Untangling Segments in the Plane

Bastien Rivier

Université Clermont Auvergne, EDSPI, and LIMOS

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Outline

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Motivation: Untangling TSP Tours

- 2d Euclidean TSP ($\mathcal{NP}$-hard):

Input: A set of n points called *cities*.
Output: The shortest *tour*
(polygon whose vertices are the cities).

- Heuristics generate tours with crossings.
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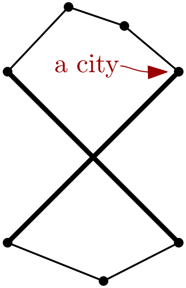

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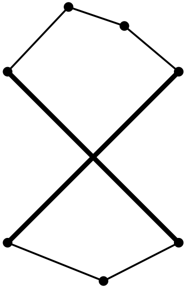

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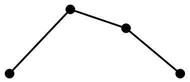

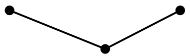

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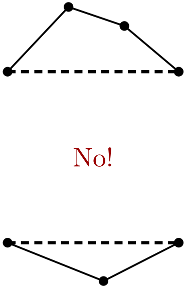

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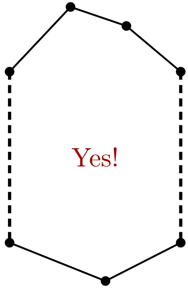

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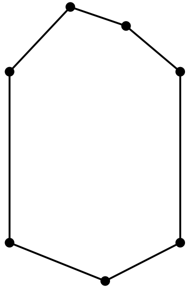

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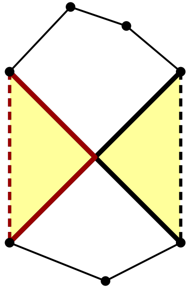

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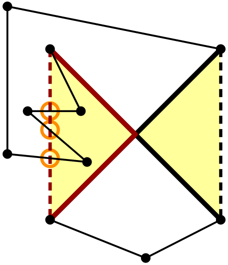
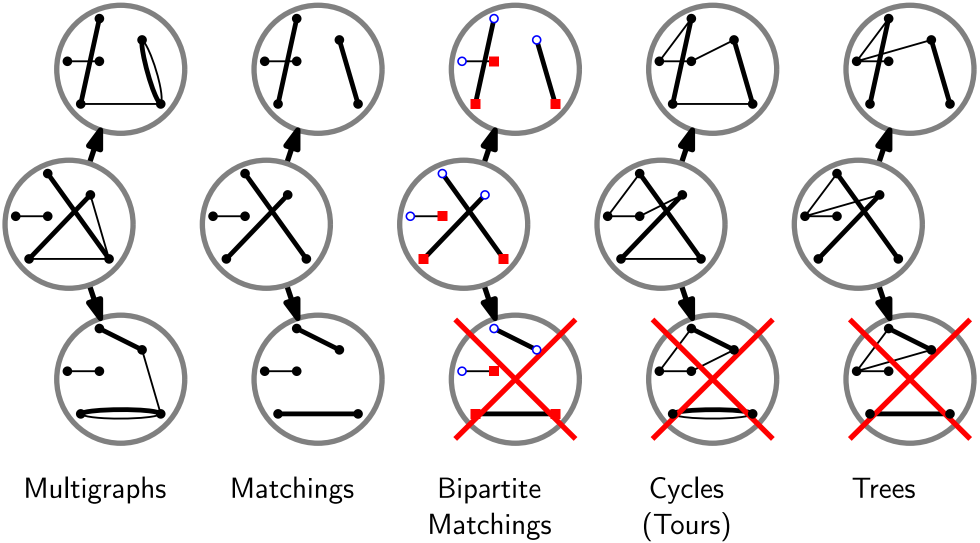


Outline

Flip Versions: from Tours to Segments



Outline

An infinite flip sequence?

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- Measuring progress with a potential, i.e., an integer function which is:
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- *Untangle sequence*: flip sequence ending with no crossing.

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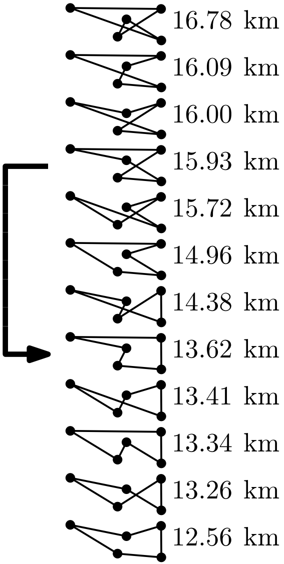

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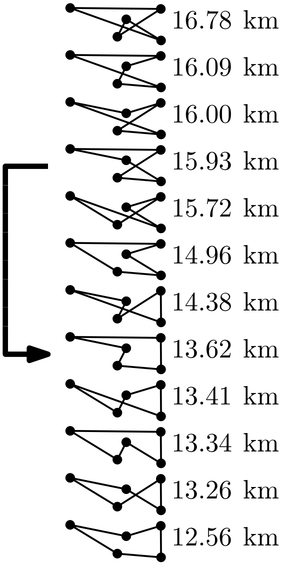

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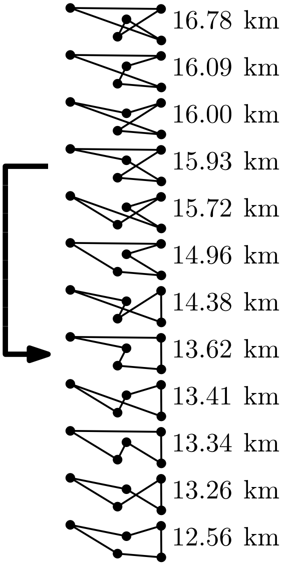
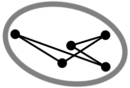


Long and Short Untangle Sequences

- The removal choice may impact the number of flips.
- The insertion choice may impact the number of flips.



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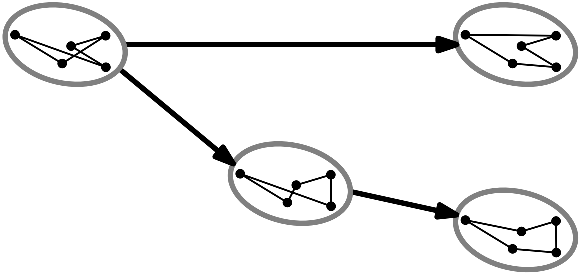

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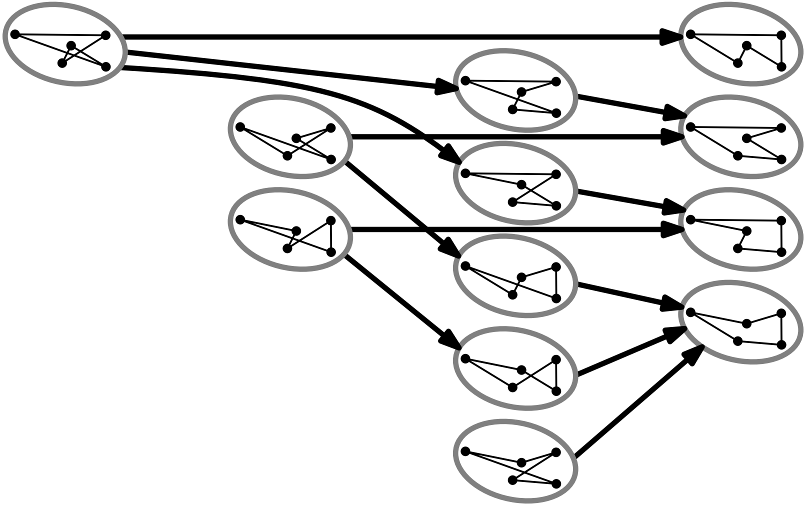

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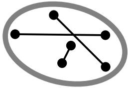

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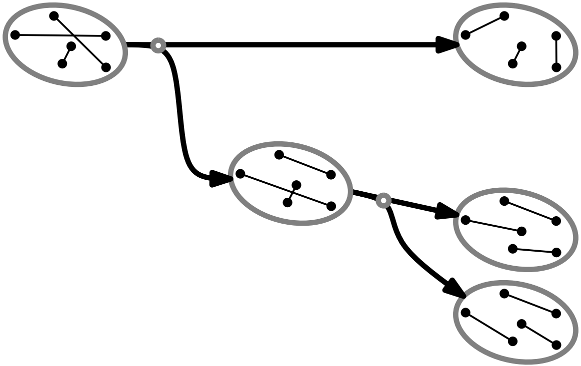

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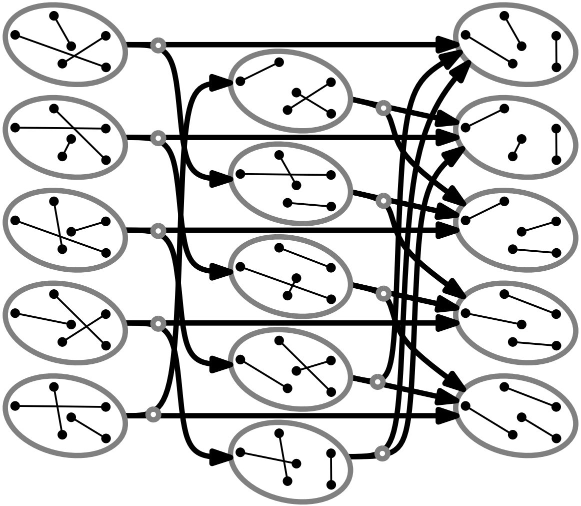

Outline

The Unknown $\mathbf{d}$

Imagine 2 perfect *players* performing Removal/Insertion/N$\emptyset$ choices flip by flip:

- the *adversary* maximizing the number of flips (choosing the n segments to untangle),
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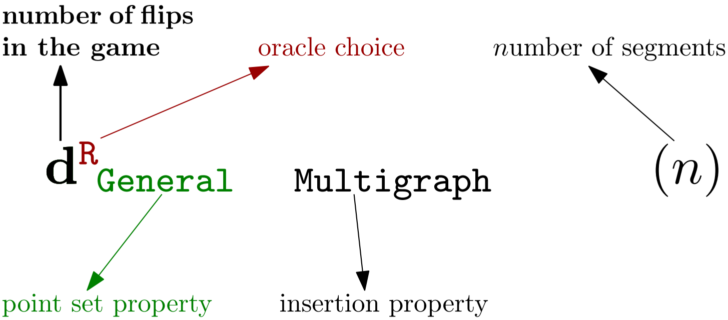

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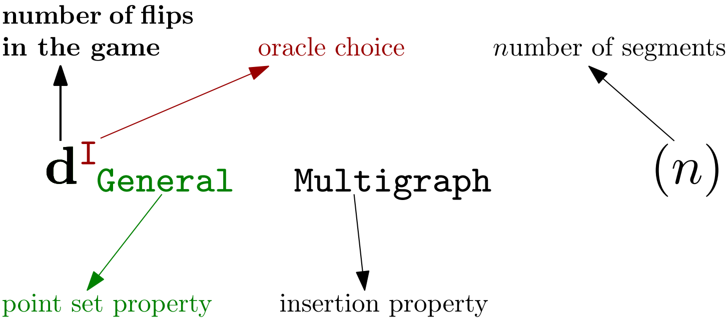

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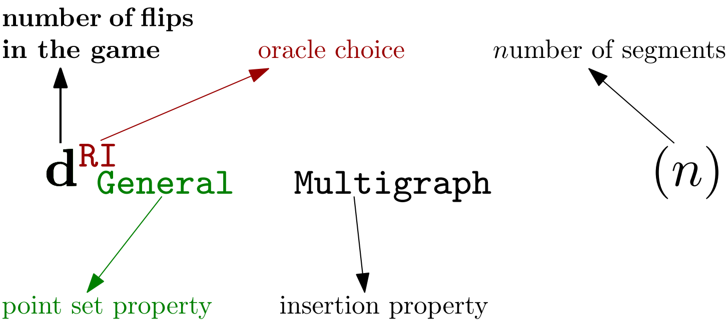

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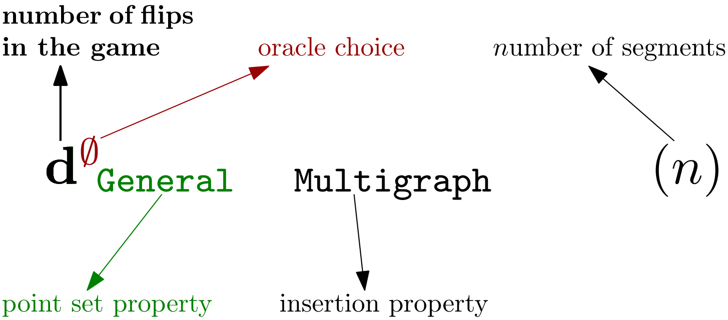

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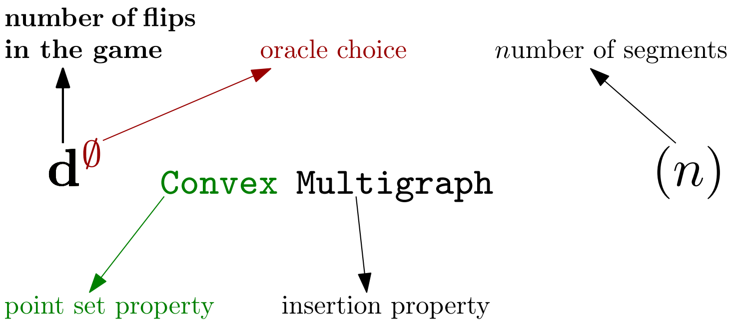

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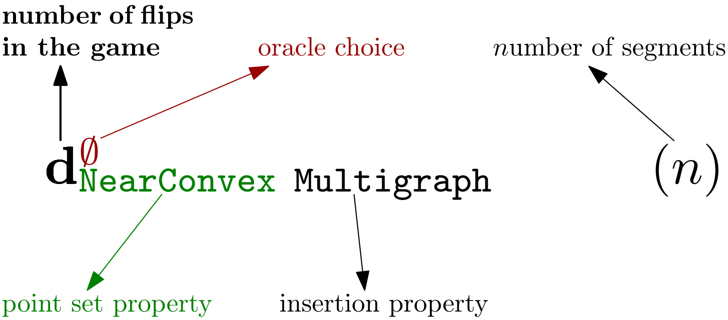

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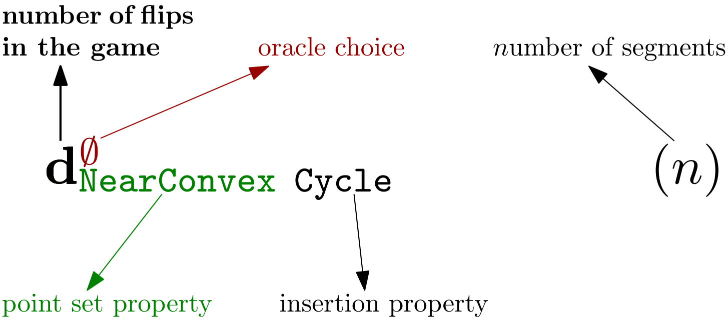

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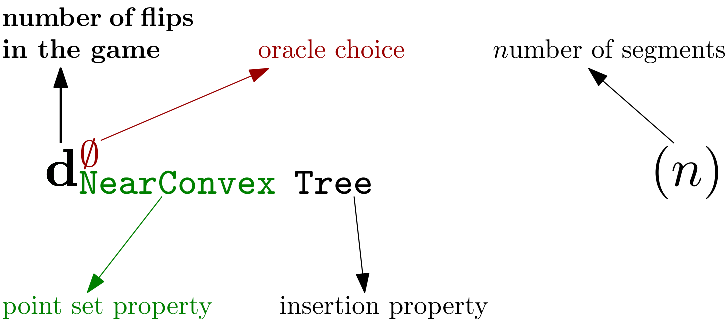

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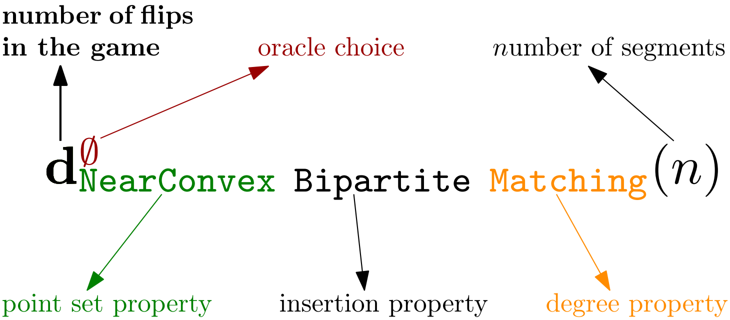

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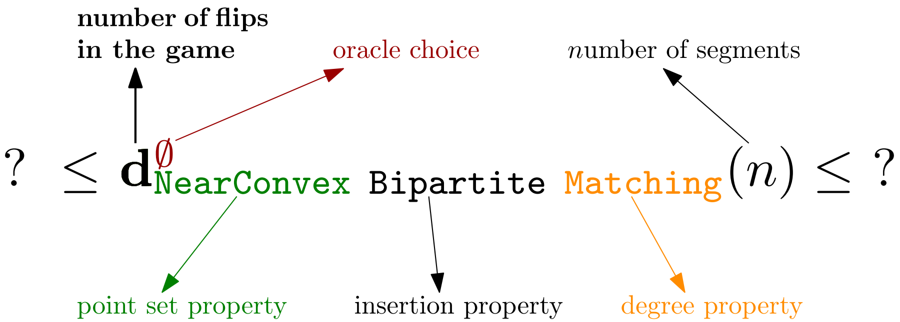

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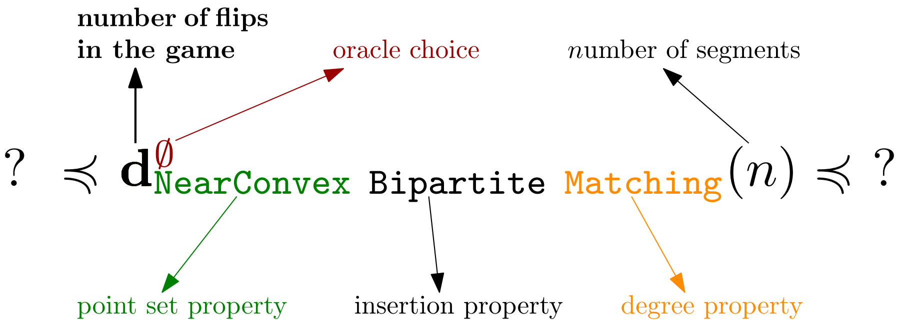

The Unknown $\mathbf{d}$: Formal Definition

Imagine 2 perfect *players* performing Removal/Insertion/N$\emptyset$ choices flip by flip:

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Π : conjunction of the point set, insertion, and degree properties.

S : the n segments to untangle.

$\mathtt{r}$: a removal strategy.

$\mathtt{i}$: an insertion strategy.

k : the number of flips to untangle S with the strategies $\mathtt{r}, \mathtt{i}$.

$$\mathbf{d}_\Pi^\emptyset(n) = \max_S \max_{\mathtt{r}} \max_{\mathtt{i}} \; k(S, \mathtt{r}, \mathtt{i})$$

$$\mathbf{d}_\Pi^{\mathtt{R}}(n) = \max_S \min_{\mathtt{r}} \max_{\mathtt{i}} \; k(S, \mathtt{r}, \mathtt{i})$$

$$\mathbf{d}_\Pi^{\mathtt{I}}(n) = \max_S \max_{\mathtt{r}} \min_{\mathtt{i}} \; k(S, \mathtt{r}, \mathtt{i})$$

(defined if insertion property is empty)

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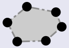
Folklore: Convex n^2 Upper Bound

Theorem (3.2.2)



$$\mathbf{d}^{\emptyset}_{\texttt{Convex Multigraph}}(n) \leq \binom{n}{2} \preccurlyeq n^2$$

Proving $\mathbf{d}^{\emptyset}_{\mathtt{Convex}}(n) \leq \binom{n}{2}$: Intuitive

- A *crossing*: an intersecting pair of segments with no endpoint in the intersection.
- $\chi_{\text{crossings}}(S)$: number of crossings in the multiset of segments S.
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Introduction
Literature
Folklore
1980
2007, 2009
2016
2019
Contribution
Conclusion

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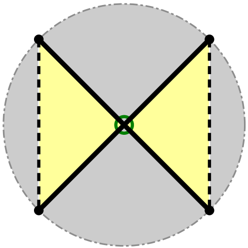

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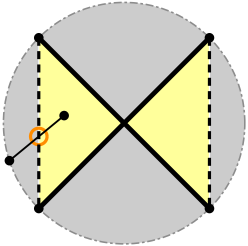

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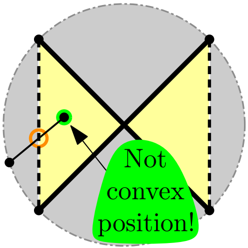


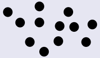
Outline

1980: General n^3 Upper Bound

[*Untangling a Traveling Salesman Tour in the Plane* –
Jan Van Leeuwen, Anneke A. Schoone]

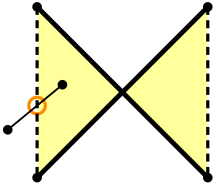
- P: the point set.

Theorem (3.1.3)



$$\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n) \leq \frac{1}{2} n \binom{|P|}{2} \preccurlyeq n\,|P|^2 \preccurlyeq n^3$$

Proof of $\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n) \leq \frac{1}{2}n\binom{|P|}{2}$: from Segments to Lines



- Λ_ℓ: number of segments crossed by the line ℓ
- A flip decreases Λ_ℓ by 0,
- L: the $\binom{|P|}{2}$ lines through two points of P.
- $\Lambda_L = \sum_{\ell \in L} \Lambda_\ell$
- At most n crossings per line $\implies \Lambda_L \leq n\binom{|P|}{2}$.
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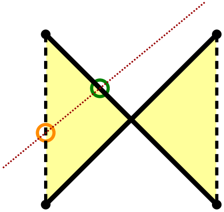

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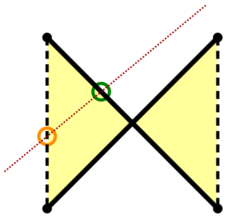
Folklore
1980
2007, 2009
2016
2019
Contribution
Conclusion

- Λ_ℓ: number of segments crossed by the line ℓ
- A flip decreases Λ_ℓ by 0,
- L: the $\binom{|P|}{2}$ lines through two points of P.
- $\Lambda_L = \sum_{\ell \in L} \Lambda_\ell$
- At most n crossings per line $\implies \Lambda_L \leq n\binom{|P|}{2}$.
- Λ_L decreases by at least 2 at each flip.



Proof of $\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n) \leq \frac{1}{2}n\binom{|P|}{2}$: from Segments to Lines

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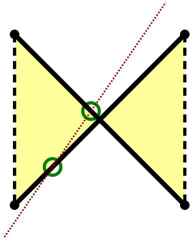

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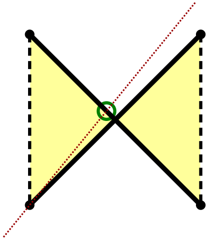

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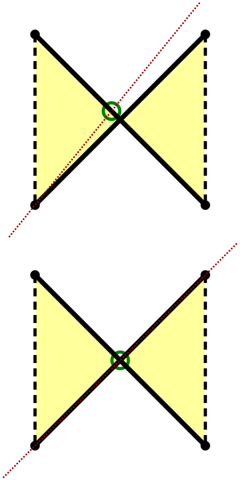

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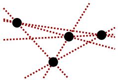

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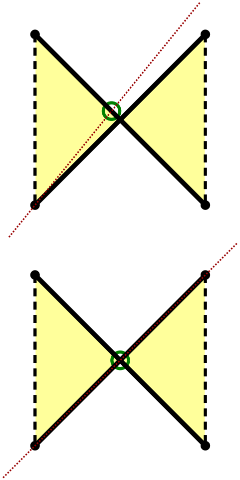

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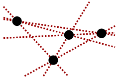

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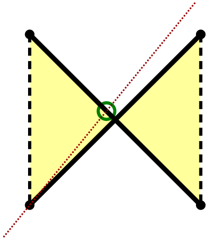

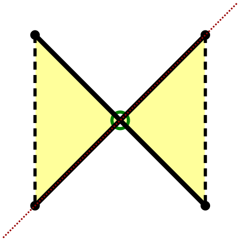

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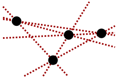

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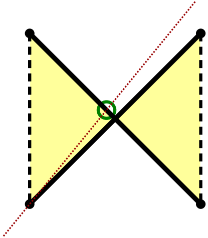
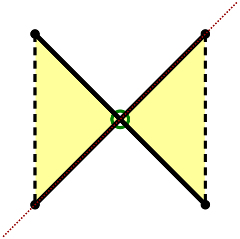




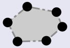
Outline

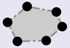
2007, 2009: Exact Value of $\mathbf{d}^{\mathrm{R}}_{\texttt{Convex Cycle}}(n)$

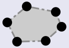
[*The Number of Flips Required to Obtain Non-crossing Convex Cycles* –
Yoshiaki Oda, Mamoru Watanabe]

[*On the Maximum Switching Number to Obtain Non-crossing Convex Cycles* –
Ro–Yu Wu, Jou–Ming Chang, Jia–Huei Lin]

Theorem (3.2.4; 3.2.7; 3.2.9)

$n-2 \leq \mathbf{d}^{\mathrm{R}}_{\texttt{Convex Cycle}}(n)$  *for* $n \geq 7$

 $\mathbf{d}^{\mathrm{R}}_{\texttt{Convex Cycle}}(n) \leq 2n-7$ *for* $n \geq 7$

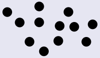
 $\mathbf{d}^{\mathrm{R}}_{\texttt{Convex Cycle}}(n) \leq n-2$ *for* $n \geq 7$

Outline

2016: Insertion Power; Easy Lower Bounds

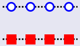
[*Flip Distance to a Non-crossing Perfect Matching* – Édouard Bonnet, Tillmann Miltzow]

Theorem (3.1.4; 3.2.1; 3.2.12; 3.2.12; 3.2.12; 3.2.12)

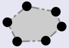


$$\mathbf{d}^{\mathtt{I}}_{\mathtt{Multigraph}}(n) \leq \frac{n}{2}(|P|-2) \preccurlyeq n\,|P| \preccurlyeq n^2$$

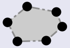
$$n^2 \preccurlyeq \binom{n}{2} \leq \mathbf{d}^{\emptyset}_{\mathtt{Convex\ Permutation\ Matching}}(n)$$



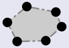
$$n \preccurlyeq n-1 \leq \mathbf{d}^{\mathtt{RI}}_{\mathtt{Convex\ Matching}}(n)$$



$$n \preccurlyeq n-1 \leq \mathbf{d}^{\mathtt{R}}_{\mathtt{Convex\ Bipartite\ Matching}}(n)$$

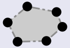


$$n \preccurlyeq \frac{n}{2}-1 \leq \mathbf{d}^{\mathtt{R}}_{\mathtt{Convex\ Cycle}}(n)$$



for even n

$$n \preccurlyeq \frac{n-1}{2} \leq \mathbf{d}^{\mathtt{R}}_{\mathtt{Convex\ Tree}}(n)$$



for odd n

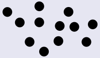
Outline

2019: Various Upper Bounds

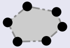
[*Flip Distance to some Plane Configurations* –
Ahmad Biniaz, Anil Maheshwari, Michiel Smid]

- P: the point set.
- $\sigma(P)$: the spread of P, i.e., the ratio between the distance of farthest and the closest pair of points.
- $\sqrt{n} \preccurlyeq \sigma(P)$

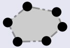
Theorem (3.1.5; 3.2.2; 3.2.10; 3.2.11; 3.2.13; 3.3.1)



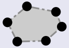
$$\mathbf{d}^{\mathtt{I}}_{\mathtt{Multigraph}}(n) \preccurlyeq n\sigma(P)$$



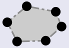
$$\mathbf{d}^{\emptyset}_{\mathtt{Convex\ Multigraph}}(n) \leq \binom{n}{2} \preccurlyeq n^2$$



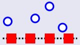
$$\mathbf{d}^{\mathtt{R}}_{\mathtt{Convex\ Bipartite\ Matching}}(n) \leq 2n-3 \preccurlyeq n$$



$$\mathbf{d}^{\mathtt{R}}_{\mathtt{Convex\ Tree}}(n) \preccurlyeq n\log n$$



$$\mathbf{d}^{\mathtt{RI}}_{\mathtt{Convex\ Multigraph}}(n) \leq n-1 \preccurlyeq n$$



$$\mathbf{d}^{\mathtt{R}}_{\mathtt{Redonaline\ Matching}}(n) \leq n(n-1) \preccurlyeq n^2$$

Outline

Contribution Papers

[1]: *Complexity Results on Untangling Red-Blue Matchings* –
Arun Kumar Das, Sandip Das, Guilherme D. da Fonseca, Yan Gerard, Bastien Rivier
(LATIN 2022 & Computational Geometry 2022).

[2]: *On the Longest Flip Sequence to Untangle Segments in the Plane* –
Guilherme D. da Fonseca, Yan Gerard, Bastien Rivier
(WALCOM 2023).

[3]: *Short Flip Sequences to Untangle Segments in the Plane* –
Guilherme D. da Fonseca, Yan Gerard, Bastien Rivier
(WALCOM 2024).

Outline

Intractability of the Shortest Untangle Sequence

Problem (1)

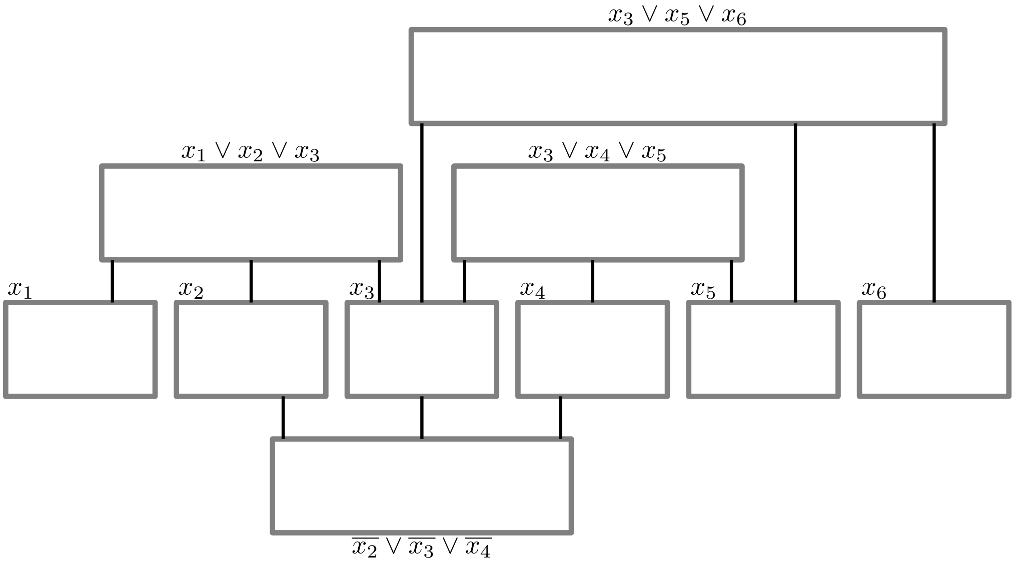
Let $\alpha \geq 1$ be a constant.
Input: S, a set of segments with rational coordinates forming a bipartite matching.
Output: An untangle sequence starting at S of length at most α times that of the shortest untangle sequence of S.

Theorem (8.0.1 [1])

Problem 1 is $\mathcal{NP}$-hard for all $\alpha \geq 1$.

Proof of Intractability: Reduce Rectilinear Planar Monotone 3-SAT

Introduction
Literature
Contribution
Intractability
Upper Bounds
Lower Bounds
Reductions
Conclusion



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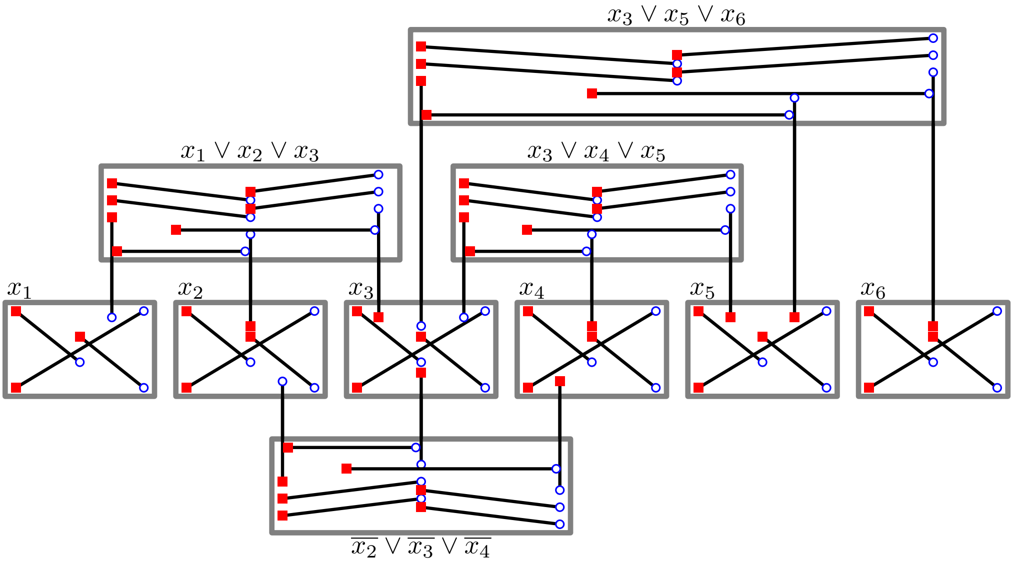

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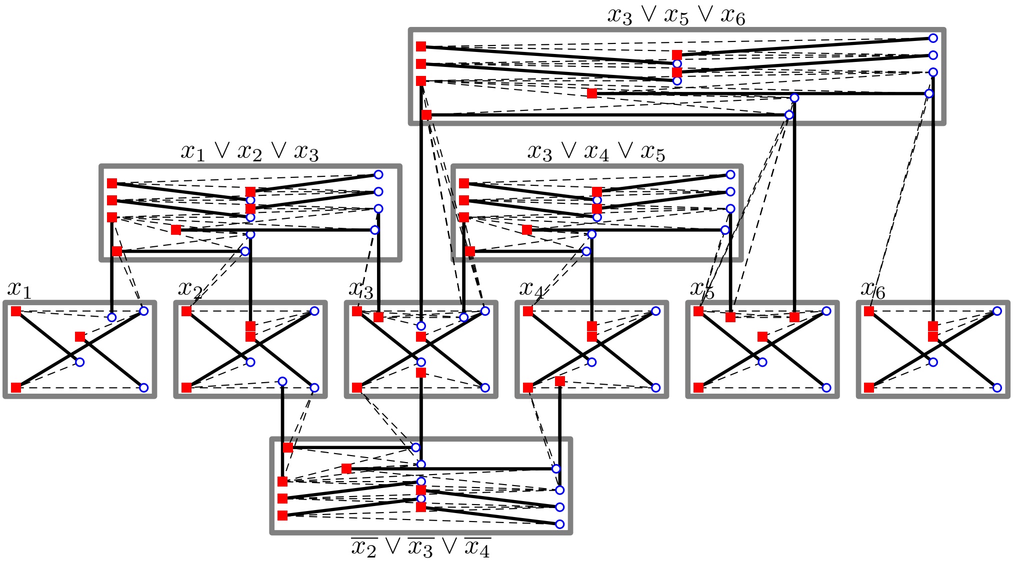
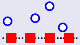


Outline

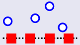
Outline

Algorithm; Improved Cubic Upper Bound

Theorem (5.8.1 [1]; 4.4.1 [1])

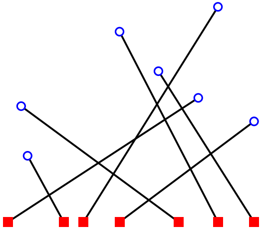


$$\mathbf{d}^{\mathtt{R}}_{\mathtt{Redonaline\ Matching}}(n) \leq \binom{n}{2} \preccurlyeq n^2$$



$$\mathbf{d}^{\emptyset}_{\mathtt{Redonaline\ Matching}}(n) \leq \binom{n}{2}\frac{n+4}{6} \preccurlyeq n^3$$

Proof of $\mathbf{d}^{\mathrm{R}}_{\texttt{Redonaline Matching}}(n) \leq \binom{n}{2}$: Removal Strategy



Algorithm: `Recursively flip`

- s_1, `the segment with crossings and with the topmost blue endpoint,`
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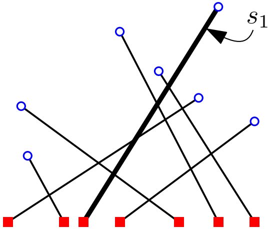

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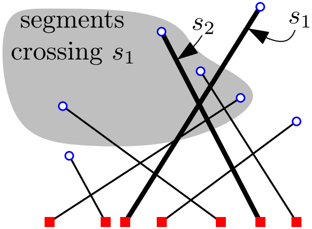

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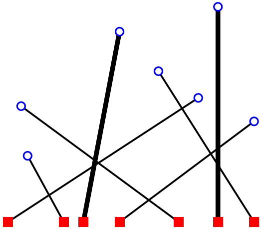

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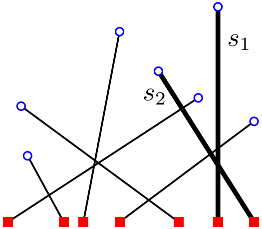

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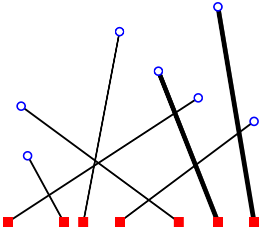

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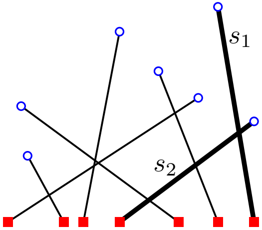

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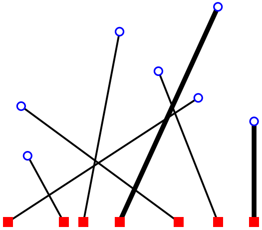

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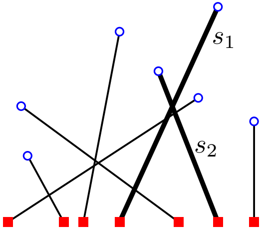

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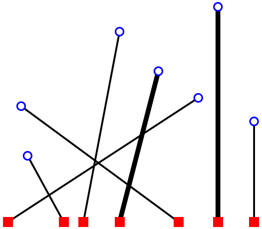

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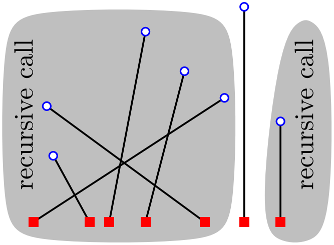

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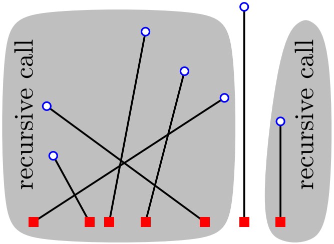

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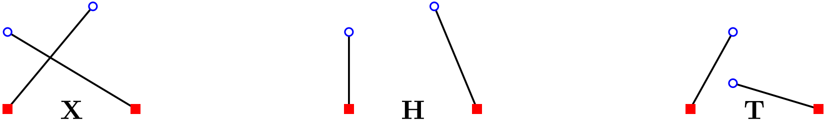
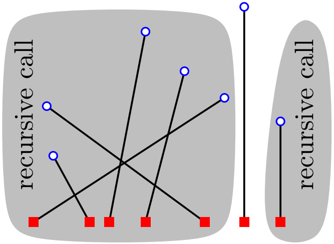
- The $\binom{n}{2}$ pairs of segments are in one of the following states.



- Does the number of $\mathbf{H}$-pairs always increase?
  - No, in general.
  - Yes, in the algorithm.

Proof of $\mathbf{d}^{\mathrm{R}}_{\texttt{Redonaline Matching}}(n) \leq \binom{n}{2}$: Removal Strategy



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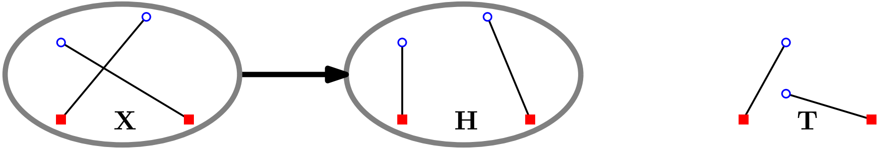

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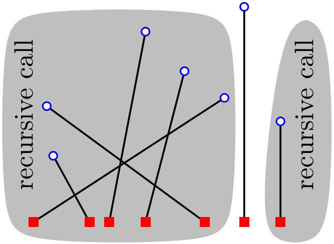

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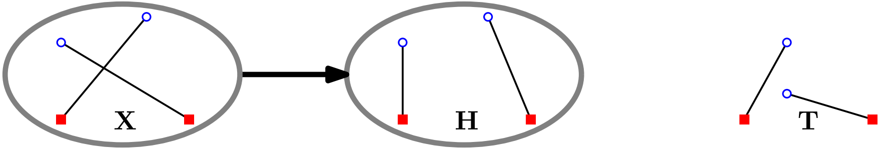

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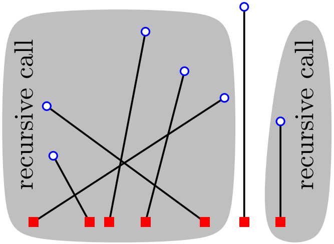

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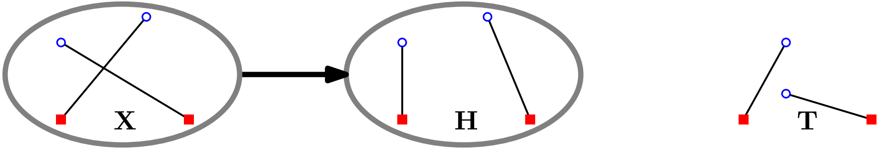

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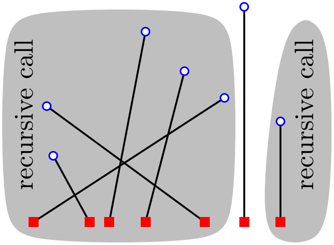

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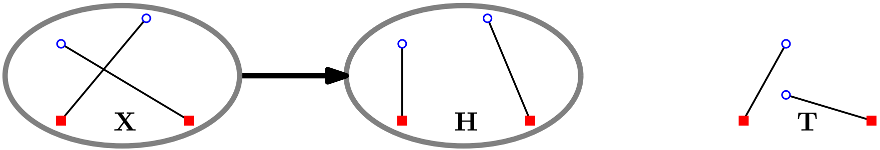

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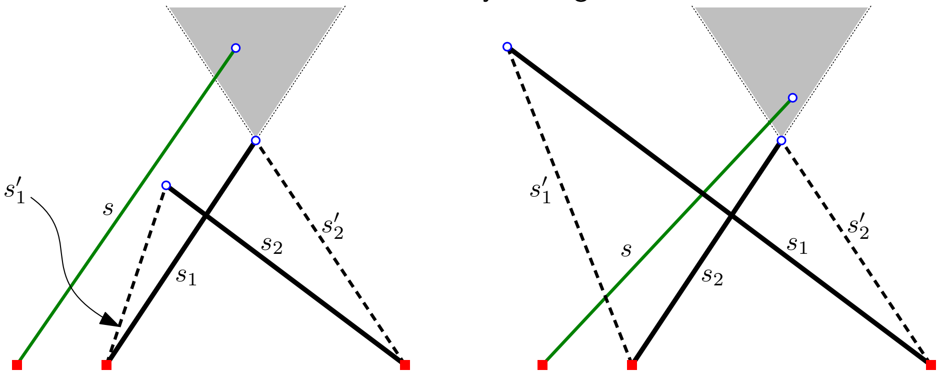
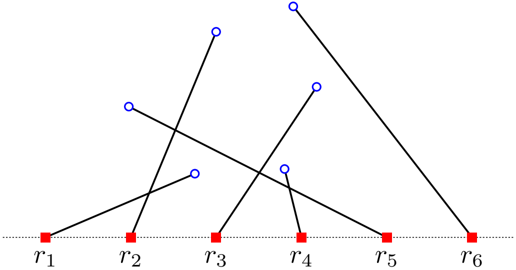
Proof of $\mathbf{d}^{\mathrm{R}}_{\texttt{Redonaline Matching}}(n) \le \binom{n}{2}$: Case Analysis

The number of $\mathbf{H}$-pairs does not increase in 2 cases which are avoided by the algorithm:



Case 1			Case 2		
s_1, s_2: $\mathbf{X}$	$\twoheadrightarrow$	s'_1, s'_2: $\mathbf{H}$	s_1, s_2: $\mathbf{X}$	$\twoheadrightarrow$	s'_1, s'_2: $\mathbf{H}$
s, s_1: $\mathbf{H}$	$\rightarrow$	s, s'_1 : $\mathbf{T}$	s, s_1: $\mathbf{X}$	$\rightarrow$	s, s'_1 : $\mathbf{X}$
s, s_2: $\mathbf{T}$	$\rightarrow$	s, s'_2 : $\mathbf{T}$	s, s_2: $\mathbf{H}$	$\rightarrow$	s, s'_2 : $\mathbf{T}$

Proof of $\mathbf{d}^{\emptyset}_{\texttt{Redonaline Matching}}(n) \leq \binom{n}{2}\frac{n+4}{6}$: Potential



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- *k-observed crossings*: pairs of segments whose projection cross.
- Crossing k-relevant pairs $\implies$ k-observed crossing.
- Φ_k: Number of k-relevant pairs forming *k-observed crossings*.

$$\Phi_k \leq k(n-k+1) - 1$$

- Φ_k decreases at each flip of a k-relevant pair, i.e., at each swap of an inversion in w.
- $\sum_{k=1}^{n} \Phi_k$ is bounded and decreases by at least 2 at each flip.

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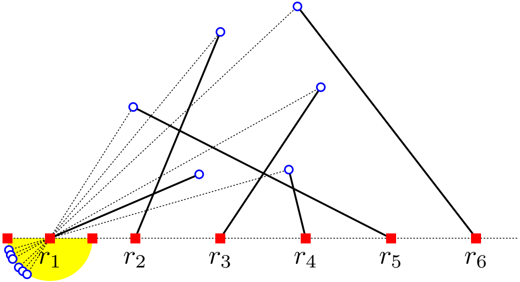

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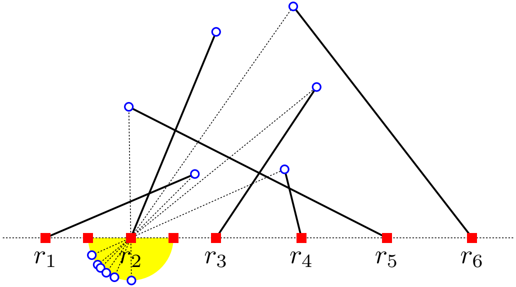

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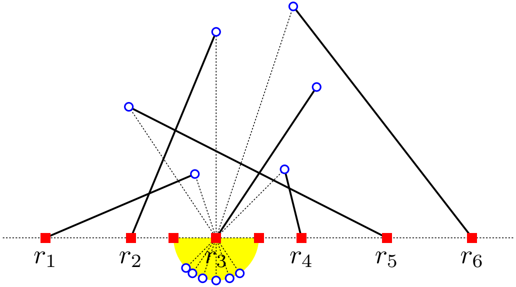

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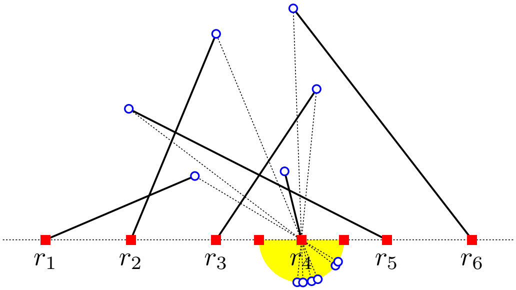

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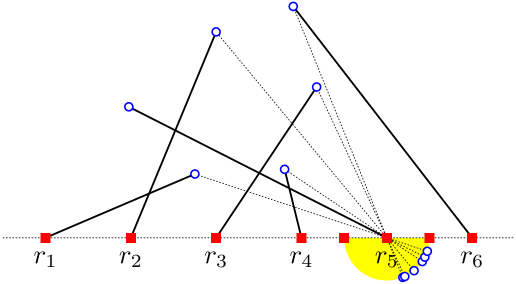

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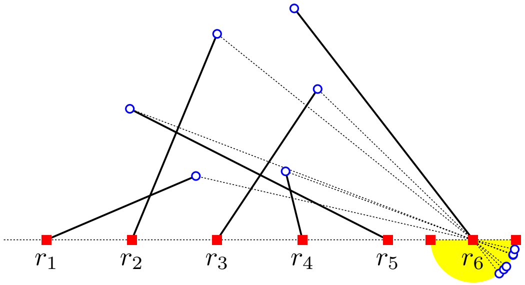

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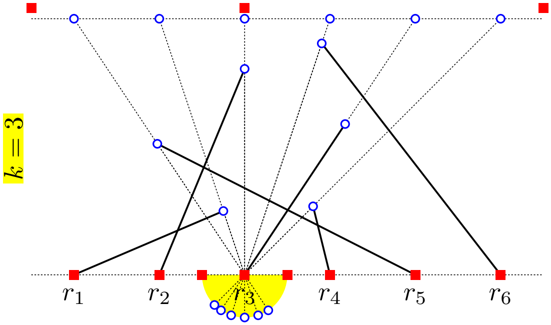

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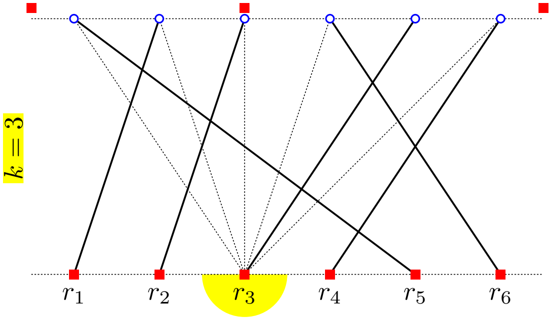

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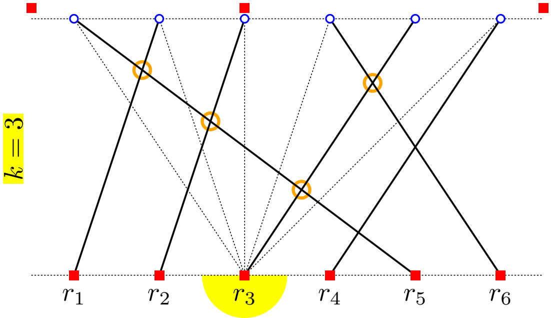

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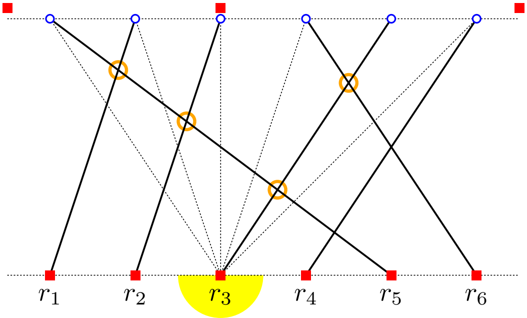
$k = 3$



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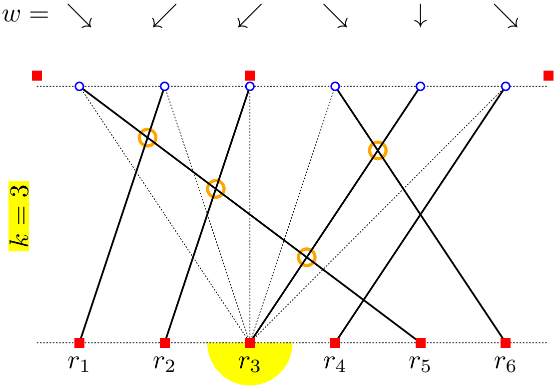

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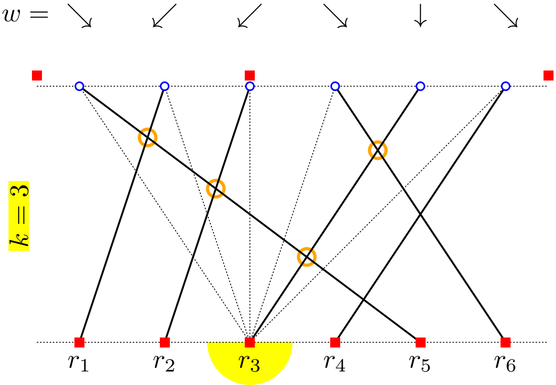

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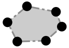
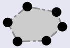
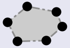
Outline

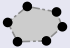
Convex Bounds

- C: the point set in convex position. 

Theorem (5.2.1 [3]; 5.3.1; 6.1.1 [3])

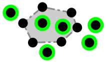
 $\mathbf{d}^{\mathrm{R}}_{\texttt{Convex Multigraph}}(n) \preccurlyeq n \log |C| \preccurlyeq n \log n$

 $\mathbf{d}^{\mathrm{R}}_{\texttt{Convex Tree}}(n) \leq 3n - 8 \preccurlyeq n$ *for* $n \geq 3$

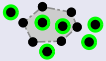
 $\mathbf{d}^{\mathrm{I}}_{\texttt{Convex Multigraph}}(n) \preccurlyeq n \log |C| \preccurlyeq n \log n$

Outline

From Convex n^2 to General n^3 Upper Bound

- $P = C \cup T$: the point set. 
- C is in convex position.
- t: sum of the degrees of the points in T.

Theorem (4.3.1 [2])

 $\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n, t) \preccurlyeq tn^2$

Proof of $\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n,t) \preccurlyeq tn^2$: a Mixed Potential

$$\Phi = \overbrace{\underbrace{\chi_{\text{crossings}}}} + \overbrace{\underbrace{\Lambda_{L'}}}$$

- L': lines through at least one non-convex point.

- Case 1. If $\chi_{\text{crossings}}$ decreases, then so does Φ (because $\Lambda_{L'}$ does not increase) ✓
- Case 2. If not:
  - Case 2.1. If p or t is non-convex: ✓
  - Case 2.2. If, say, r is non-convex: ✓
  - Case 2.3. The remaining p, q, s, t are convex:

Proof of $\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n,t) \preccurlyeq tn^2$: a Mixed Potential

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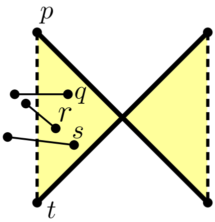

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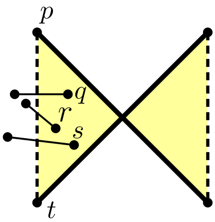

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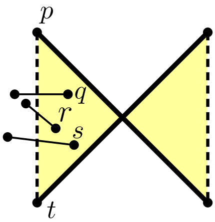

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- L': lines through at least one non-convex point.

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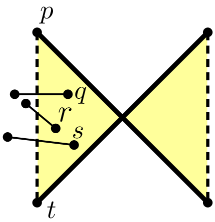
- Case 2. If not:
  - Case 2.1. If p or t is non-convex: ✓
  - Case 2.2. If, say, r is non-convex: ✓
  - Case 2.3. The remaining p, q, s, t are convex:



Proof of $\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n,t) \preccurlyeq tn^2$: a Mixed Potential

$$\Phi = \overbrace{\underbrace{\chi_{\text{crossings}}}_{\text{may not decrease!}}}^{\preccurlyeq n^2} + \overbrace{\underbrace{\Lambda_{L'}}_{\text{compensate } \chi_{\text{crossings}}?}}^{\preccurlyeq n^2 t} \preccurlyeq n^2 t$$

- L': lines through at least one non-convex point.
- Case 1. If $\chi_{\text{crossings}}$ decreases, then so does Φ (because $\Lambda_{L'}$ does not increase) ✓
- Case 2. If not:
  - Case 2.1. If p or t is non-convex: ✓
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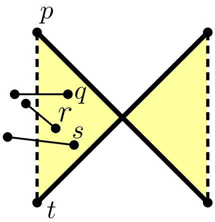
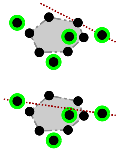


Proof of $\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n,t) \preccurlyeq tn^2$: a Mixed Potential

$$\Phi = \overbrace{\underbrace{\chi_{\text{crossings}}}_{\text{may not decrease!}}}^{\preccurlyeq n^2} + \overbrace{\underbrace{\Lambda_{L'}}_{\text{compensate } \chi_{\text{crossings}}?}}^{\preccurlyeq n^2 t} \preccurlyeq n^2 t$$



- L': $|$lines through at least one non-convex point.$| \preccurlyeq nt$

- Case 1. If $\chi_{\text{crossings}}$ decreases, then so does Φ (because $\Lambda_{L'}$ does not increase) ✓
- Case 2. If not:
  - Case 2.1. If p or t is non-convex: ✓
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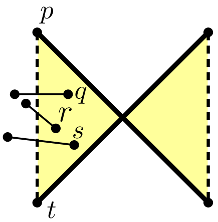
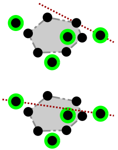
  - Case 2.3. The remaining p, q, s, t are convex:



Proof of $\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n,t) \preccurlyeq tn^2$: a Mixed Potential

$$\Phi = \overbrace{\underbrace{\chi_{\text{crossings}}}_{\text{may not decrease!}}}^{\preccurlyeq n^2} + \overbrace{\underbrace{\Lambda_{L'}}_{\text{compensate } \chi_{\text{crossings}}?}}^{\preccurlyeq n^2 t} \preccurlyeq n^2 t$$



- L': $|$lines through at least one non-convex point.$| \preccurlyeq nt$
- Case 1. If $\chi_{\text{crossings}}$ decreases, then so does Φ (because $\Lambda_{L'}$ does not increase) ✓
- Case 2. If not:
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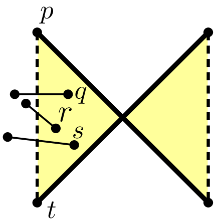
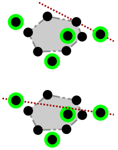


Proof of $\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n,t) \preccurlyeq tn^2$: a Mixed Potential

$$\Phi = \overbrace{\underbrace{\chi_{\text{crossings}}}_{\text{may not decrease!}}}^{\preccurlyeq n^2} + \overbrace{\underbrace{\Lambda_{L'}}_{\text{compensate } \chi_{\text{crossings}}?}}^{\preccurlyeq n^2 t} \preccurlyeq n^2 t$$



- L': $|$lines through at least one non-convex point.$| \preccurlyeq nt$

- Case 1. If $\chi_{\text{crossings}}$ decreases, then so does Φ (because $\Lambda_{L'}$ does not increase) ✓
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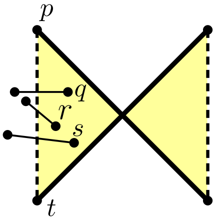
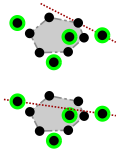


Proof of $\mathbf{d}^{\emptyset}_{\mathtt{Multigraph}}(n,t) \preccurlyeq tn^2$: a Mixed Potential

$$\Phi = \overbrace{\underbrace{\chi_{\text{crossings}}}_{\text{may not decrease!}}}^{\preccurlyeq n^2} + \overbrace{\underbrace{\Lambda_{L'}}_{\text{compensate } \chi_{\text{crossings}}?}}^{\preccurlyeq n^2 t} \preccurlyeq n^2 t$$



- L': $|$lines through at least one non-convex point.$| \preccurlyeq nt$

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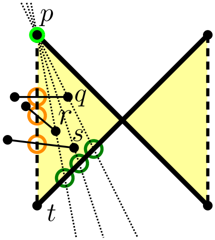
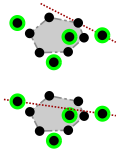
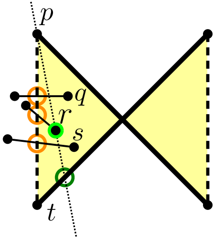


Proof of $\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n,t) \preccurlyeq tn^2$: a Mixed Potential

$$\Phi = \underbrace{\overbrace{\chi_{\text{crossings}}}^{\preccurlyeq n^2}}_{\text{may not decrease!}} + \underbrace{\overbrace{\Lambda_{L'}}^{\preccurlyeq n^2 t}}_{\text{compensate } \chi_{\text{crossings}}?} \preccurlyeq n^2 t$$



- L': |lines through at least one non-convex point.| $\preccurlyeq nt$



- Case 1. If $\chi_{\text{crossings}}$ decreases, then so does Φ (because $\Lambda_{L'}$ does not increase) ✓
- Case 2. If not:
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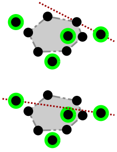
  - Case 2.2. If, say, r is non-convex: ✓
  - Case 2.3. The remaining p, q, s, t are convex:

Proof of $\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n,t) \preccurlyeq tn^2$: a Mixed Potential

$$\Phi = \overbrace{\underbrace{\chi_{\text{crossings}}}_{\text{may not decrease!}}}^{\preccurlyeq n^2} + \overbrace{\underbrace{\Lambda_{L'}}_{\text{compensate } \chi_{\text{crossings}}?}}^{\preccurlyeq n^2 t} \preccurlyeq n^2 t$$



- L': |lines through at least one non-convex point.| $\preccurlyeq nt$

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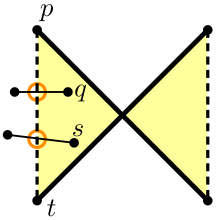
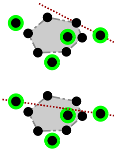
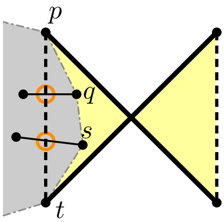


Proof of $\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n,t) \preccurlyeq tn^2$: a Mixed Potential

$$\Phi = \overbrace{\underbrace{\chi_{\text{crossings}}}_{\text{may not decrease!}}}^{\preccurlyeq n^2} + \overbrace{\underbrace{\Lambda_{L'}}_{\text{compensate } \chi_{\text{crossings}}?}}^{\preccurlyeq n^2 t} \preccurlyeq n^2 t$$



- L': $|$lines through at least one non-convex point.$| \preccurlyeq nt$



- Case 1. If $\chi_{\text{crossings}}$ decreases, then so does Φ (because $\Lambda_{L'}$ does not increase) ✓
- Case 2. If not:
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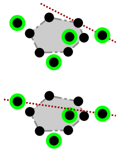
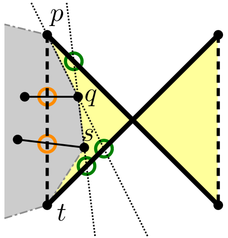
  - Case 2.3. The remaining p, q, s, t are convex:

Proof of $\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n,t) \preccurlyeq tn^2$: a Mixed Potential

$$\Phi = \overbrace{\underbrace{\chi_{\text{crossings}}}_{\text{may not decrease!}}}^{\preccurlyeq n^2} + \overbrace{\underbrace{\Lambda_{L'}}_{\text{compensate } \chi_{\text{crossings}}?}}^{\preccurlyeq n^2 t} \preccurlyeq n^2 t$$



- L': $|$lines through at least one non-convex point.$|$ $\preccurlyeq nt$

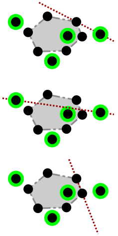


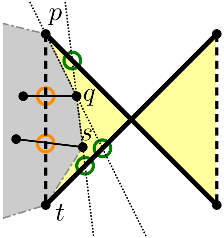
- Case 1. If $\chi_{\text{crossings}}$ decreases, then so does Φ (because $\Lambda_{L'}$ does not increase) ✓
- Case 2. If not:
  - Case 2.1. If p or t is non-convex: ✓
  - Case 2.2. If, say, r is non-convex: ✓
  - Case 2.3. The remaining p, q, s, t are convex:

Proof of $\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n,t) \preccurlyeq tn^2$: a Mixed Potential

$$\Phi = \overbrace{\underbrace{\chi_{\text{crossings}}}_{\text{may not decrease!}}}^{\preccurlyeq n^2} + \overbrace{\underbrace{\Lambda_{L'}}_{\text{compensate } \chi_{\text{crossings}}?}}^{\preccurlyeq n^2 t} \preccurlyeq n^2 t$$

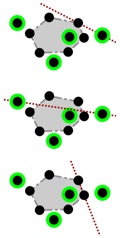
- L': $|$lines through at least one non-convex point.$|$ $\preccurlyeq nt$
  $\cup$ $|$lines through two consecutive convex points.$|$ $\preccurlyeq n$
- Case 1. If $\chi_{\text{crossings}}$ decreases, then so does Φ (because $\Lambda_{L'}$ does not increase) ✓
- Case 2. If not:
  - Case 2.1. If p or t is non-convex: ✓
  - Case 2.2. If, say, r is non-convex: ✓
  - Case 2.3. The remaining p, q, s, t are convex:





Proof of $\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n,t) \preccurlyeq tn^2$: a Mixed Potential

$$\Phi = \overbrace{\underbrace{\chi_{\text{crossings}}}_{\text{may not decrease!}}}^{\preccurlyeq n^2} + \overbrace{\underbrace{\Lambda_{L'}}_{\text{compensate } \chi_{\text{crossings}}?}}^{\preccurlyeq n^2 t} \preccurlyeq n^2 t$$



- L': $|$lines through at least one non-convex point.$|$ $\preccurlyeq nt$
  $\cup$ $|$lines through two consecutive convex points.$|$ $\preccurlyeq n$
- Case 1. If $\chi_{\text{crossings}}$ decreases, then so does Φ (because $\Lambda_{L'}$ does not increase) ✓
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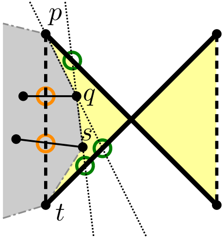
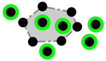
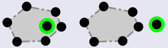
  - Case 2.2. If, say, r is non-convex: ✓
  - Case 2.3. The remaining p, q, s, t are convex: ✓

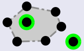


Adding Non-Convex Points One by One, with Removal Choice

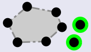
- $P = C \cup T$: the point set. 
- C is in convex position.
- t: sum of the degrees of the points in T.

Theorem (5.4.2 [3]; 5.5.2 [3]; 5.6.1 [3]; 5.7.1 [3])

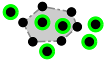
 $\mathbf{d}^{\mathrm{R}}_{|\mathtt{T}|=1\ \mathtt{Multigraph}}(n,t) \preccurlyeq n \log |C| + tn \preccurlyeq n \log n + tn$

 $\mathbf{d}^{\mathrm{R}}_{\mathtt{Inout\ Multigraph}}(n,t) \preccurlyeq t^2 n + n \log n$

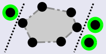
 $\mathbf{d}^{\mathrm{R}}_{\mathtt{Inin\ Multigraph}}(n,t) \preccurlyeq tn + n \log n$

 $\mathbf{d}^{\mathrm{R}}_{\mathtt{Outout\ Multigraph}}(n,t) \preccurlyeq 2^t n \log n$

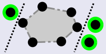
Near Convex with Insertion Choice

- $P = C \cup T$: the point set. 
- C is in convex position.
- t: sum of the degrees of the points in T.

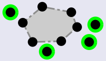
Theorem (6.2.1 [3]; 7.1.1 [3]; 7.2.3 [3])



$$\mathbf{d}^{\mathtt{I}}_{\mathtt{Separated\ Multigraph}}(n,t) \preccurlyeq t\,|P| \log |C| + n \log |C| \preccurlyeq tn \log n$$



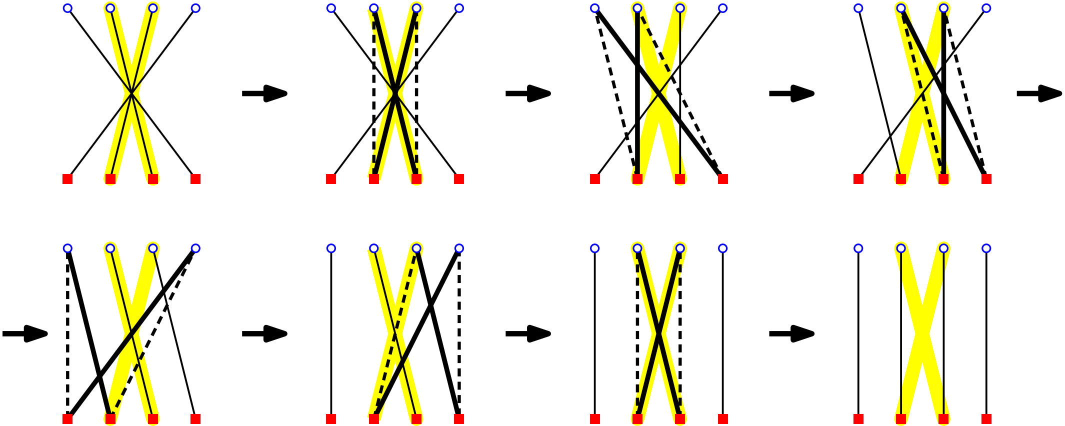
$$\mathbf{d}^{\mathtt{RI}}_{\mathtt{Separated\ Multigraph}}(n,t) \preccurlyeq n + t\,|P| \preccurlyeq tn$$



$$\mathbf{d}^{\mathtt{RI}}_{\mathtt{Allout\ Matching}}(n,t) \preccurlyeq t^3 n$$

Outline

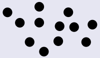
The Same Flip Used Multiple Times in a Sequence



Counting Flips without Multiplicity

Theorem (4.5.1 [2])

In the `Multigraph` *version, any untangle sequence of* n *segments has* $O(n^{8/3})$ *distinct flips, i.e. :*



$$\left\{\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n)\right\}_{\textit{distinct}} \preccurlyeq n^{8/3}.$$

Proof of $\{\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n)\}_{\text{distinct}} \preccurlyeq n^{8/3}$: Balancing Argument

- There are $O(\frac{n^3}{k})$ flips decreasing Λ_L by at least k.
- There are $O(n^2k^2)$ flips decreasing Λ_L by less than k: we enumerate them by sweeping a line.
- We choose $k = n^{1/3}$.

Proof of $\{\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n)\}_{\text{distinct}} \preccurlyeq n^{8/3}$: Balancing Argument

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Proof of $\{\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n)\}_{\text{distinct}} \preccurlyeq n^{8/3}$: Balancing Argument

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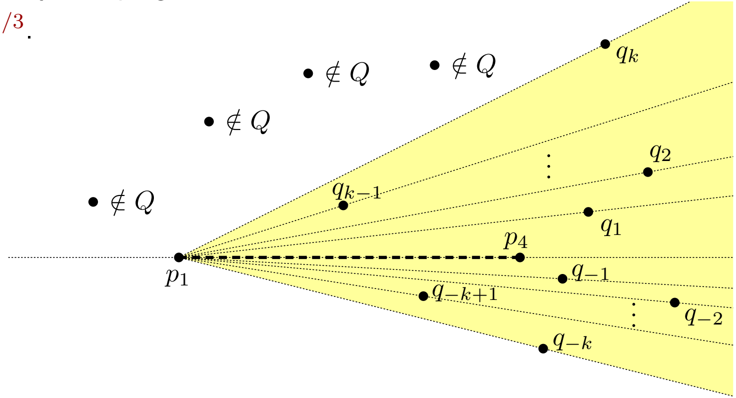
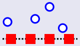


Outline

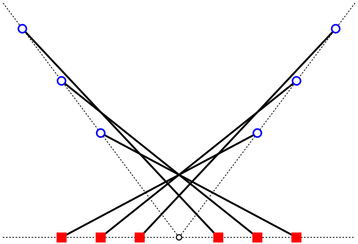
Outline

No Choice Lower Bound: Butterfly

Theorem (4.2.1 [1])

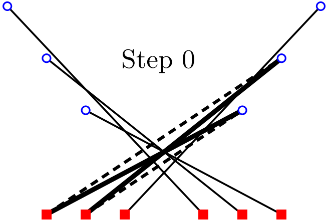
$$n^2 \precsim \frac{3}{2}\binom{n}{2} - \frac{n}{4} \leq \mathbf{d}^{\emptyset}_{\texttt{Redonaline Matching}}(n)$$  *for even* n

Proof of $\frac{3}{2}\binom{n}{2} - \frac{n}{4} \leq \mathbf{d}^{\emptyset}_{\texttt{Redonaline Matching}}(n)$: Butterfly



- Example of an untangle sequence of $n = 6$ segments using more than $\binom{n}{2} = 15$ flips.
- No *shortcut*.
- Half the pairs of segments are flipped twice, i.e., $\mathbf{X} \twoheadrightarrow \mathbf{H} \rightarrow \mathbf{T} \rightarrow \mathbf{X} \twoheadrightarrow \mathbf{H}$.
- Bubble sort on the 3 segments from the 3 leftmost red points.
- 6 $\mathbf{H}$-pairs turn into $\mathbf{T}$-pairs, i.e., 6 $\mathbf{H} \rightarrow \mathbf{T}$.
- 2 $\mathbf{H} \rightarrow \mathbf{T}$ and 2 $\mathbf{T} \rightarrow \mathbf{X}$.
- 4 $\mathbf{T} \rightarrow \mathbf{X}$.

Proof of $\frac{3}{2}\binom{n}{2} - \frac{n}{4} \leq \mathbf{d}^{\emptyset}_{\texttt{Redonaline Matching}}(n)$: Butterfly



- Example of an untangle sequence of $n = 6$ segments using more than $\binom{n}{2} = 15$ flips.
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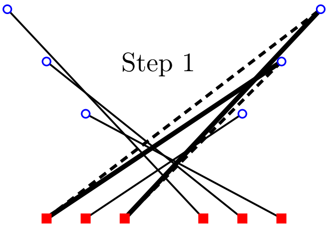
- Bubble sort on the 3 segments from the 3 leftmost red points.
- 6 $\mathbf{H}$-pairs turn into $\mathbf{T}$-pairs, i.e., 6 $\mathbf{H} \rightarrow \mathbf{T}$.
- 2 $\mathbf{H} \rightarrow \mathbf{T}$ and 2 $\mathbf{T} \rightarrow \mathbf{X}$.
- 4 $\mathbf{T} \rightarrow \mathbf{X}$.

Proof of $\frac{3}{2}\binom{n}{2} - \frac{n}{4} \leq \mathbf{d}^{\emptyset}_{\texttt{Redonaline Matching}}(n)$: Butterfly



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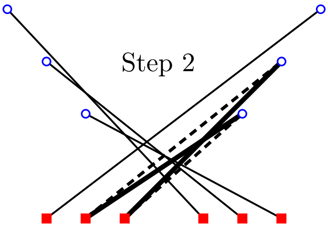

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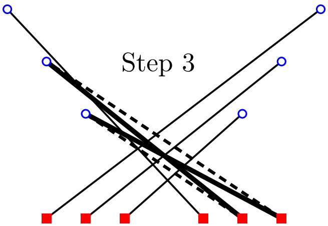

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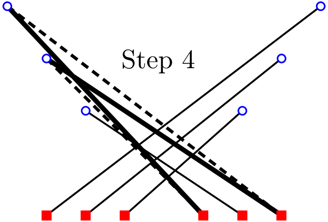

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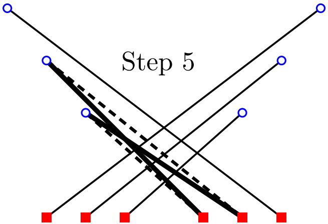

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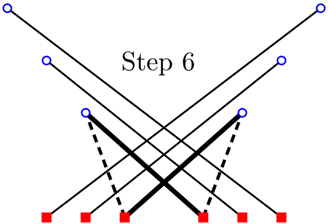

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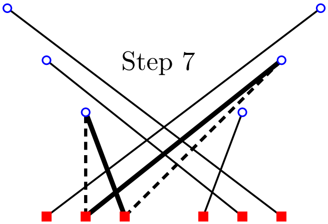

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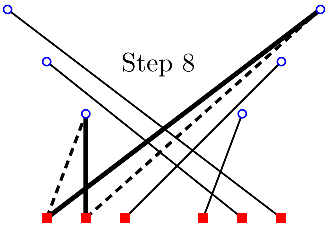

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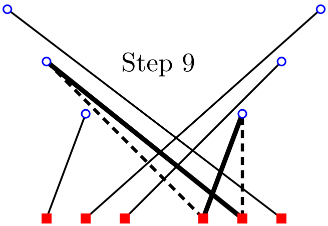

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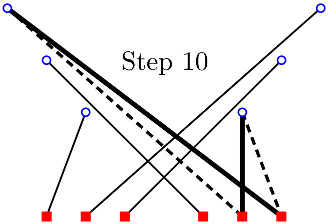

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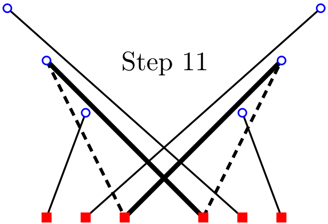

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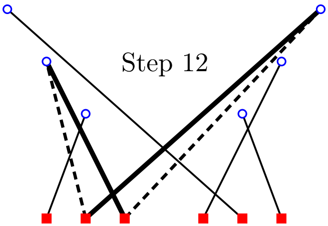

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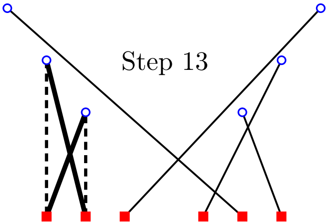

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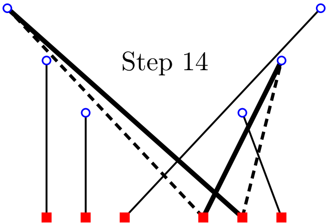

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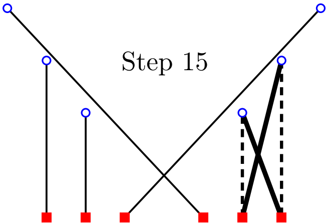

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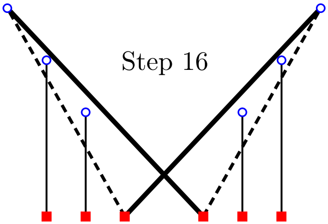

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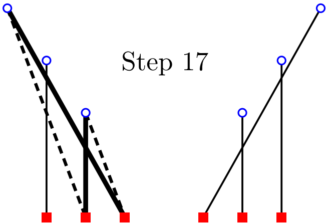

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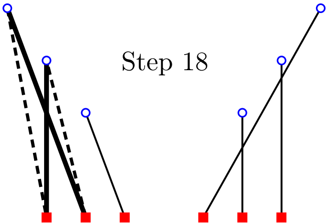

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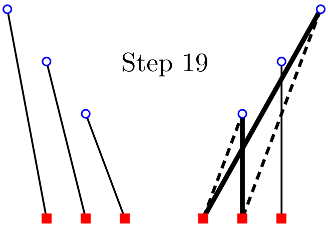

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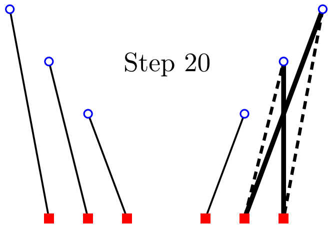

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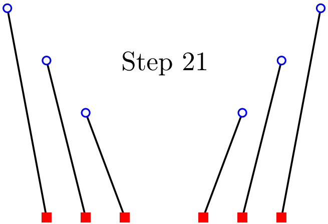

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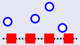
Outline

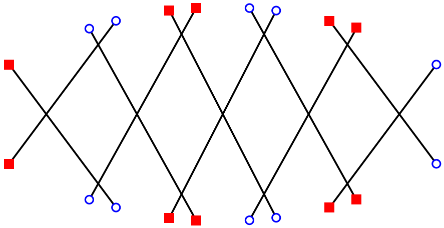
Removal Choice Lower Bound: Fence

Theorem (5.1.1 [1])

$$n \precsim \frac{3}{2}n - 2 \leq \mathbf{d}^{\mathtt{R}}_{\mathtt{Convex\ Bipartite\ Matching}}(n)$$  *for even* n

Proof of $\frac{3}{2}n - 2 \leq \mathbf{d}^{\mathrm{R}}_{\texttt{Convex Bipartite Matching}}(n)$: Fence

- Any untangle sequence of a *fence* uses one flip per crossing.



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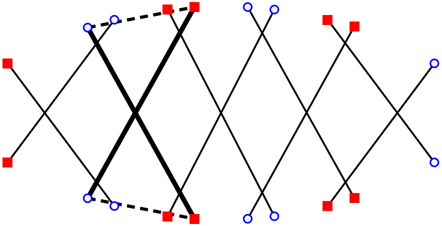
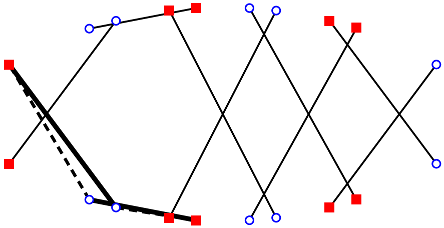


Proof of $\frac{3}{2}n - 2 \leq \mathbf{d}^{\mathrm{R}}_{\texttt{Convex Bipartite Matching}}(n)$: Fence

- Any untangle sequence of a *fence* or a *derived fence* uses one flip per crossing.



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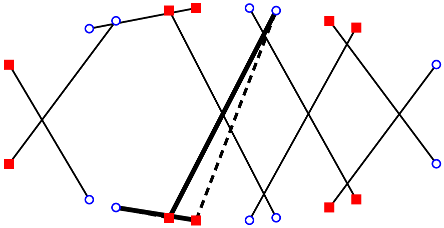
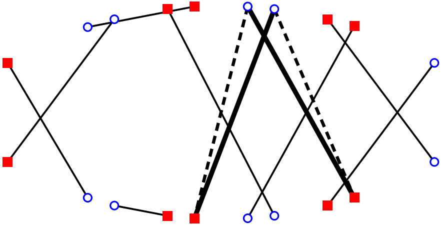
- Any untangle sequence of a *fence* or a *derived fence*
uses one flip per crossing.



Proof of $\frac{3}{2}n - 2 \leq \mathbf{d}^{\mathrm{R}}_{\texttt{Convex Bipartite Matching}}(n)$: Fence

- Any untangle sequence of a *fence* or a *derived fence* uses one flip per crossing.



Proof of $\frac{3}{2}n - 2 \leq \mathbf{d}^{\mathtt{R}}_{\mathtt{Convex\ Bipartite\ Matching}}(n)$: Fence

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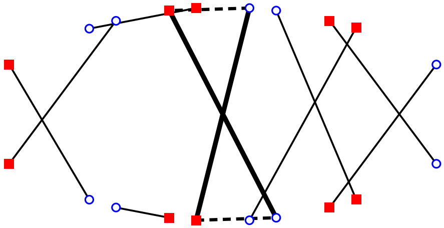

Proof of $\frac{3}{2}n - 2 \leq \mathbf{d}^{\mathtt{R}}_{\mathtt{Convex\ Bipartite\ Matching}}(n)$: Fence

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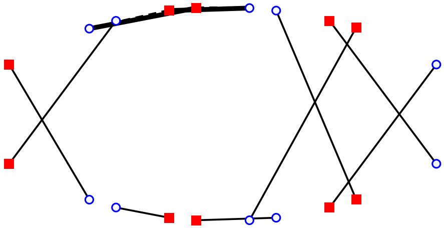

Proof of $\frac{3}{2}n - 2 \leq \mathbf{d}^{\mathtt{R}}_{\mathtt{Convex\ Bipartite\ Matching}}(n)$: Fence

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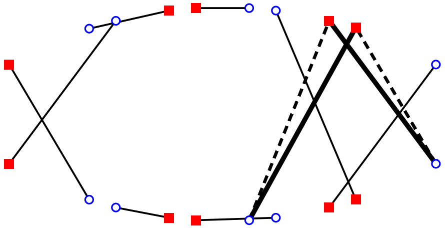

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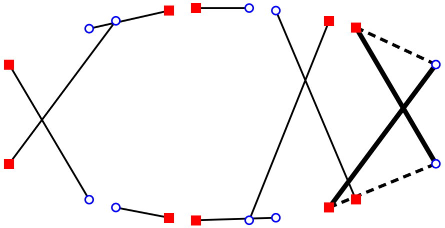

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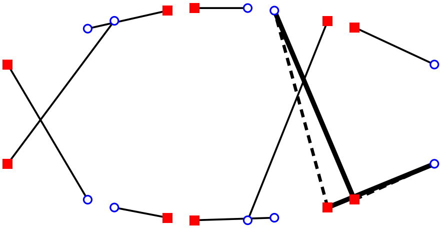

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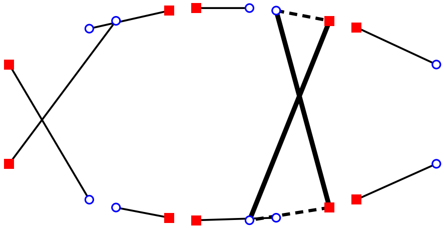

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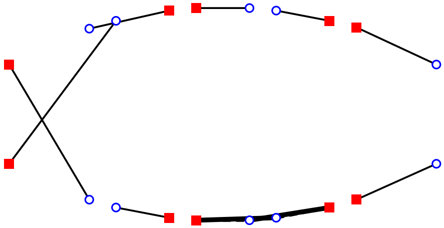

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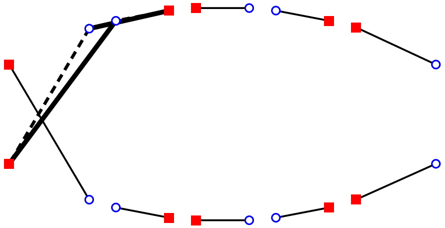

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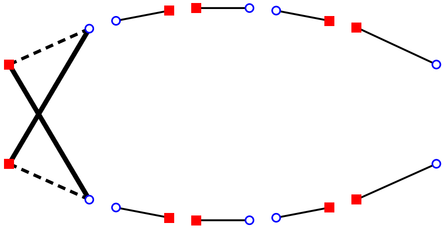

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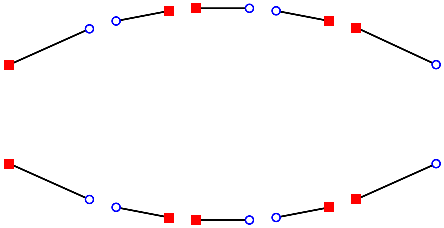

Outline

Outline

Trivial Reductions

Lemma (2.3.1 [2]; 2.3.2 [2]; 2.3.3 [2])

The following inequalities hold for any non-negative integer n, and for any two properties Π, Π' such that $\Pi \implies \Pi'$, and for any Choices $\in \{\emptyset, \mathtt{R}, \mathtt{I}, \mathtt{RI}\}$.

$$\mathbf{d}_{\Pi}^{\mathtt{RI}}(n) \leq \begin{Bmatrix} \mathbf{d}_{\Pi}^{\mathtt{R}}(n) \\ \mathbf{d}_{\Pi}^{\mathtt{I}}(n) \end{Bmatrix} \leq \mathbf{d}_{\Pi}^{\emptyset}(n) \quad \textit{(choice reductions)}$$

$$\mathbf{d}_{\Pi}^{\textit{Choices}}(n) \leq \mathbf{d}_{\Pi'}^{\textit{Choices}}(n) \quad \textit{(property reductions)}$$

$$\begin{aligned} \mathbf{d}_{\Pi\ \mathtt{Matching}}^{\mathtt{RI}}(n) &\leq \mathbf{d}_{\Pi\ \mathtt{Bipartite\ Matching}}^{\mathtt{R}}(n) \\ \mathbf{d}_{\Pi\ \mathtt{Matching}}^{\mathtt{I}}(n) &\leq \mathbf{d}_{\Pi\ \mathtt{Bipartite\ Matching}}^{\emptyset}(n) \end{aligned} \quad \textit{(transfer reductions)}$$

Outline

No Choice Reductions

Theorem (4.1.1 [2])

For all n and for Π being either the empty property or the Convex *property:*

$$\mathbf{d}^{\emptyset}_{\Pi\ \texttt{Multigraph}}(n) = \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(n),$$

$$2\mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Bipartite Matching}}(2n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(2n),$$

$$2\mathbf{d}^{\emptyset}_{\Pi\ \texttt{Bipartite Matching}}(n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Cycle}}(3n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(3n),$$

$$2\mathbf{d}^{\emptyset}_{\Pi\ \texttt{Bipartite Matching}}(n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Tree}}(3n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(3n).$$

- Given a flip sequence of the left-hand-side of an inequality, we build a flip sequence of the right-hand-side of the inequality.
- Immediate for black $\leq$.

No Choice Reductions

Theorem (4.1.1 [2])

For all n and for Π being either the empty property or the Convex *property:*

$$\begin{aligned}
\mathbf{d}^{\emptyset}_{\Pi\ \texttt{Multigraph}}(n) &\leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Multigraph}}(n),\\
2\mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(n) &\leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Bipartite Matching}}(2n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(2n),\\
2\mathbf{d}^{\emptyset}_{\Pi\ \texttt{Bipartite Matching}}(n) &\leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Cycle}}(3n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(3n),\\
2\mathbf{d}^{\emptyset}_{\Pi\ \texttt{Bipartite Matching}}(n) &\leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Tree}}(3n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(3n).
\end{aligned}$$

- Given a flip sequence of the left-hand-side of an inequality, we build a flip sequence of the right-hand-side of the inequality.
- Immediate for black $\leq$.

No Choice Reductions

Theorem (4.1.1 [2])

For all n and for Π being either the empty property or the `Convex` *property:*

$$\mathbf{d}^{\emptyset}_{\Pi\ \texttt{Multigraph}}(n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Multigraph}}(n),$$

$$2\mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Bipartite Matching}}(2n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(2n),$$

$$2\mathbf{d}^{\emptyset}_{\Pi\ \texttt{Bipartite Matching}}(n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Cycle}}(3n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(3n),$$

$$2\mathbf{d}^{\emptyset}_{\Pi\ \texttt{Bipartite Matching}}(n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Tree}}(3n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(3n).$$

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- Immediate for black $\leq$.

No Choice Reductions

Theorem (4.1.1 [2])

For all n and for Π being either the empty property or the Convex *property:*

$$\mathbf{d}^{\emptyset}_{\Pi\ \texttt{Multigraph}}(n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Multigraph}}(n),$$

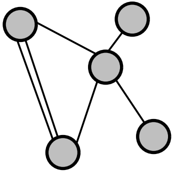
$$2\mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Bipartite Matching}}(2n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(2n),$$

$$2\mathbf{d}^{\emptyset}_{\Pi\ \texttt{Bipartite Matching}}(n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Cycle}}(3n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(3n),$$

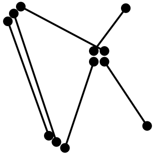
$$2\mathbf{d}^{\emptyset}_{\Pi\ \texttt{Bipartite Matching}}(n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Tree}}(3n) \leq \mathbf{d}^{\emptyset}_{\Pi\ \texttt{Matching}}(3n).$$

- Given a flip sequence of the left-hand-side of an inequality, we build a flip sequence of the right-hand-side of the inequality.
- Immediate for black $\leq$.

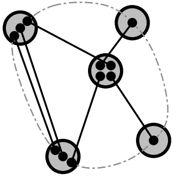
Proof of $\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n) \leq \mathbf{d}^{\emptyset}_{\texttt{Matching}}(n)$



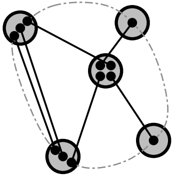
Proof of $\mathbf{d}^{\emptyset}_{\texttt{Multigraph}}(n) \leq \mathbf{d}^{\emptyset}_{\texttt{Matching}}(n)$

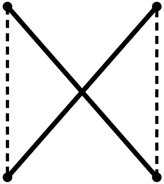


Proof of $\mathbf{d}^{\emptyset}_{\texttt{Convex Multigraph}}(n) \leq \mathbf{d}^{\emptyset}_{\texttt{Convex Matching}}(n)$

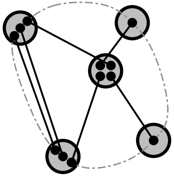


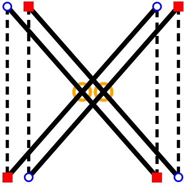
Proof of $2\mathbf{d}^{\emptyset}_{\texttt{Matching}}(n) \leq \mathbf{d}^{\emptyset}_{\texttt{Bipartite Matching}}(2n)$



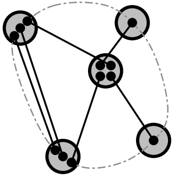


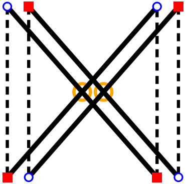
Proof of $2\mathbf{d}^{\emptyset}_{\texttt{Matching}}(n) \leq \mathbf{d}^{\emptyset}_{\texttt{Bipartite Matching}}(2n)$

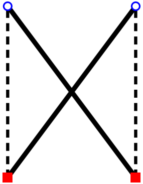




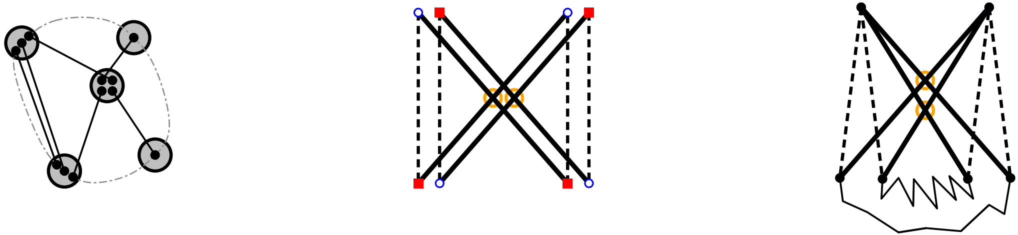
Proof of $2\mathbf{d}^{\emptyset}_{\texttt{Bipartite Matching}}(n) \leq \mathbf{d}^{\emptyset}_{\texttt{Cycle}}(3n)$



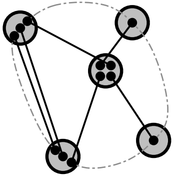


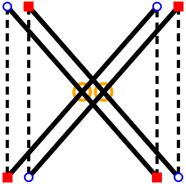


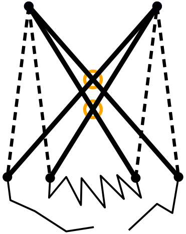
Proof of $2\mathbf{d}^{\emptyset}_{\texttt{Bipartite Matching}}(n) \leq \mathbf{d}^{\emptyset}_{\texttt{Cycle}}(3n)$



Proof of $2\mathbf{d}^{\emptyset}_{\texttt{Bipartite Matching}}(n) \leq \mathbf{d}^{\emptyset}_{\texttt{Tree}}(3n)$







Outline

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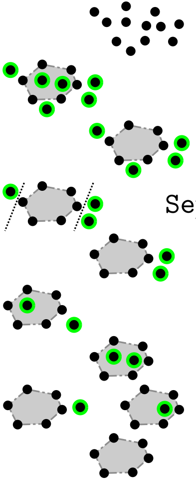
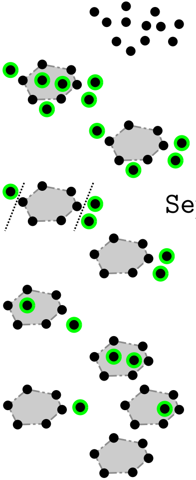
Asymptotic Bounds for Multigraphs (Part 1)



General	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq n^2$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq \boldsymbol{n^2}$
C ∪ T	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq n^2$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq n^2$
Allout	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq n^2$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq n^2$
Separated	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq \boldsymbol{tn}$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq \boldsymbol{tn \log n}$
Outout	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq tn$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq tn \log n$
Inout	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq t^2 n$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq n^2$
Inin	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq tn$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq n^2$
$\lvert \mathrm{T} \rvert = 1$	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq tn$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq n^2$
Convex	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq \boldsymbol{n}$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq \boldsymbol{n \log n}$

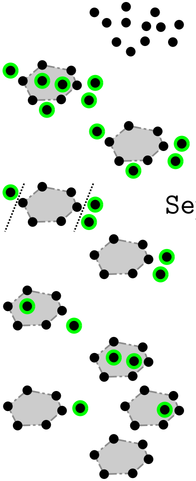
Asymptotic Bounds for Multigraphs (Part 2)



General	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n^3$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq \boldsymbol{n^3}$
C ∪ T	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq tn^2$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq \boldsymbol{tn^2}$
Allout	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq tn^2$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Separated	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq tn^2$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Outout	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq \boldsymbol{2^t n \log n}$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Inout	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq \boldsymbol{t^2 n + n \log n}$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Inin	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq \boldsymbol{tn + n \log n}$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
$\lvert \mathrm{T} \rvert = 1$	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq \boldsymbol{tn + n \log n}$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Convex	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq \boldsymbol{n \log n}$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq \boldsymbol{n^2}$

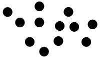
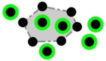
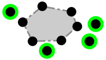
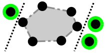
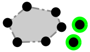
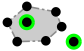
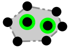
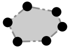
Outline

Asymptotic Bounds for Matchings (Part 1)



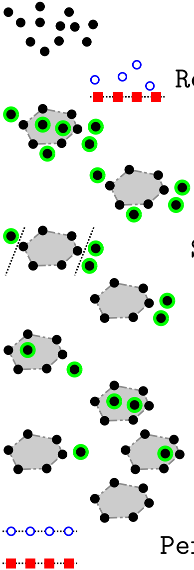
General	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq n^2$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq n^2$
$\mathtt{C} \cup \mathtt{T}$	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq n^2$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq n^2$
Allout	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq \boldsymbol{t^3 n}$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq n^2$
Separated	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq tn$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq tn \log n$
Outout	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq n$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq n \log n$
Inout	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq n$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq n^2$
Inin	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq n$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq n^2$
$\lvert \mathtt{T} \rvert = 1$	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq n$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq n^2$
Convex	$n \preccurlyeq \mathrm{d}^{\mathrm{RI}} \preccurlyeq n$	$n \preccurlyeq \mathrm{d}^{\mathrm{I}} \preccurlyeq n \log n$

Asymptotic Bounds for Matchings (Part 2)

	General	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n^3$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq n^3$
	C ∪ T	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq tn^2$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
	Allout	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq tn^2$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
	Separated	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq tn^2$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
	Outout	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n \log n$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
	Inout	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n \log n$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
	Inin	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n \log n$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
	$\lvert \mathrm{T} \rvert = 1$	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n \log n$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
	Convex	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n \log n$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq n^2$

Outline

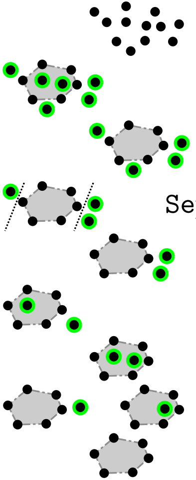
Asymptotic Bounds for Bipartite Matchings



General	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n^3$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq n^3$
Redonaline	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq \boldsymbol{n^2}$	$\boldsymbol{n^2} \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq \boldsymbol{n^3}$
C ∪ T	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq tn^2$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Allout	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq tn^2$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Separated	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq tn^2$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Outout	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Inout	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Inin	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
$\lvert\mathrm{T}\rvert = 1$	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Convex	$\boldsymbol{n} \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq \boldsymbol{n}$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq n^2$
Permutation	$\boldsymbol{n} \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n$	$\boldsymbol{n^2} \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq n^2$

Outline

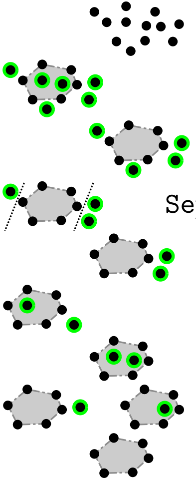
Asymptotic Bounds for Cycles



General	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n^3$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq n^3$
C ∪ T	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq tn^2$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Allout	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq tn^2$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Separated	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq tn^2$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Outout	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Inout	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Inin	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
$\lvert T \rvert = 1$	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Convex	$\boldsymbol{n} \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq \boldsymbol{n}$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq n^2$

Outline

Asymptotic Bounds for Trees



General	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq n^3$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq n^3$
C ∪ T	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq tn^2$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Allout	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq tn^2$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Separated	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq tn^2$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Outout	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq 2^t n$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Inout	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq t^2 n$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Inin	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq tn$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
$\lvert T \rvert = 1$	$n \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq tn$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq tn^2$
Convex	$\boldsymbol{n} \preccurlyeq \mathrm{d}^{\mathrm{R}} \preccurlyeq \boldsymbol{n}$	$n^2 \preccurlyeq \mathrm{d}^{\emptyset} \preccurlyeq n^2$

Outline

Big Conjectures: Tight Convex Bounds?

$$n^2 \preccurlyeq \mathbf{d}^{\emptyset}_{\texttt{Multigraph}} \preccurlyeq n^2 \preccurlyeq n^3$$

$$n \preccurlyeq \mathbf{d}^{\texttt{R}}_{\texttt{Multigraph}} \preccurlyeq n \text{ or } n \log n \preccurlyeq n^3$$

$$n \preccurlyeq \mathbf{d}^{\texttt{I}}_{\texttt{Multigraph}} \preccurlyeq n \text{ or } n \log n \preccurlyeq n^2$$

$$n \preccurlyeq \mathbf{d}^{\texttt{RI}}_{\texttt{Multigraph}} \preccurlyeq n \preccurlyeq n^2$$

Similar NP-Hard Problems?

We know $\mathcal{NP}$-hardness for:

- The shortest untangle sequence in the `Bipartite Matching` version.

We conjecture $\mathcal{NP}$-hardness for:

- The shortest untangle sequence in all other versions.
- The longest untangle sequence in all versions.

We do not know $\mathcal{NP}$-hardness for:

- The shortest/longest untangle sequence in any version for `Convex` point sets.

Near-Convex: Smooth Transitions without Point Set Restrictions?

- Smooth transitions between `Convex` and `General` point sets?
- No restriction on the number/position of non-convex points?

Miscellaneous Questions

- Which bound is tight?

$$n \preccurlyeq \mathbf{d}^{\mathtt{R}}_{\mathtt{Convex\ Multigraph}}(n) \preccurlyeq n \log n$$

$$n \preccurlyeq \mathbf{d}^{\mathtt{I}}_{\mathtt{Convex\ Multigraph}}(n) \preccurlyeq n \log n$$

- Why a bound specific to Matching?
- A sub-quadratic upper bound on the reuse of a given flip?
- Removal choice to control flip reuse? ($\rightarrow$ sub-cubic upper bound on $\mathbf{d}^{\mathtt{R}}_{\mathtt{Multigraph}}$)
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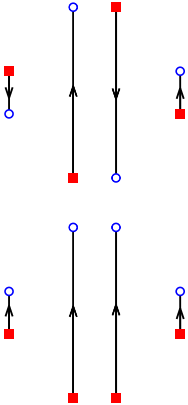
Outline

Arrows and Shortcuts



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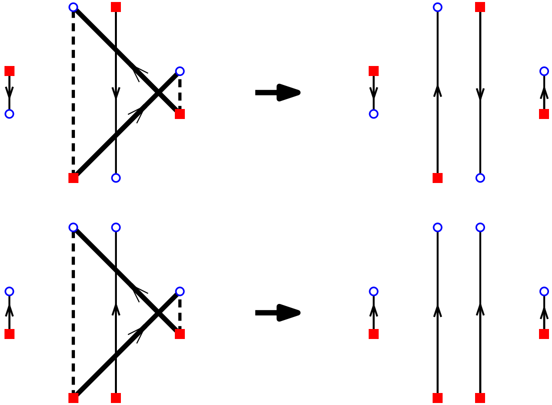

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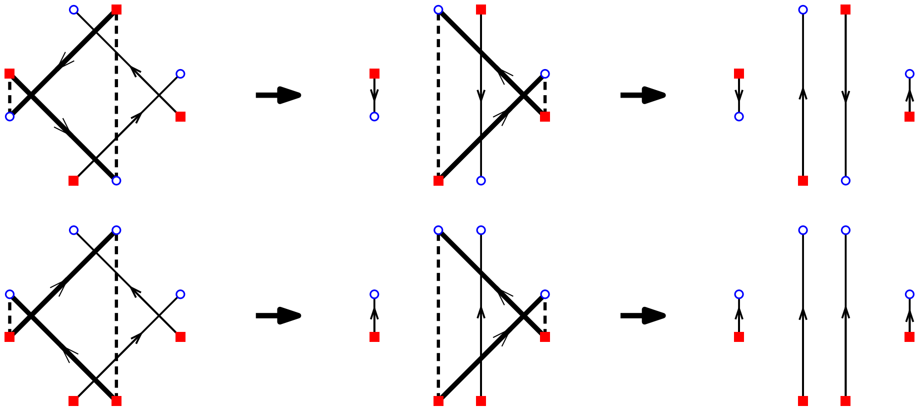

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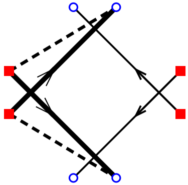

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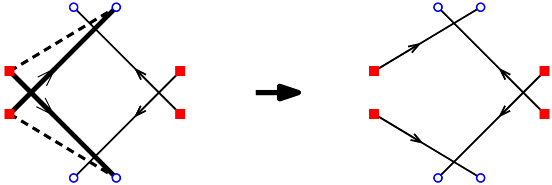

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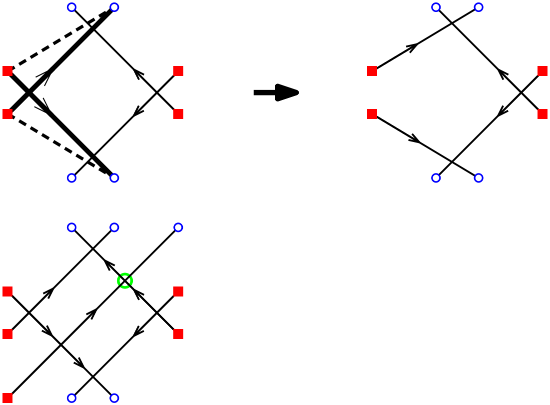

Swapping Flips via State Tracking: $\mathbf{d}^{\mathrm{R}}_{\texttt{Bipartite Matching}} \preccurlyeq n^2$?

- A labeled bipartite matching = a permutation.
- A flip = a special transposition.
- Example of a flip sequence:

$$(1\ 2)(3\ 4)(2\ 3)$$

- Swapping two transpositions:

$$(ab)(ab) = \mathrm{Id} \quad (1)$$
$$(ab)(cd) = (cd)(ab) \quad (2)$$
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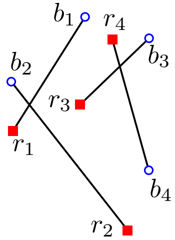
- Is it possible to swap and cancel flips?
- Yes, in our experiments on the butterfly.



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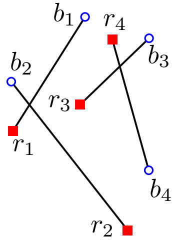

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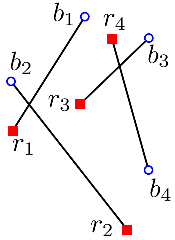

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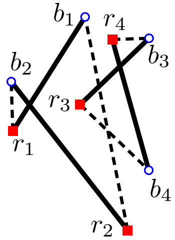

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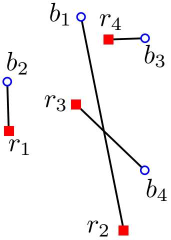

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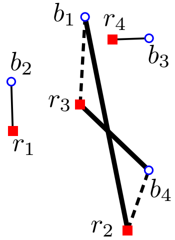

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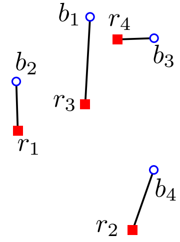

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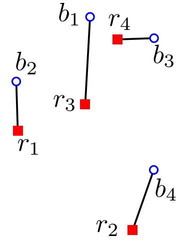

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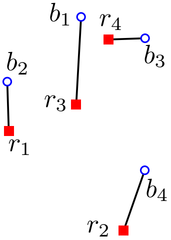

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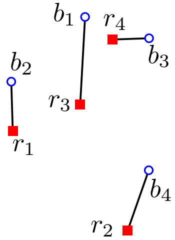

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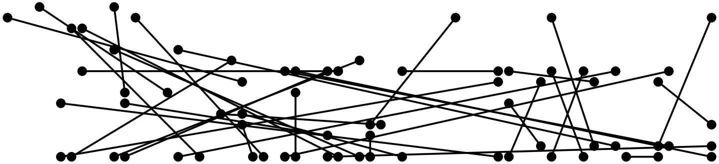
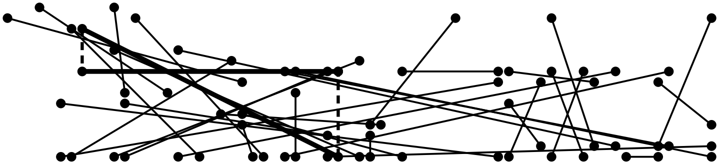
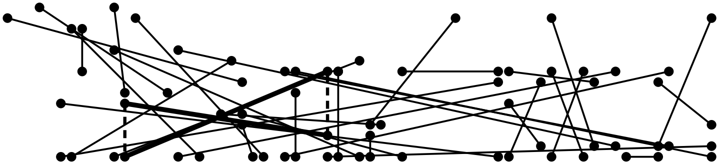
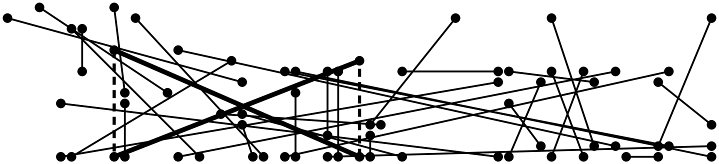
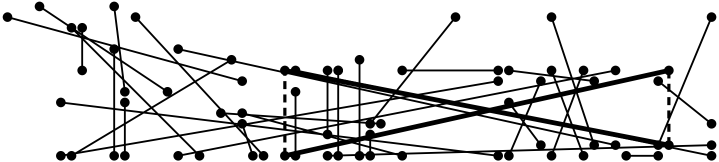
- Is it possible to swap and cancel flips?
- Yes, in our experiments on the butterfly.

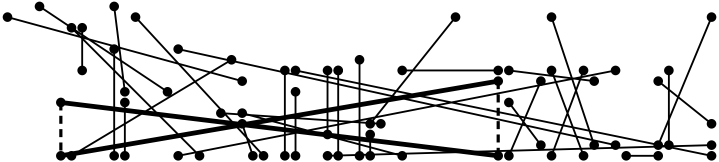


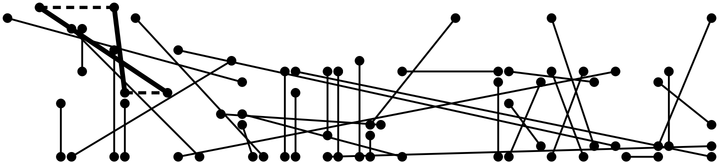


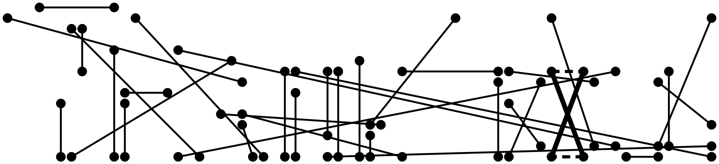


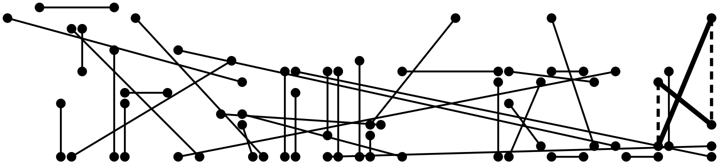


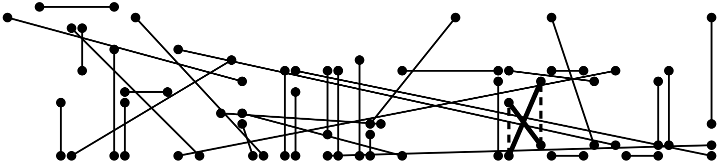


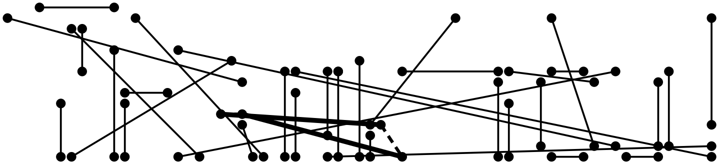


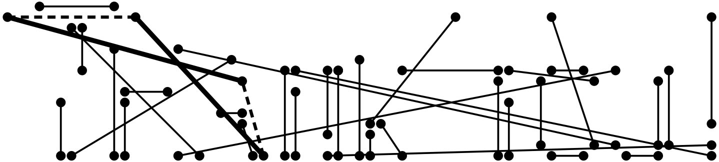


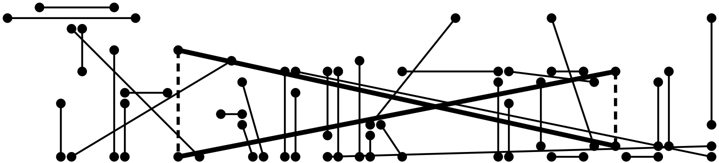


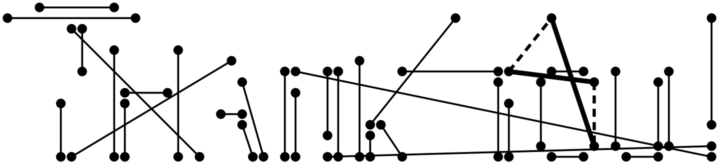


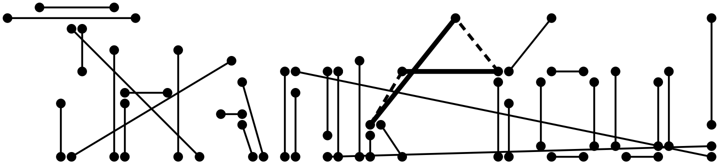


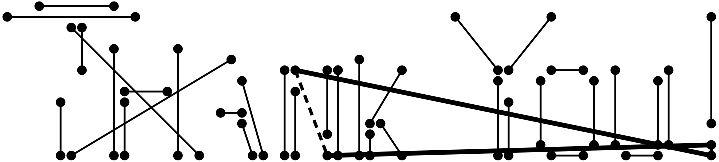








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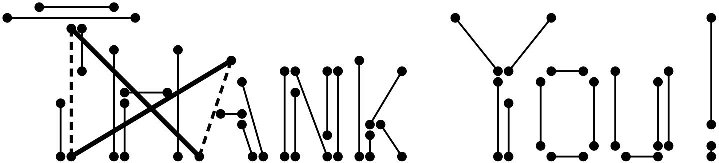
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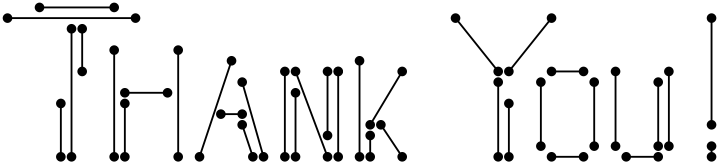
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