

# The Combinatorial Complexity of Approximating Polytopes

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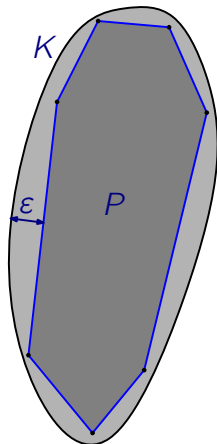
SoCG 2016

# Polytope Approximation

## Problem description:

- Input: convex body  $K$  in  $d$ -dimensional space and parameter  $\varepsilon$
- Output: *succinct* polytope  $P$  which  $\varepsilon$ -approximates  $K$
- Question: how succinct can  $P$  be?

- **$\varepsilon$ -approximate**: Hausdorff distance  $\varepsilon \cdot \text{diam}(K)$
- **Succinct**: Low *combinatorial complexity* (to be defined)
- Assume w.l.o.g. that  $\text{diam}(K) = 1$
- Dimension  $d$  is a constant ( $2^d/\varepsilon = O(1/\varepsilon)$ )



# Uniform vs. Nonuniform Bounds

## Nonuniform bounds:

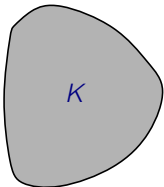
- Hold for  $\varepsilon \leq \varepsilon_0$ , where  $\varepsilon_0$  **depends on the input  $K$**
- Example: Gruber [Gru93] bounds the complexity  $n$  using the Gaussian curvature  $\kappa$  of the input

$$n = (1/\varepsilon)^{(d-1)/2} \int_{\partial K} \sqrt{\kappa(x)} dx$$

## Uniform bounds: (our case)

- Hold for  $\varepsilon \leq \varepsilon_0$ , where  $\varepsilon_0$  is a **constant**
- Example: Dudley [Dud74] and Bronshteyn and Ivanov [BI74] bound the maximum number of facets/vertices as a function of  $\varepsilon$ ,  $d$ , and the diameter of the input

# Bronshiteyn and Ivanov's Approximation

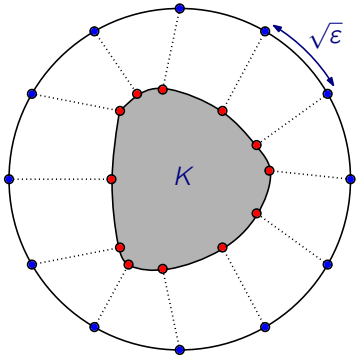


- 1 Surround  $K$  by a sphere of radius 2
- 2 Distribute points on the sphere with distance  $\sim \sqrt{\epsilon}$
- 3 Take the nearest neighbor on  $K$  for each point
- 4 Make  $P$  the **convex hull** of the points

Bronshiteyn and Ivanov, 1974:

A convex body  $K$  of diameter 1 can be  $\epsilon$ -approximated by a polytope  $P$  with  $O(1/\epsilon^{(d-1)/2})$  **vertices**.

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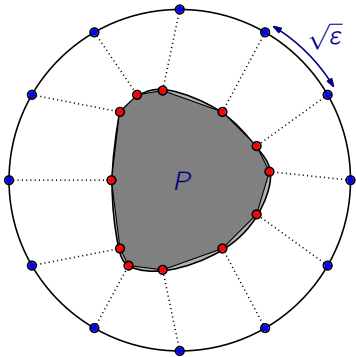


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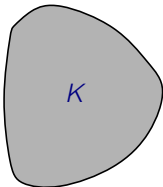


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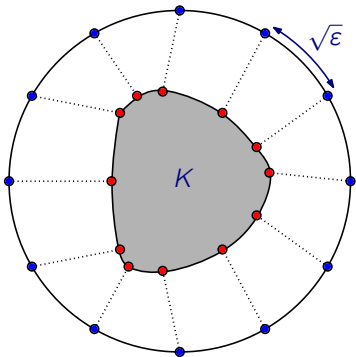


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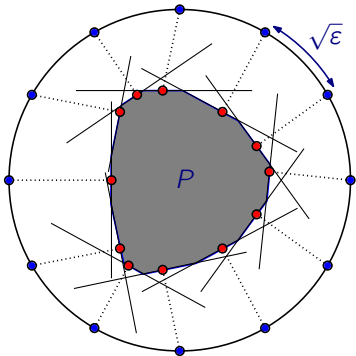
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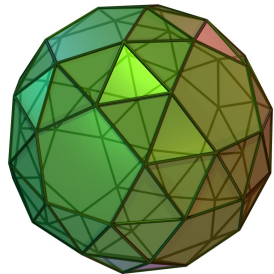
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# Combinatorial Complexity

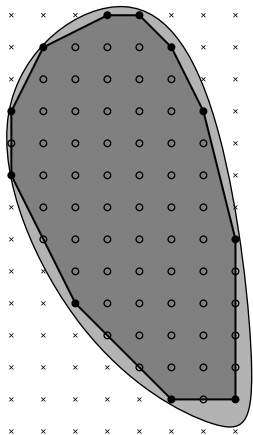
- Faces have different dimensions:
  - 0-face: *vertex*
  - 1-face: *edge*
  - ⋮
  - $(d - 1)$ -face: *facet*
- **Combinatorial complexity:**  
sum of number of  $k$ -faces  $k = 0, \dots, d - 1$



## Upper bound theorem

A polytope with  $n$  vertices (or facets) has combinatorial complexity  $O(n^{\lfloor d/2 \rfloor})$ .

# Approximation of Low Combinatorial Complexity



Lattice polytope

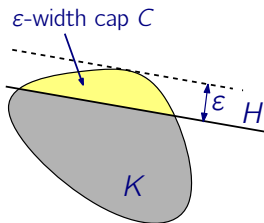
- Bronshteyn and Ivanov: optimal number of **vertices**  $O(1/\varepsilon^{(d-1)/2})$
- Dudley: same optimal number of **facets**
- Upper bound theorem: their combinatorial complexity is  $O(1/\varepsilon^{d^2/4})$   
(Maybe it's much better, but we don't know how to prove)

## Combinatorial complexity

- Best known bound: Roughly  $O(1/\varepsilon^{d-2})$  using lattice polytopes [And63]
- Our bound<sup>a</sup>:  $\tilde{O}(1/\varepsilon^{(d-1)/2})$

<sup>a</sup>  $\tilde{O}$  hides  $\log^{(d-1)/2} \frac{1}{\varepsilon}$

# Caps

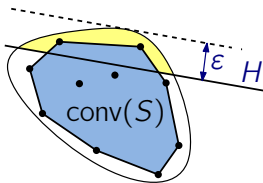


- **Cap**: intersection of  $K$  and a halfspace  $H$
- **Width**: measured perpendicular to  $H$
- A set  $S$  of points **stabs** all  $\epsilon$ -width caps if for every  $\epsilon$ -width cap  $C$  we have  $C \cap S \neq \emptyset$

## Approximation via Hitting Sets

If a point set  $S \subset K$  stabs all  $\epsilon$ -width caps, then  $\text{conv}(S)$  is an  $\epsilon$ -approximation to  $K$ .

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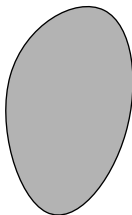
# Bounding Complexity via Witness-Collector

We identify two sets of regions:

- $\mathcal{W}$ : witnesses
- $\mathcal{C}$ : collectors, one per witness

that satisfy:

- (1) Each witness contributes **one point** to  $S$
- (2) Any halfspace  $H$  either:
  - **Deep**: contains a witness, or
  - **Shallow**:  $H \cap K$  is contained within a collector
- (3) Each collector contains  $O(1)$  points of  $S$



Witness-Collector Complexity Bound [Devillers et al. 2013]

Given a set of witnesses and collectors satisfying the above properties, the combinatorial complexity of the  $\text{conv}(S)$  is  $O(|\mathcal{C}|)$ .

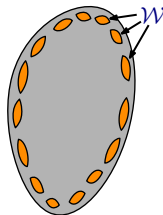
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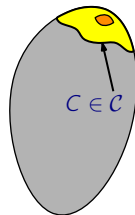
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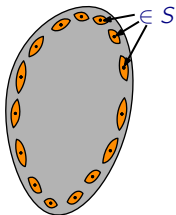
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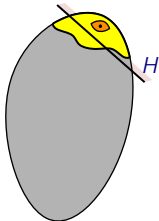
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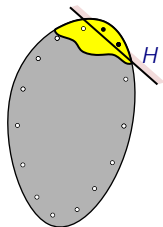
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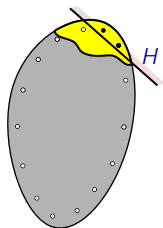
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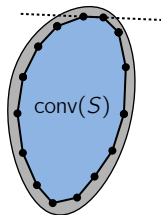
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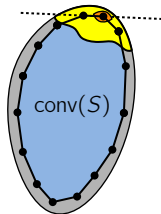
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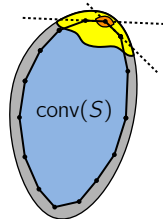
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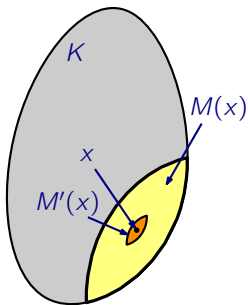
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# Macbeath Regions



## Macbeath Regions [Macbeath 52]

Given a convex body  $K$ ,  $x \in K$ , and  $\lambda > 0$ :

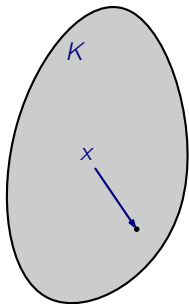
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## Properties:

- $M(x)$  resembles the minimum volume cap containing  $x$
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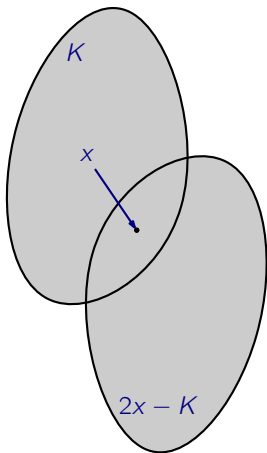
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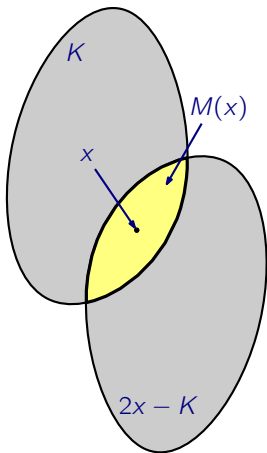
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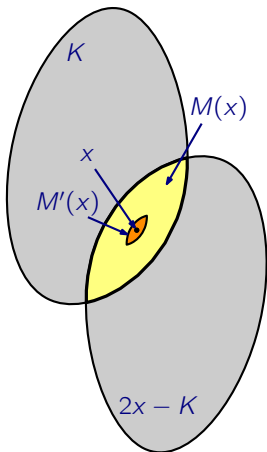
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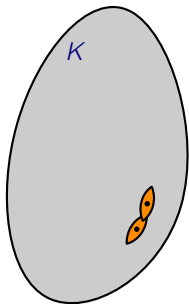
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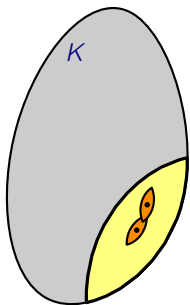
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# Economical Cap Covering

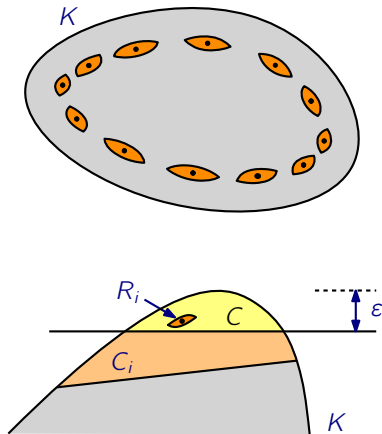
## Economical Cap Covering

Given:

- $K$ : convex body
- $\varepsilon$ : small positive parameter

There exists [ELR70, Bar07]:

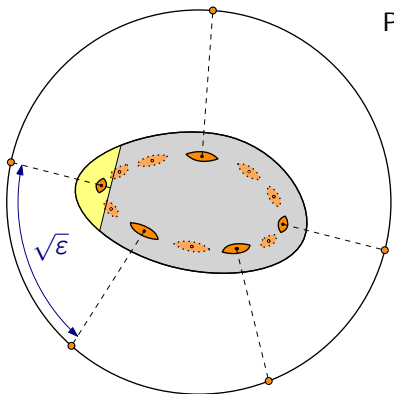
- Macbeath regions  $R_1, \dots, R_k$
- Caps  $C_1, \dots, C_k$  of width  $\Theta(\varepsilon)$
- $k = O(1/\varepsilon^{(d-1)/2})$  (new)
- For every cap  $C$ , there is  $i$ :
  - $R_i \subseteq C$  if width  $\geq \varepsilon$
  - $C \subseteq C_i$  if width  $\leq \varepsilon$



# Cardinality Bound

Proof strategy:

- Prune Macbeath regions that are too close to each other (by increasing volume)
- A constant fraction of the regions are pruned
- Project the centers of the regions onto the Dudley ball (perpendicularly to the corresponding cap)
- Show that the pairwise distance in the Dudley ball is at least  $\sqrt{\epsilon}$





# Approximating with ECC

## Economical Cap Covering

Given:

- $K$ : convex body
- $\varepsilon$ : small positive parameter

There exists [ELR70, Bar07]:

- Macbeath regions  $R_1, \dots, R_k$
- Caps  $C_1, \dots, C_k$  of width  $\Theta(\varepsilon)$
- $k = O(1/\varepsilon^{(d-1)/2})$  (new)
- For every cap  $C$ , there is  $i$ :
  - $R_i \subseteq C$  if width  $\geq \varepsilon$
  - $C \subseteq C_i$  if width  $\leq \varepsilon$

Procedure:

- Create an ECC
- Place a point inside each Macbeath region
- Compute the convex hull

Approximation:

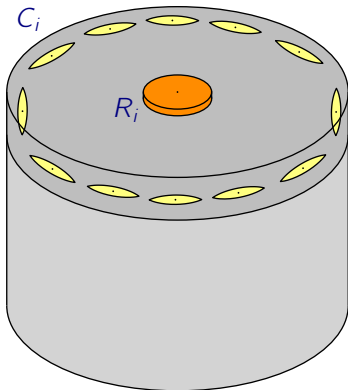
- Every  $\varepsilon$ -width cap is stabbed

Complexity:

- $O(1/\varepsilon^{(d-1)/2})$  vertices
- Combinatorial complexity:  $O(1/\varepsilon^{d^2/4})$  by the upper bound theorem

# Better Bound on the Combinatorial Complexity

What if we use the Macbeath regions as witnesses and the caps as collectors?

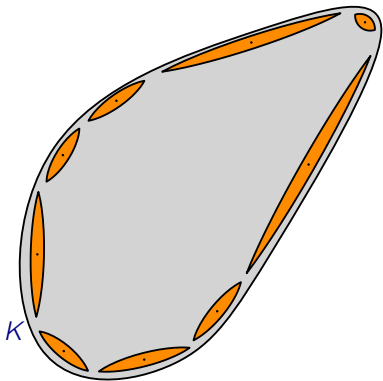


- Does not work for the traditional construction (for fixed width)
- We need one additional property:  
For all  $i$ ,  $C_i$  intersects a constant number of bodies  $R_j$

# Stratified Construction

- Organize the Macbeath regions in  $O(\log \frac{1}{\varepsilon})$  layers by *volume*
- Larger volume Macbeath regions in outer layers
- Approximation error increases from  $\varepsilon$  to  $O(\varepsilon \log \frac{1}{\varepsilon})$
- Scale  $\varepsilon$  accordingly:  $\hat{\varepsilon} = \frac{\varepsilon}{\log \frac{1}{\varepsilon}}$
- Number of regions grows to

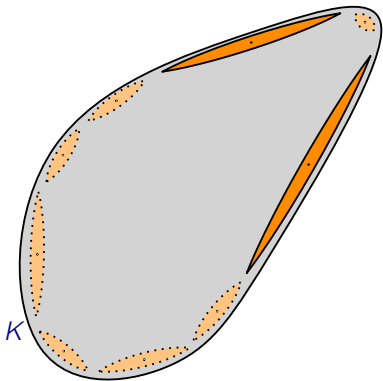
$$O\left(\left(\frac{1}{\hat{\varepsilon}}\right)^{\frac{d-1}{2}}\right) = O\left(\left(\frac{\log \frac{1}{\varepsilon}}{\varepsilon}\right)^{\frac{d-1}{2}}\right)$$



# Stratified Construction

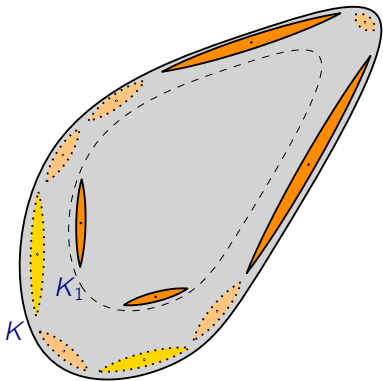
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- Scale  $\varepsilon$  accordingly:  $\hat{\varepsilon} = \frac{\varepsilon}{\log \frac{1}{\varepsilon}}$
- Number of regions grows to

$$O\left(\left(\frac{1}{\hat{\varepsilon}}\right)^{\frac{d-1}{2}}\right) = O\left(\left(\frac{\log \frac{1}{\varepsilon}}{\varepsilon}\right)^{\frac{d-1}{2}}\right)$$



# Stratified Construction

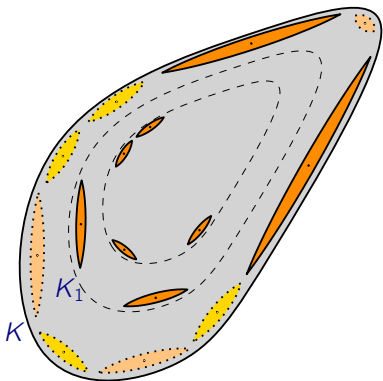
- Organize the Macbeath regions in  $O(\log \frac{1}{\varepsilon})$  layers by *volume*
- Larger volume Macbeath regions in outer layers
- Approximation error increases from  $\varepsilon$  to  $O(\varepsilon \log \frac{1}{\varepsilon})$
- Scale  $\varepsilon$  accordingly:  $\hat{\varepsilon} = \frac{\varepsilon}{\log \frac{1}{\varepsilon}}$
- Number of regions grows to



$$O\left(\left(\frac{1}{\hat{\varepsilon}}\right)^{\frac{d-1}{2}}\right) = O\left(\left(\frac{\log \frac{1}{\varepsilon}}{\varepsilon}\right)^{\frac{d-1}{2}}\right)$$

# Stratified Construction

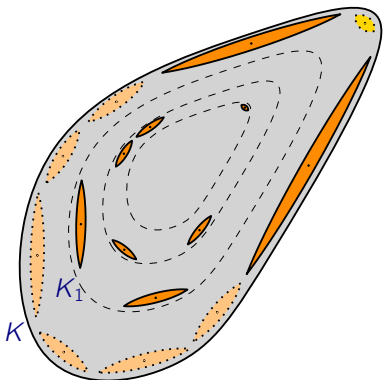
- Organize the Macbeath regions in  $O(\log \frac{1}{\epsilon})$  layers by *volume*
- Larger volume Macbeath regions in outer layers
- Approximation error increases from  $\epsilon$  to  $O(\epsilon \log \frac{1}{\epsilon})$
- Scale  $\epsilon$  accordingly:  $\hat{\epsilon} = \frac{\epsilon}{\log \frac{1}{\epsilon}}$
- Number of regions grows to



$$O\left(\left(\frac{1}{\hat{\epsilon}}\right)^{\frac{d-1}{2}}\right) = O\left(\left(\frac{\log \frac{1}{\epsilon}}{\epsilon}\right)^{\frac{d-1}{2}}\right)$$

# Stratified Construction

- Organize the Macbeath regions in  $O(\log \frac{1}{\epsilon})$  layers by *volume*
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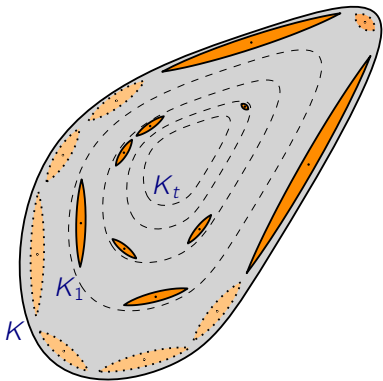


$$O\left(\left(\frac{1}{\hat{\epsilon}}\right)^{\frac{d-1}{2}}\right) = O\left(\left(\frac{\log \frac{1}{\epsilon}}{\epsilon}\right)^{\frac{d-1}{2}}\right)$$

# Stratified Construction

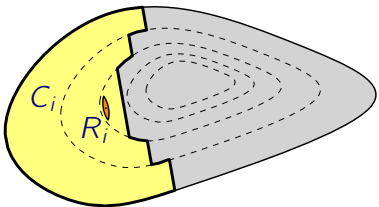
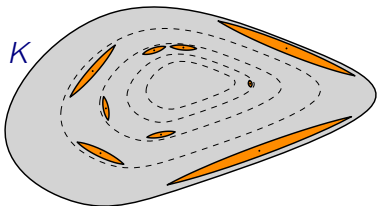
- Organize the Macbeath regions in  $O(\log \frac{1}{\epsilon})$  layers by *volume*
- Larger volume Macbeath regions in outer layers
- Approximation error increases from  $\epsilon$  to  $O(\epsilon \log \frac{1}{\epsilon})$
- Scale  $\epsilon$  accordingly:  $\hat{\epsilon} = \frac{\epsilon}{\log \frac{1}{\epsilon}}$
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$$O\left(\left(\frac{1}{\hat{\epsilon}}\right)^{\frac{d-1}{2}}\right) = O\left(\left(\frac{\log \frac{1}{\epsilon}}{\epsilon}\right)^{\frac{d-1}{2}}\right)$$





# Stratified Economical Cap Covering



## Stratified ECC

Given:

- $K$ : convex body
- $\varepsilon$ : small positive parameter

There exists:

- Convex bodies  $R_1, \dots, R_k$
- Regions  $C_1, \dots, C_k$
- $k = \tilde{O}(1/\varepsilon^{(d-1)/2})$
- Every  $\varepsilon$ -width cap  $C$  contains an  $R_i$
- For every cap  $C$ , either  $R_i \subseteq C$  or  $C \subseteq C_i$  for some  $i$
- For all  $i$ ,  $C_i$  intersects a constant number of bodies  $R_j$

# Comparing Economical Cap Coverings

## Economical Cap Covering

Given:  $K, \varepsilon$

There exists:

- Convex bodies  $R_1, \dots, R_k$
- Caps  $C_1, \dots, C_k$  of width  $\Theta(\varepsilon)$
- $k = O(1/\varepsilon^{(d-1)/2})$
- For every cap  $C$ , there is  $i$ :
  - $R_i \subseteq C$  if width  $\geq \varepsilon$
  - $C \subseteq C_i$  if width  $\leq \varepsilon$

## Stratified ECC

Given:  $K, \varepsilon$

There exists:

- Convex bodies  $R_1, \dots, R_k$
- Non-convex regions  $C_1, \dots, C_k$
- $k = \tilde{O}(1/\varepsilon^{(d-1)/2})$
- For every cap  $C$ , there is  $i$ :
  - $R_i \subseteq C$  if width  $\geq \varepsilon$
  - $C \subseteq C_i$  if no  $R_j \subseteq C$
- For all  $i$ ,  $C_i$  intersects a constant number of bodies  $R_j$

# Succinct Approximation

Procedure:

- Create a stratified ECC
- Place a point inside each  $R_i$
- Compute the convex hull

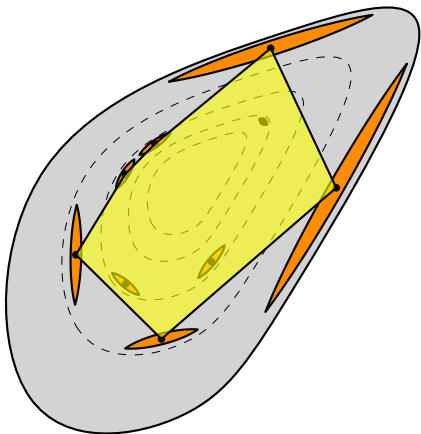
Approximation:

- Every  $\varepsilon$ -width cap is stabbed

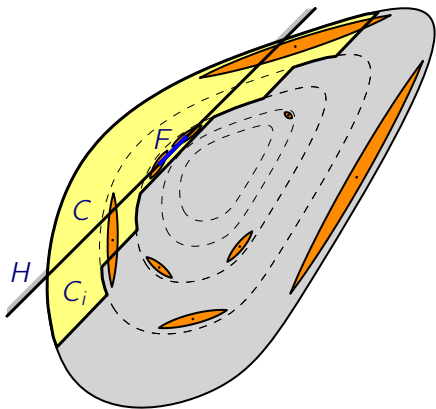
Complexity:

- $\tilde{O}(1/\varepsilon^{(d-1)/2})$  vertices
- Combinatorial complexity:  
 $\tilde{O}(1/\varepsilon^{(d-1)/2})$

Why?



# Counting the Faces



- $R_j$  are witnesses and  $C_i$  are collectors
- Charge each face  $F$  to a  $C_i$  and show each  $C_i$  receives  $O(1)$  charges
- $H$ : supporting halfspace of  $F$
- $C = K \cap H$  contains no  $R_j$
- Some  $C_i$  contains  $C$
- Charge  $F$  to  $C_i$
- $C_i$  intersects  $O(1)$  bodies  $R_j$   
 →  $C_i$  contains  $O(1)$  points  
 →  $O(1)$  faces charge  $C_i$

# Conclusion and Open Problems

## Our result:

First near-optimal bound on the combinatorial complexity of an  $\varepsilon$ -approximating polytope in  $\mathbb{R}^d$ :

$$O\left(\left(\frac{\log \frac{1}{\varepsilon}}{\varepsilon}\right)^{\frac{d-1}{2}}\right) = \tilde{O}\left(\left(\frac{1}{\varepsilon}\right)^{\frac{d-1}{2}}\right)$$

## Open problems:

- How long does it take to actually build the approximation?
- Can we analyze Dudley's construction?
- Can we get rid of the  $\log$  factors?
- Can we compete against the optimal combinatorial complexity for the given  $K$ ?

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Sculpture by Antony Gormley

Thank you!