Designing Linear-Time Approximation Algorithms for Unit Disk Graphs

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Unit Disk Graphs

- **Unit disk graph**: Intersection graph of unit-disks in the plane
- Applications in wireless networks
- Neither planar nor perfect:
  - $K_i$ and $C_i$ are UDGs for all $i$
- Recognition: NP-Hard, $\exists \mathbb{R}$-complete
  - Doubly exponential algorithm exists
- Vertex coordinates (disk centers) are real numbers
Approximation Algorithms

Two types of algorithms:
- Geometric: vertex coordinates
  Edge \( pq \) if \( \|pq\| \leq 2 \)
- Graph-based: adjacency information only

PTASs for several problems:
(even without geometry)
- Minimum Dominating Set
- Maximum (Weight) Independent Set
- Minimum (Weight) Vertex Cover
- ...

PTASs have high complexity:
\( O(n^{10}) \) to 4-approximate the minimum dominating set
Our goal:

What approximation factor can we achieve in near-linear time?

- For geometric algorithms: $O(n \log^{O(1)} n) = \tilde{O}(n)$ time
- For graph algorithms: $O((n + m) \log^{O(1)} n) = \tilde{O}(n + m)$ time

$n$: number of vertices
$m$: number of edges
Two Optimization Problems

- **Independent Set**: Subset of points with minimum distance $> 2$
- **Maximum Independent Set (MIS)**: Maximize cardinality
- **Dominating Set**: Subset of points $D$ such that all input points are within distance at most 2 from a point in $D$
- **Minimum Dominating Set (MDS)**: Minimize cardinality
Greedy Algorithms

Maximal Independent Set

- Maximal independent set gives a 5-approximation to both:
  - Maximum independent set
  - Minimum dominating set
- Can be computed in $O(n + m)$ time

\[
I \leftarrow \emptyset
\]

For each $v \in V(G)$:

\[
I \leftarrow I \cup \{v\}
\]

Remove $v$ and its neighbors from $G$
Geometric Version

- Takes $O(n)$ time using $O(1)$-time hashing
- Hash points into grid
- Cells of diameter 2
- Algorithm:
  
  \[
  I \leftarrow \emptyset \\
  \text{For each } v \in V(G):
  \]
  
  \[
  I \leftarrow I \cup \{v\}
  \]

  Empty $v$’s cell
  Remove $v$’s neighbors from cells nearby

- Each point is examined at most 25 times (cells nearby)
Approximation for Maximum Independent Set

- Unit disk graph: no induced $K_{1,6}$
- $I^*$: optimal solution
- $I$: algorithm solution (maximal independent set)
- For each vertex $v$ added to $I$ at most 5 neighbors of $v$ in $I^*$ are removed
- Conclusion: $|I^*| \leq 5|I|$
- 5-approximation
Improvement for Maximum Independent Set

- Sort vertices from left to right
- Run the same algorithm
- Right neighbors form 3 cliques
- $3$-approximation for MIS
- $O(n \log n)$ time to sort
- Much slower without geometry
Approximation for Minimum Dominating Set

- Unit disk graph: no induced $K_{1,6}$
- $D^*$: optimal solution
- $D$: algorithm solution (maximal independent set)
- Each vertex $v$ in $D$ has at most 5 neighbors in $D^*$
- Conclusion: $|D| \leq 5|D^*|$
- 5-approximation
- Sorting won’t help!
Local Search

Local Search

- Build a suboptimal solution $S$
- Find two *small* sets $L_0, L_1$
- Say $|L_0|, |L_1| < k$
- Make $S \leftarrow (S \setminus L_0) \cup L_1$
- Verify that $S$ is feasible
- Maximization: use $|L_0| < |L_1|$
- Minimization: use $|L_0| > |L_1|$
- Repeat until no further improvement possible
Local Search for Minimum Dominating Set

Irreducible corona:

\( D \): independent dominating set

\( C \subset D \) is a **corona** centered at vertex \( c \) if:

- \( |C| = 5 \)
- \( C \) is an independent set
- \( c \) is adjacent to all \( c \)

\( C, c \) is **reducible** if \( D \setminus C \cup \{c\} \) is a dominating set

Reducible corona:

**Theorem**

If \( D \) has no reducible corona, then \( D \) is a \(\frac{44}{9}\)-approximation to the minimum dominating set.

- Such \( D \) can be computed in \( O(n + m) \) time without geometry or \( O(n \log n) \) time with geometry
Lower Bound of 4.8 (against 4.89 UB)

\[ OPT = 5 \]
\[ |D| = 5 \cdot 4 + 4 = 24 \]
Proof Technique

Several geometric results needed:

- Lemma 1 (Pál 1921): If a set of points $P$ has diameter 1, then $P$ can be enclosed by a circle of radius $1/\sqrt{3}$.

- Lemma 2 (Fodor 2007): The radius of the smallest circle enclosing 13 points with mutual distance $\geq 1$ is $(1 + \sqrt{5})/2$.

- Lemma 3 (Fejes Tóth 1953): Every packing of two or more congruent disks in a convex region has density at most $\pi/\sqrt{12}$.

- Lemma 4: The closed neighborhood of a clique in a unit disk graph contains at most 12 independent vertices.

- Lemma 5: The closed $d$-neighborhood of a vertex in a unit disk graph contains at most $\pi(2d + 1)^2/\sqrt{12}$ independent vertices, for integer $d \geq 1$. 
Strip Decomposition

Independent Set with Strips (Quadratic)

- Break the problem into horizontal strips of height 2
- Solve MIS for each strip exactly
- Return maximum among all even or odd strips

Good: 2-approximation even for weighted version

Bad: Don’t know how to solve MIS exactly for each strip in $\tilde{O}(n)$ time (but $O(n^2 \log n)$ is possible)
Use strips of height at most $\sqrt{3}$

Resulting graphs are co-comparability

Solve each strip exactly

Separation 2 between strips

Multiple shifts need to be considered

Approximation factor: $1 + \frac{2}{\sqrt{3}} + \varepsilon < 2.16$
Exact MIS Inside a Strip

- Height of the strip: $\sqrt{3}$
- Dynamic programming
- $v_1, \ldots, v_n$: vertices sorted by $x$ coordinate
- For $k$ from 1 to $n$:
  \[ f(k) = \text{maximum independent set of } v_1, \ldots, v_k \]
- Recurrence (cocomparability graph):
  \[ f(k) = 1 + \max_{i<k \text{ and } \|v_i v_k\|>2} f(i) \]
- Query uses semi-dynamic data structure: \( O(\log^2 n) \) time per query
- MIS can be solved in \( O(n \log^2 n) \) time
- Extends to weighted version (extra \( O(\log n) \) factor)
Shifting Coresets

(1) Break the original problem into subproblems of $O(1)$ diameter (shifting strategy)

(2) Build a coreset with $O(1)$ points for each subproblem, which gives an $\alpha$-approximation to the subproblem

(3) Solve the coreset optimally

(4) Combine the solutions into an $(\alpha + \varepsilon)$-approximation
Breaking Independent Set instance into $O(1)$-diameter subproblems (shifting strategy):

- Set $k$ to smallest integer with $\left(\frac{k}{k-2}\right)^2 \geq 1 + \frac{\varepsilon}{4}$
- Use grids of size $2k$
- Create $k^2$ shifted grids with even origins
- Contract grid cells by 1 in all directions
- Each contracted cell is a subproblem
Analysis of Shifting Strategy

- Contracted cells are distance 2 apart: union preserves independence
- 4-approximation in yellow area
- Yellow area gets much bigger than white area as $k \to \infty$
- Expected number of OPT points in white area is small
- Maximum is larger than expectation
Constant-Diameter Coreset for MIS

- **Coreset**: Subset with $O(1)$ points that approximates the original solution
- **Algorithm**: 
  - Create grid with cells of diameter \(0.29 < (2 - \sqrt{2})/2\)
  - Select a point of maximum weight inside each cell (coreset)
  - Find the optimal independent set among the selected points
- We need to prove it gives a 4-approximation!
Proof of 4-Approximation of MIS

- Consider the optimal independent set
- Moving points by at most 0.29, we obtain a planar graph
- Planar graphs are 4-colorable
- The color of maximum weight is a 4-approximation
Lower Bound of 3.25

- $P_1$: Set of points from the figure
- $P_2$: Multiply coordinates from $P_1$ by $(1 + \varepsilon)$ and weights by $(1 - \varepsilon)$
- $P_1 \cup P_2$ gives a lower bound of 3.25
  - $P_2$ is independent
  - MWIS: $P_2$, with $w(P_2) \approx 3.25$
  - Coreset: $P_1$
  - $P_1$ has MWIS with weight 1
Minimum Dominating Set Algorithm

- Break the problem into subproblems of $O(1)$ diameter using the shifting strategy
- Cells need to be expanded rather than contracted
- We’ll present only the coreset
Constant-Diameter Coreset for MDS

Algorithm:
- Create grid with cells of diameter 0.24
- Select the points of min and max x and y coordinates
- Find the optimal dominating set using the coreset points, but dominating every point
- We need to prove it’s a 4-approximation!
Proof of 4-Approximation of MDS

- Either point $p$ from OPT is in the coreset (great!)
- Or there are points $q_1, q_2$ near $p$ with angle $\geq 90^\circ$
- We dominate all points dominated by $p$ using at most 4 points $q_1, q_2, q_3, q_4$
Lower Bound of 4

- 4-approximation
- Optimal solution
- Remaining disks
**Conclusion**

**Greedy:**
- 5-approximation to IS and DS in linear time with or without geometry
- 3-approximation to IS in $O(n \log n)$ time with geometry

**Local search:**
- $44/9$-approximation to DS in $O(n + m)$ time without geometry
- $44/9$-approximation to DS in $O(n \log n)$ time with geometry

**Strip decomposition:**
- 2.16-approximation to IS in $O(n \log^2 n)$ time with geometry
- Generalizes to weighted version in $O(n \log^3 n)$ time

**Shifting coresets:**
- $(4 + \varepsilon)$-approximation to IS and DS in $O(n)$ time with geometry
- Generalizes to weighted version for IS
Open Problems

- Other techniques that yield near-linear-time approximation algorithms?
- Can we prove inapproximability in near-linear-time?
- Can we improve the analysis of existing algorithms?
- Can we do better than $3$-approximation for the chromatic number (greedy)?
- Maximum independent set without geometry better than greedy?
- Minimum *weight* dominating set?
- Intersection of other shapes: general disks, pseudo-disks, line segments, axis-aligned rectangles...
Thank you!

Photo by Gilbert Garcin