Optimal Area-Sensitive Bounds for Polytope Approximation

Sunil Arya
Hong Kong University of Science and Technology

Guilherme D. da Fonseca
Universidade Federal do Estado do Rio de Janeiro (UniRio)

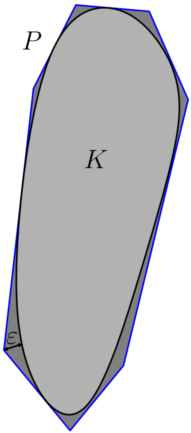
David M. Mount
University of Maryland, College Park

SoCG 2012, Chapel Hill, North Carolina

Polytope Approximation

Problem description:

- Input: convex body K in d-dimensional space and parameter ε
- Output: polytope P which ε-approximates K with a small number of facets (alternatively, vertices)



- Focus on Hausdorff metric in Euclidean spaces of constant dimension d
- Assume (without loss of generality) that $\mathrm{diam}(K) = 1$
- Assume the width of K is at least ε. Otherwise, the instance can be reduced (by projection) to a lower dimensional space

Uniform vs. Nonuniform Bounds

- Several algorithms find the "best" polytope for a given input [Cl93]
- How good is this best polytope?

Nonuniform bounds:

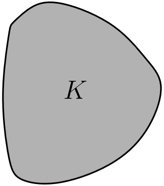
- Hold for $\varepsilon \leq \varepsilon_0$, where ε_0 depends on the input
- Example: Gruber [Gru93] bounds the complexity n using the Gaussian curvature κ of the input

$$n = (1/\varepsilon)^{(d-1)/2} \int_{\partial K} \sqrt{\kappa(x)}\, dx$$

Uniform bounds:

- Hold for $\varepsilon \leq \varepsilon_0$, where ε_0 is a constant
- Example: Dudley [Dud74] and Bronshteyn and Ivanov [BI76] bound the maximum number of facets/vertices as a function of ε, d, and the diameter of the input

Dudley's Approximation

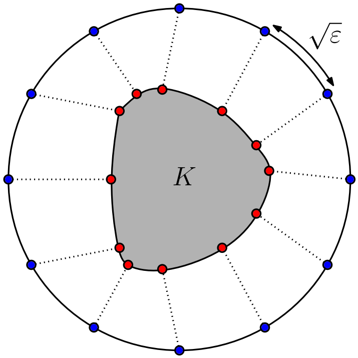


Dudley, 1974:

A convex body K of diameter 1 can be ε-approximated by a polytope P with $O(1/\varepsilon^{(d-1)/2})$ facets.

- Dudley's approximation is the best possible for balls
- It oversamples areas of very high and very low curvatures
- Intuition: A skinny body should be easier to approximate

Dudley's Approximation

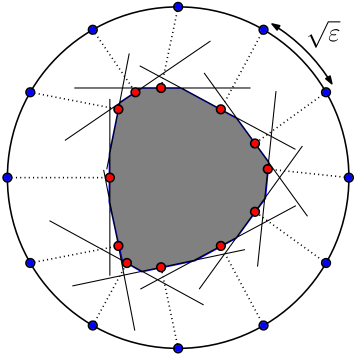


Dudley, 1974:

A convex body K of diameter 1 can be ε-approximated by a polytope P with $O(1/\varepsilon^{(d-1)/2})$ facets.

- Dudley's approximation is the best possible for balls
- It oversamples areas of very high and very low curvatures
- Intuition: A skinny body should be easier to approximate

Dudley's Approximation



Dudley, 1974:

A convex body K of diameter 1 can be ε-approximated by a polytope P with $O(1/\varepsilon^{(d-1)/2})$ facets.

- Dudley's approximation is the best possible for balls
- It oversamples areas of very high and very low curvatures
- Intuition: A skinny body should be easier to approximate

Previous Result: Area-Sensitive Polytope Approximation

Better uniform bound for *skinny* bodies [AFM12]:

A convex body K can be ε-approximated by a polytope P with $O(\sqrt{\text{area}(K)} \log(\text{area}(K)/\varepsilon)/\varepsilon^{(d-1)/2})$ facets.

Compared to Dudley's bound:

- Uses area instead of diameter
- Significant improvement for skinny bodies
- Suboptimal by a log factor

Our Result: Optimal Polytope Approximation

Optimal area-based bound:

A convex body K can be ε-approximated by a polytope P with $O(\sqrt{\text{area}(K)}/\varepsilon^{(d-1)/2})$ facets (alternatively, vertices).

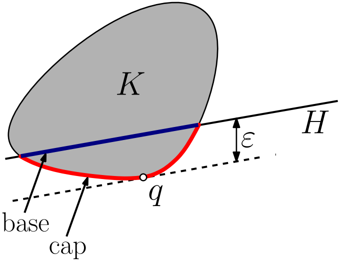
Compared to Dudley's bound:

- Uses area instead of diameter
- Bounds match when the body is fat

Compared to our previous bound:

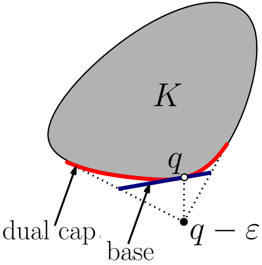
- No log factor: bound is optimal
- Sampling uses Macbeath regions instead of ε-nets
- Requires a combination of functional duality and polarity

Caps and ε-Caps



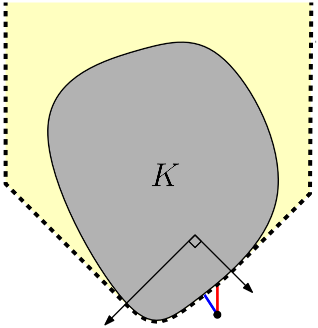
- Cap: intersection of the boundary of K and a halfspace H
- Width: maximum vertical distance between a point in C and ∂H
- ε-cap: Cap of width ε
- Base: $\partial H \cap K$

Dual Caps and ε-Dual Caps



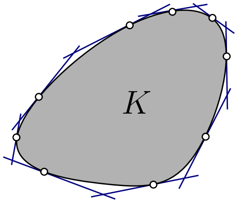
- Dual cap: portion of the boundary of K visible from a given point
- Width: vertical distance between the point and K
- ε-dual cap: dual cap of width ε
- Dual of an ε-cap
- Base: see figure

Splitting K



- ε-caps and ε-dual caps are defined in terms of vertical distances
- When slopes are bounded, vertical distances approximate Euclidean distances
- We partition K into $2d$ regions with bounded slopes
- Each region is extended and then cropped vertically to handle boundary conditions

Approximation by Stabbing ε-Dual Caps



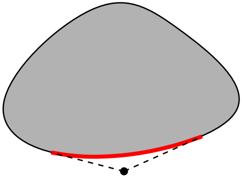
- A set of points N stabs all ε-dual caps if every ε-dual cap D has $D \cap N \neq \emptyset$

Lemma:

If a set N of points stabs all ε-dual caps, then the polytope defined by tangent hyperplanes constructed at the points of N is an ε-approximation to (the bottom portion of) K.

- We divide the ε-dual caps in two categories: large and small
- We stab each category separately

Stabbing Large ε-Dual Caps



Large ε-dual cap D:

$\text{area}(D) \geq \sqrt{\text{area}(K)} \cdot \varepsilon^{(d-1)/2}$

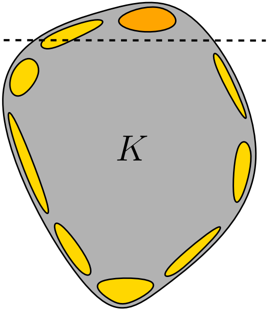
Simple solution [AFM12]:

- Use ε-nets
- A random point on the boundary of K is likely to stab D
- Dual caps have bounded VC-dimension
- Introduces a log factor

Better sampling:

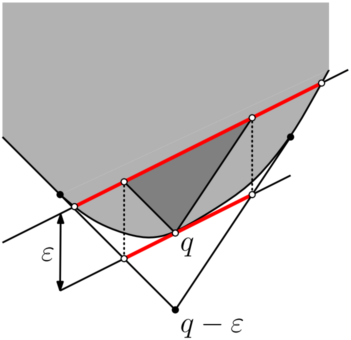
- Use Macbeath regions
- Avoids the log factor

Macbeath Regions



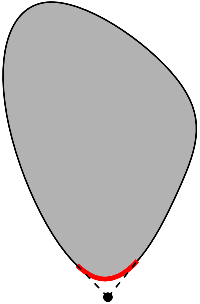
- Given:
  - K: convex body
  - v: parameter $0 < v < \mathrm{vol}(K)$
- There exists [Mac52]:
  - Set $\mathcal{M}$ of disjoint convex bodies inside K
  - Each $M \in \mathcal{M}$ has $\mathrm{vol}(M) = \Theta(v)$
  - Every cap C with $\mathrm{vol}(C) = v$ contains a region $M \in \mathcal{M}$
  - Also, a constant factor scaling of M contains C
- Macbeath regions: convex bodies $M \in \mathcal{M}$

From Caps to Dual Caps



- Macbeath regions are suited to caps, not dual caps
- We can associate large ε-dual caps with large ε-caps
- We stab large ε-caps using Macbeath regions

Small ε-Dual Caps



Small ε-dual cap D:

$\text{area}(D) < \sqrt{\text{area}(K)} \cdot \varepsilon^{(d-1)/2}$

- Area can be as small as ε^{d-1}
- Hard to stab directly by random sampling
- Key insight: In the dual body, small ε-dual caps become large caps

Functional Duality and Polarity

Functional duality

Maps point $(a_1, \ldots, a_d)$ to hyperplane $x_d = a_1 x_1 + \cdots + a_{d-1} x_{d-1} - a_d$

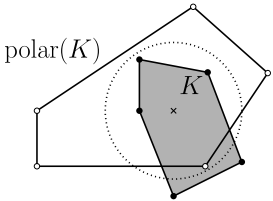
- Widely used in computational geometry
- Preserves vertical distances

Polarity

Maps point $(a_1, \ldots, a_d)$ to hyperplane $a_1 x_1 + \cdots + a_d x_d = 1$

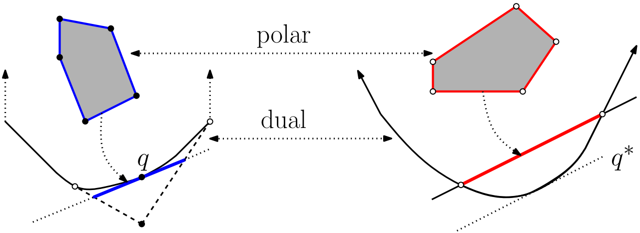
- Widely used in convex and combinatorial geometry
- Allows for concepts such as the polar body and Mahler volume

Polar Body and Mahler Volume



- K: convex body
- Polar body of K: convex hull of the polar of the suporting hyperplanes of K
- Mahler volume of K: product of the volume of K and the volume of $polar(K)$
- The Mahler volume of K is bounded below by a constant [Kup08]

Stabbing Small ε-Dual Caps



Lemma:

The base of an ε-dual cap in the primal is the polar of the base of the corresponding cap in the dual, scaled by a factor of ε.

- By Mahler volume considerations, small ε-dual caps in the primal correspond to large ε-caps in the dual
- We stab such caps in the dual using Macbeath regions

Conclusion and Open Problems

Our results:

- We obtain optimal area-sensitive bounds for polytope approximation
- We use Macbeath regions instead of ε-nets for sampling

Open problems:

- Our proofs are existential (a log factor approximation can be built using [Cl93]). How can the construction be made efficient?
- Our results only hold for the whole convex body. Can they be extended to patches? (Related results in [AFM12].)

Bibliography

- [AFM12] S. Arya, G. D. da Fonseca, and D. M. Mount. Polytope Approximation and the Mahler Volume. *In Proc. ACM-SIAM Symposium on Discrete Algorithms*, pages 29–42, 2012.
- [BI76] E. M. Bronshteyn and L. D. Ivanov. The approximation of convex sets by polyhedra. *Siberian Math. J.*, 16:852–853, 1976.
- [Cl93] K. L. Clarkson. Algorithms for polytope covering and approximation. *WADS*, 246–252, 1993.
- [Dud74] R. M. Dudley. Metric entropy of some classes of sets with differentiable boundaries. *Approx. Theory*, 10(3):227–236, 1974.
- [Gru93] P. M. Gruber. Asymptotic estimates for best and stepwise approximation of convex bodies. *I. Forum Math.*, 5:521–537, 1993.
- [Kup08] G. Kuperberg. From the Mahler conjecture to Gauss linking integrals. *Geometric And Functional Analysis*, 18:870–892, 2008.
- [Mac52] A. M. Macbeath. A theorem on non-homogeneous lattices. *Annals of Mathematics*, 54:431–438, 1952.

Sculpture by Antony Gormley.

Thank you!