Designing Linear-Time Approximation Algorithms for Unit Disk Graphs

Guilherme D. da Fonseca
Aix Marseille Université and LIS

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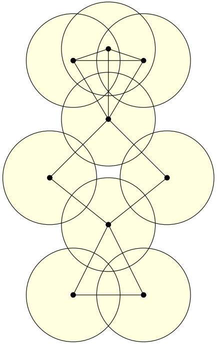
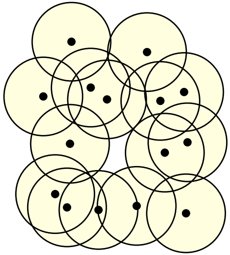
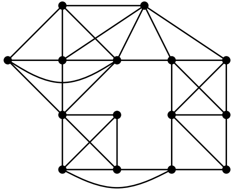
Unit Disk Graphs



- *Unit disk graph*: Intersection graph of unit-disks in the plane
- Applications in wireless networks
- Neither planar nor perfect:
  K_i and C_i are UDGs for all i
- Recognition: NP-Hard, $\exists\mathbb{R}$-complete
  Doubly exponential algorithm exists
- Vertex coordinates (disk centers) are real numbers

Approximation Algorithms





- Two types of algorithms:
  - Geometric: vertex coordinates
    Edge pq if $\|pq\| \leq 2$
  - Graph-based: adjacency information only
- PTASs for several problems:
  (even without geometry)
  - Minimum Dominating Set
  - Maximum (Weight) Independent Set
  - Minimum (Weight) Vertex Cover
  - ...
- PTASs have high complexity:
  $O(n^{10})$ to 4-approximate the *minimum dominating set*

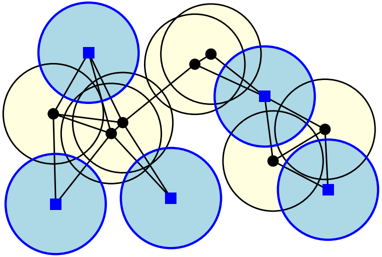
Near-Linear-Time Approximation

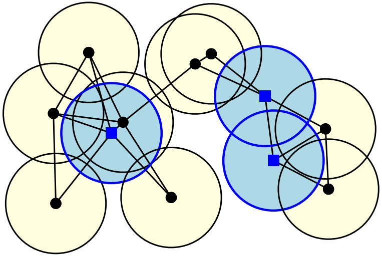
Our goal:

What approximation factor can we achieve in near-linear time?

- For geometric algorithms: $O(n \log^{O(1)} n) = \widetilde{O}(n)$ time
- For graph algorithms: $O((n+m) \log^{O(1)} n) = \widetilde{O}(n+m)$ time
- n: number of vertices
- m: number of edges

Two Optimization Problems





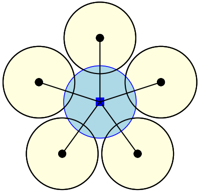
- *Independent Set*: Subset of points with minimum distance > 2
- *Maximum* Independent Set (MIS): Maximize cardinality

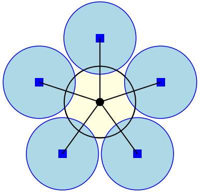
- *Dominating Set*: Subset of points D such that all input points are within distance at most 2 from a point in D
- *Minimum* Dominating Set (MDS): Minimize cardinality

Greedy Algorithms

M. V. Marathe, H. Breu, H. B. Hunt III, S. S. Ravi, and D. J. Rosenkrantz. Simple heuristics for unit disk graphs. *Networks*, 25(2):59–68, 1995.

Maximal Independent Set





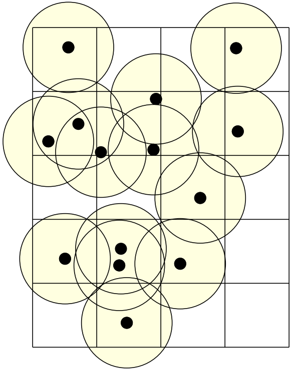
- *Maximal* independent set gives a 5-approximation to both:
  - Maximum independent set
  - Minimum dominating set
- Can be computed in $O(n+m)$ time

  $I \leftarrow \emptyset$
  For each $v \in V(G)$:
  $\quad I \leftarrow I \cup \{v\}$
  $\quad$Remove v and its neighbors from G

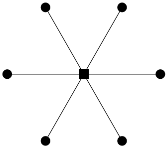
Geometric Version

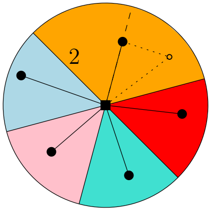
- Takes $O(n)$ time using $O(1)$-time hashing
- Hash points into grid
- Cells of diameter 2
- Algorithm:

  $I \leftarrow \emptyset$
  For each $v \in V(G)$:
    $I \leftarrow I \cup \{v\}$
    Empty v's cell
    Remove v's neighbors from cells nearby
- Each point is examined at most 25 times (cells nearby)



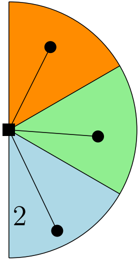
Approximation for Maximum Independent Set





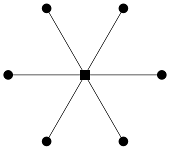
- Unit disk graph: no induced $K_{1,6}$
- I^*: optimal solution
- I: algorithm solution (maximal independent set)
- For each vertex v added to I
  at most 5 neighbors of v in I^* are removed
- Conclusion: $|I^*| \leq 5|I|$
- 5-approximation

Improvement for Maximum Independent Set

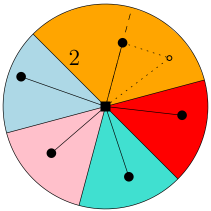


- Sort vertices from left to right
- Run the same algorithm
- Right neighbors form 3 cliques
- 3-approximation for MIS
- $O(n \log n)$ time to sort
- Much slower without geometry

Approximation for Minimum Dominating Set



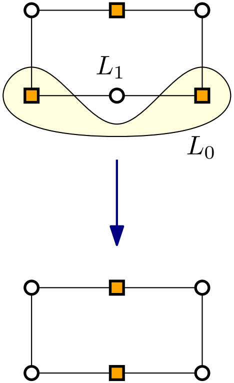
- Unit disk graph: no induced $K_{1,6}$
- D^*: optimal solution
- D: algorithm solution (maximal independent set)
- Each vertex v in D has at most 5 neighbors in D^*
- Conclusion: $|D| \leq 5|D^*|$
- 5-approximation
- Sorting won't help!



Local Search

Guilherme D. da Fonseca, Celina M.H. de Figueiredo, Vinícius G. Pereira de Sá, and Raphael C.S. Machado. Efficient sub-5 approximations for minimum dominating sets in unit disk graphs. *Theoretical Computer Science*, 540–541:70–81, 2014.

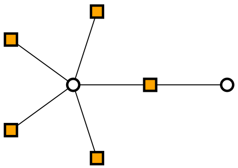
Local Search



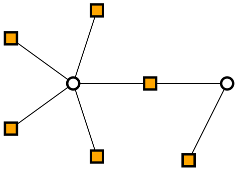
- Build a suboptimal solution S
- Find two *small* sets L_0, L_1
- Say $|L_0|, |L_1| < k$
- Make $S \leftarrow (S \setminus L_0) \cup L_1$
- Verify that S is feasible
- Maximization: use $|L_0| < |L_1|$
- Minimization: use $|L_0| > |L_1|$
- Repeat until no further improvement possible

Local Search for Minimum Dominating Set

Irreducible corona:



Reducible corona:



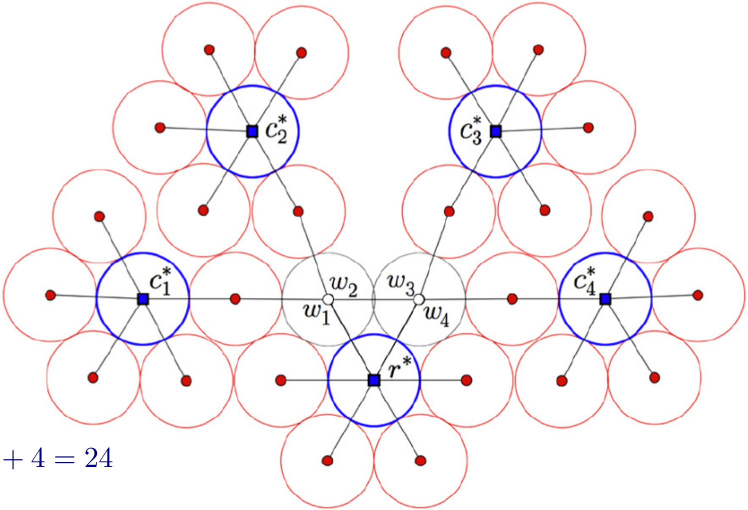
- D: independent dominating set
- $C \subset D$ is a corona centered at vertex c if:
  - $|C| = 5$
  - C is an independent set
  - c is adjacent to all c
- C, c is reducible if $D \setminus C \cup \{c\}$ is a dominating set

Theorem

If D has no reducible corona, then D is a $44/9$-approximation to the minimun dominating set.

- Such D can be computed in $O(n + m)$ time without geometry or $O(n \log n)$ time with geometry

Lower Bound of 4.8 (against 4.89 UB)



$OPT = 5$

$|D| = 5 \cdot 4 + 4 = 24$

Proof Technique

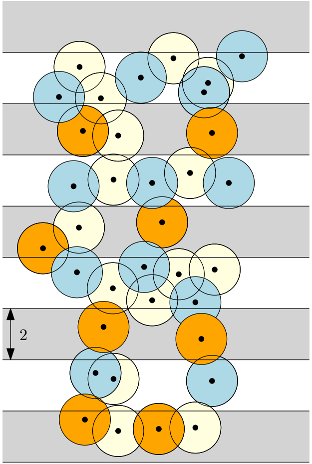
Several geometric results needed:

- Lemma 1 (Pál 1921): If a set of points P has diameter 1, then P can be enclosed by a circle of radius $1/\sqrt{3}$.
- Lemma 2 (Fodor 2007): The radius of the smallest circle enclosing 13 points with mutual distance ≥ 1 is $(1+\sqrt{5})/2$.
- Lemma 3 (Fejes Tóth 1953): Every packing of two or more congruent disks in a convex region has density at most $\pi/\sqrt{12}$.
- Lemma 4: The closed neighborhood of a clique in a unit disk graph contains at most 12 independent vertices.
- Lemma 5: The closed d-neighborhood of a vertex in a unit disk graph contains at most $\pi(2d+1)^2/\sqrt{12}$ independent vertices, for integer $d \geq 1$.

Strip Decomposition

Gautam K. Das, Guilherme D. da Fonseca, and Ramesh K. Jallu. Efficient Independent Set Approximation in Unit Disk Graphs. *Discrete Applied Mathematics*, to appear.

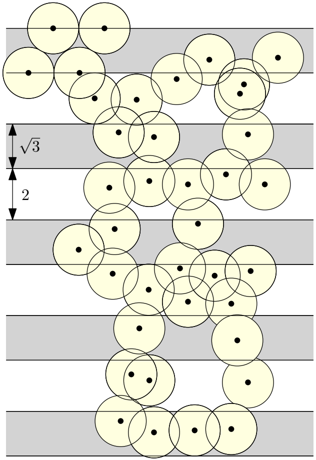
Independent Set with Strips (Quadratic)



- Break the problem into horizontal strips of height 2
- Solve MIS for each strip exactly
- Return maximum among all even or odd strips

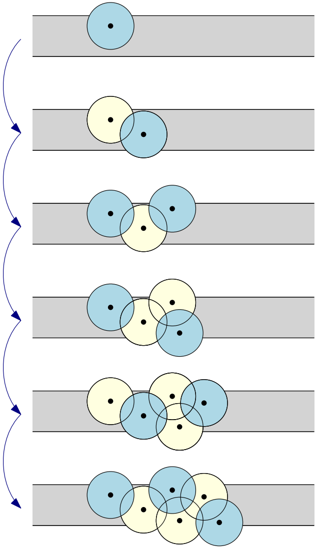
- Good: 2-approximation even for weighted version
- Bad: Don't know how to solve MIS exactly for each strip in $\widetilde{O}(n)$ time (but $O(n^2 \log n)$ is possible)

Independent Set with Strips (Linear)



- Use strips of height at most $\sqrt{3}$
- Resulting graphs are co-comparability
- Solve each strip exactly
- Separation 2 between strips
- Multiple shifts need to be considered
- Approximation factor: $1 + \frac{2}{\sqrt{3}} + \varepsilon < 2.16$

Exact MIS Inside a Strip

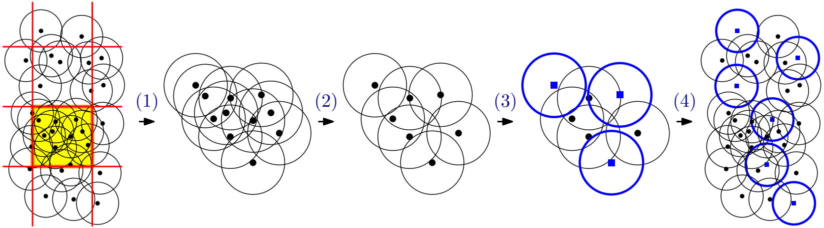


- Height of the strip: $\sqrt{3}$
- Dynamic programming
- $v_1, \ldots, v_n$: vertices sorted by x coordinate
- For k from 1 to n:
  $f(k)$ = maximum independent set of $v_1, \ldots, v_k$
- Recurrence (cocomparability graph):
  $f(k) = 1 + \max_{i<k \text{ and } \|v_i v_k\|>2} f(i)$
- Query uses semi-dynamic data structure:
  $O(\log^2 n)$ time per query
- MIS can be solved in $O(n \log^2 n)$ time
- Extends to weighted version (extra $O(\log n)$ factor)

Shifting Coresets

Guilherme D. da Fonseca, Vinícius G. Pereira de Sá, and Celina M.H. de Figueiredo. Shifting Coresets: Obtaining Linear-Time Approximations for Unit Disk Graphs and Other Geometric Intersection Graphs. *International Journal on Computational Geometry and Applications*, 27(4):255–276, 2017.

Shifting Coresets



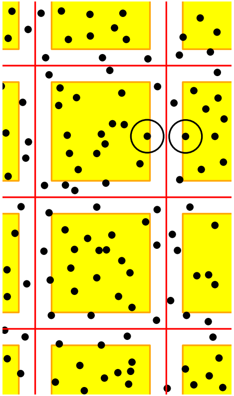
(1) Break the original problem into subproblems of $O(1)$ diameter (shifting strategy)

(2) Build a coreset with $O(1)$ points for each subproblem, which gives an α-approximation to the subproblem

(3) Solve the coreset optimally

(4) Combine the solutions into an $(\alpha + \varepsilon)$-approximation

Breaking IS into Subproblems

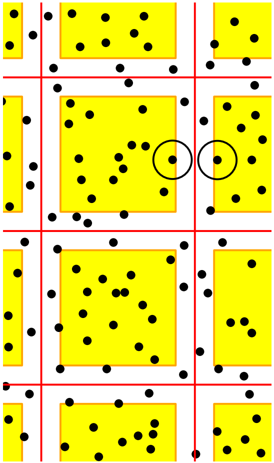


Break Independent Set instance into $O(1)$-diameter subproblems (shifting strategy):

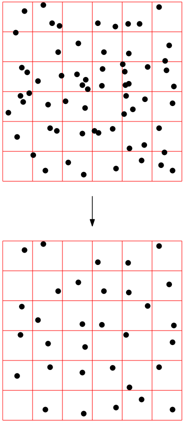
- Set k to smallest integer with $\left(\frac{k}{k-2}\right)^2 \geq 1 + \frac{\varepsilon}{4}$
- Use grids of size $2k$
- Create k^2 shifted grids with even origins
- Contract grid cells by 1 in all directions
- Each contracted cell is a subproblem

Analysis of Shifting Strategy

- Contracted cells are distance 2 apart: union preserves independence
- 4-approximation in yellow area
- Yellow area gets much bigger than white area as $k \to \infty$
- Expected number of OPT points in white area is small
- Maximum is larger than expectation

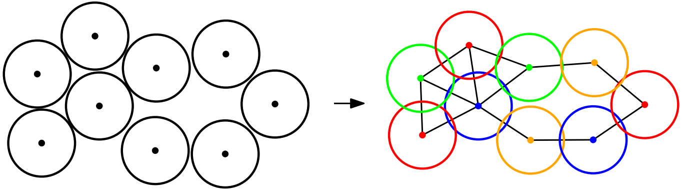


Constant-Diameter Coreset for MIS



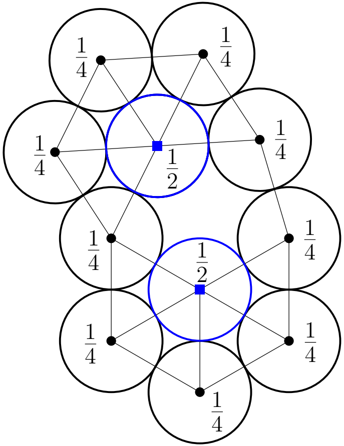
- *Coreset*: Subset with $O(1)$ points that approximates the original solution
- Algorithm:
  - Create grid with cells of diameter $0.29 < (2 - \sqrt{2})/2$
  - Select a point of maximum weight inside each cell (coreset)
  - Find the optimal independent set among the selected points
- We need to prove it gives a 4-approximation!

Proof of 4-Approximation of MIS



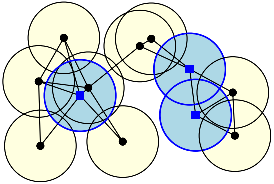
- Consider the optimal independent set
- Moving points by at most 0.29, we obtain a planar graph
- Planar graphs are 4-colorable
- The color of maximum weight is a 4-approximation

Lower Bound of 3.25



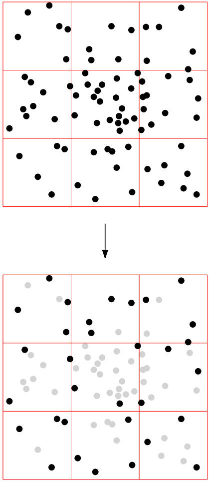
- P_1: Set of points from the figure
- P_2: Multiply coordinates from P_1 by $(1+\varepsilon)$ and weights by $(1-\varepsilon)$
- $P_1 \cup P_2$ gives a lowerbound of 3.25
  - P_2 is independent
  - MWIS: P_2, with $w(P_2) \approx 3.25$
  - Coreset: P_1
  - P_1 has MWIS with weight 1

Minimum Dominating Set Algorithm



- Break the problem into subproblems of $O(1)$ diameter using the shifting strategy
- Cells need to be expanded rather than contracted
- We'll present only the coreset

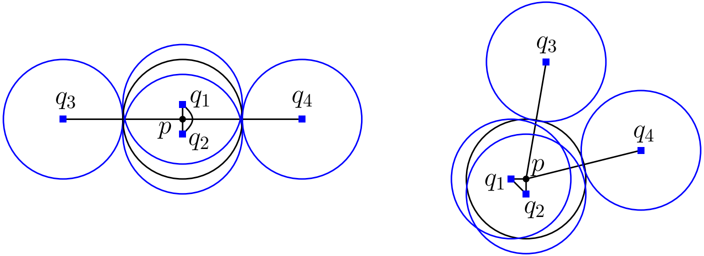
Constant-Diameter Coreset for MDS



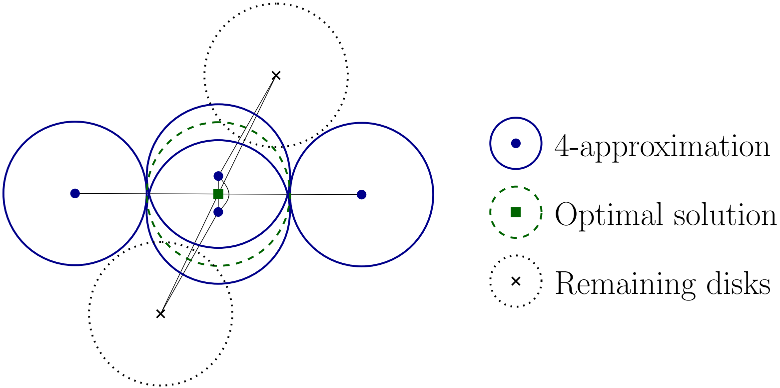
- Algorithm:
  - Create grid with cells of diameter 0.24
  - Select the points of $\min$ and $\max$ x and y coordinates
  - Find the optimal dominating set using the coreset points, but dominating every point
- We need to prove it's a 4-approximation!

Proof of 4-Approximation of MDS

- Either point p from OPT is in the coreset (great!)
- Or there are points q_1, q_2 near p with angle $\geq 90°$
- We dominate all points dominated by p using at most 4 points q_1, q_2, q_3, q_4



Lower Bound of 4



Conclusion

Greedy:

- 5-approximation to IS and DS in linear time with or without geometry
- 3-approximation to IS in $O(n \log n)$ time with geometry

Local search:

- $44/9$-approximation to DS in $O(n+m)$ time without geometry
- $44/9$-approximation to DS in $O(n \log n)$ time with geometry

Strip decomposition:

- 2.16-approximation to IS in $O(n \log^2 n)$ time with geometry
- Generalizes to weighted version in $O(n \log^3 n)$ time

Shifting coresets:

- $(4+\varepsilon)$-approximation to IS and DS in $O(n)$ time with geometry
- Generalizes to weighted version for IS

Open Problems

- Other techniques that yield near-linear-time appproximation algorithms?
- Can we prove inaproximability in near-linear-time?
- Can we improve the analysis of existing algorithms?
- Can we do better than 3-approximation for the chromatic number (greedy)?
- Maximum independent set without geometry better than greedy?
- Minimum *weight* dominating set?
- Intersection of other shapes: general disks, pseudo-disks, line segments, axis-aligned rectangles...

Thank you!

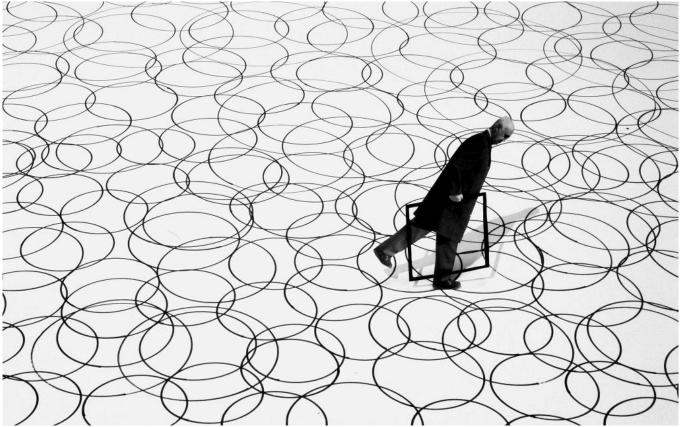


Photo by Gilbert Garcin