

A Bicriteria Approximation Scheme for the Minimum Well-Separated Pair Decomposition

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Disclaimer: This work has been produced with heavy AI use in all its phases.

Well-Separated Pair Decomposition

Problem

Background

Results

Overview

Shifting

Rounding

Local Growth

Doubling

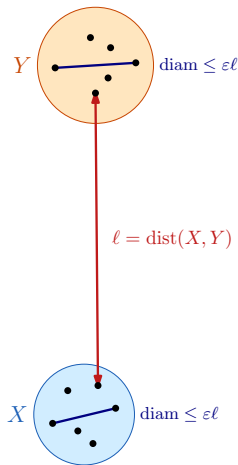
Open Problems

Bibliography

Definitions:

- P : set of n points in a metric space, $\varepsilon > 0$
- **Dumbbell** $\{X, Y\}$: two disjoint nonempty subsets of P , its **sides**
- **ε -separated**:
 $\max\{\text{diam}(X), \text{diam}(Y)\} \leq \varepsilon \cdot \text{dist}(X, Y)$

- A **pair** is $\{p, q\} \subseteq P$, $p \neq q$
- $\{X, Y\}$ **covers** the pairs $\{x, y\}$ with $x \in X$, $y \in Y$
- **ε -WSPD**: a family $\mathcal{W}$ of ε -separated dumbbells covering all $\binom{n}{2}$ pairs; size $|\mathcal{W}|$
- **Cover WSPD**: pairs may be covered several times;
partition WSPD: exactly once



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Background

Results

Overview

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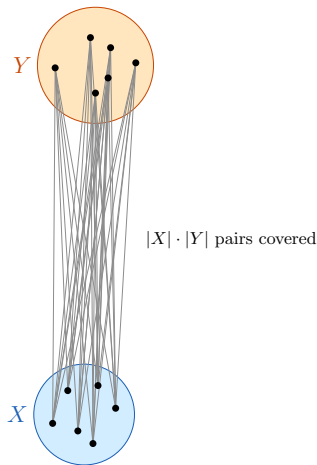
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Background

Results

Overview

Shifting

Rounding

Local Growth

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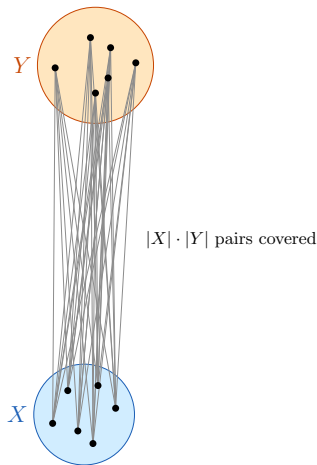
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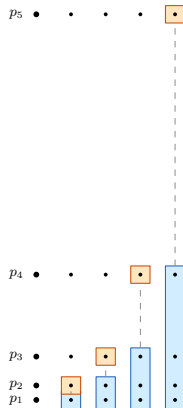


- Basic tool for proximity problems: spanners, approximate MST, nearest neighbors, distance oracles [CK95, NS07]
- In $\mathbb{R}^d$: partition ε -WSPD of size $O(n/\varepsilon^d)$ in $O(n \log n + n/\varepsilon^d)$ time [CK95]
- Similar bounds in doubling metrics [HM06]
- Many inputs admit **much smaller** WSPDs

Problem MINWSPD:

Given P and $\varepsilon \in (0, 1)$, compute an ε -WSPD of minimum size OPT_ε .

- Cover version: OPT_ε is at most the minimum size of a partition ε -WSPD



Exponentially spread points:

$n - 1$ dumbbells

$\{\{p_1, \dots, p_i\}, \{p_{i+1}\}\}$

What Was Known

Problem

Background

Results

Overview

Shifting

Rounding

Local Growth

Doubling

Open Problems

Bibliography

- MINWSPD is **NP-hard** in the plane when ε depends on n [BCH+26]
- General metrics, $\varepsilon = 1/3$: no n^c -approximation for some $c > 0$, unless $P = NP$ [BCH+26]
- On the line [BCH+26]: a PTAS by local search [MR10], an $O(n \log n)$ -time 3-approximation, a 10-approximation for partitions
- Doubling dimension κ : the net-tree WSPD [HM06] is a $2^{O(\kappa)}$ -**approximation** [BCH+26]
- Every metric: size 7OPT_ε with separation 33ε [HRR25]
(bicriteria, constants from the proof)

Question [BCH+26]:

Is there a $(1 + \delta)$ -**approximation**, or a simple $O(1)$ -approximation, in $\mathbb{R}^d$ for $d \geq 2$?

Biclique covers:

- A dumbbell is a biclique of the complete graph on P
- MINWSPD is a **restricted** biclique cover problem
- General bipartite graphs: hard within $n^{1-\gamma}$ [CHHK14]

At Least $n - 1$ Dumbbells

Problem

Background

Results

Overview

Shifting

Rounding

Local Growth

Doubling

Open Problems

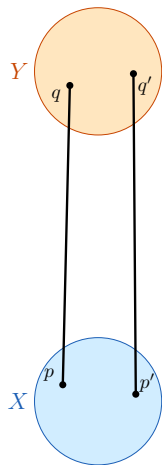
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- Partition WSPDs: **Graham–Pollak** [GP71], K_n needs $n - 1$ disjoint bicliques
- Covers: K_n is covered by $\lceil \log_2 n \rceil$ bicliques

Lemma (adapting [CK93], partitions in $\mathbb{R}^d$):

In every metric space and for every $\varepsilon \in (0, 1)$, every ε -WSPD has at least $n - 1$ dumbbells.

- T : minimum spanning tree. Map each edge pq of T to a dumbbell $\{X, Y\}$ with $p \in X$, $q \in Y$
- Two edges with $|pq| \leq |p'q'|$ in the same dumbbell: $|pp'| \leq \text{diam}(X) \leq \varepsilon|pq| < |p'q'|$, and so is $|qq'|$
- **Bottleneck property**: the tree paths $p' \rightarrow p$ and $q \rightarrow q'$ use only edges shorter than $p'q'$
- With pq , they connect p' to q' in $T - p'q'$: a **cycle**



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Problem

Background

Results

Overview

Shifting

Rounding

Local Growth

Doubling

Open Problems

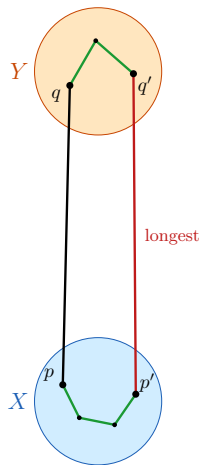
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Bicriteria approximation scheme (Euclidean and doubling metrics):

- For fixed d and $\delta \in (0, 1]$: a $(1 + \delta)\varepsilon$ -**WSPD** of size at most $(1 + \delta)\text{OPT}_\varepsilon$
- Running time $f(d, \delta) \cdot n^6$, independent of ε , with $f(d, \delta) = 2^{2^{O(d(1/\delta + \log(d/\delta)))}}$
- Same guarantees in doubling metrics

Setting	Size	Separation	Time	Reference
$\mathbb{R}^1$	$(1 + \delta) \text{OPT}_\varepsilon$	ε	$n^{O(1/\delta^2)}$	[BCH+26]
$\mathbb{R}^1$	3OPT_ε	ε	$O(n \log n)$	[BCH+26]
Doubling dim. κ	$2^{O(\kappa)} \text{OPT}_\varepsilon$	ε	$2^{O(\kappa)} (n \log n + \text{OPT})$ exp.	[BCH+26, HM06]
Any metric	7OPT_ε	33ε	polynomial	[HRR25]
$\mathbb{R}^d$, fixed d	$(1 + \delta) \text{OPT}_\varepsilon$	$(1 + \delta)\varepsilon$	$f(d, \delta) n^6$	new
Doubling dim. κ	$(1 + \delta) \text{OPT}_\varepsilon$	$(1 + \delta)\varepsilon$	$f(\kappa, \delta) n^6$	new

- The row of [HRR25] requires $\varepsilon < 1/33$
- Equivalently: an ε -WSPD of size at most $(1 + \delta) \text{OPT}_{\varepsilon/(1+\delta)}$

Compatibility Graph

Problem

Background

Results

Overview

Shifting

Rounding

Local Growth

Doubling

Open Problems

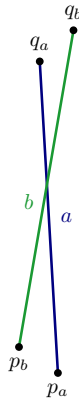
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- Pairs a, b are **compatible** if some ε -separated dumbbell covers both
- **Heredity**: sub-dumbbells of ε -separated dumbbells are ε -separated
- Test $\{\{p_a, p_b\}, \{q_a, q_b\}\}$ and $\{\{p_a, q_b\}, \{q_a, p_b\}\}$: $O(1)$ distances
- **Compatibility graph** H : one vertex per pair, one edge per compatible pair of pairs
- The pairs covered by a dumbbell form a **clique** of H

Reformulation:

MINWSPD is **clique cover** in H ,
restricted to the cliques realized by dumbbells.

- **Locality**: if an ε -separated $\{X, Y\}$ covers a pair e of length $s(e)$, then $\text{dist}(X, Y) \leq s(e) \leq (1 + 2\varepsilon)\text{dist}(X, Y)$



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Problem

Background

Results

Overview

Shifting

Rounding

Local Growth

Doubling

Open Problems

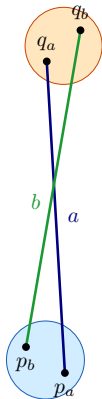
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The Algorithm in Three Layers

Problem

Background

Results

Overview

Shifting

Rounding

Local Growth

Doubling

Open Problems

Bibliography

- Parameters: $\zeta = \delta/3$, $m = \lceil 1/\zeta \rceil$, $\varepsilon' = (1 + 2\zeta)\varepsilon$, $\Lambda = (1 + 2\varepsilon)^m \leq 3^m$

1 Shifting over scales [HM85]

Split the pairs by length into **bands**: the lengths in a band lie in $[\sigma, \Lambda\sigma)$

Costs a factor $1 + 1/m \leq 1 + \zeta$ in the size

2 Rounding to a grid of cell side $\Theta(\zeta\varepsilon\sigma)$ in the band $[\sigma, \Lambda\sigma)$

Local problems become **finite**

The only step that relaxes ε to ε'

3 Local growth in H inside each band [NH05,NHK08], as for clique partition in unit disk graphs [PS12]

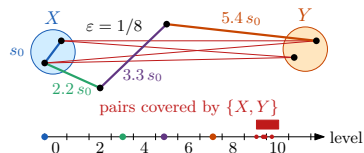
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Result:

A $(1 + 2\zeta)\varepsilon$ -WSPD of size at most $(1 + \zeta)^2 \text{OPT}_\varepsilon \leq (1 + 3\zeta) \text{OPT}_\varepsilon = (1 + \delta) \text{OPT}_\varepsilon$.

Shifting over Scales

- Scale $s(e)$: length of the pair e ; $s_0 = \min_e s(e)$
- **Level**: $\lambda(e) = \lfloor \log_{1+2\epsilon}(s(e)/s_0) \rfloor$
- **Band** E_j : pairs with $\lfloor \lambda(e)/m \rfloor = j$ (m consecutive levels); scales in $[\sigma_j, \Lambda\sigma_j]$
- At most $\binom{n}{2}$ nonempty bands, **independent of the spread**; solve each band, output the union
- Problem: a dumbbell that **straddles** a boundary is paid for twice
- **Offset** $o \in \{0, \dots, m-1\}$ shifts the bands: E_j^o has $\lfloor (\lambda(e) - o)/m \rfloor = j$; output the **best offset**



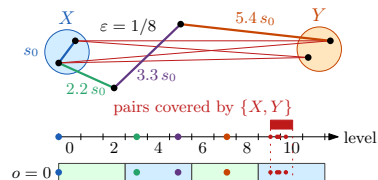
Lemma:

Some offset o has $\sum_j \theta_\epsilon(E_j^o) \leq (1 + 1/m) \text{OPT}_\epsilon$.

- By locality, the pairs of a dumbbell have **two consecutive levels**: it is split for only **one** of the m offsets

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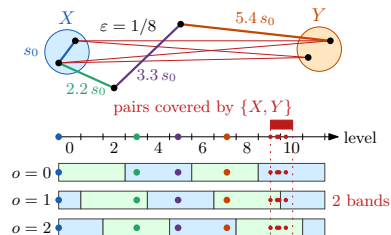
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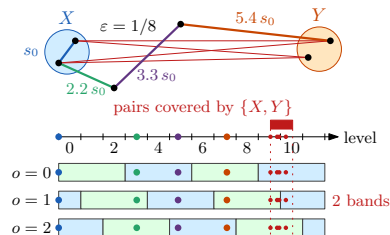
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Rounding to a Grid

Problem

Background

Results

Overview

Shifting

Rounding

Local Growth

Doubling

Open Problems

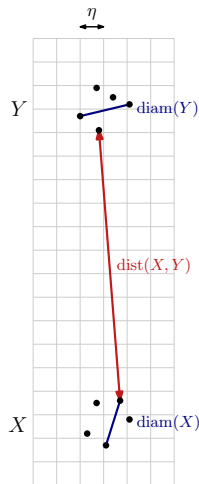
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- **Cell dumbbell** $\{A, B\}$: two sets of cells with $\max\{\text{diam}(\bigcup A), \text{diam}(\bigcup B)\} \leq \epsilon' \text{dist}(\bigcup A, \bigcup B)$
- $\theta_{\square}(S)$: minimum number of cell dumbbells covering a set S of pairs of the band

Lemma:

If $\{X, Y\}$ is ϵ -separated and covers a pair of the band, then the cells A and B meeting X and Y form a cell dumbbell. Hence $\theta_{\square}(S) \leq \theta_{\epsilon}(S)$.

- $\ell = \text{dist}(X, Y) \geq \sigma/3$, so $\eta\sqrt{d} \leq \zeta\epsilon\ell/4$:
 $\text{diam} \leq \epsilon\ell(1 + \frac{\zeta}{2})$ and $\text{dist} \geq \ell(1 - \frac{\zeta}{2})$
- $1 + \frac{\zeta}{2} \leq (1 + 2\zeta)(1 - \frac{\zeta}{2})$ for $\zeta \leq \frac{1}{2}$



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Problem

Background

Results

Overview

Shifting

Rounding

Local Growth

Doubling

Open Problems

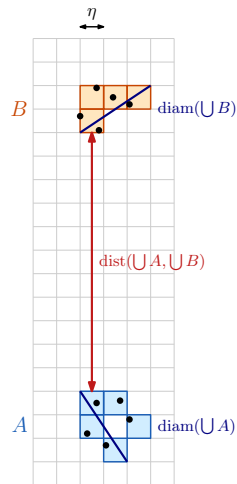
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- S : pairs of the band with endpoints in two balls of radius $R = \rho \varepsilon \Lambda \sigma$
- Q : cells that contain an endpoint of a pair of S
- $|Q| \leq K = 2(24\sqrt{d} \rho \Lambda / \zeta + 2)^d$

Lemma:

- 1 $\theta_{\square}(S) \leq K^2$
- 2 $\theta_{\square}(S)$ and an optimal family of cell dumbbells can be computed in $g(d, \zeta, \rho) + O(|S| \log |S|)$ time

- (1): the two cells $\{c\}, \{c'\}$ of a pair form a cell dumbbell
- (2): restrict the sides to Q ; a side lies in a cube of $K_0 = O(\sqrt{d} \Lambda / \zeta)^d$ cells around any of its cells
- (2): by sorting, replace S by its at most K^2 distinct pairs of cells, then **brute force**
- The scale $\varepsilon \sigma$ cancels in R/η , and $\Lambda \leq 3^m$: no dependence on n or ε

Local Growth

Problem
Background
Results
Overview
Shifting
Rounding

Local Growth

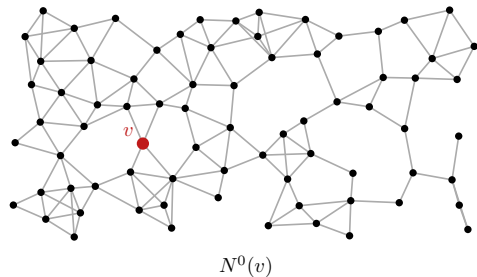
Doubling
Open Problems
Bibliography

- U : uncovered pairs of the band E
- $N^r(v)$: pairs at hop distance at most r from v in $H[U]$
- Vertices of H are **pairs** of points

While $U \neq \emptyset$:

- 1 Pick any $v \in U$
- 2 $r \leftarrow$ smallest integer with $\theta_{\square}(N^{r+1}(v)) \leq (1 + \zeta) \theta_{\square}(N^r(v))$
- 3 Add a minimum cover of $N^{r+1}(v)$ by cell dumbbells; remove the pairs it covers from U

- **Core** of round i : $C_i = N^{r_i}(v_i)$
- The **ring** $N^{r_i+1}(v_i) \setminus C_i$ is also removed



Local Growth

Problem
Background
Results
Overview
Shifting
Rounding

Local Growth

Doubling

Open Problems

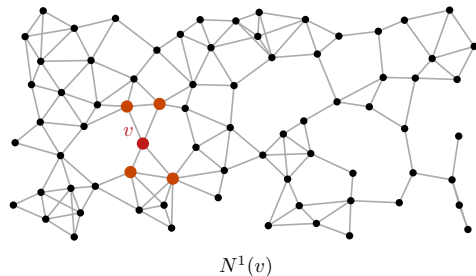
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Problem
Background
Results
Overview
Shifting
Rounding

Local Growth

Doubling

Open Problems

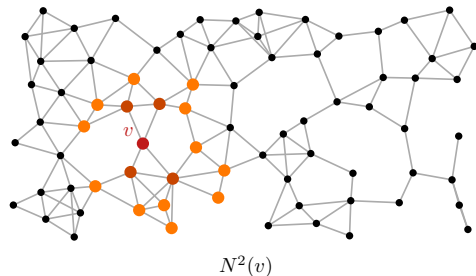
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Problem
Background
Results
Overview
Shifting
Rounding

Local Growth

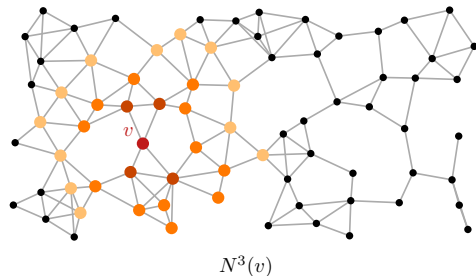
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Open Problems
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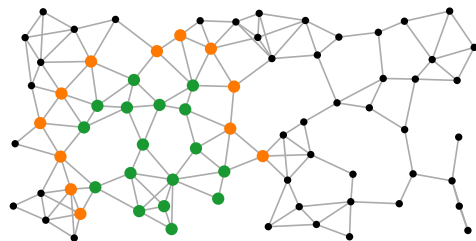


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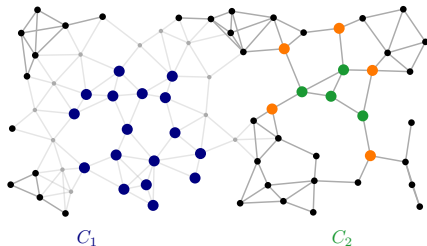
- 1 Pick any $v \in U$
- 2 $r \leftarrow$ smallest integer with $\theta_{\square}(N^{r+1}(v)) \leq (1 + \zeta) \theta_{\square}(N^r(v))$
- 3 Add a minimum cover of $N^{r+1}(v)$ by cell dumbbells; remove the pairs it covers from U

- **Core** of round i : $C_i = N^{r_i}(v_i)$
- The **ring** $N^{r_i+1}(v_i) \setminus C_i$ is also removed



core $N^r(v)$ and ring $N^{r+1}(v) \setminus N^r(v)$

Cores Are Independent



Lemma:

$\sum_i \theta_{\square}(C_i) \leq \theta_{\varepsilon}(E)$, and hence the band is covered by at most $(1 + \zeta) \theta_{\varepsilon}(E)$ cell dumbbells.

- $\mathcal{W}^*$: minimum cover of E by ε -separated dumbbells
- No $Z \in \mathcal{W}^*$ covers $a \in C_i$ and $b \in C_k$, $i < k$:
 - a, b compatible and both in U_i
 - so $b \in N^{r_i+1}(v_i)$, removed in round i
- The sets $\mathcal{W}_i^*$ of dumbbells covering pairs of C_i are **disjoint**
- $|\mathcal{W}_i^*| \geq \theta_{\varepsilon}(C_i) \geq \theta_{\square}(C_i)$ by rounding
- Round i pays $\theta_{\square}(N^{r_i+1}(v_i)) \leq (1 + \zeta) \theta_{\square}(C_i)$

The Growth Stops

Problem

Background

Results

Overview

Shifting

Rounding

Local Growth

Doubling

Open Problems

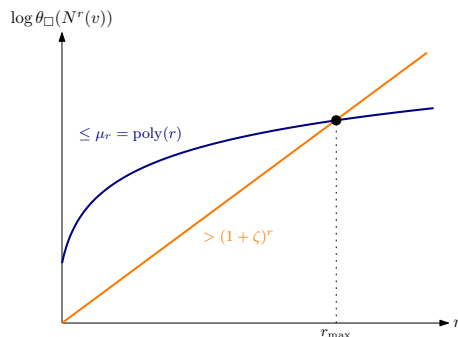
Bibliography

- Compatible pairs of the band have endpoints within $\varepsilon\Lambda\sigma$ of each other
- So $N^r(v)$ lies in two balls of radius $r\varepsilon\Lambda\sigma$ around the endpoints of v
- Local problems: **polynomial** upper bound
 $\theta_{\square}(N^r(v)) \leq \mu_r = 4(26\sqrt{d}(r+1)\Lambda/\zeta)^{2d}$
- While step 2 fails: **exponential** lower bound
 $\theta_{\square}(N^r(v)) > (1+\zeta)^r$

Lemma:

The growth stops at radius

$$r \leq r_{\max} = O((d/\zeta)(1/\zeta + \log(d/\zeta))).$$



Theorem:

Let $d \geq 1$ and $\delta \in (0, 1]$. Given $P \subset \mathbb{R}^d$ with n points and $\varepsilon \in (0, 1)$, one can compute in $f(d, \delta) \cdot n^6$ time a $(1 + 2\delta/3)\varepsilon$ -WSPD $\mathcal{W}$ of P with $|\mathcal{W}| \leq (1 + \delta)\text{OPT}_\varepsilon$.

- Size, with o^* the offset of the shifting lemma:

$$|\mathcal{W}| \leq \sum_j |\mathcal{W}_j^{o^*}| \leq (1 + \zeta) \sum_j \theta_\varepsilon(E_j^{o^*}) \leq (1 + \zeta)^2 \text{OPT}_\varepsilon \leq (1 + \delta) \text{OPT}_\varepsilon$$

- Time: H has $O(n^4)$ edges; per offset, $O(n^2)$ rounds, each with one breadth-first search in $O(n^4)$ time and $O(r_{\max})$ local problems
- In fixed-dimensional $\mathbb{R}^d$: improves the factors $(7, 33)$ of [HRR25] to $(1 + \delta, 1 + \delta)$ (their result holds in every metric space)
- The time does not depend on ε or on the spread

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Why Relax the Separation?

Problem

Background

Results

Overview

Shifting

Rounding

Local Growth

Doubling

Open Problems

Bibliography

- Rounding is the **only** step that relaxes ε
- Suppose we could decide in $|S|^{h(k)}$ time whether a local set S is covered by k **ε -separated** dumbbells
- Then shifting, cores and growth work with θ_ε in place of $\theta_\square$
- A grid of side $\varepsilon\sigma/(4\sqrt{d})$ still gives $\theta_\varepsilon(S) \leq K'^2$
- Result: a **PTAS** for OPT_ε without relaxing ε , in $n^{O(h(K'^2))}$ time
- Clique partition in unit disk graphs: local solvers use that optimal clusterings in the plane can be **separated by lines** [CRW91, PS12]

Open:

A separability property for optimal WSPDs, even in the plane?

- Doubling dimension $\leq \kappa$: every ball of radius $2R$ is covered by 2^κ balls of radius R (for P itself)
- Shifting, cores and growth only use the **triangle inequality**
- Replace the grid by a **net**: $P_\eta \subseteq P$ maximal with pairwise distances $> \eta = \zeta \varepsilon \sigma / 48$
- **Cell** $\Gamma(q)$: points assigned to a nearest net point q ; $\text{diam}(\Gamma(q)) \leq 2\eta$
- Cell dumbbell: sets $A, B \subseteq P_\eta$ with $\{\Gamma(A), \Gamma(B)\}$ ε' -separated, tested on the points
- Packing: $|P_\eta \cap \mathbb{B}(z, R)| \leq (4R/\eta)^\kappa$ for $z \in P$, $R \geq \eta$

Theorem:

For P of doubling dimension κ : a $(1 + 2\delta/3)\varepsilon$ -WSPD of size at most $(1 + \delta)\text{OPT}_\varepsilon$ in $f(\kappa, \delta) \cdot n^6$ time, where $f(\kappa, \delta) = 2^{2^{O(\kappa(1/\delta + \log(\kappa/\delta)))}}$. The algorithm does not need to know κ .

- The net-tree WSPD has size $2^{O(\kappa)}\text{OPT}_\varepsilon$ [HM06,BCH+26]: relaxing the separation by $1 + \delta$ brings the size factor down to $1 + \delta$

- A PTAS that does **not relax** ε ?
For instance, via a separability lemma for optimal WSPDs in the plane
- An approximation scheme in **near-linear** time?
- Does local search exchanging up to k dumbbells give a $(1 + O(1/\sqrt{k}))$ -approximation in the plane, as for pseudodisks [MR10], low-density objects [HQ17], and on the line [BCH+26]?
- Is MINWSPD **NP-hard** on the line [BCH+26], or in the plane for a fixed ε ?
- An approximation scheme for the **partition** version?

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Thank you!