Approximate Convex Intersection Detection with Applications to Width and Minkowski Sums

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Results

1. Approximate polytope intersection in $O(\mathrm{polylog}\frac{1}{\varepsilon})$ time
   - Given two preprocessed polytopes
   - Storage: $O(1/\varepsilon^{(d-1)/2})$
2. Approximation to Minkowski sum in $O(n \log \frac{1}{\varepsilon} + 1/\varepsilon^{(d-1)/2+\alpha})$ time
   - Any $\alpha > 0$
   - Previously $O(n + 1/\varepsilon^{d-1})$
3. Width approximation in $O(n \log \frac{1}{\varepsilon} + 1/\varepsilon^{(d-1)/2+\alpha})$ time
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Directional Width

Exact directional width

Given:

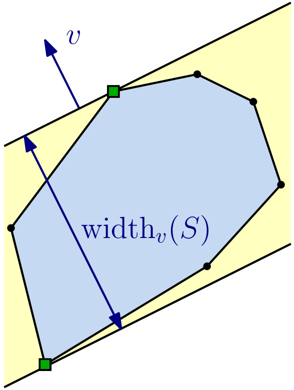
- S: set of n points in $\mathbb{R}^d$
- v: unit vector

Define $\mathrm{width}_v(S)$:

- Smallest distance between two hypeplanes orthogonal to v enclosing S

Approximate directional width:

- Given $\varepsilon > 0$
- Find points $p, q \in S$ with $\mathrm{width}_v(\{p, q\}) \geq (1 - \varepsilon)\, \mathrm{width}_v(S)$



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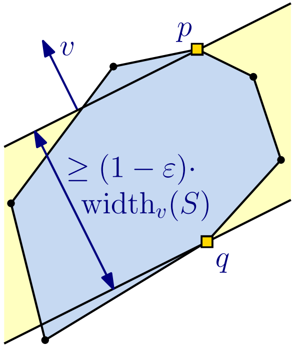

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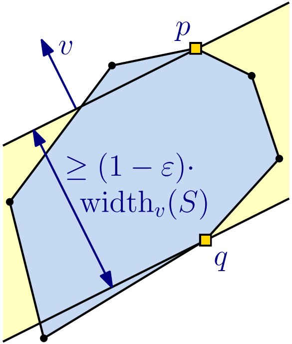
Data Structure used as a Black Box



Preprocess into a data structure: [AFM17a,AFM17b]

- S: set of n points in $\mathbb{R}^d$
- ε: small positive parameter

Given query vector v:

- Answer approximate directional width

Complexity of directional width

- Query time: $O(\log^2(1/\varepsilon))$
- Storage: $O\left(1/\varepsilon^{\frac{d-1}{2}}\right)$
- Preprocessing time: $O\left(n \log \frac{1}{\varepsilon} + 1/\varepsilon^{\frac{d-1}{2}+\alpha}\right)$ for any $\alpha > 0$

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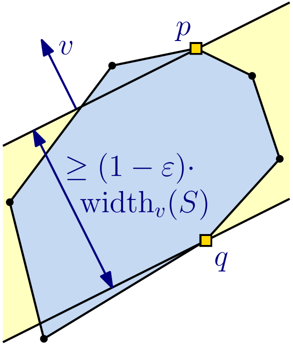

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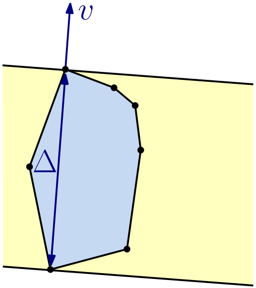
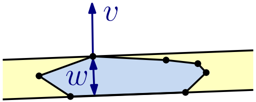
Diameter vs Width

- Diameter: $\max_v \mathrm{width}_v(S)$
- Width: $\min_v \mathrm{width}_v(S)$
- Diameter: Approximated using $O(1/\varepsilon^{\frac{d-1}{2}})$ directional width queries [Cha02]
  Time: $O\left(n \log \frac{1}{\varepsilon} + 1/\varepsilon^{\frac{d-1}{2}+\alpha}\right)$ [AFM17b,Cha17]
- Width: Known algorithms take $O(n + 1/\varepsilon^{d-1})$ time [Cha02,Cha06]
- Can we approximate the width faster?





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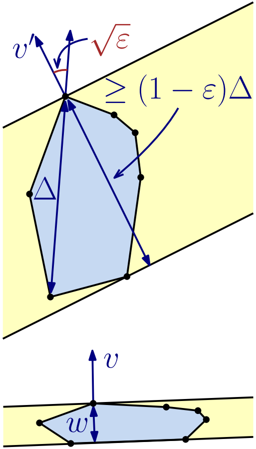

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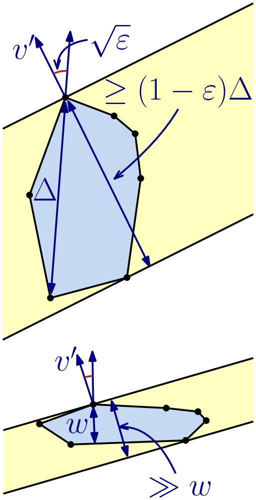
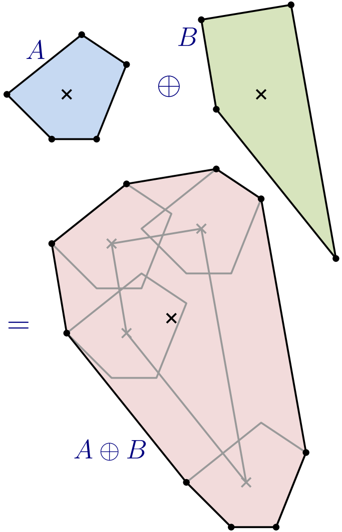


Minkowski Sum



Minkowski sum

- A, B: Sets of points
- $A \oplus B = \{p + q : p \in A,\ q \in B\}$

- Applications: motion planning, CAD, biology, engineering...
- Slow to compute: $O(n^2)$
- What if we approximate?

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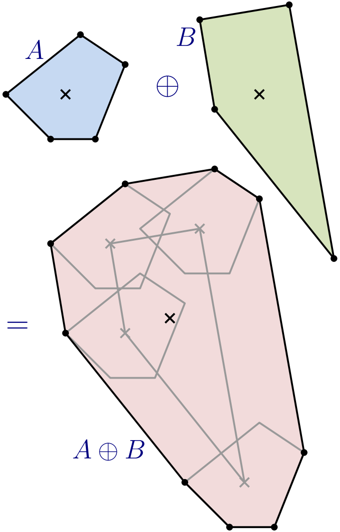

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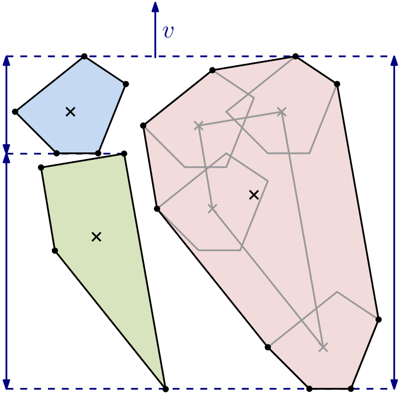
Important Properties

1. $\mathrm{width}_v(A \oplus B) = \mathrm{width}_v(A) + \mathrm{width}_v(B)$
   - We can query $A \oplus B$ using data structures for A and B
2. Width of A: Min $\|v\|$ for $v \in \partial(A \oplus (-A))$
   - Easy if $A \oplus (-A)$ is represented by hyperplanes
3. $A \cap B \neq \emptyset \iff O \in A \oplus -B$
   - We'll use in the next slide



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Build hyperplane representation of $A \oplus -A$ using only directional width queries

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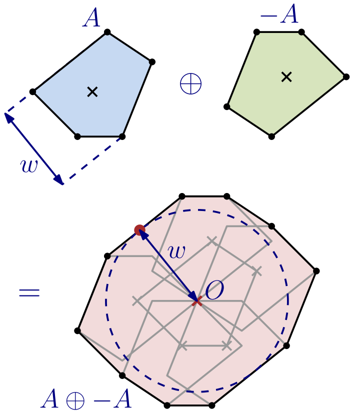

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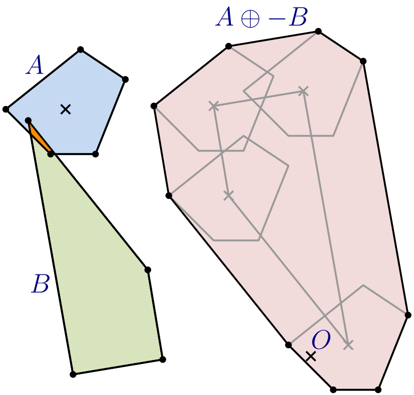

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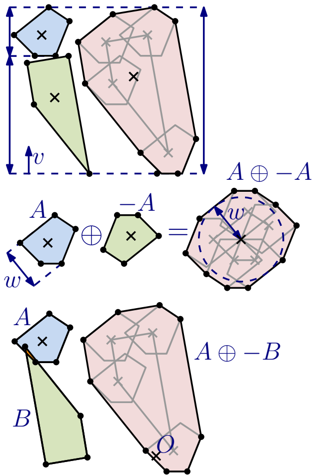

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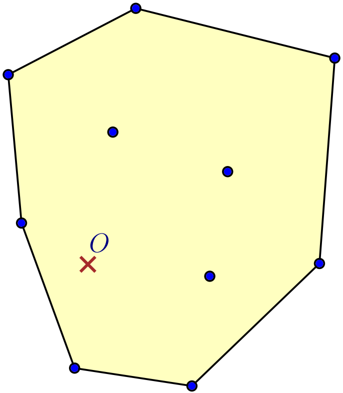
Polytope Intersection



Property 3

$A \cap B \neq \emptyset \iff O \in A \oplus -B$

- S: set of points
- Question: Is $O \in \mathrm{conv}(S)$?
- Classic linear programming problem
- Faster approximate solution after preprocessing?
- Look at the dual

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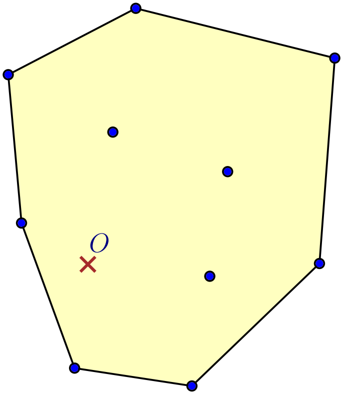

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Point-Hyperplane Duality

Duality

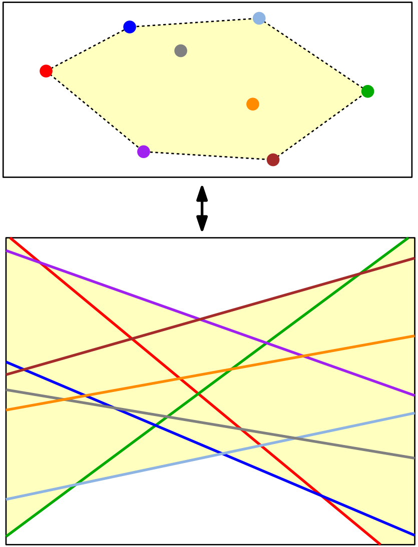
Point $(p_1, \ldots, p_d)$ maps to hyperplane $x_d = p_1 x_1 + \cdots + p_{d-1} x_{d-1} - p_d$

We want to solve:

- Primal: $O \in \text{conv}(S)$
- Dual: hyperplane $O^* : x_d = 0$ between upper and lower envelopes

We have access to:

- Primal: directional width
- Dual: vertical ray shooting



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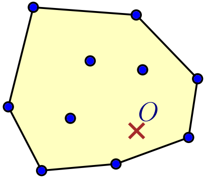

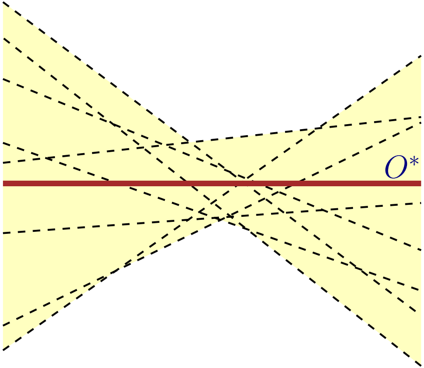

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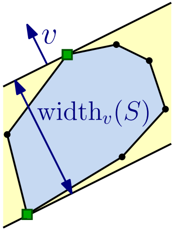
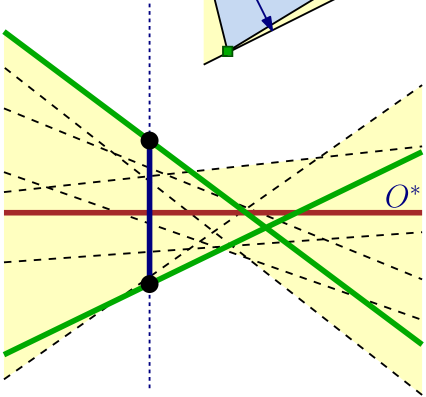
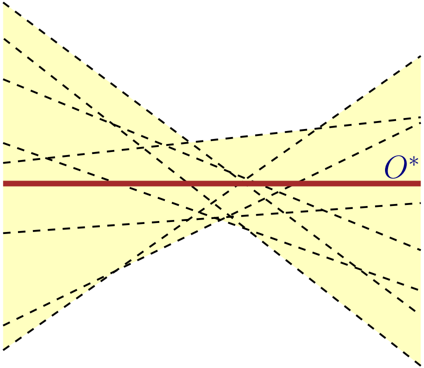




One-Dimensional Convex Minimization



- Upper envelope is convex
- Minimize convex function using evaluations
- Slope at most c
- Binary search:
  - Sample 4 points
  - Recurse $2/3$ (or $1/3$) interval containing smallest sample
  - Stop with interval size ε/c
- $O(\log \frac{1}{\varepsilon})$ time for $f : [0,1] \to \mathbb{R}$

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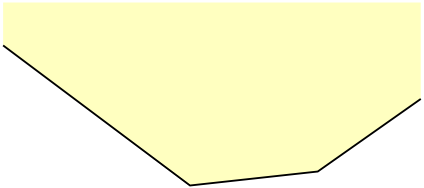

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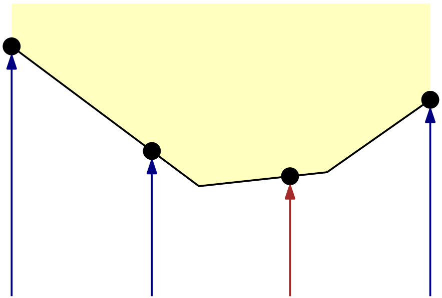

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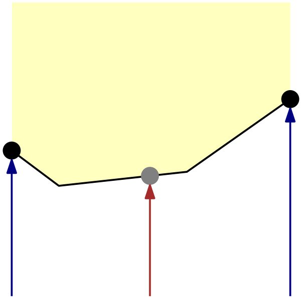

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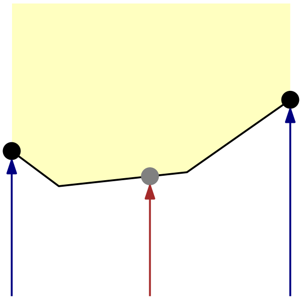

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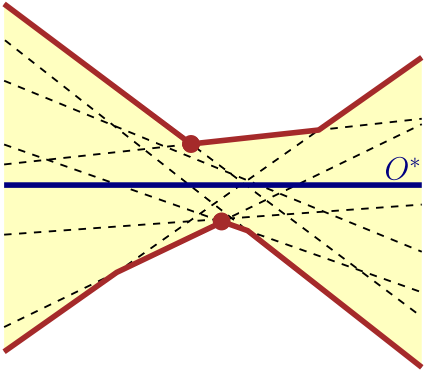

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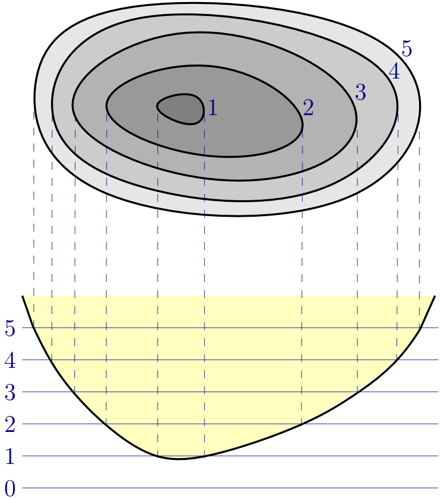
d-Dimensional Convex Minimization



$$g(x_1) = \min_{x_2,\ldots,x_d \in [0,1]^{d-1}} f(x_1, \ldots, x_d)$$

- $g : [0,1] \to \mathbb{R}$ is convex
- Minimize $g(\cdot)$
- Solve $(d-1)$-dimensional minimization to evaluate $g(\cdot)$
  - $t(1) = O(\log \frac{1}{\varepsilon})$
  - $t(d) = t(d-1) \cdot t(1)$
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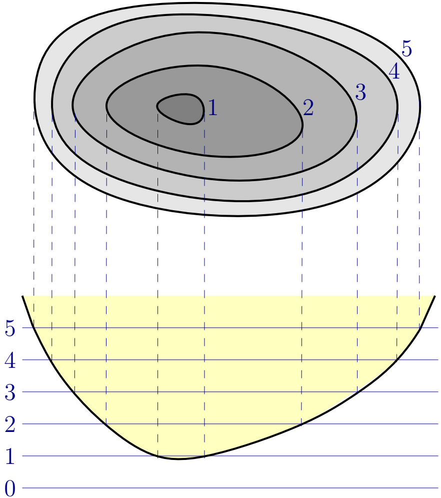

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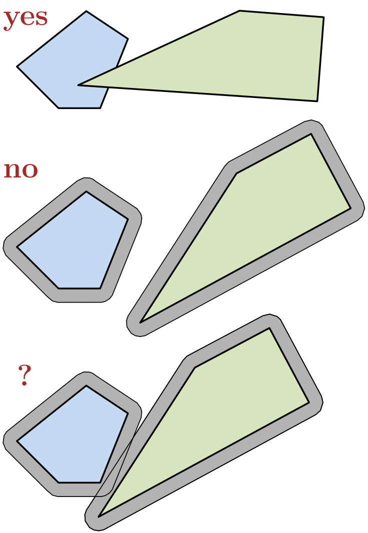
Approximate Polytope Intersection

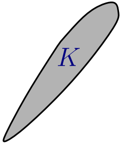
- If A intersects B: answer **yes**
- If the distance between A and B is more than $\varepsilon \cdot (\mathrm{diam}(A) + \mathrm{diam}(B))$: answer **no**
- Otherwise either answer is ok

(1) Approximate polytope intersection

- Query time: $O(\mathrm{polylog}\frac{1}{\varepsilon})$
- Storage: $O(1/\varepsilon^{(d-1)/2})$
- Preprocessing time: $O(n \log \frac{1}{\varepsilon} + 1/\varepsilon^{(d-1)/2+\alpha})$, for any $\alpha > 0$



Dudley Approximation



Dudley's result: [Dud74]

A convex body K of diameter 1 can be ε-approximated by a polytope P with $O(1/\varepsilon^{\frac{d-1}{2}})$ facets.

- Fatten K into K'
- Ball B of radius $2 \cdot \mathrm{diam}(K')$
- $\sqrt{\varepsilon}$-net N on B
- Closest point on K' for each point in N
- P' bounded by tangent hyperplanes
- Unfatten P' into P

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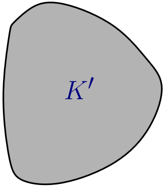

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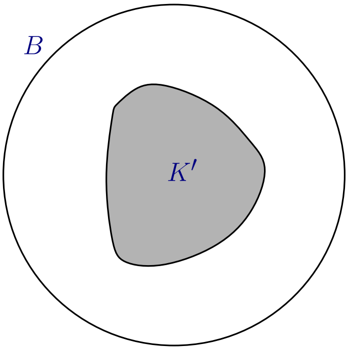

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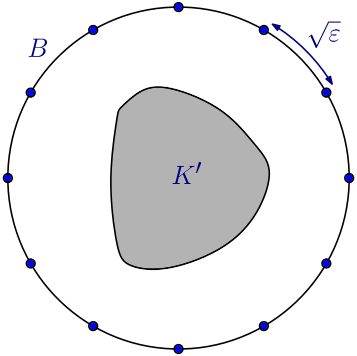

Dudley's result: [Dud74]

A convex body K of diameter 1 can be ε-approximated by a polytope P with $O(1/\varepsilon^{\frac{d-1}{2}})$ facets.

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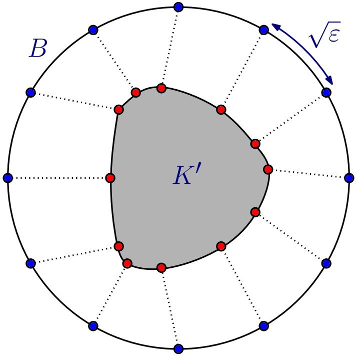

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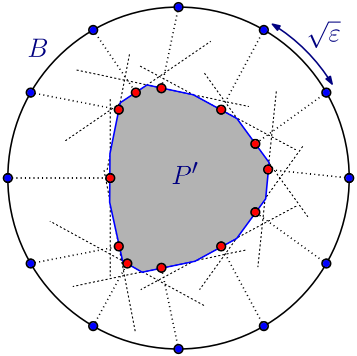

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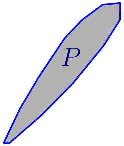

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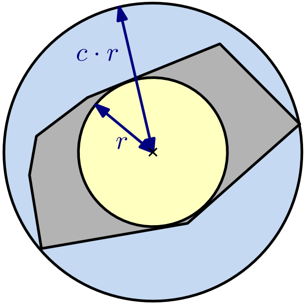
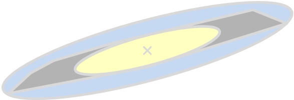
Fatness and John Ellipsoid



Fatness

A convex body K is fat if it is sandwiched between balls of radii r and $c \cdot r$ for some constant c that does not depend on K



Fatten by scaling John Ellipsoid to a ball:

John Ellipsoid [Joh48]

For every convex body K in $\mathbb{R}^d$, there exist ellipsoids E_1, E_2 such that $E_1 \subseteq K \subseteq E_2$ and E_2 is a d-scaling of E_1

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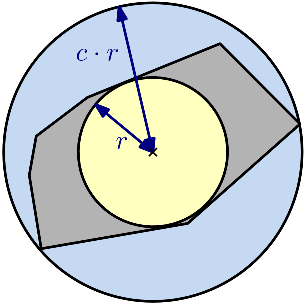

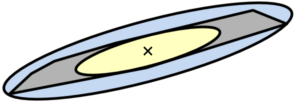

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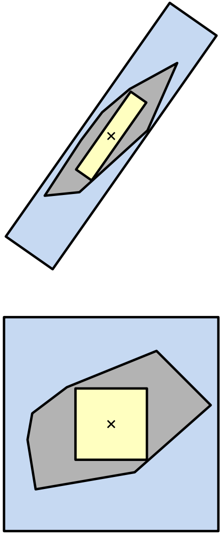
Fattening Minkowski Sums



- Minkowski sum of ellipsoids is not an ellipsoid
- It follows from John that:

For every convex body K in $\mathbb{R}^d$, there exist rectangles R_1, R_2 such that $R_1 \subseteq K \subseteq R_2$ and R_2 is a $(3d/2)$-scaling of R_1

- Store $R_1(A)$ with A
- For $A \oplus B$ use $R_1(A) \oplus R_1(B)$
- $R_1(A) \oplus R_1(B)$ has $O(1)$ vertices
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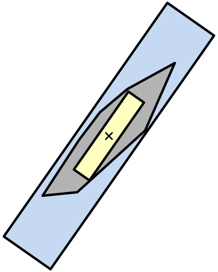

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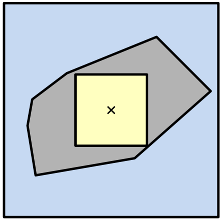

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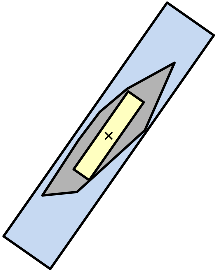

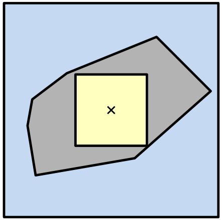

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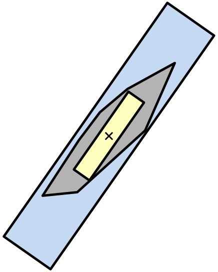

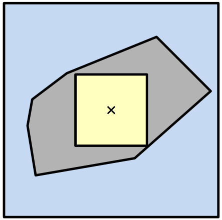

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Closest Point

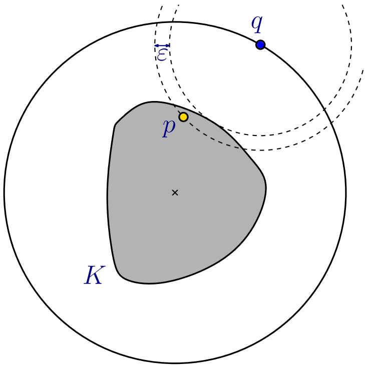
Approximate closest point

Given:

- K: preprocessed polytope
- q: query point with $\mathrm{dist}(q, K) = \Theta(1)$

Find:

- $p \in K$ with $\|pq\| \leq \mathrm{dist}(q, K) + \varepsilon$



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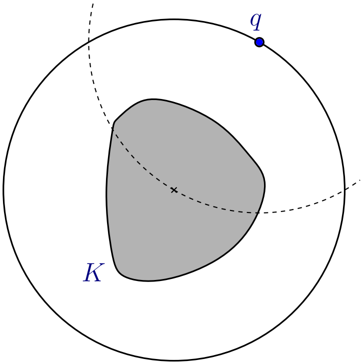

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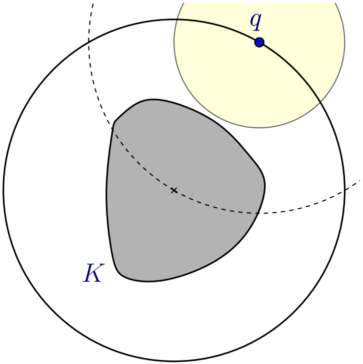

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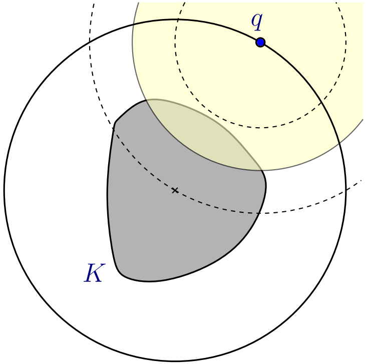

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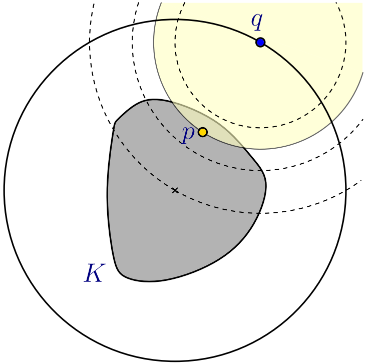

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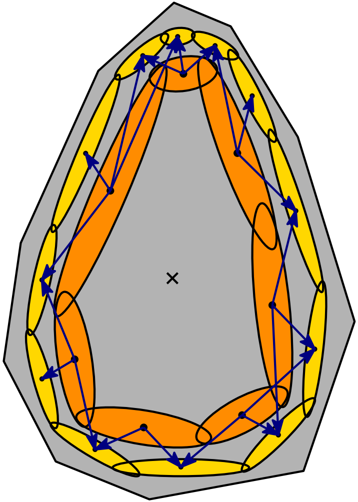
Minkowski Sum Approximation



- Build directional width data structures for A and B
- Let $K = A \oplus B$
- Run Dudley's algorithm
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(2) Minkowski sum approximation

Time: $O(n \log \frac{1}{\varepsilon} + 1/\varepsilon^{(d-1)/2+\alpha})$, for any $\alpha > 0$

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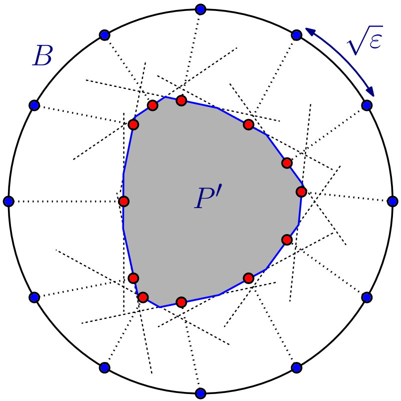

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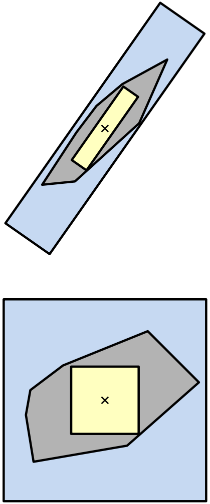

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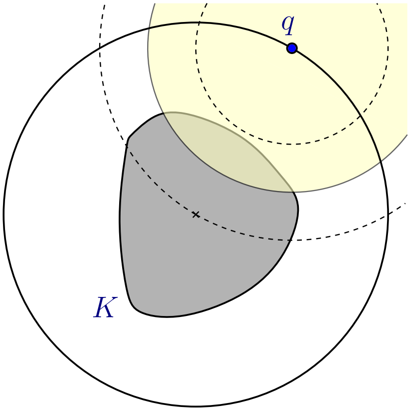

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Width Approximation

- Compute Dudley of $A \oplus -A$
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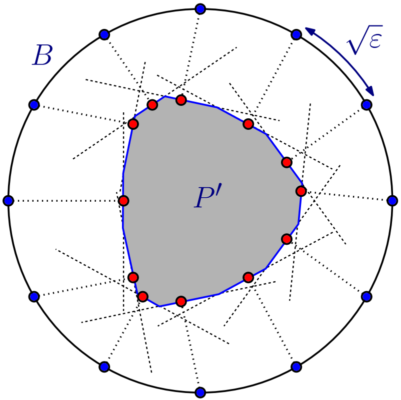

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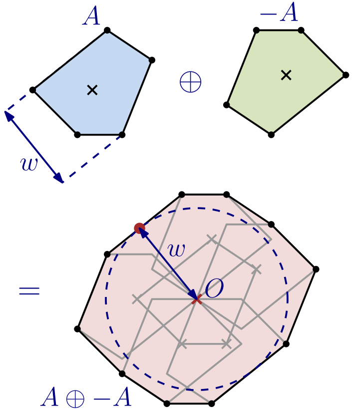

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Conclusion

Using appproximate directional width we solved:

1. Approximate polytope intersection queries in $O(\mathrm{polylog}\frac{1}{\varepsilon})$ time with $O(1/\varepsilon^{(d-1)/2})$ storage
2. Approximation to Minkowski sum in $O(n \log \frac{1}{\varepsilon} + 1/\varepsilon^{(d-1)/2+\alpha})$ time
3. Width approximation in $O(n \log \frac{1}{\varepsilon} + 1/\varepsilon^{(d-1)/2+\alpha})$ time

Open problems:

- Remove the $1/\varepsilon^{\alpha}$ factor
- Lower bounds (or improved upper bounds): Is $1/\varepsilon^{(d-1)/2}$ necessary?
- Diameter for non-Euclidean metrics
- Approximate the separation depth

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Painting by Tomma Abts

Thank you!