Complexity Results on Untangling Red-Blue Matchings

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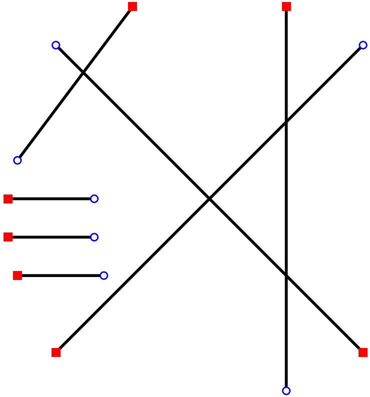
Section 1

Introduction

Crossing-Free Matchings

- R: Set of n red points
- B: Set of n blue points

- **Question 1:** Can we match R to B using **non-crossing** line segments?
- **Answer:** Yes, and we can compute in $O(n \log n)$ time [HS92]



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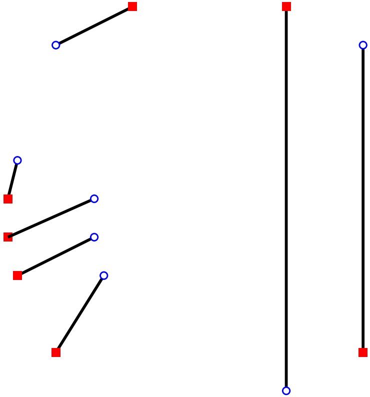
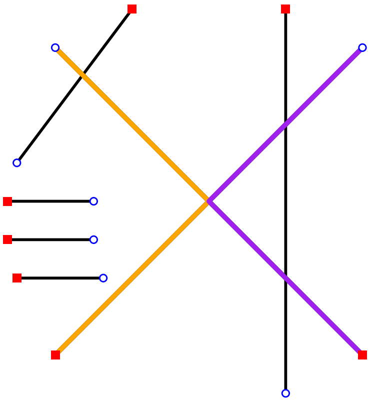


Flips



- A **flip** replaces a crossing pair of segments by non-crossing segments
- **Question 2:** Will **flipping** always **lead to** a **crossing-free** matching?
- **Answer:** Yes, the total **Euclidean length decreases** and there are *only* $n!$ possible matchings
- **Question 3: How many** flips?
- **Answer:** Hard to say...

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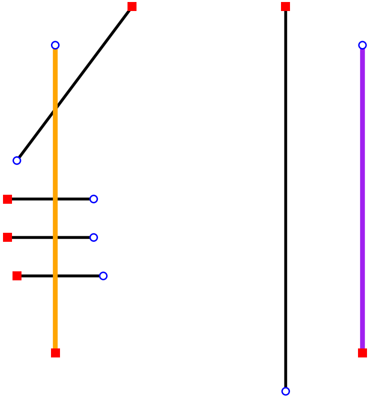

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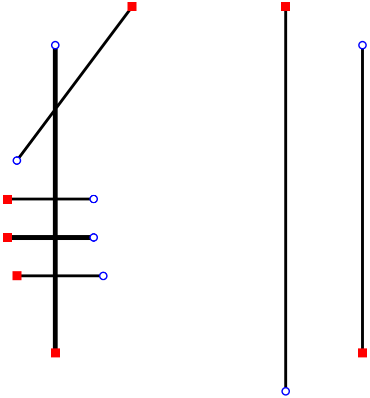

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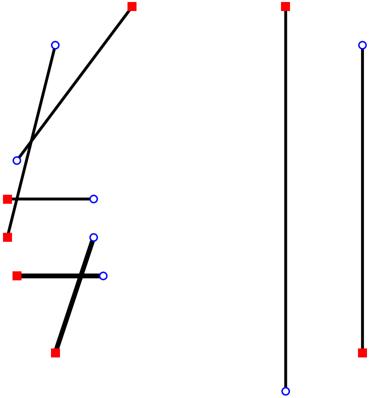

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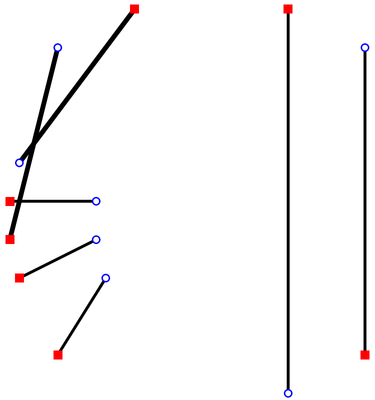

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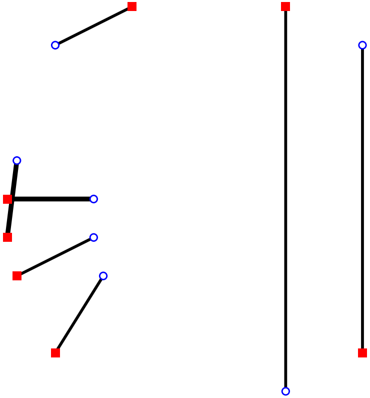

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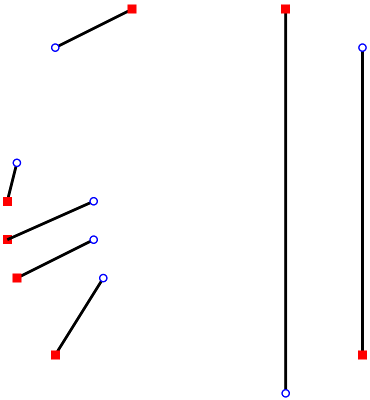

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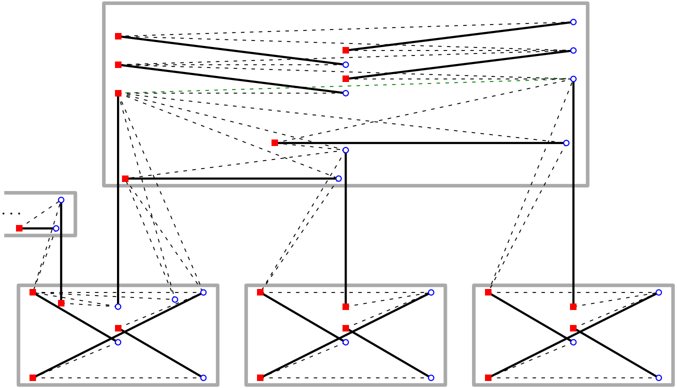
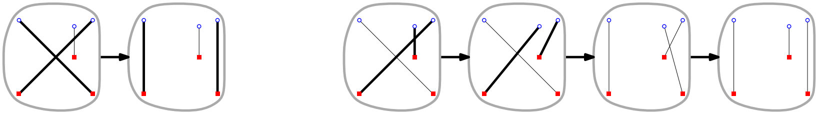
NP-Hardness

- We show that finding the **minimum number of flips** is **NP-hard**
- In fact, even a constant-factor **approximation** is **NP-hard**



Little $\mathbf{d}$ and Big $\mathbf{D}$

- Some untangle sequences are shorter than others

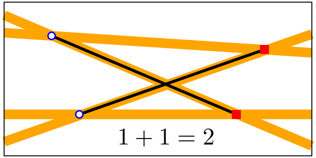


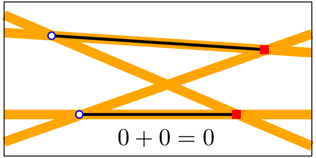
- $\mathbf{d}(n)$: length of the **shortest** untangle sequence (worst-case)
- $\mathbf{D}(n)$: length of the **longest** untangle sequence
- Clearly: $\mathbf{d}(n) \leq \mathbf{D}(n)$

Cubic Upper Bound for $\mathbf{D}(n)$ [BM16, vL81]

- Consider n^2 lines by red-blue pairs
- Potential of segment s:
  number of **lines properly crossing** s
- Potential of matching:
  sum of the segment potentials
- Initial potential $\leq n(n-1)^2$
- Flip reduces potential by at least 2
- Hence,

$$\mathbf{D}(n) \leq \frac{n(n-1)^2}{2} = \binom{n}{2}(n-1)$$





History of Bounds

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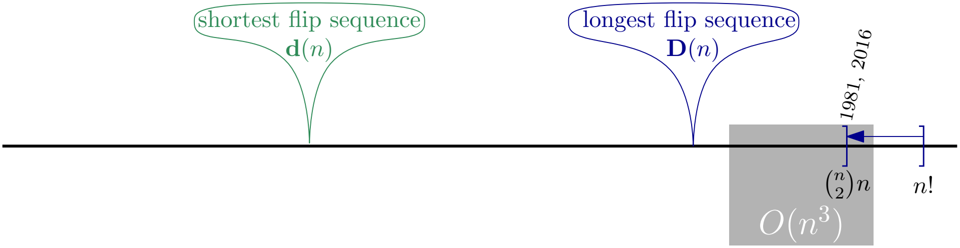

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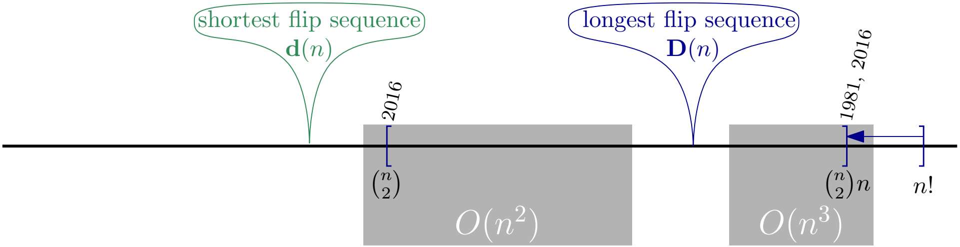

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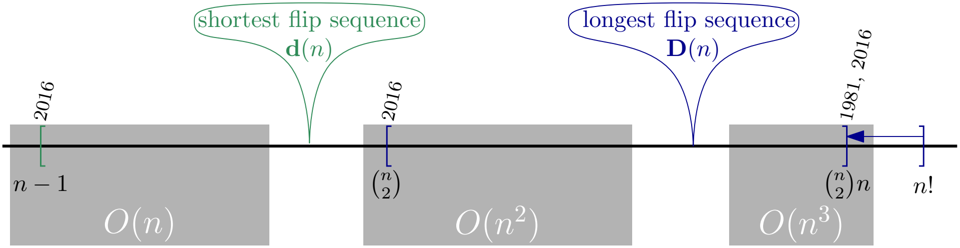

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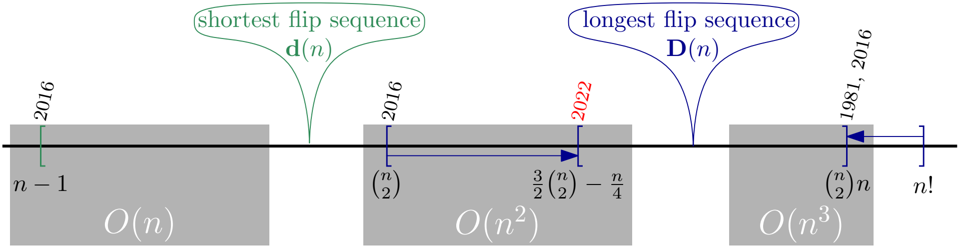

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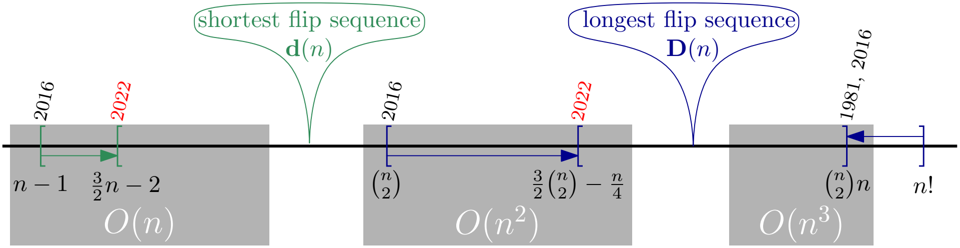
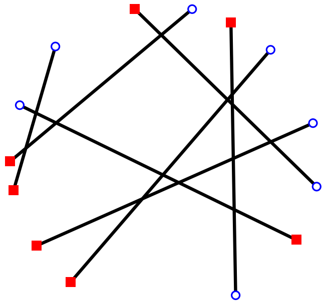
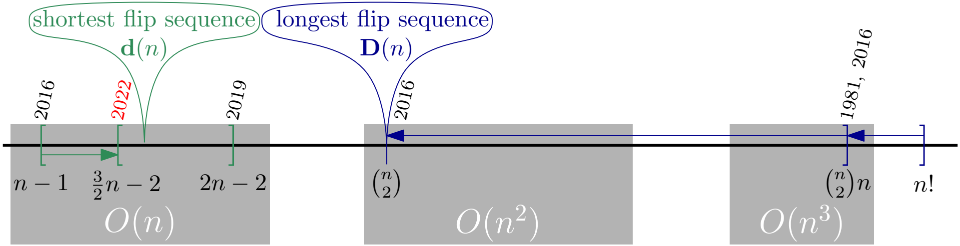


Convex Case

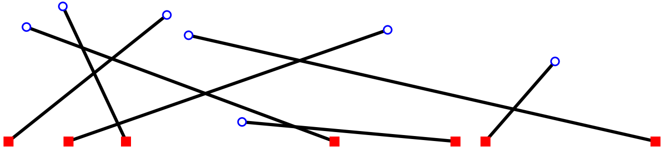
- General bounds have large gaps
- What if the points are in convex position?
  - Number of crossings decreases at each flip
  - Tight bounds for $\mathbf{D}(n)$
  - Almost tight bounds for $\mathbf{d}(n)$

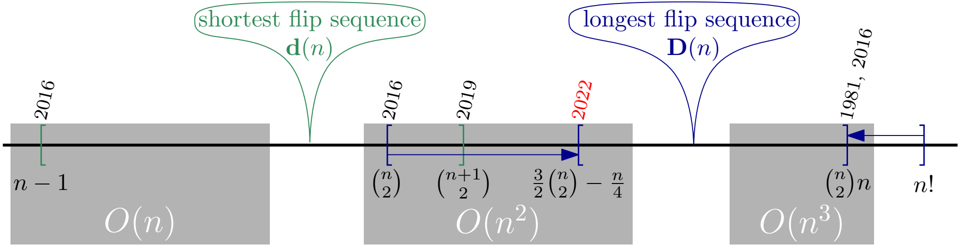




Red-on-a-Line Case

- What if the red points are colinear?





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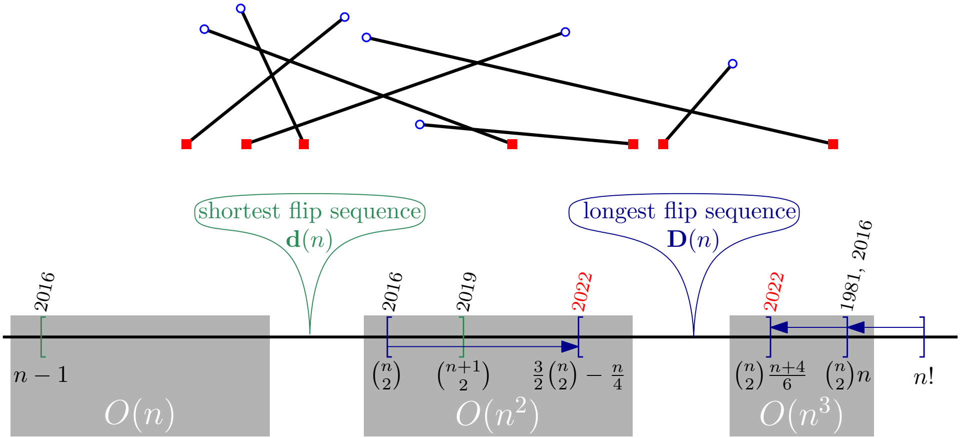

Red-on-a-Line Case

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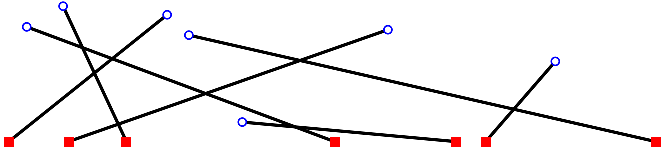
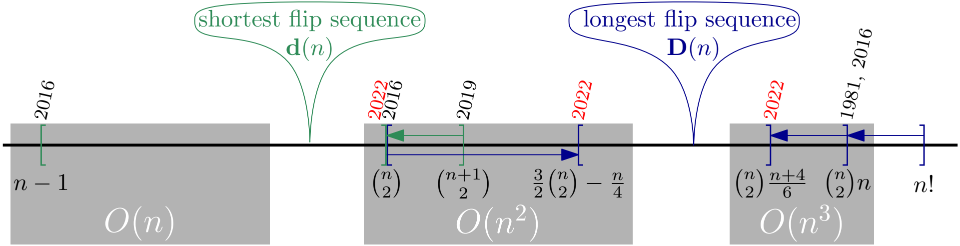




State of the Art Bounds

$\mathbf{d}(n)$ bounds	lower	upper
general	$\frac{3}{2}n - 2$, new	$\binom{n}{2}(n-1)$, [BM16, vL81]
convex	$\frac{3}{2}n - 2$, new	$2n - 2$, [BMS19]
red-on-a-line	$n - 1$, [BM16]	$\binom{n}{2}$, new
$\mathbf{D}(n)$ **bounds**	**lower**	**upper**
general	$\frac{3}{2}\binom{n}{2} - \frac{n}{4}$, new	$\binom{n}{2}(n-1)$, [BM16, vL81]
convex	$\binom{n}{2}$, [BM16]	$\binom{n}{2}$, [BMS19]
red-on-a-line	$\frac{3}{2}\binom{n}{2} - \frac{n}{4}$, new	$\binom{n}{2}\frac{n+4}{6}$, new

Section 2

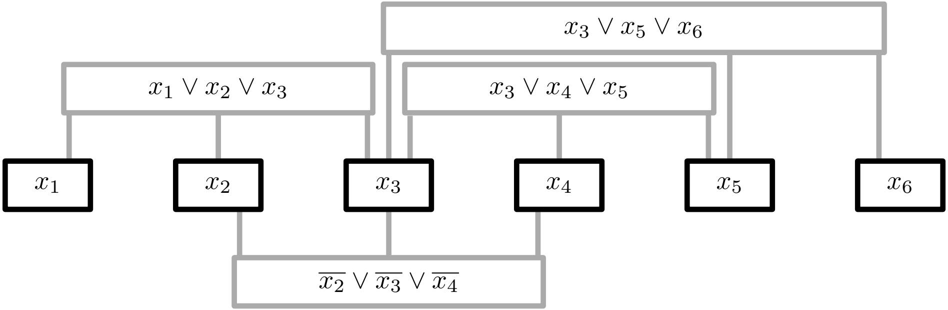
NP-Hard

Problem:

Input: Matching M, integer k
Output: Is there an untangle sequence of length at most k?

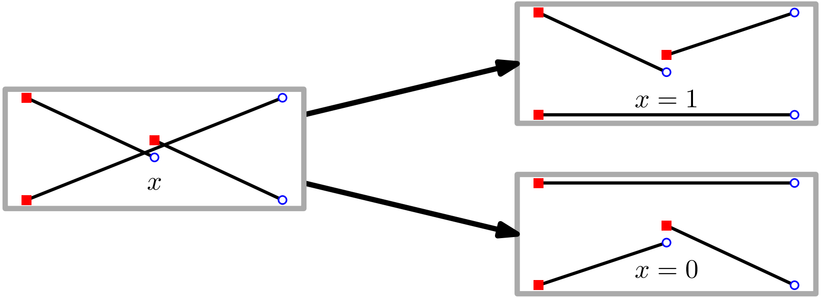
RPM 3-SAT

- Variation of 3-SAT [DBK12]:
  - Clauses are **all positive** or **all negative**
  - **Planar** orthogonal drawing



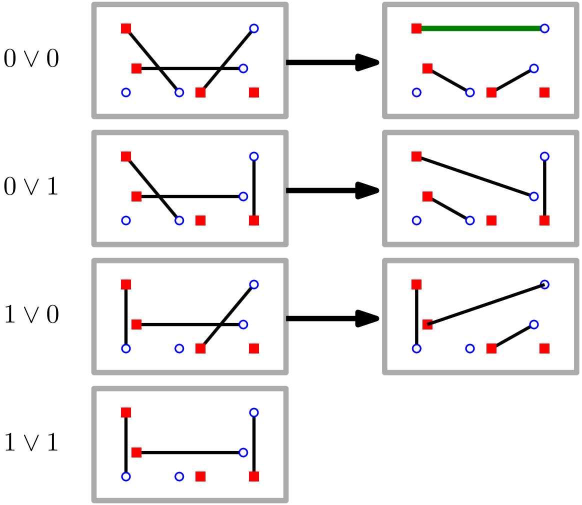
Variable Gadget

- Variable gadgets can be flipped to be *true* or *false*



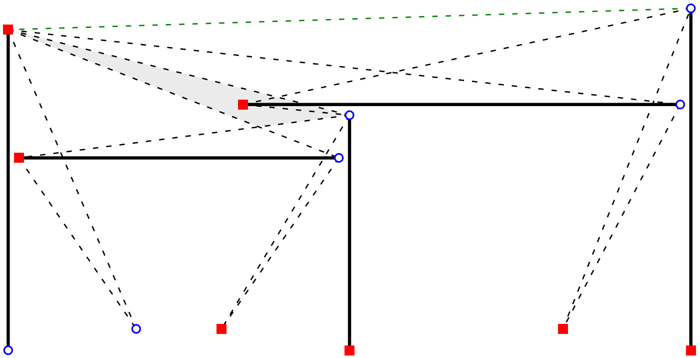
OR Gadget

- Used to build clauses



Clause Gadget

- Clause gadgets are 2 OR gadgets to have 3 inputs
- Clause gadgets connect to variable gadgets



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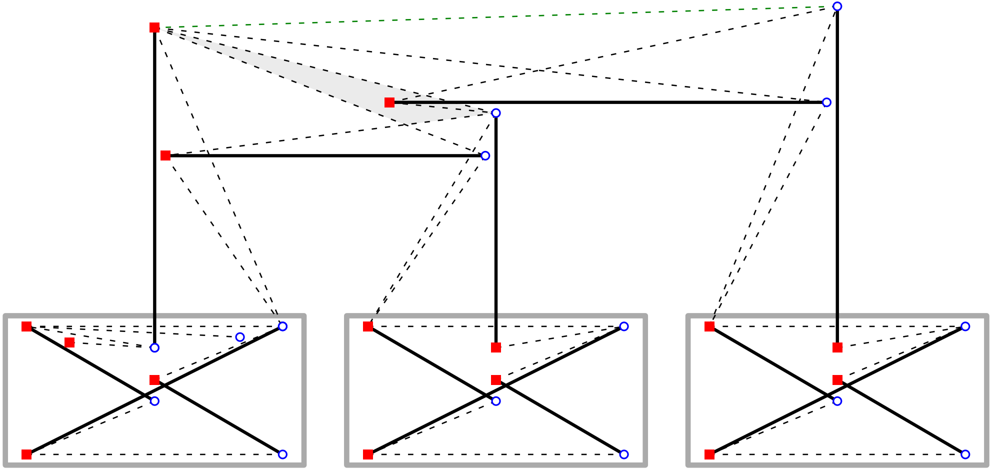
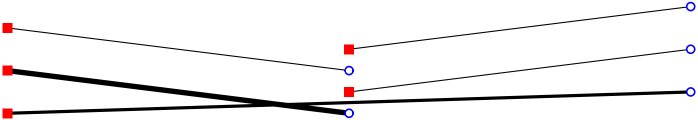


Padding Gadget

- Padding gadgets serve to **increase the number of flips**
- Each clause has a padding gadget
- If the **clause** is **not satisfied**, the **padding** gadget is **flipped**



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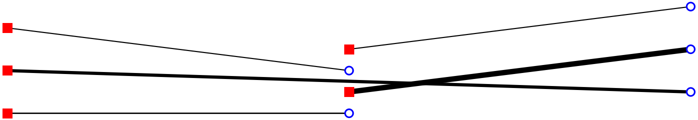

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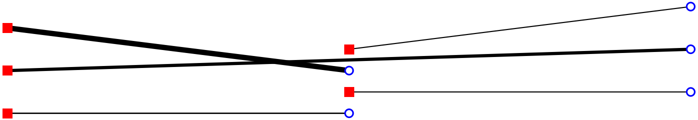

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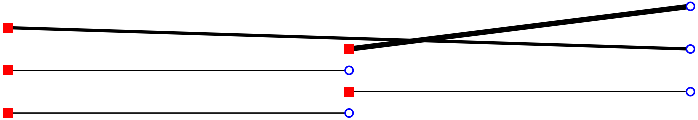

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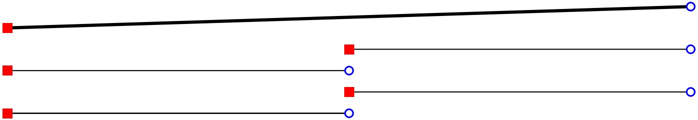
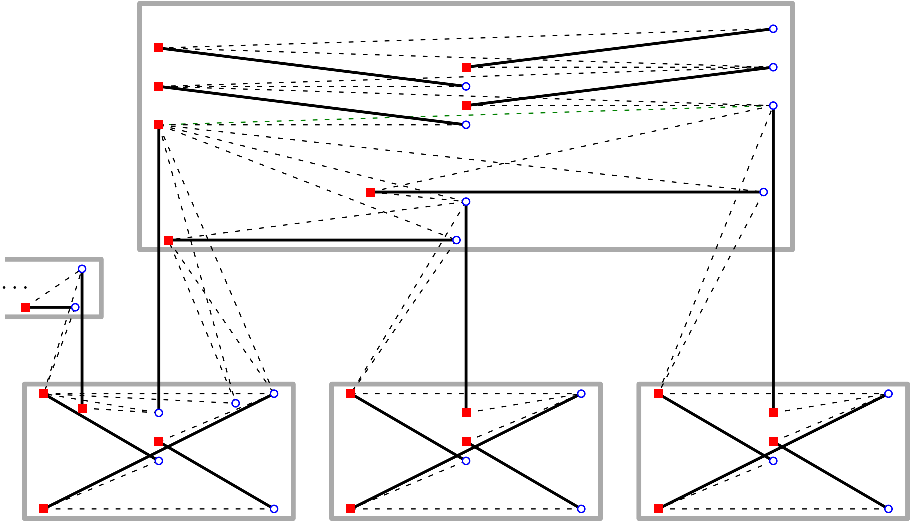


All Gadgets Together



Section 3

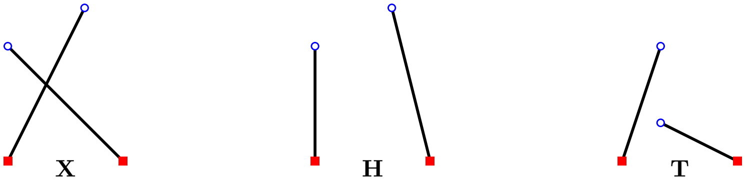
$$\mathbf{d}(n) \leq \binom{n}{2}$$

for the red-on-a-line case

Introduction
NP-Hard
$\mathbf{d}(n) \leq$
States
Algorithm
Analysis
$\mathbf{d}(n) \geq$
$\mathbf{D}(n) \leq$
$\mathbf{D}(n) \geq$
Conclusion

$\mathbf{X}$, $\mathbf{H}$, and $\mathbf{T}$ States

- Each pair of segments defines a state:



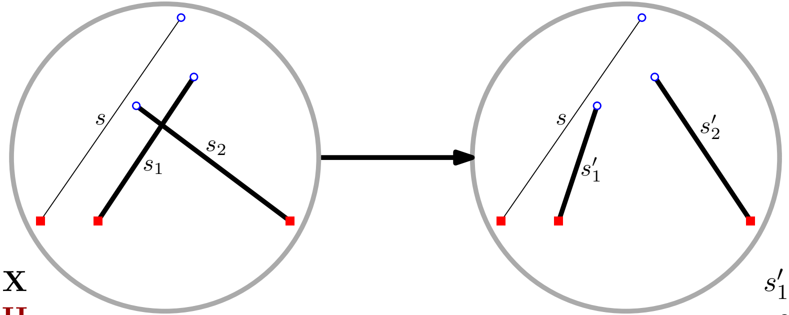
Convex case:

- No $\mathbf{T}$ states
- Every flip increases $|\mathbf{H}|$
- Hence at most $\binom{n}{2}$ flips

What if the points are **not** in **convex** position?

$|\mathbf{H}|$ May Not Increase

In general, $|\mathbf{H}|$ may **not increase**:



s_1, s_2	**X**
s, s_1	**H**
s, s_2	**T**

s'_1, s'_2	**H**
s, s'_1	**T**
s, s'_2	**T**

- Multiple copies of s would make $|\mathbf{H}|$ decrease
- $|\mathbf{H}|$ decreases if the upper cone is empty

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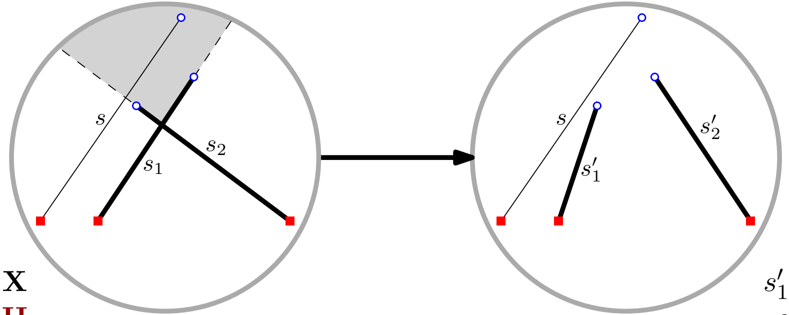

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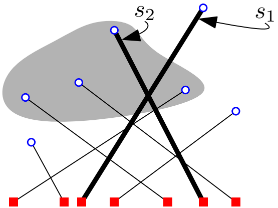
Algorithm for Red-on-a-Line

- To bound $\mathbf{d}(n)$ we can **choose** segments to **flip**
- **Top segment**: segment with the topmost blue point

Algorithm for Red-on-a-Line

- Always flip top segment s_1 with top segment s_2 among segments that cross s_1
- If s_1 has no crossing, solve both sides of s_1 recursively



- Since the line containing s_1 crosses no segment, **both sides** are **independent**

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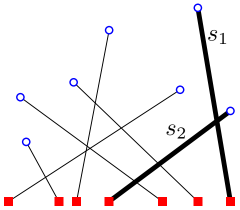

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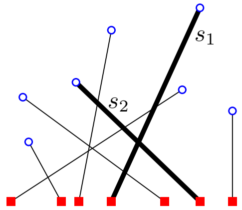

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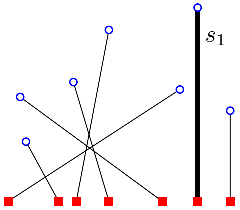

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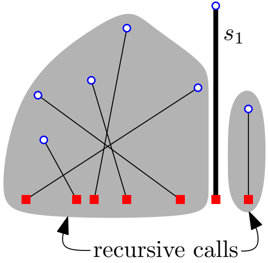

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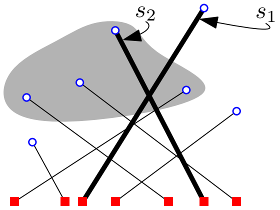
Analysis of the Number of Flips



Lemma:

Flipping the top segments s_1, s_2 increases the number of $\mathbf{H}$ pairs

- We do not count the $\mathbf{H}$ pairs between different recursive calls
- Total number of pairs: $\binom{n}{2}$
- Hence,

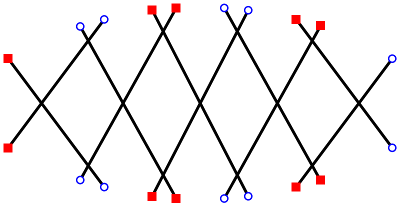
$$\mathbf{d}(n) \leq \binom{n}{2}$$

Section 4

$$\mathbf{d}(n) \geq \frac{3}{2}n - 2$$

for the convex case

Fence



10 segments, 13 crossings

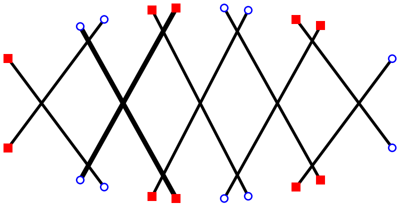
- Fence with n segments has

$$\frac{3}{2}n - 2 \text{ crossings}$$

Lemma:

Every flip starting from a fence reduces the number of crossings by 1

Fence



10 segments, 13 crossings

- Fence with n segments has

$$\frac{3}{2}n - 2 \text{ crossings}$$

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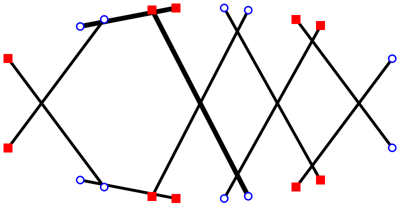

10 segments, 12 crossings

- Fence with n segments has

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Lemma:

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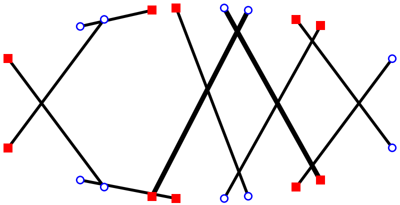

10 segments, 11 crossings

- Fence with n segments has

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Lemma:

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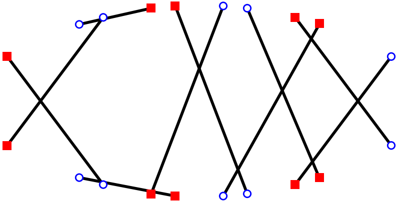
Fence



10 segments, 10 crossings

- Fence with n segments has

$$\frac{3}{2}n - 2 \text{ crossings}$$

Lemma:

Every flip starting from a fence reduces the number of crossings by 1

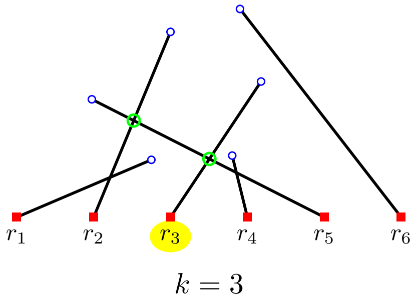
Section 5

$$\mathbf{D}(n) \leq \binom{n}{2}\frac{n+4}{6}$$

for the red-on-a-line case

k-crossings and k-observed crossings

- Red points numbered from left to right $r_1, \ldots, r_n$ and consider r_k
- k-**pair**: Pair of segments with red points r_i, r_j and $i \leq k \leq j$
- Project blue points from r_k
- k-**observed crossing**: projected segments cross
- crossing k-pairs are k-observed crossing



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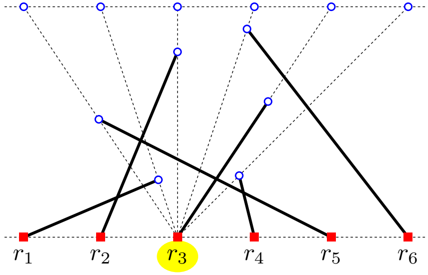

$k = 3$

k-crossings and k-observed crossings

- Red points numbered from left to right $r_1, \ldots, r_n$ and consider r_k
- k-**pair**: Pair of segments with red points r_i, r_j and $i \leq k \leq j$
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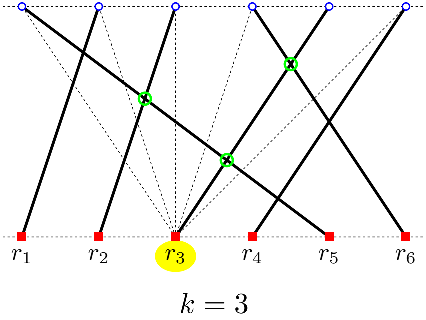
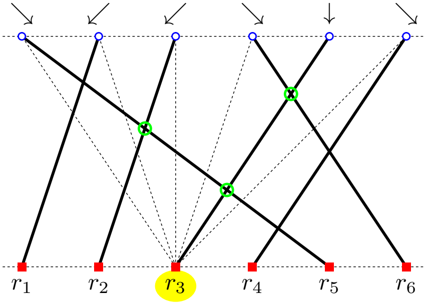


Potential



- Φ_k: Number of k-pairs forming k-observed crossings
- By number of pairs i, j with $i \neq j$ and $1 \leq i \leq k \leq j \leq n$:

$$\Phi_k \leq k(n-k+1) - 1$$
$$= k(n+1) - k^2 - 1$$

Lemma:

Φ_k decreases for each flipped k-crossing

Sum

- Φ: Sum of Φ_k

$$\Phi = \sum_{k=1}^{n} \Phi_k \leq \sum_{k=1}^{n} \left(k(n+1) - k^2 - 1\right) = (n+1)\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2 - n = \binom{n}{2}\frac{n+4}{3}$$

- Φ decreases by at least 2 for each flip
  (1 unit for k corresponding to each red point of the flip)
- Hence, for red-on-a line $\mathbf{D}(n) \leq \binom{n}{2}\frac{n+4}{6}$
- Compare to $\mathbf{D}(n) \leq \binom{n}{2}(n-1)$ in general

Section 6

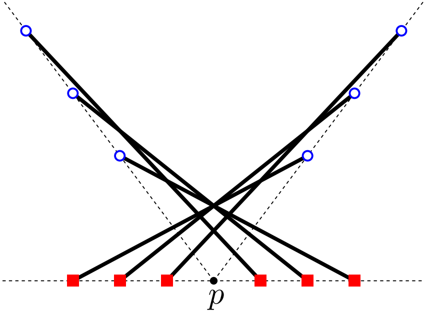
$$\mathbf{D}(n) \geq \frac{3}{2}\binom{n}{2} - \frac{n}{4}$$

for the red-on-a-line case

Butterfly

$$\mathbf{D}(n) \geq \frac{3}{2}\binom{n}{2} - \frac{n}{4}$$

- Example with $n = 6$ and 21 flips



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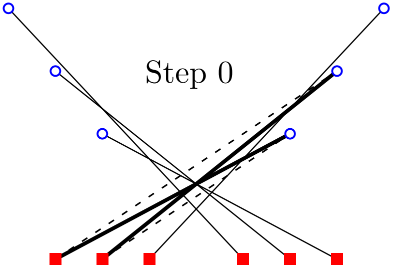

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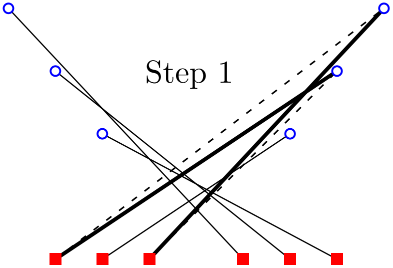

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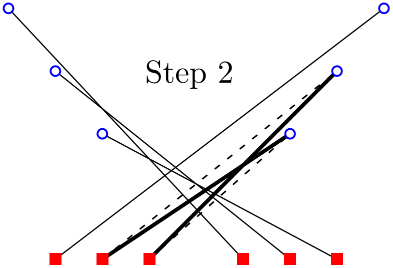

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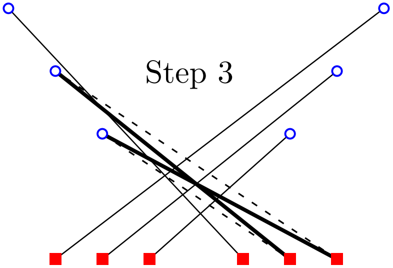

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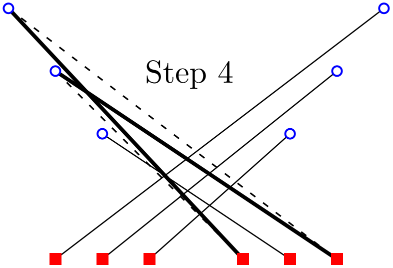

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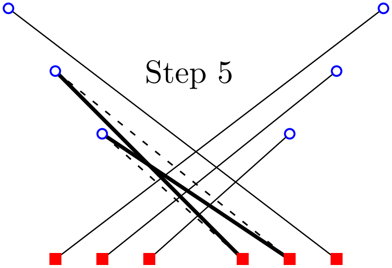

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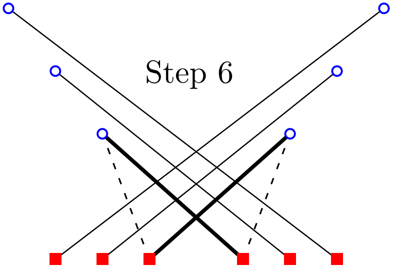

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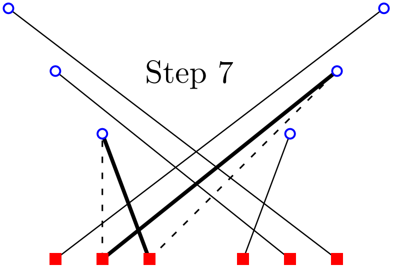

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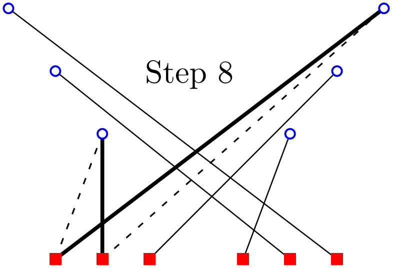

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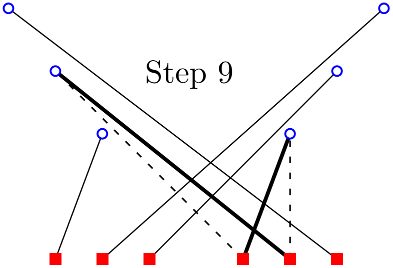

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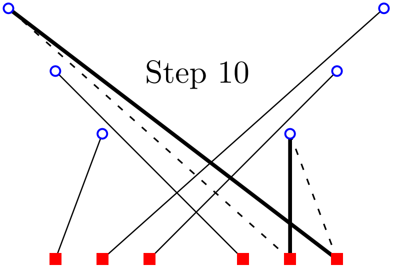

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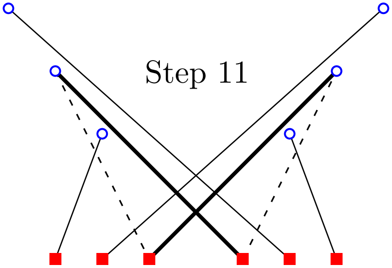

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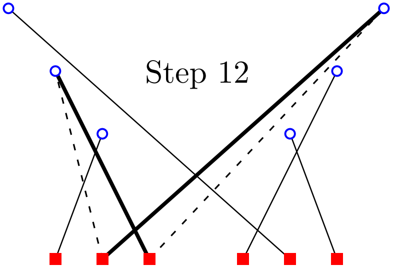

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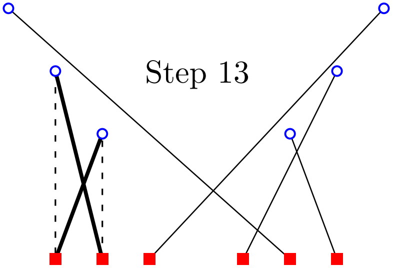

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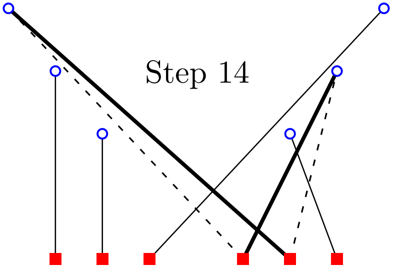

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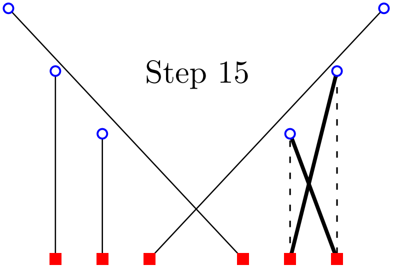

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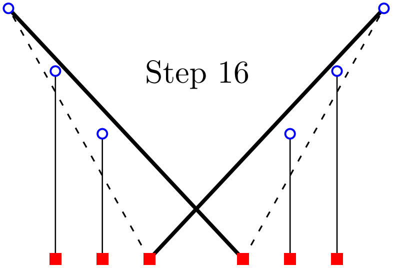

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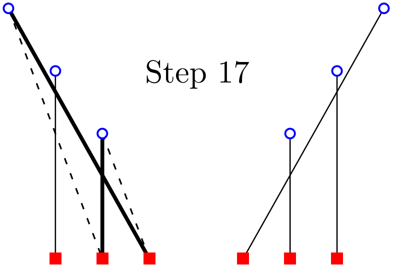

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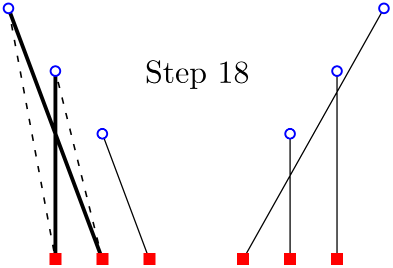

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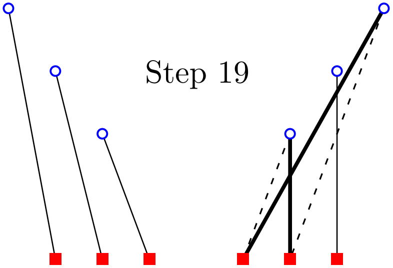

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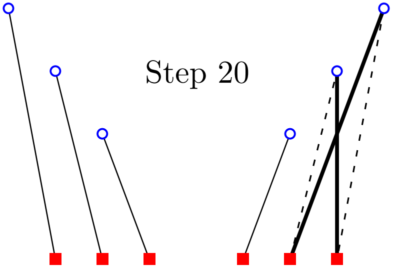

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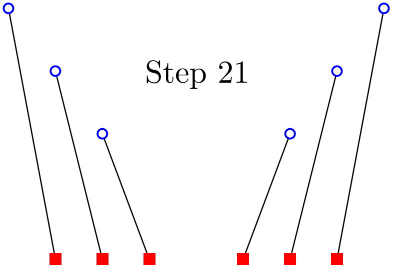

Section 7

Conclusion

Other NP-Complete Problems?

1. The shortest flip sequence...
   1. for a **TSP** tour?
   2. for a **red-on-a-line** matching?
   3. for a **convex** instance?
2. What about the **longest** flip sequence?

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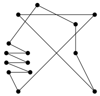

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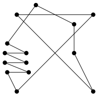

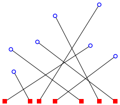

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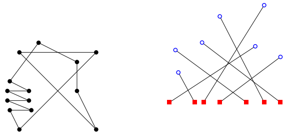

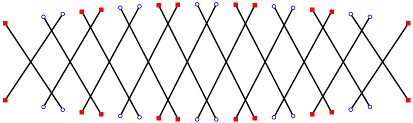

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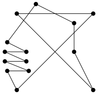
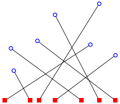
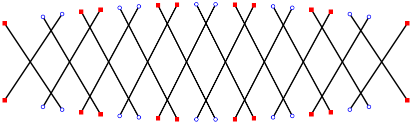
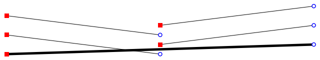






Better Bounds?



1. Better bounds if the input is a **crossing-free** matching **plus** an extra **segment**?
2. Can we avoid **flipping** the **same pair of segments** twice?
3. What is the number of flips involving a fixed **point**? A fixed **segment**?
4. What about **non-bipartite** matchings?
5. What about **TSP** tours?

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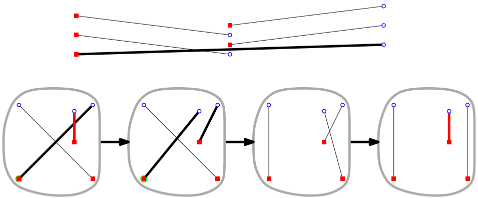

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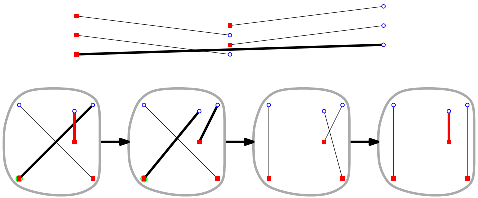

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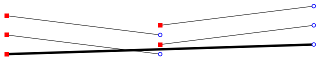

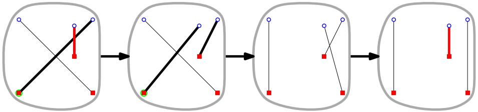

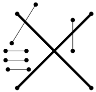

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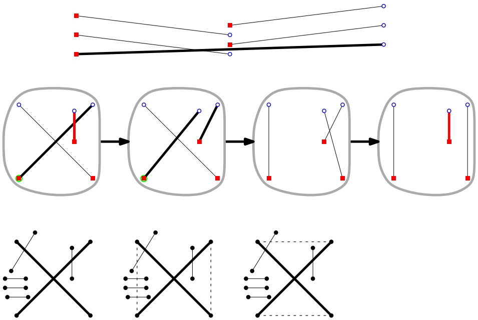

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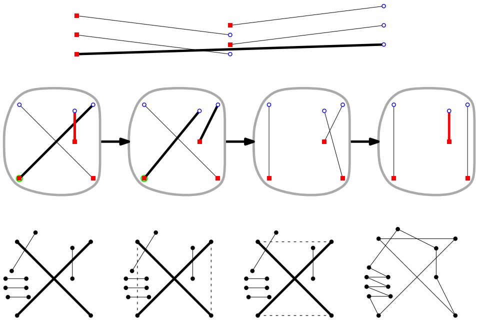

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Thank You

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