Optimal Bound on the Combinatorial Complexity of Approximating Polytopes

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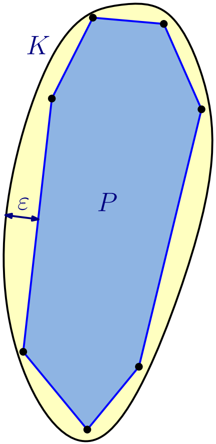
SODA 2020

Introduction

Convex Approximation:

- Given a convex body K in $\mathbb{R}^d$ and parameter $\varepsilon > 0$, compute a polytope P of low combinatorial complexity that ε-approximates K

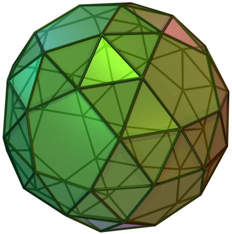


- ε-approximate: Hausdorff distance $\leq \varepsilon \cdot \mathrm{diam}(K)$
- Assume w.l.o.g. that K is fat and $\mathrm{diam}(K) = 1$
- Dimension d is a constant

Combinatorial Complexity

- Combinatorial complexity:
  Sum of the number of k-faces $k = 0, \ldots, d-1$
- Can ε-approximate with:
  - $O(1/\varepsilon^{(d-1)/2})$ facets [Dudley 1974]
  - $O(1/\varepsilon^{(d-1)/2})$ vertices [Bronshteyn & Ivanov 1974]

  Both bounds are tight
  No known construction achieves both simultaneously



- A polytope with n vertices (facets) can have combinatorial complexity $\Theta(n^{\lfloor d/2 \rfloor})$
- Dudley's and Bronshteyn-Ivanov's constructions may suffer much higher combinatorial complexity

Main Result

Question: Is it possible to ε-approximate any convex body by a convex polytope of total combinatorial complexity $O(1/\varepsilon^{(d-1)/2})$?

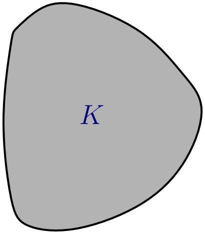
Earlier Results:

- Yes, but the polyhedron might be nonconvex [Erickson 2003]
- Yes, but the curvature must be bounded [Clarkson 2006]
- Well, almost: $O\left(\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)^{\frac{d-1}{2}}\right)$ [AFM 2017]

Main Result:

Can ε-approximate any convex body with total combinatorial complexity $O\left(\frac{1}{\varepsilon}\right)^{\frac{d-1}{2}}$

Bronshteyn and Ivanov's Approximation



1. Surround K by a sphere of radius 2
2. Distribute points on the sphere with distance $\sim \sqrt{\varepsilon}$
3. Take the nearest neighbor on K for each point
4. Make P the convex hull of the points

Bronshteyn and Ivanov, 1974:

A convex body K of diameter 1 can be ε-approximated by a polytope P with $O(1/\varepsilon^{(d-1)/2})$ vertices.

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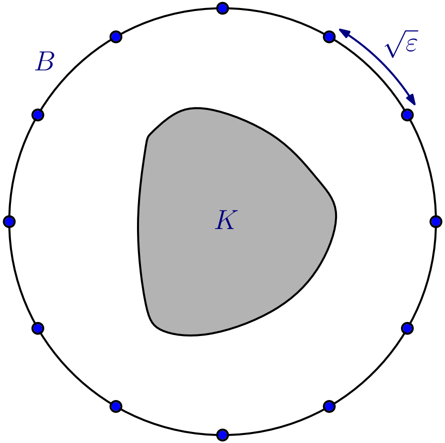

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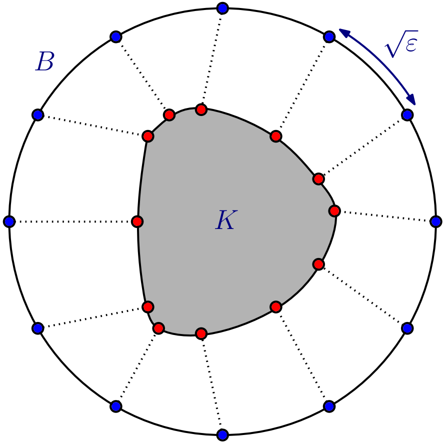

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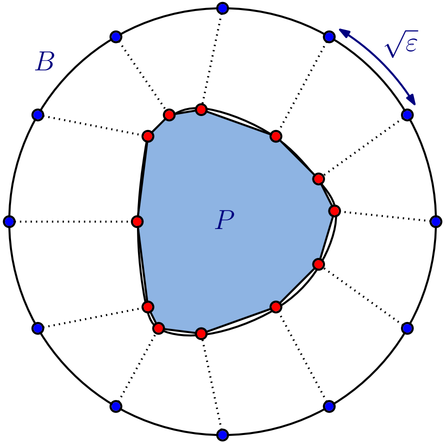

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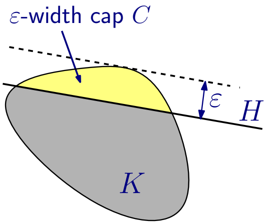

Convex approximation is a covering problem

- Cap: intersection of K and a halfspace H
- Width: measured perpendicular to H

Hitting Caps

If a point set $S \subset K$ is a hitting set for all ε-width caps, then $\mathrm{conv}(S)$ is an ε-approximation to K

- Bronshteyn & Ivanov: $|S| = O(1/\varepsilon^{(d-1)/2})$ possible
- Total complexity may be much larger
- To bound the total complexity, we need more structure

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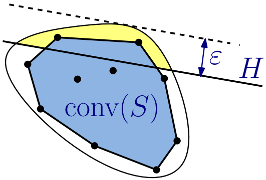

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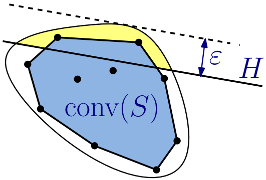

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The Witness-Collector Method

We identify two sets of regions:

- $\mathcal{W}$: witnesses
- $\mathcal{C}$: collectors, one per witness

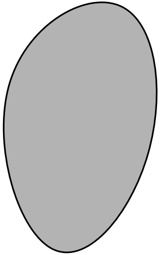
such that:

(1) Each witness contributes one point to S

(2) Any halfspace H is either:

- Deep: contains a witness, or
- Shallow: $H \cap K$ is contained within a collector

(3) Each collector contains $O(1)$ points of S



Witness-Collector Complexity Bound [Devillers et al. 2013]

Given a set of witnesses and collectors satisfying the above properties, the combinatorial complexity of the $\mathrm{conv}(S)$ is $O(|\mathcal{C}|)$.

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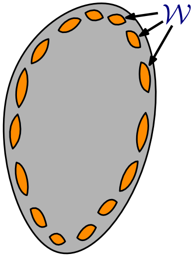

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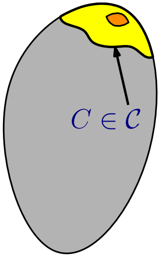

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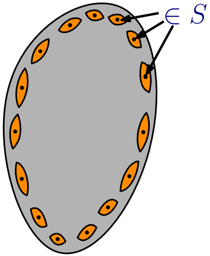

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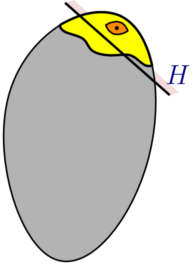

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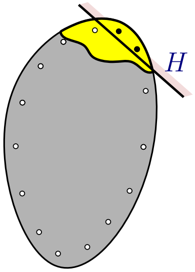

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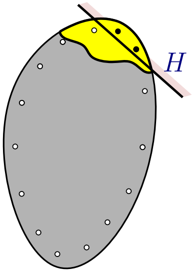

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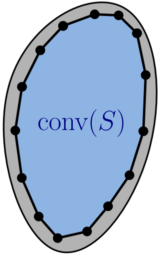

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Witness-Collector Properties

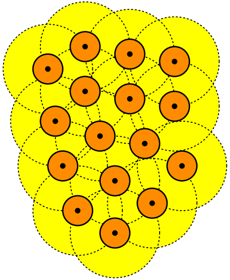
Desirable properties:

- Packing: Witnesses are disjoint (for packing arguments)
- Covering: Collectors cover K's boundary
- Similarity: Small expansion of witness covers its collector
- Locality: If two potential witnesses overlap, a constant factor expansion of one encloses the other

How about ε-caps (and their expansions)?

Unfortunately, caps are not local

$\Rightarrow$ No constant expansion of W_1 will contain W_2



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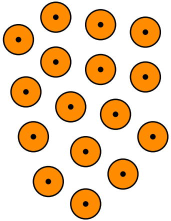

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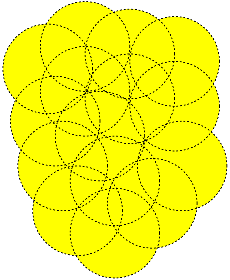

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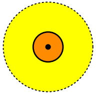

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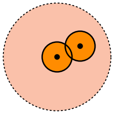

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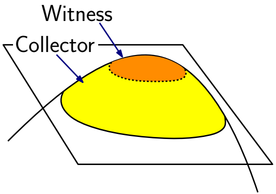

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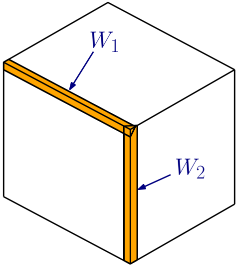

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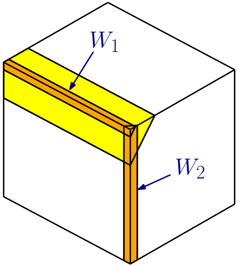

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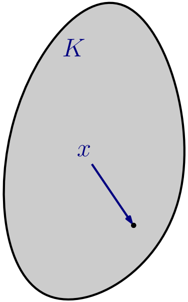
Macbeath Regions - Definition



Macbeath Regions [Macbeath 1952]

Given a convex body K and $x \in K$:

$$M(x) = K \cap (2x - K)$$

- $M(x)$: largest centrally symmetric convex body inside K, centered at x
- Let $M^\lambda(x)$ be a scaling of $M(x)$ by a factor λ
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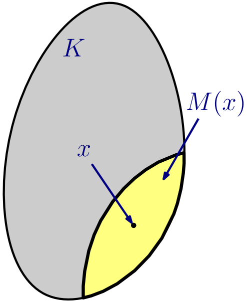

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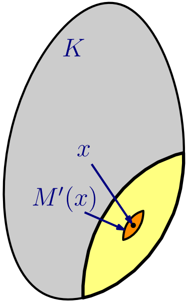

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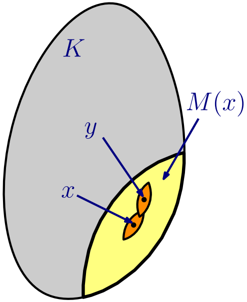
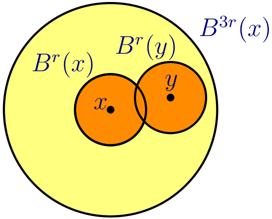
Macbeath Regions - Properties



- **Locality**: For $\lambda \leq 1/5$, if $M^{\lambda}(x)$ and $M^{\lambda}(y)$ overlap, then $M^{\lambda}(y) \subseteq M^{4\lambda}(x)$

  (Like balls of radius r: $B^r(x)$ and $B^r(y)$ overlap, then $B^r(y) \subseteq B^{3r}(x)$)



- **Similarity**: $M^{3d}(x)$ covers x's minimum volume cap
- **Approximation**: If $\lambda \leq 1$ and x is at distance ε from ∂K, then for all $y \in M^{\lambda}(x)$: y is at distance $O(\varepsilon)$ from ∂K

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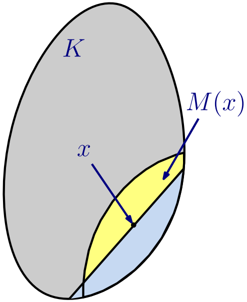

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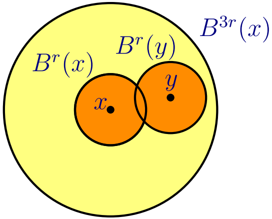

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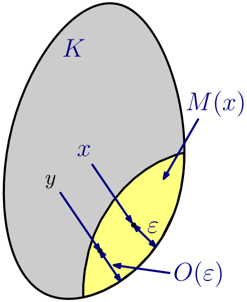

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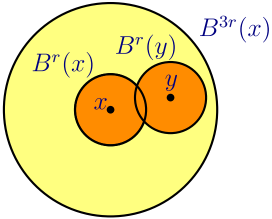

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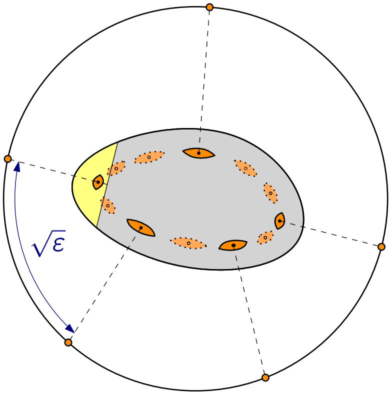
Cardinality Bound



Economical Cap Cover [AFM 2017]

Let $K \subset \mathbb{R}^d$ be a convex body of diameter 1. A set of disjoint Macbeath regions $M'(x)$ placed at points x at distance ε from the boundary of K has $O(1/\varepsilon^{(d-1)/2})$ Macbeath regions.

Proof strategy:

- Prune Macbeath regions that are too close to each other (by increasing volume)
- A constant fraction of the regions are pruned
- Project the centers of the regions onto the Dudley ball (perpendicularly to the corresponding cap)
- Show that the pairwise distance in the Dudley ball is at least $\sqrt{\varepsilon}$

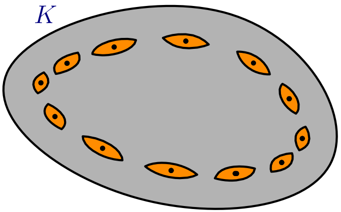
Economical Cap Covering (ECC)

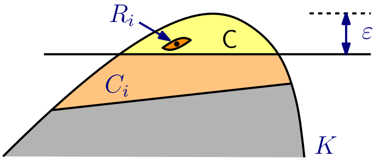
Can we use Macbeath regions as a basis for Witness-Collector?

Economical Cap Cover [AFM 2017]

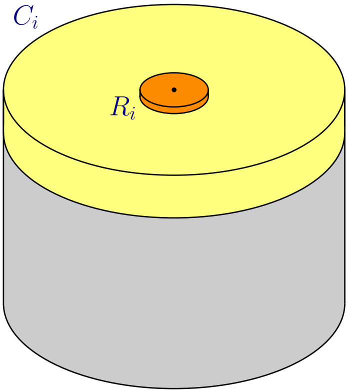
Given K and $\varepsilon > 0$, there exists:

- Macbeath regions $R_1, \ldots, R_k$, for $k = O(1/\varepsilon^{(d-1)/2})$
- Caps $C_1, \ldots, C_k$ of width $\Theta(\varepsilon)$
- For every cap C of K, there is i such that either:
  - Deep: $R_i \subseteq C$
  - Shallow: $C \subseteq C_i$





Are we there yet?



- This suggests a Witness-Collector system:
  - Witnesses: R_i
  - Collectors: C_i
  - Points: Centers of the R_i's
- Witness-Collector conditions (1) and (2) are satisfied
- But condition (3):
  Each collector contains $O(1)$ points of S
  ...does not hold

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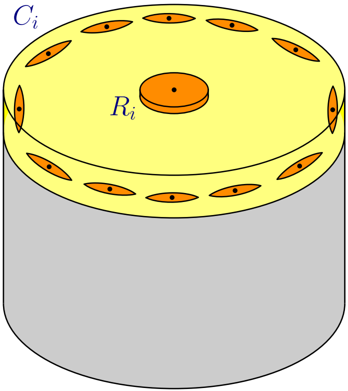

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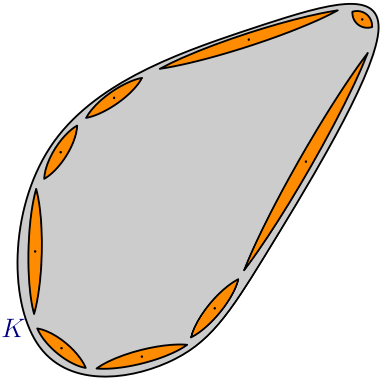
Stratified Construction



Our earlier (suboptimal) solution [AFM 2017]:

- Partition witnesses in $O(\log \frac{1}{\varepsilon})$ layers by volume
- Larger-volume regions in outer layers
- Why it works?
  - Witness overlaps collector $\Rightarrow$ high volume
  - Apply a packing argument to bound them
- But, error increases from ε to $O(\varepsilon \log \frac{1}{\varepsilon})$
- Number of regions grows to

$$O\left(\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)^{\frac{d-1}{2}}\right)$$

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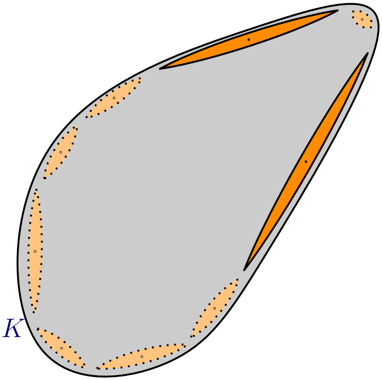

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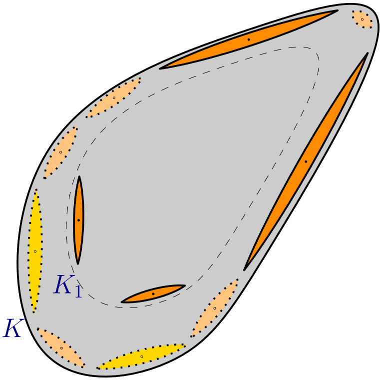

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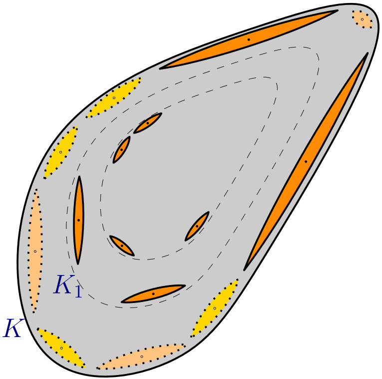

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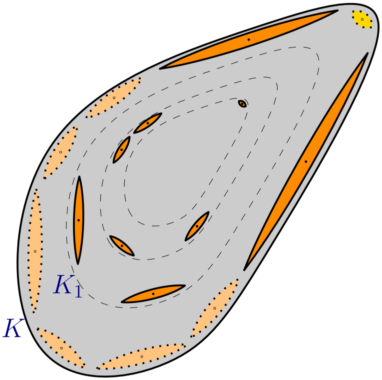

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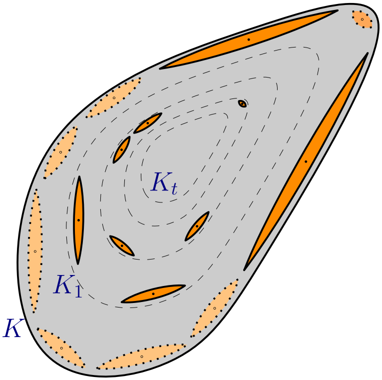

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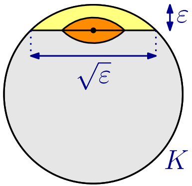
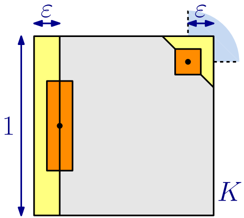
New Insights

Our optimal construction is based primarily on two new ideas:

- Volume-sensitive bound on the number of Macbeath regions in the ECC
- A volume-inverting correspondence between Macbeath regions in K and its polar K^*

Volume-Sensitive Bound on Macbeath Regions





Number of Macbeath regions differs by volume:

- Ball has $\Theta(1/\varepsilon^{(d-1)/2})$ regions of volume $\Theta(\varepsilon^{(d+1)/2})$
- Volumes of Macbeath regions vary from $\Theta(\varepsilon^d)$ to $\Theta(\varepsilon)$
- Since K has unit diameter, the portion of K within ε of its boundary has volume $\Theta(\varepsilon)$
- By a packing argument, the number of disjoint Macbeath regions of volume at least v is $O(\varepsilon/v)$
- What about Macbeath regions of small volume?

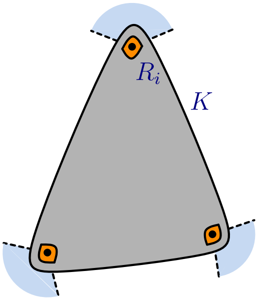
Macbeath Regions in the Polar

A packing argument cannot be used to bound the number of small Macbeath regions (volume from ε^d up to $\varepsilon^{(d+1)/2}$)

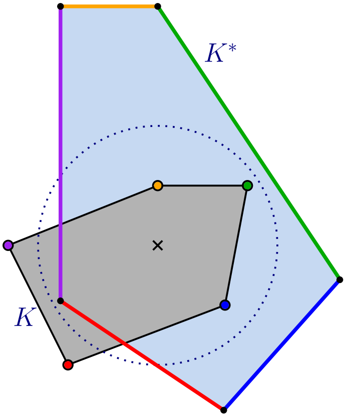
But still, there cannot be many of them!

- Consider a Macbeath region of volume $O(\varepsilon^d)$
- Such a tiny Macbeath region must be close to a portion of K's boundary with high curvature
- By convexity, K's boundary curvature is bounded
- Therefore, the number of such Macbeath regions is $O(1)$

How to generalize this intuition to all small Macbeath regions?

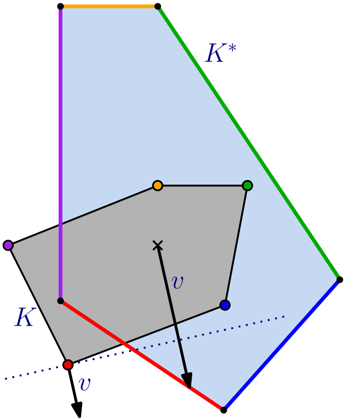


Polar Body



- K: convex body
- Polar K^*:
  points p such that $p \cdot q \leq 1$ for $q \in K$
- High curvature maps to low curvature
- If origin is well-centered, then the product of the volumes (Mahler volume) is between two constants
- In K: extreme point in direction v
- In K^*: ray shooting in direction v from origin

Polar Body



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- In K: extreme point in direction v
- In K^*: ray shooting in direction v
  from origin

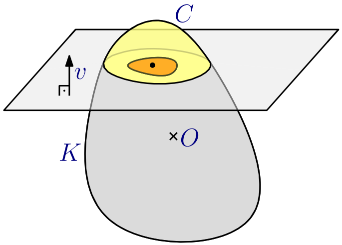
Caps in the Polar

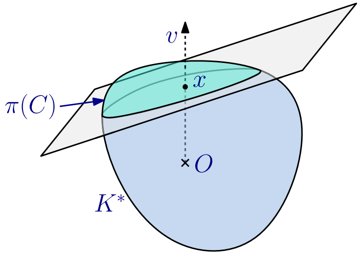
Correspondence between caps in K and K^*:

- Let C be an ε-width cap of K with base orthogonal to v
  (extremal query in direction v with error ε)
- In K^*, shoot a ray in direction v from the origin and let x be a point distance ε before the boundary
  (ray shooting query in direction v with error ε)
- Let $\pi(C)$ be the minimum volume cap containing point x

Polar Relationship for Caps

The polar of the base of C scaled by ε is a constant approximation of the base of $\pi(C)$.



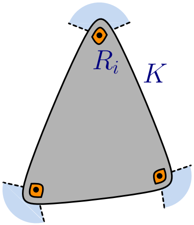


Volume-Sensitive Bound

- Mahler volume inequalities imply that the product of the base areas is $\Theta(\varepsilon^{d-1})$
- Hence, for each ε-width Macbeath region R of K, there exists a Macbeath region R^* in K^*, such that

$$\text{vol}(R) \cdot \text{vol}(R^*) = \Theta(\varepsilon^{d+1})$$

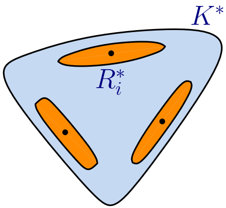
- Transforms small-volume Macbeath regions in K to large-volume Macbeath regions in K^*.
- Packing argument in K^* bounds their number



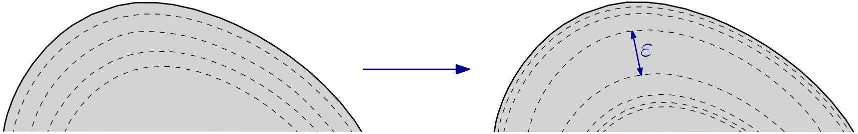
Volume-Sensitive Bound

Let $\mathcal{C}$ be a set of caps of K of width $\Theta(\varepsilon)$ and volume $\Theta(v)$, such that the Macbeath regions $M'(x)$ centered at the centroids x of the bases of these caps are disjoint. Then

$$|\mathcal{C}| = O\left(\min\left(\frac{\varepsilon}{v}, \frac{v}{\varepsilon^d}\right)\right).$$



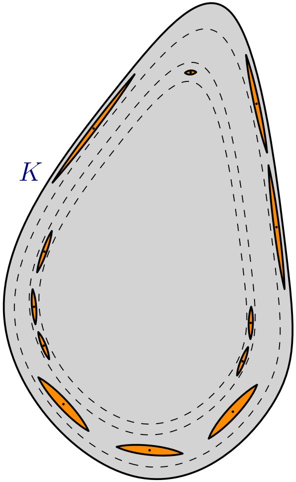
Varying Layer Thickness



We make layers have different thicknesses:

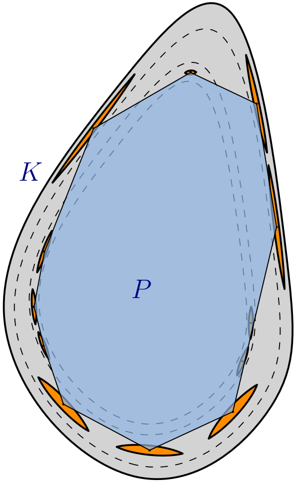
- Intermediate layer with thickness ε:
  - Regions of volume: $\varepsilon^{(d+1)/2}$
  - Number of regions: $1/\varepsilon^{(d-1)/2}$
- A layer k layers away from the intermediate will have thickness ε/k^2:
  - Regions of original volume: $2^k\varepsilon^{(d+1)/2}$ or $\varepsilon^{(d+1)/2}/2^k$
  - Number of regions if thickness ε: $1/(2^k\varepsilon^{(d-1)/2})$
  - Number of regions becomes: $1/(2^k(\varepsilon/k^2)^{(d-1)/2}) = (k^{d-1}/2^k)/\varepsilon^{(d-1)/2}$
- Totals:
  - Number of regions: $\sum_k (k^{d-1}/2^k)/\varepsilon^{(d-1)/2} = 1/\varepsilon^{(d-1)/2}$
  - Thickness: $\sum_k \varepsilon/k^2 = \varepsilon$

Construction



- Build Macbeath regions in layers:
  - Outer layers responsible for larger Macbeath regions
  - Inner layers responsible for smaller Macbeath regions (volume doubles for each layer out)
  - Intermediate layer has thickness ε (responsible for regions of volume $\Theta(\varepsilon^{(d+1)/2})$)
  - k layers away, thickness decreases to ε/k^2
- Place a point inside each Macbeath region (anywhere)
- Take the convex hull

Construction



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Conclusion and Open Problems

There are many details omitted...

Main Result:

- An optimal $O(1/\varepsilon^{(d-1)/2})$ bound on the combinatorial complexity of an ε-approximating polytope in $\mathbb{R}^d$.
- Key contributions:
  - A volume-sensitive bound on the number of disjoint Macbeath regions
  - A volume-inverting correspondence between Macbeath regions in a convex body and its polar body (a local version of the Mahler volume)

Open problems:

- Efficient construction?
- Can simple constructions (e.g., Dudley or Bronshteyn & Ivanov) provide any guarantees on combinatorial complexity?
- Instance optimality? (As opposed to worst-case optimality)

Bibliography

[AFM17] S. Arya and G. D. da Fonseca and D. M. Mount, On the Combinatorial Complexity of Approximating Polytopes, *Discr. and Comp. Geom*, 58:849–870, 2017.

[Bar07] I. Bárány. Random polytopes, convex bodies, and approximation. In W. Weil, editor, *Stochastic Geometry*, volume 1892 of *Lect. Notes in Math.*, pages 77–118, 2007.

[BI74] E. M. Bronshteyn and L. D. Ivanov. The approximation of convex sets by polyhedra. *Siberian Math. J.*, 16:852–853, 1976. (Original in Russian: 1974.)

[Cla06] K. L. Clarkson, Building triangulations using ε-nets, In *Proc. 38th Annu. ACM STOC*, 2006, 326–335.

[DGG13] O. Devillers, M. Glisse, and X. Goaoc. Complexity analysis of random geometric structures made simpler. In *Proc. 29th Annu. Sympos. Comput. Geom.*, pages 167–176, 2013.

[Dud74] R. M. Dudley. Metric entropy of some classes of sets with differentiable boundaries. *Approx. Theory*, 10(3):227–236, 1974.

[ELR70] G. Ewald, D. G. Larman, and C. A. Rogers. The directions of the line segments and of the r-dimensional balls on the boundary of a convex body in Euclidean space. *Mathematika*, 17:1–20, 1970.

[Gru93] P. M. Gruber. Asymptotic estimates for best and stepwise approximation of convex bodies. *I. Forum Math.*, 5:521–537, 1993.

[Mac52] A. M. Macbeath. A theorem on non-homogeneous lattices. *Annals of Mathematics*, 54:431–438, 1952.

Sculpture by Yoann Crépin

Thank you!