{\partial w}$ up to terms of higher order. The vector field F corresponds to $u = 1, v = 0$ and produces the abnormals, hence $\beta_1 = 0,\ \beta_2 = -\frac{2}{3},\ \gamma_2 = 0.$

Therefore, up to higher order terms, we have

$$\begin{cases} F &= \frac{\partial}{\partial x_1} + \frac{x_2}{2}\frac{\partial}{\partial y} - \frac{x_2^2}{3}\frac{\partial}{\partial z} + \frac{\gamma_1 x_1 x_2}{2}\frac{\partial}{\partial w}, \\ G &= \frac{\partial}{\partial x_2} - \frac{x_1}{2}\frac{\partial}{\partial y} + \frac{x_1 x_2}{3}\frac{\partial}{\partial z} - \frac{\gamma_1 x_2^2}{2}\frac{\partial}{\partial w}, \end{cases}$$

or equivalently

$$(3.4)\qquad \begin{cases} \dot x_1 &= u \\ \dot x_2 &= v \\ \dot y &= \frac{x_2}{2}u - \frac{x_1}{2}v \\ \dot z &= -\frac{2}{3}x_2\left(\frac{x_2}{2}u - \frac{x_1}{2}v\right) \\ \dot w &= \gamma_1 x_1\left(\frac{x_2}{2}u - \frac{x_1}{2}v\right). \end{cases}$$

Let ω be the unique (along Γ) one-form such that $\omega(\Delta') = 0,\ \omega(\dot\Gamma) = 1,\ \omega(J) = 1$.

Then $d\omega|_{\Delta'} = dx_2\wedge dy - \frac{3}{2}\gamma_1 dx_1\wedge dy$, hence $-\frac{2}{3}d\omega(F,[F,G]) = \gamma_1 = \omega([F[F,G]])$.

Lemma 2. *The values of $\omega([F,[F,G]])$ at Γ do not depend on the choice of F apart from the abnormal trajectories.*

Proof. For another choice $\widetilde F = F + \alpha F + \beta G,\ \widetilde G = G + \gamma F + \delta G$ we get

$$\omega([\widetilde F,[\widetilde F,\widetilde G]]) = d\omega(\widetilde F,[\widetilde F,\widetilde G]) + [\widetilde F,\widetilde G]\omega(\widetilde F) - \widetilde F\omega([\widetilde F,\widetilde G]),$$

where the last two terms vanish. Clearly, $[\widetilde F,\widetilde G] = [F,G](1+\alpha)(1+\delta) + H$, where $H \in \Delta$, and also $[\widetilde F,[\widetilde F,\widetilde G]] = [F(1+\alpha)+\beta G,\ [F,G](1+\alpha)(1+\delta)+H] = [F,[F,G]](1+\alpha)^2(1+\delta) + [G,[F,G]]\beta(1+\alpha)(1+\delta) + \tilde H$, where $\tilde H \in \Delta'$, and $\beta|_\Gamma = 0$.

Hence, $\omega([\widetilde F,[\widetilde F,\widetilde G]]) = \omega([F,[F,G]])$ at the points of Γ, as required.

So reparamerizing z and therefore the surface S, the lowest order terms in normal form of the system become as follows

$$\text{(3.5)} \qquad \begin{cases} \dot{x}_1 &= u \\ \dot{x}_2 &= v \\ \dot{y} &= \frac{x_2}{2}u - \frac{x_1}{2}v \\ \\ \dot{z} &= x_2\left(\frac{x_2}{2}u - \frac{x_1}{2}v\right) \\ \\ \dot{w} &= \gamma_1 x_1\left(\frac{x_2}{2}u - \frac{x_1}{2}v\right). \end{cases}$$

Notice, that the normal form 3.5 differs (after an appropriate reparametrization of Γ) from that for the "car with a trailer" system from [4]

$$\text{(3.6)} \qquad \begin{cases} \dot{x}_1 &= u \\ \dot{x}_2 &= v \\ \dot{y} &= \frac{x_2}{2}u - \frac{x_1}{2}v \\ \\ \dot{w} &= x_1\left(\frac{x_2}{2}u - \frac{x_1}{2}v\right). \end{cases}$$

only by an extra constraint equation for z.

Prorosition 2. *The optimal ε-interpolation for 3.5 is given by the same syntesis ("periodic elastica") as for (3.6) involving Jacobi elliptic functions (see* [4]*):*

$$v = 1 - 2\ \mathrm{dn}\left(K\left(1 + \frac{4t}{e}\right)\right)^2,$$

$$u = -2\ \mathrm{dn}\left(K\left(1 + \frac{4t}{e}\right)\right)\mathrm{sn}\left(K\left(1 + \frac{4t}{e}\right)\right)\sin(\frac{\varphi_0}{2}),$$

where K is the quarter period of the elliptic functions such that $2\ \mathrm{Eam}(K) = K$, *and approximately* $\varphi_0 = 130^o$.

Proof. Since the periodic elastica l provides the maximum of $w|_\varepsilon$ satisfying the constrains $y\,|_0\ = y|_\varepsilon = 0$, it will be sufficient to show that for this solution the equality $z|_\varepsilon - z|_0 = 0$ also holds. We have

$$z|_\varepsilon - z_0 = \int_l x_2 dy = yx_2\,|_0^\varepsilon - \int_l y dx_2 = -\int_0^\varepsilon yvdt$$

$$= -\int_0^\varepsilon y\left(1 - 2dn\left(K\left(1 + \frac{4t}{\varepsilon}\right)\right)\right)^2 dt = 0$$

since the elliptic function has period $\frac{\varepsilon}{2}$, is even with respect to the point $\frac{\varepsilon}{2}$, and the the function $y(t)$ is odd with respect to the same point.

4. Logarithmic singular points for the estimates

In this section we prove a lemma, which provides the entropy estimation in presence of singular points on Γ. The lemma is a strong - equivalence analog of the result from [12], which deals with weak equivalence only. Obviously, similar results hold in more general setting. However, for the sake of shortness, we state and prove the lemma only for one-step-bracket generating and $(2,3)$ cases.

The proposition 1 and our normal form techniques (see [1] to [7]) imply the following

Prorosition 3. *If for a generic system with $p = k_1$ a point $m \in \Gamma$ is singular then in some normal coordinates (see* [2, 4]*) the nilpotent approximation of the system near $m = \{w = 0\}$ in the ε-tube around Γ has the form:*

$$\left\{\begin{array}{rcl} \dot{x}_i &=& u_i + O(\varepsilon^2), \quad i = 1, \ldots, k \\ \dot{y}_j &=& u^* L_j(w)x + O(\varepsilon^2), \quad j = 1, \ldots, k_1 - 1 \\ \dot{w} &=& u^*(wM(w))x + O(\varepsilon^3). \end{array}\right. \tag{4.1}$$

Here $M(t), L_j(t)$ are skew-symmetric matrices smoothly depending on t with values at the origin spanning the space Ω, and $*$ means the transposition.

In the $(2,3)$ case

Prorosition 4. *If for a generic system with $k = 2,\ p = 3$ a point $m \in \Gamma$ is singular then in some normal coordinates the nilpotent approximation of the system near $m = \{w = 0\}$ in the ε-tube around Γ has the form:*

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& u + O(\varepsilon^2) \\ \dot{x}_2 &=& v + O(\varepsilon^2) \\ \dot{y} &=& \frac{x_2}{2}u - \frac{x_1}{2}v + O(\varepsilon^3) \\ \dot{z} &=& -\frac{2}{3}x_2\left(\frac{x_2}{2}u - \frac{x_1}{2}v\right) + O(\varepsilon^4) \\ \dot{w} &=& (w\gamma_1(w)x_1 + a(w)x_2)\left(\frac{x_2}{2}u - \frac{x_1}{2}v\right) + O(\varepsilon^4). \end{array}\right. \tag{4.2}$$

Here $\gamma_1(w), a(w)$ are smooth functions in w, $\gamma_1(0) \neq 0$.

These normal forms and the definitions of the system invariants [2, 4] imply the following.

Corollary 1. *For a generic singular point there are well defined derivatives of the invariants $\chi(w)$, in the first bracket generating case, and γ_1 in $(2,3)$ case. Moreover in the first bracket generating case when $k = 4$ there is a well defined limit for $w \to 0$ of the invariant $\rho(w)$.*

Lemma 3. *In the above generic cases the entropy asymptotic of the piece of $\Gamma : [-1,1] \mapsto \mathbb{R}^n$ containing a single singular point at the origin is given by:*

$$E \asymp -\frac{8\pi \ln \varepsilon}{|\chi'(0)|\varepsilon^2}$$

- in the one step bracket generating case with $k = 2$, or $k = 3$;

$$E \asymp -\frac{4\pi(3-2|\rho(0)|)\ln\varepsilon}{|\chi'(0)|\varepsilon^2}$$

- in the one step bracket generating case with $k=4$; *and, finally,*

$$E \asymp -\frac{4\pi\ln\varepsilon}{3\sigma|\gamma_1'(0)|\varepsilon^3}$$

in the $(2,3)$ *case.*

Proof. Consider a half of the curve $\Gamma^+[0,1]\to\mathbb{R}^n$. Set $s=2$ in the one-step-bracket generating case, and $s=3$ for $2,3$ case. Introduce a metric on Γ, say, determined by parameter $t\in[0,1]$. Split Γ into two segments $S_0=\{t: t\in[0,\varepsilon]\}$ and the remaining piece S_1 and estimate the entropy separetely along each of them.

The space $\Delta^{(s)}$ coincides with the ambient tangent space at the origin and hence in its vicinity. Hence, by the sub-Riemannian ball-box theorem ([11, 12, 13]) there is a constant K_0 such that the ε-sub-Riemannian ball centered at the origin contains the $K_0\varepsilon^s$ ordinary Riemannian ball around the origin. Take now a value $\varepsilon_0>0$ such that for any $t\in[0,\varepsilon_0]$ and for any $\varepsilon\le\varepsilon_0$ the $\frac{K}{2}\varepsilon^s$-Riemannian ball centered at $\Gamma(t)$ is contained in the ε-sub-Riemannian ball around the point $\Gamma(t)$. For arbitrary $\varepsilon\le\varepsilon_0$ dividing the segment $[0,\varepsilon]$ into $\frac{2}{K}\varepsilon^{1-s}$ pieces, each of the length $\frac{K}{2}\varepsilon^s$, there exist an admissible curve ε-interpolating the segment $t\in[0,\varepsilon]$ of Γ of total entropy $\frac{2}{K}\varepsilon^{1-s}$.

Now we will estimate the entropy over segment S_1 similarly to the proof of Lemma 4 in [4].

Divide the segment $S_1[\varepsilon,1]$ into pieces of length ε. For arbitrary piece $\theta=[t_j,t_{j+1}]$ besides the genuine system $\mathcal{S}$ defined by normal form from proposition 3 (respectively, 4) with $w=t\in\theta$ consider an approximate system $\mathcal{S}_*$ defined by the same normal form 3 or 4 with the coefficients of all terms (like L_j, γ_1, etc.) being fixed at the boundary instant t_j. Take an ε-interpolation of θ by a sequence of admissible arcs g_i with end-points on Γ and of sub-Riemannian length ε of one of these systems. An admissible arc $\widetilde{g}_i$ of the other system with the the same initial data at the left end-point and the same control as g_i deviates from the right end-point of g_i by the vector δ_i whose normal form coordinate components are uniformly bounded by $\|x\|\le c_1\varepsilon^2$, $\|y\|\le c_2\varepsilon^3$, $\|w\|\le c_3\varepsilon^3$ in the one-step-bracket generating case, and by $\|x\|\le c_4\varepsilon^2$, $\|y\|\le c_5\varepsilon^3$, $\|w\|\le c_6\varepsilon^4$ in the $(2,3)$ case. Here $c_1,\dots,c_6$ -are certain constants uniform for the entire Γ. Applying the sub-Riemannian ball-box theorem adapted to the normal coordinates, the end-points of δ_i can be joined by an auxiliary admissible arc $\widetilde{\delta}_i$ of the length $c\varepsilon^{1+q}$ with some positive constants q (in our cases $q=1/4$) and c. The system of combined arcs $\widetilde{g}_i$ and $\widetilde{\delta}_i$ forms a $\varepsilon(1+c\varepsilon^q)$ interpolation of θ with the same number of segments but admissible with respect to the other system.

The entropy of the system $\mathcal{S}_*$ restricted to each θ is provided by either by the expression

$$E_{\theta,\mathcal{S}_*}\asymp(\varepsilon)\frac{4\pi|t_{j+1}-t_j|}{|\chi(\theta)|\varepsilon^2}\quad\text{either by}\quad\asymp\frac{2\pi(3-|\rho(\theta)|)|t_{j+1}-t_j|}{|\chi(\theta)|\varepsilon^2}$$

in the one-step-bracket generating case (see [2, 4]) or by

$$E_{\theta,\mathcal{S}_*}\asymp\frac{2\pi|t_{j+1}-t_j|}{3\sigma|\gamma_1(\theta)|\varepsilon^2}$$

in $(2,3)$ case (section 3). So, summing up over all θ-pieces we get either

$$E_{\mathcal{S}_*}(\varepsilon) \asymp \int\limits_{\varepsilon}^{1} \frac{4\pi dt}{|\chi(t)|\varepsilon^2}$$

or the other respective integral in the other cases. This implies either

$$E_{\mathcal{S}_*}(\varepsilon) \asymp -\frac{4\pi \ln \varepsilon}{|\chi'(0)|\varepsilon^2},$$

or another respective appropriate expression.

In fact, the inverse of the integrand which we denote by $f(t)$, is smooth and vanishes at the origin. Hence in some vicinity of the origin $f(t) = f'(0)t(1 + th(t))$ and $\frac{1}{f(t)} = \frac{1}{f'(0)t} - \widetilde{h}(t)$ with some smooth functions $h(t)$, $\widetilde{h}(t)$. So,

$$\int\limits_{\varepsilon}^{1} \frac{dt}{f(t)} = \int\limits_{\varepsilon}^{1} \frac{dt}{f'(0)t} - \int\limits_{\varepsilon}^{1} \frac{\widetilde{h}(t)dt}{f'(0)} \asymp -\frac{\ln \varepsilon}{f'(0)},$$

when $\varepsilon \to 0$.

On the other hand, the constructed systems of arcs show (similarly to Lemma 4 from [4]) that $E_{\mathcal{S}_*}(\varepsilon(1 + k\varepsilon^q)^2) \succeq E_{\mathcal{S}}(\varepsilon(1 + c\varepsilon^q)) \succeq E_{\mathcal{S}_*}(\varepsilon)$. Hence $E_{\mathcal{S}}(\varepsilon) \asymp E_{\mathcal{S}_*}(\varepsilon)$. Combining this with the estimates over S_0, which is relatively negligible, and multiplying by 2 for entire Γ, crossing the singular hypersurface, the required estimates hold.

5. Generic one-step-bracket generating cases with $k = 4$ and $p < 6$

5.1. **Normal forms.** Denote by $P_y = \mathbb{R}\{L_1, \ldots, L_s\}$ and by $P = \mathbb{R}\{M, L_1, \ldots, L_s\}$ the vector subspaces associated to $\mathcal{A}$. These matrices are defined up to a linear transformation G of the space Ω of $k \times k$ skew symmetric matrices, which preserve $\mathcal{A}$, and an orthogonal transformation U of the distribution Δ. This pair acts on $\{M, L_i\}$ as follows:

$$M \mapsto G(U^{-1}MU), \qquad L_i \mapsto G(U^{-1}L_iU).$$

A normal form of the system is an appropriate choice of the matrices from the orbit of the action.

We describe now normal forms of the families $\mathcal{A}$ when the distribution is four dimensional. We will use standard representation of a skew-symmetric 4×4 matrice as a sum of pure quaternions, generated by i, j, k :

$$i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and anti-quaternions generated by $\hat{\imath}, \hat{\jmath}, \hat{k}$, with:

$$\hat{\imath} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \hat{\jmath} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \hat{k} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Recall that, skew-symmetric matrices, which have double eigenvalues form two 3-dimensional mutually transversal subspaces Q (of pure quaternions) and $\hat{Q}$ of (anti-quaternions).

The case when $p = 6$ is maximal possible, that is when the first brackets form a free nilpotent algebra, was treated in [4]. In particular, the normal form is $M = i + \rho\widehat{i}, \quad \{L_i\} = \{\widehat{i}, \ , j, \ \widehat{j}, \ k, \ \widehat{k}\}$ where the invariant ρ satisfies $0 \leq \rho \leq 1$.

Prorosition 5. *The normal form of a generic system with $k = 4, \quad p = 5$ is*

$$M = \alpha\widehat{i} + \beta\widehat{j}; \ \{L\} = \{i + \rho_1\widehat{i}, \ j + \rho_2\widehat{j}, \ k, \ \widehat{k}\} \tag{5.1}$$

with $\alpha, \beta, \rho_1, \rho_2 \in \mathbb{R}$.

Proof. The intersections of P_y^4 with each of Q and $\hat{Q}$ are at least one-dimensional. Up to an appropriate action of the orthogonal group of Δ which splits into the product of the conjugations with quaternions and with anti-quaternions, we can always assume, that $P_y^4 \cap Q$ contains the vector k, and $P_y^4 \cap \widehat{Q}$ contains $\widehat{k}$.

Now the two-dimensional factor-space $P_y^4/\mathbb{R}\{k, \widehat{k}\}$ is spanned by two operators

$$L_1^* = a_{11}i + a_{12}j + b_{11}\widehat{i} + b_{12}\widehat{j},$$

$$L_2^* = a_{21}i + a_{22}j + b_{21}\widehat{i} + b_{22}\widehat{j}$$

with some constants $a_{i,j}, \quad b_{i,j}$ which we organize into two matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \text{ and } B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

An orthogonal transformation O_1 of Δ which preserves $k, \widehat{k}, \widehat{i}, \widehat{j}$ acts as a rotation in i, j plane. Denote by O_2 a similar rotation of $\widehat{i}, \widehat{j}$ plane. A change of basis in $\mathbb{R}\{L_1^*, L_2^*\}$ acts on A, B as the multiplication from the left by a non-degenerate 2×2 matrix V. Hence, the triple acts on A, B as $A, \ B \mapsto VAO_1, \ VBO_2$.

Assume the rank of A is 2. Due to the condition that L_1^*, L_2^* are independent, the other cases are similar or trivial.

Represent the matrix $A^{-1}B$ as the product $A^{-1}B = O_1 \mathrm{Diag} O_2^{-1}$ with some diagonal matrix Diag and orthogonal $O_1, \ O_2$.

Take $V = O_1^{-1}A^{-1}$, then $VAO_1 = \mathrm{Id}$, and $VBO_2 = O_1^{-1}A^{-1}BO_2 = \mathrm{Diag}$, and get the required normal form.

Clearly, the values of $\alpha, \beta, \rho_1, \rho_2$ are invariants of the system. In fact, modulo P_y, the matrix M takes the form $M = \alpha\widehat{i} + \beta\widehat{j}$.

Remark 1. *Similar arguments justify the following normal form for the generic system with $k = 4, \quad p = 4$:*

$$M = \alpha\widehat{i} + \beta\widehat{j} + \gamma\widehat{k},$$

$$\{L_i\} = \left\{ i + \rho_1\widehat{i}; \quad j + \rho_2\widehat{j}; \quad k + \rho_3\widehat{k}; \right\}.$$

5.2. **Entropy exponent estimations for $k = 4,\ p = 5$.** According to the normal form from the previous subsection up to a multiplication of M by a scalar factor we can take

$$M = \cos\theta\widehat{i} + \sin\theta\widehat{j},\quad L_1 = i + \rho_1\widehat{i},\quad L_2 = j + \rho_2\widehat{j},\quad L_3 = k,\quad L_4 = \widehat{k}.$$

The formula

$$\|S\| = \left(x^2 + y^2 + z^2\right)^{\frac{1}{2}} + \left(\widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2\right)^{\frac{1}{2}}$$

for the norm of the matrix $S = xi + yj + zk + \widehat{x}\widehat{i} + \widehat{y}\widehat{j} + \widehat{z}\widehat{k}$

implies that the entropy exponent takes the form

$$\chi = \min_{\lambda}\left(\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)^{\frac{1}{2}} + \left((\cos\theta + \rho_1\lambda_1)^2 + (\sin\theta + \rho_2\lambda_2)^2 + \lambda_4^2\right)^{\frac{1}{2}}\right) \tag{5.2}$$

Prorosition 6. *The system is strictly convex if and only if the values of $\rho_1,\ \rho_2$ satisfy the inequality*

$$\frac{\rho_1^2 - 1}{1 - \rho_2^2} > 0$$

(the unit separates the squares). In this case the value of χ is

$$\chi = |\cos(\theta + \xi)|$$

where $\tan\xi = \left(\frac{\rho_1^2 - 1}{1 - \rho_2^2}\right)^{\frac{1}{2}}$.

In the non-strictly convex case (when the inequality fails) the value of χ is the maximum of 1 *and* $\frac{\cos^2\theta}{\rho_1^2} + \frac{\sin^2\theta}{\rho_2^2}$.

Proof.

Clearly, to minimize the expression (5.2) we have to set $\lambda_3 = \lambda_4 = 0$.

The function

$$N = \left(\lambda_1^2 + \lambda_2^2\right)^{\frac{1}{2}} + \left((\cos\theta + \rho_1\lambda_1)^2 + (\sin\theta + \rho_2\lambda_2)^2\right)^{\frac{1}{2}}$$

fails to be smooth at $\lambda_1 = \lambda_2 = 0$ with the value $N_1 = 1$, and at $\lambda_1 = -\frac{\cos\theta}{\rho_1},\ \ \lambda_2 = -\frac{\sin\theta}{\rho_2}$ with the value $N_2 = \frac{\cos^2\theta}{\rho_1^2} + \frac{\sin^2\theta}{\rho_2^2}$. Outside these points differentiation of N with respect to λ_1 and λ_2 provides a smooth critical point $(\lambda_1^*,\ \lambda_2^*)$ given by the formulas

$$\lambda_1^* = \cos\theta\frac{c^*\rho_1}{1 - c^*\rho_1^2},\quad \lambda_2^* = \sin\theta\frac{c^*\rho_2}{1 - c^*\rho_2^2}, \tag{5.3}$$

where

$$c^* = \frac{1 - \left(\frac{\rho_2^2 - 1}{1 - \rho_1^2}\right)^{\frac{1}{2}}\tan\theta}{\rho_2^2 - \left(\frac{\rho_2^2 - 1}{1 - \rho_1^2}\right)^{\frac{1}{2}}\rho_1^2},$$

which hold in some subset D in (ρ_1, ρ_2).

The respective value of N takes the form

$$N_* = \left((\lambda_1^*)^2 + (\lambda_2^*)^2\right)^{\frac{1}{2}} + \left(\left(\frac{\lambda_1^*}{c^*\rho_1}\right)^2 + \left(\frac{\lambda_2^*}{c^*\rho_2}\right)^2\right)^{\frac{1}{2}} = |\cos(\theta + \xi)|$$

where $\tan\xi = \left(\frac{\rho_1^2 - 1}{1 - \rho_2^2}\right)^{\frac{1}{2}}$.

Recall that the system is strictly convex if and only if there exist a pair of mutually orthogonal vectors x, u such that $xMu = \chi,\ xL_iu = 0$, that is the bracket $[x, u]$ is co-linear to the tangent vector $\dot{\Gamma}$ of the program curve.

The required vectors $u = (u_1, \dots, u_4),\quad x = (x_1, \dots, x_4)$ satisfy the equations $xku = x\widehat{k}u = 0$, which provide $u_1x_4 - u_4x_1 = u_2x_3 - u_3x_2 = 0$. Hence, $u_4 = au_1,\ x_4 = ax_1,\ u_3 = bu_2,\ x_3 = bu_2$ for some constants a, b except for some obvious cases (e.g., $u_1 = 0, x_1 = 0$) which can be treated similarly.

Orthogonality $u \cdot x = 0$ and conditions $\|u\| = \|x\| = 1$ imply

$$x_1 = -u_2\left(\frac{1+a^2}{1+b^2}\right)^{\frac{1}{2}},\quad x_2 = u_1\left(\frac{1+a^2}{1+b^2}\right)^{\frac{1}{2}}.$$

Finally, the equations $x(i + \rho_1\widehat{i})u = 0$ and $x(j + \rho_2\widehat{j})u = 0$ imply that

$$\frac{(1+\rho_1)(1+\rho_2)}{(1-\rho_2)(\rho_1-1)} \geq 0.$$

The latter implies that the unit lies between ρ_1^2 and ρ_2^2 as required.

Calculating the values of

$$a^2 = \frac{(1+\rho_1)(1+\rho_2)}{(1-\rho_1)(\rho_2-1)},\quad b^2 = \frac{(1-\rho_1)(1+\rho_2)}{(1+\rho_1)(\rho_2-1)},$$

a sequence of nice simplifications yields $\chi = xM^*u = |\cos(\theta + \xi)|$ which proves the proposition.

Remark 2. *If both $\rho_i^2,\ i = 1, 2$ are either greater or smaller than 1, the value χ is either 1 or $\frac{\cos^2\theta}{\rho_1^2} + \frac{\sin^2\theta}{\rho_2^2}$. If all the invariants are constant along Γ the optimal synthesis in this case can be found solving maximization problem with 4 constrains (similar to the case $k = 4,\ p = 6$, solved in* [4]*). However, to get easy practically reasonable estimation of the entropy exponent it will be sufficient to apply the projection Lemma 4 from the following section.*

6. Generic systems with $k = 5$

Prorosition 7. *A generic system with $k = 5$, and $p = 10$ which corresponds to a free nilpotent algebra of 1-brackets on 5-dimensional distribution has the following normal form:*

$$M = \alpha\hat{i};\quad \{L_i\} = \{i + \rho\hat{i},\ \hat{j},\ j,\ \hat{k},\ k,\ dx_s \wedge dx_5,\ s = 1, 2, 3, 4\}.$$

Here we write any skew-symmetric form on $\mathbb{R}^5$ as a linear combination $\sum a_{ij}dx_i \wedge dx_j$ of external two-forms on the tangent space equipped with some orthogonal coordinates $x_1, \dots, x_5$. To underline the relation with the case $k = 4$ we keep the quaternion notations for the following forms in $x_1, \dots x_4$ only: $i = dx_2 \wedge dx_1 + dx_4 \wedge dx_3,\quad \hat{i} = dx_2 \wedge dx_1 - dx_4 \wedge dx_3$, etc.

Proof. The hyperplane P_y which in this case has dimension 9 is determined by a single linear equation on the coefficients a_{ij} of the form

$$P_y = \{a_{ij} : \sum_{i,j} a_{ij}p_{ij} = 0\}.$$

Here p_{ij} form a skew-symetric set of constants. Let $b = (b^1, \dots, b^5)$ be a kernel vector of the skew-symmetric matrix (p_{ij}):

$$p_{ij} b^j = 0, \quad i = 1, \dots, 5.$$

The orthogonal change of coordinates $(x_i) \mapsto (x'_i)$ such that $x'_5 = \sum_j b^j x_j$ has the following property. For an arbitrary linear form $G(x'_1, \dots, x'_4)$, the external product $dG \wedge dx'_5$ belongs to P_y. In fact, $dx'_5 \wedge dx_j = b^j dx_i \wedge dx_j$. Hence the coefficients a_{ij} of the forms $\omega_k = dx'_5 \wedge dx_k$ are

$$a_{ij}(\omega_k) = b^i \delta^j_k,$$

and $\sum_{ij} a_{ij}(\omega_k) p_{ij} = \sum_i b^i p_{ik} = 0$, as required.

The projection of P_y to the space of 4×4 skew-symmetric matrices in $x'_1, \dots, x'_4$ along the subspace spanned by the forms $dx'_5 \wedge dx'_i$ is a hyperplane in the 6-dimensional space. Now the normal form reduction obtained for matrices in four variables provides

$$M = a\widehat{i},\ L_1 = i + \rho\widehat{i},\ L_2 = j,\ L_3 = \widehat{j},\ L_4 = k,\ L_5 = \widehat{k},$$

together with $L_6 = dx_5 \wedge dx_1,\ \dots,\ L_9 = dx_5 \wedge dx_4$. The proof is complete.

Remark 3. *Using the normal coordinates along Γ associated with the coordinates from Proposition 7 the entropy estimation reduces to the problem with $k = 4$ dimensional distribution only.*

In fact, in these coordinates the nilpotent approximation splits into a subsystem which depend only on $x' = (x_1, \dots, x_4)$, and $u' = (u_1, \dots, u_4)$ and further 5 constraint equations which involve x_5, u_5.

$$\left\{ \begin{array}{ccll} \dot{x}' & = & u' & \\ \dot{x}_5 & = & u_5 & \\ \dot{y}_i & = & u'^* L'_i i x', & i = 1, \dots, 5 \\ \dot{w} & = & u'^* M x' & \\ \dot{y}_{5+j} & = & u_5 x'_j - x_5 u'_j, & j = 1, \dots, 4 \end{array} \right. \tag{6.1}$$

Clearly, the associated interpolation problem to find a trajectory of sub-Riemannian length ε which maximizes $w|_\varepsilon - w|_0$, and which joins the points of Γ, has optimal solution with $u_5(t) = 0,\ x_5(t) = 0$, since the projection to x', u', w variables of any admissible trajectory is an admissible trajectory of the reduced nilpotent problem with 4-dimensional distribution. According to Lemma 4 from [4], the entropy estimations of the one-step bracket generating system and that of its nilpotent approximation coincide.

Remark 4. *Notice that a similar reduction from k to $k-1$ is valid for any odd k. For instance, the entropy estimation for everywhere one-step-bracket generating system with $k = 3$, and $p = 3$ is provided by the respective contact nilpotent $k = 2, p = 1$ reduced system.*

As we have seen, in the non-complete cases with $p < \frac{1}{2}k(k-1)$ the normal form of the nilpotent approximation depends on several invariants, and precise entropy estimation in the non-strictly convex setting requires long calculations. Practically, it seems reasonable to have easier approximative formulas. The following lemma 4 combined with the results of ([4]) and of these sections provide a comparison of the non-complete systems with a properly chosen complete extension.

Lemma 4. *Given an affine s-dimensional family of skew-symmetric matrices M, $L_1, \ldots, L_s$ of size $n\times n$ there exist auxiliary skew-symmetric matrices $Z_{s+1}, \ldots, Z_{m-1}$ where $m = \frac{n(n-1)}{2}$ such that M, L_i, Z_j form a basis in the ambient space of skew-symmetric matrices of the given size and*

$$\chi(M, L_i) = \min_{\lambda} \|M + \sum \lambda_i L_i\| = \min_{\lambda,\mu} \|M + \sum \lambda_i L_i + \sum \mu_j Z_j\| \tag{6.2}$$

$$\lambda \in \mathbb{R}^s, \mu \in \mathbb{R}^{m-s-1}.$$

In other words, we want to underline that any one-step-bracket-generating problem is a projection of a system with free algebra of the first brackets.

The results of ([4]) on a free algebra with 4-dimensional distribution in $\mathbb{R}^{10}$ provide an ε- interpolating strategy which shows that the complexity (entropy) lies inside the range $[e, \frac{3}{2}e]$ where $e = \int_\Gamma \frac{2ds}{\varepsilon^2 \chi(s)}$ - is the minimal estimation which is attained in the strictly convex case.

Since χ is preserved under projection, the strategy and the estimations hold also for any non-free system. Notice that the projection of an admissible trajectory is of course an admissible trajectory of the projected system.

Proof. We will use induction upon the number of extra matrices. It is sufficient to show that if Z is linearly independent with M, $L_1, \ldots, L_s$ then for some value of $\tau \in \mathbb{R}$ the matrix $\widehat{Z} = Z + \tau M$ satisfies the equality

$$\min_{\lambda \in \mathbb{R}^s} \|M + \sum \lambda_i L_i\| = \min_{\substack{\lambda \in \mathbb{R}^s, \\ \mu \in \mathbb{R}}} \|M + \sum \lambda_i L_i + \mu \widehat{Z}\|. \tag{6.3}$$

At first, notice that the right hand side of (6.3) never exceeds the left hand side. Assume that for $\tau = 0$ it is strictly less (otherwise we just take $\widehat{Z} = Z$), and the minimal value m_* of the right hand side is attained at $\mu_1, \widehat{\lambda}$. Assume also, that for $\mu = 0$ the minimum is m_0 and is attained at λ_0.

For each value of τ the function $f_\tau(\mu) = \min_{\lambda \in \mathbb{R}^s} \|M + \mu\widehat{Z} + \sum \lambda_i L_i\| = \min_{\lambda \in \mathbb{R}^s} \|M(1+\mu\tau) + \mu Z + \sum \lambda_i L_i\|$ is convex (see e.g. [2]) in μ and is continuous in μ and τ.

Its minimal value $m_\tau = \min_{\mu} f_\tau(\mu)$ and the minimum point(s) also depend continuously on τ. The value of $f_\tau(\mu)$ at $\mu = 0$ is always m_0. Let $\tau_* = \frac{2}{\mu_1}$, $\mu = -\mu_1$, and $\lambda = -\widehat{\lambda}$, then the value of

$$\|M + \mu\widehat{Z} + \sum \lambda_i L_i\| = \| - M - \mu_1 Z - \sum \widehat{\lambda}_i L_i\| = m_*.$$

So, when τ runs from 0 to τ_* the minimum point of $f_\tau(\mu)$ moves from μ_1 to the point at the other side of the origin. Hence for some τ it coincides with the origin providing the minimal value m_0, as required.

7. Elimination of triple eigenvalues

Prorosition 8. *For a generic system with $k > 5$ and $p \leq 8$ the χ value is attained at a matrix with at most 4 dimensional maximal eigenvalue subspace and hence the optimal strategy for the restriction of the system to this subspace of the distribution provides the entropy estimation interval* $\left[\int\limits_{\Gamma} \frac{4\pi ds}{\varepsilon^2\chi(s)}, \frac{3}{2}\int\limits_{\Gamma} \frac{4\pi ds}{\varepsilon^2\chi(s)}\right]$.

Proof. The following lemma (5) implies that for generic Motion Planning Problems with $p \leq 8$ the affine subspaces $M + \sum \lambda_i L_i$ can meet the subvariety Σ_3 of the skew-symmetric $n \times n$ matrices with triple eigenvalue only at isolated points on the program curve Γ. These isolated points make no contribution to the entropy (complexity) since χ remains positive and continuous in their vicinity.

In a neighbourhood of a point $m \in \Gamma$ outside this intersection, similarly to [1, 2], and to the previous section, there exists a normal coordinate system along Γ such that the normal form of the system contains a subsystem depending only on 4 coordinates x, such that the optimal interpolation corresponds to zero values of the remaining coordinates.

Lemma 5. *The subvariety Σ_3 has codimension 8 in the space of all $n \times n$ skew-symmetric matrices.*

Proof of the lemma. The stationary subgroup $S_A \subset \mathcal{O}(n)$ of the skew-symmetric matrix $A = \mathrm{diag}(B, B, B, C)$ where $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and C has distinct eigenvalues different from $\pm\sqrt{(-1)}$ consists of the exponentials of skew-symmetric matrices commuting with A and has dimension 9 in the orthogonal group $\mathcal{O}(n)$. So, the dimension of the $\mathcal{O}(n)$-orbit of A is $\frac{n(n-1)}{2} - 9$ and the variety σ_3 which at a regular point is the union of the orbits of λA, $\lambda \in \mathbb{R}$ has codimension 8.

8. (3,4) - case

Generic problem for a sub-Riemannian system with a growth vector (3,3,1) has the following normal form of its nilpotent approximation.

$$\dot{x}_1 = u_1,\ \dot{x}_2 = u_2,\ \dot{x}_3 = u_3,\ \dot{y}_i = x^* J_i u, \tag{8.1}$$

where $i = 1, 2, 3$ and J_i - is the basis of skew-symmetric 3×3 forms in x, and finally

$$\dot{w} = \sum_{i=1,2,3} Q_i(x) u_i, \tag{8.2}$$

where Q_i are quadratic forms in x.

The optimal synthesis here is given by an interpolating curve γ in the x space, which has length ε, starts and terminates at the origin, has zero integrals of $\dot{y}_i$ along, and maximizes the integral $\int\limits_{\gamma} \dot{w} d\gamma$. Of course, addition of a gradient of a cubic form to the vector field $Q = (Q_1,\ Q_2,\ Q_3)$ does not affect the maximizing integral.

Remark 5. *The problem can be stated in an "isoperimetric" version. Find a loop γ in $\mathbb{R}^3$ of given length, which maximizes the flux of* $\mathrm{curl}(Q)$ *through the 2-chain bounded by γ and providing the zero circulation of the vector fields $Y = (xJ_iu), i = 1, 2, 3$.*

The proof of the following statement is straightforward.

Prorosition 9. *The Hamilton equations of the Pontriagin Maximum Principle applied to the problem stated are*

$$\dot{u} = (p + Ax) \wedge u, \quad \dot{x} = u, \quad \dot{p} = 0. \tag{8.3}$$

where $Ax = \mathrm{curl}(Q)$ (hence the matrix A is traceless), p -are the impulses adjoint to the y coordinates. Along the required periodic curve they vanish.

Remarks. 1. The respective Hamiltonian function has the form $H = (p_1 + Q_1(x))^2 + (p_2 + Q_2(x))^2 + (p_3 + Q_3(x))^2$. The addition to Q_i of the gradient of the cubic form in x can be compensated by the canonical transformation $p \mapsto p + \mathrm{grad}F$ which preserves the projections of the trajectories to the x space.

2. The $\mathcal{O}(3)$ action on x induces the action on A by conjugation.

Prorosition 10. *A traceless matrix A can be reduced by an orthogonal transformation to the form*

$$A = \begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix}$$

with zero diagonal entries.

Proof. The zero level set of the quadratic form $(Ax, x) = 0$ is a cone, since a traceless matrix define a non-definite form. In canonical coordinates the equation of the cone can be written as

$$\alpha x_1^2 + \beta x_2^2 - \delta x_3^2 = 0$$

with $\alpha + \beta - \delta = 0, \quad \alpha, \ \beta, \ \delta > 0$.

Any plane section of the cone passing through the x_3 axis is a union of two straight lines. Each of them forms an angle $> \frac{\pi}{4}$ with the x_3 axis. So for any straight line ξ which belongs to the cone there is a unique choice of two other lines also belonging to the cone and forming together with ξ an orthogonal frame. In this frame the matrix A has the required form.

Lemma 6. Entropy estimation. *The maximal admissible value of $\int_\gamma \dot{w} d\gamma$ belongs to the interval $[\chi_0, 3\chi_0]$ where χ_0 is the maximum of the respective solutions (provided by elastica [4]) of the three $(2, 2)$ motion planning problems*

$$\begin{array}{rcl} \dot{x}_1 & = & u_1, \\ \dot{x}_2 & = & u_2, \\ \dot{y} & = & \frac{1}{2}u_1x_2 - \frac{1}{2}x_1u_2, \\ \dot{w} & = & a_1x_2^2u_1 + a_2x_1^2u_2 \end{array}$$

with the pairs of constants either a, b either c, d or e, f.

Proof. The flux of the vector field

$$\mathrm{curl}(Q) = (ay + bz)\frac{\partial}{\partial x} + (cy + dz)\frac{\partial}{\partial y} + (ex + fy)\frac{\partial}{\partial z}$$

through a 2- chain bounded by γ is the sum of the fluxes of the coordinate fields.

Each of these fields separately corresponds to the $(2, 2)$ integrable problem discussed in [4]. Taking a projection of an admissible curve to a coordinate plane we get an admissible curve for the $(2, 2)$ problem (with the inequality $\leq \varepsilon$ length constraint). So the result does not exceed three times the maximal value of the respective estimations of the $(2, 2)$ Motion planning problems. On the other hand, any elastica coordinate plane curve is admissible for the initial three dimensional problem. So the estimation from below is the maximum of the entropy exponents of the coordinate plane $(2, 2)$ problems.

Remark 6. *When determining the normal form of A one degree of freedom remains. Rotating the vector ξ along the cone the values of $a, b, \ldots, f$ constants vary. By an appropriate choice of ξ we get a better estimation $\max_{\xi} \chi_0$ instead of χ_0.*

References

[1] J.P. Gauthier, V.M. Zakalyukin, On the codimension one motion planning problem, Journal of Dynamical and Control Systems, 11(2005), n 1, pp.73 -89.

[2] J.P. Gauthier, V.M. Zakalyukin, On the One-Step-Bracket Generating Motion Planning Problem, Journal of Dynamical and Control Systems, 11(2005), n 2, pp. 215 - 235.

[3] J.P. Gauthier, V.M. Zakalyukin, Robot Motion Planning: A Wild Case, Proceedings of the Steklov Institute of Mathematics, 250 (2005), pp. 56 - 69.

[4] J.P. Gauthier, V.M. Zakalyukin, On the Motion Planning Problem, Complexity, Entropy and Nonholonomic Interpolation, to appear in Journal of Dynamical and Control Systems, 12(2006), 27 pp.

[5] J.P. Gauthier, V.M. Zakalyukin, Nonholonomic Interpolation: A General Methodology for Motion Planning in Robotics, to appear in the Proceedings of MTNS 2006 Conference, Kyoto, Japan, July 2006, 12 pp.

[6] A.A. Agrachev, H.E.A. Chakir, J.P. Gauthier, Subriemannian Metrics on R^3, in Geometric Control and Nonholonomic Mechanics, Mexico City 1996, pp. 29-76, Proc. Can. Math. Soc. 25, 1998.

[7] A.A. Agrachev, J.P. Gauthier, Subriemannian Metrics and Isoperimetric Problems in the Contact Case, in honor L. Pontriaguine, 90th birthday commemoration, Contemporary Maths, Tome 64, pp. 5-48, 1999 (Russian). English version: journal of Mathematical sciences, Vol 103, N°6, pp. 639-663.

[8] G. Charlot Quasi Contact SR Metrics: Normal Form in $\mathbb{R}^{2n}$, Wave Front and Caustic in $\mathbb{R}^4$; Acta Appl. Math., 74, N°3, pp. 217-263, 2002.

[9] H.E.A. Chakir, J.P. Gauthier, I.A.K. Kupka, Small Subriemannian Balls on R^3, Journal of Dynamical and Control Systems, Vol 2, N°3, , pp. 359-421, 1996.

[10] J.P. Gauthier, F.Monroy-Perez, C. Romero-Melendez, On complexity and Motion Planning for Corank one SR Metrics, COCV, v.10, 2004, p.634 - 655.

[11] M. Gromov, Carnot Caratheodory Spaces Seen from Within, Eds A. Bellaiche, J.J. Risler, Birkhauser, pp. 79-323, 1996.

[12] F. Jean, Complexity of Nonholonomic Motion Planning, International Journal on Control, Vol 74, N°8, pp 776-782, 2001.

[13] F. Jean, Entropy and Complexity of a Path in SR Geometry, COCV, v. 9, 2003, p. 485 - 506.

[14] F. Jean, E. Falbel, Measures and transverse paths in SR geometry, Journal d'Analyse Mathématique, v.91, 2003, p.231 - 246.

[15] T. Kato, Perturbation theory for Linear Operators, Springer Verlag, pp. 120-122, 1966.

[16] J.P. Laumont, S. Sekhavat, F. Lamiraux, Robot Motion Planning and Control, Lecture notes in Control and Information Sciences 229. Springer, 1998.

J.P.Gauthier, Laboratoire LE2I, UMR CNRS 5158, Université de Bourgogne, BP 47870, 21078, Dijon CEDEX, France., V.Zakalyukin, Moscow State University, 119 993, Leninski Gori,1, Moscow, Russia.