Optimal Approximate Polytope Membership

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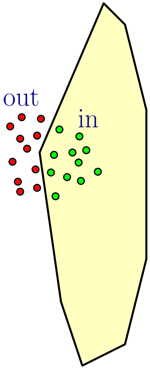
Polytope Membership Queries

Polytope Membership Queries

Given a polytope P in d-dimensional space, preprocess P to answer membership queries:

Given a point q, is $q \in P$?

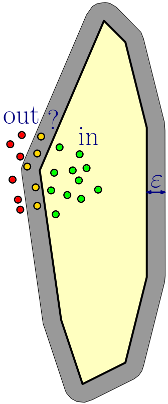
- Assume that dimension d is a constant and P is given as intersection of n halfspaces
- Dual of halfspace emptiness searching
- For $d \leq 3$
  Query time: $O(\log n)$ Storage: $O(n)$
- For $d \geq 4$
  Query time: $O(\log n)$ Storage: $O(n^{\lfloor d/2 \rfloor})$



Approximate Polytope Membership Queries

Approximate Version

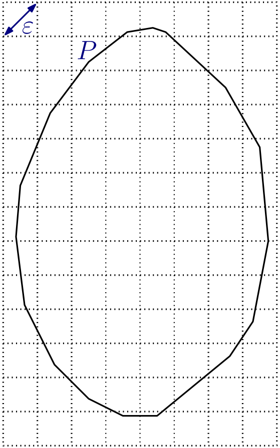
- An approximation parameter $\varepsilon > 0$ is given (at preprocessing time)
- Assume the polytope has diameter 1
- If the query point's distance from P:
  - 0: answer must be inside
  - $\geq \varepsilon$: answer must be outside
  - > 0 and $< \varepsilon$: either answer is acceptable



Previous solutions were either:

- Time-efficient
  Query time: $O(\log \frac{1}{\varepsilon})$ Storage: $O(1/\varepsilon^{d-1})$
- Space-efficient
  Query time: $\widetilde{O}(1/\varepsilon^{(d-1)/8})$ Storage: $O(1/\varepsilon^{(d-1)/2})$

Time Efficient Solution [BFP82]



- Create a grid with cells of diameter ε
- For each column, store the topmost and bottommost cells intersecting P
- Query processing:
  - Locate the column that contains q
  - Compare q with the two extreme values

Time Efficient Solution [BFP82]

- $O(1/\varepsilon^{d-1})$ columns
- Query time: $O(\log \frac{1}{\varepsilon})$ ← optimal
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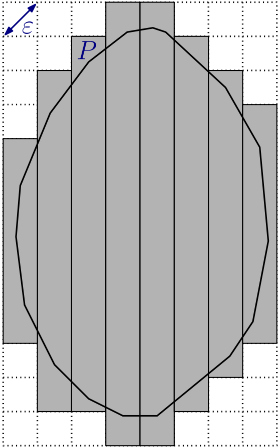

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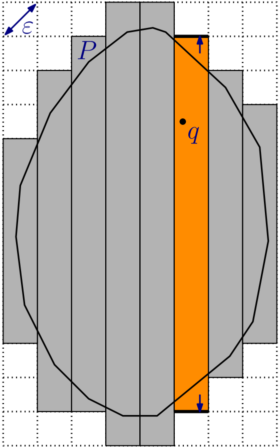

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Space Efficient Solution [AFM11, AFM12]

Preprocess:

- Input P, ε
- $t = \widetilde{O}(1/\varepsilon^{(d-1)/8})$
- $Q \leftarrow$ unit hypercube
- Split-Reduce(Q)

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- Find an ε-approximation of $Q \cap P$
- If at most t facets, then Q stores them
- Otherwise, subdivide Q and recurse

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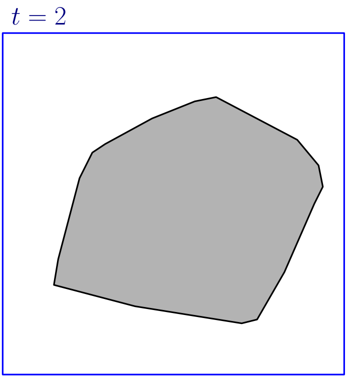

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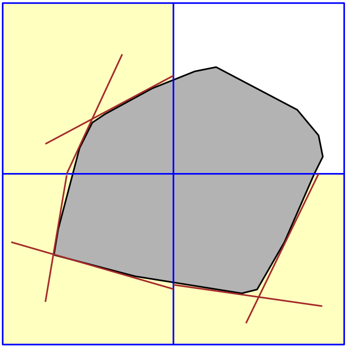

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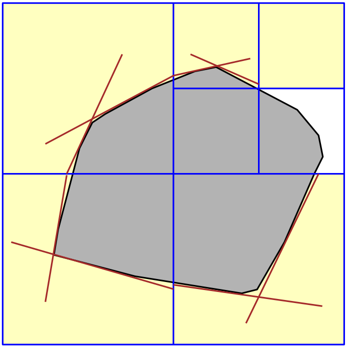

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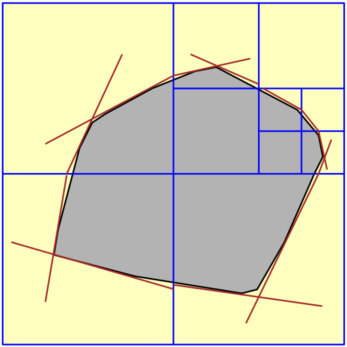

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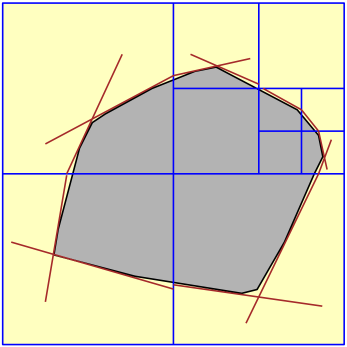

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New Results

New solution is space-efficient and time-efficient:

Approximate Polytope Membership:

Query time: $O(\log \frac{1}{\varepsilon})$ ← optimal
Storage: $O(1/\varepsilon^{(d-1)/2})$ ← optimal
(Previous storage: $O(1/\varepsilon^{d-1})$ [BFP82])

Consequence:

Approximate Nearest Neighbor Searching:

Query time: $O(\log n)$
Storage: $O(n/\varepsilon^{d/2})$
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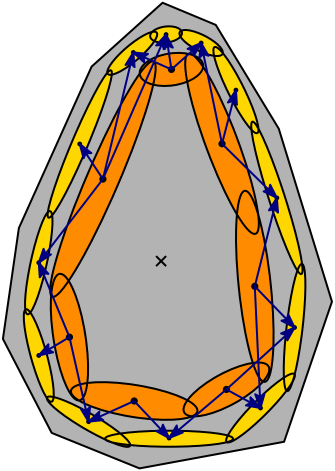
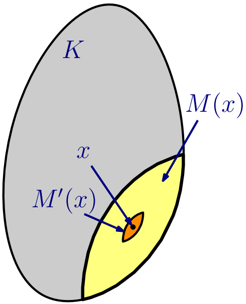
Techniques



- Previous solutions use grids and quadtrees
  - Similar width in all directions
- Our solution uses a hierarchy of Macbeath regions:
  - Adapt to the curvature of the body
  - Narrow in directions of high curvature
  - Wide in directions of low curvature

Macbeath Regions [Mac52]



Given a convex body K, $x \in K$, and $\lambda > 0$:

- $M^\lambda(x) = x + \lambda((K - x) \cap (x - K))$
- $M(x) = M^1(x)$: intersection of K and K reflected around x
- $M'(x) = M^{1/5}(x)$

Properties

- $M'(x) \cap M'(y) \neq \emptyset \Rightarrow M'(x) \subseteq M(y)$
- $y \in M'(x) \Rightarrow \delta(y) = \Theta(\delta(x))$, where $\delta(x)$: distance from x to ∂K

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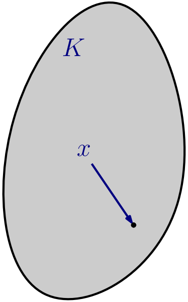

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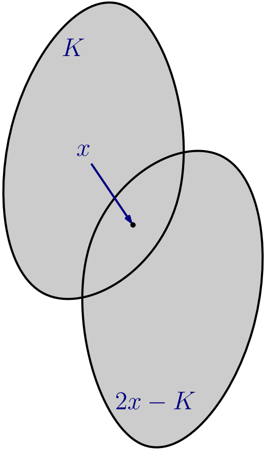

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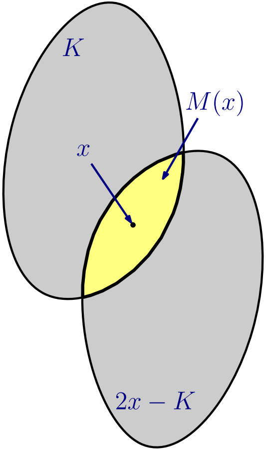

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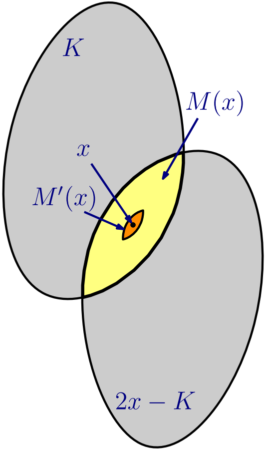

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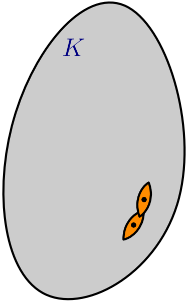

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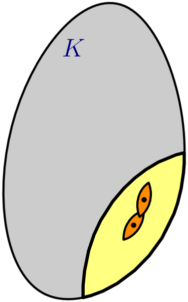

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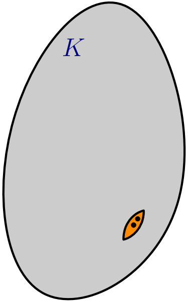

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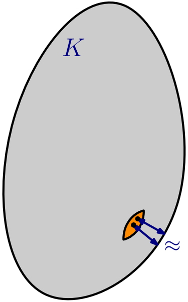

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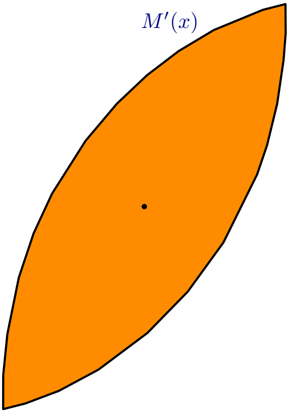

John Ellipsoid [Joh48]

For every centrally symmetric convex body K in $\mathbb{R}^d$, there exist ellipsoids E_1, E_2 such that $E_1 \subseteq K \subseteq E_2$ and E_2 is a $\sqrt{d}$-scaling of E_1

Macbeath Ellipsoid

- $E(x)$: enclosed John ellipsoid of $M'(x)$
- $M^\lambda(x) \subseteq E(x) \subseteq M'(x)$ for $\lambda = 1/(5\sqrt{d})$

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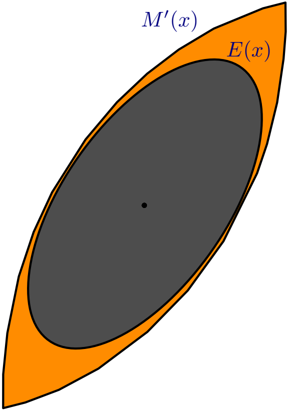

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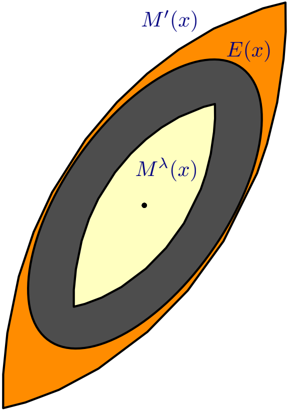

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Covering with Macbeath Ellipsoids

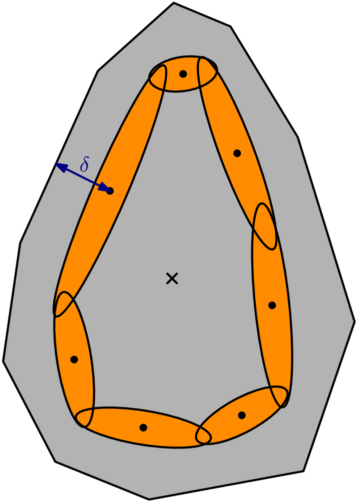
Covering (see [Bar07])

Given:

- K: convex body
- δ: small positive parameter

There exist ellipsoids $E(x_1), \ldots, E(x_k)$

- $\delta(x_1) = \cdots = \delta(x_k) = \delta$
- Cover: Every ray from the origin intersects some ellipsoid
- $k = O(1/\delta^{(d-1)/2})$ [AFM16]



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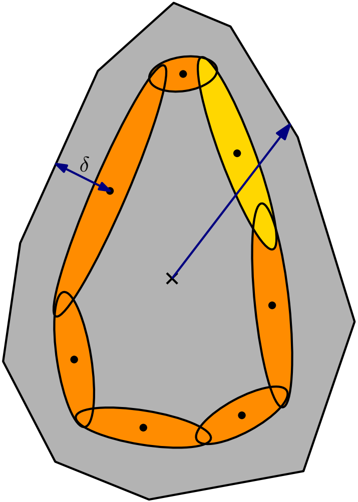

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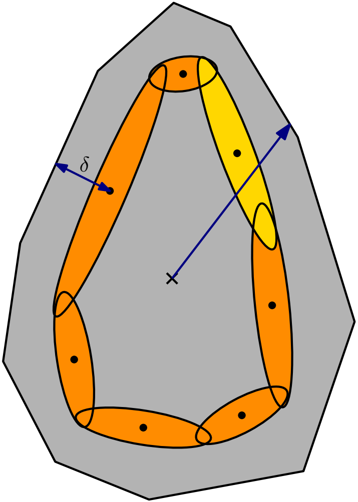
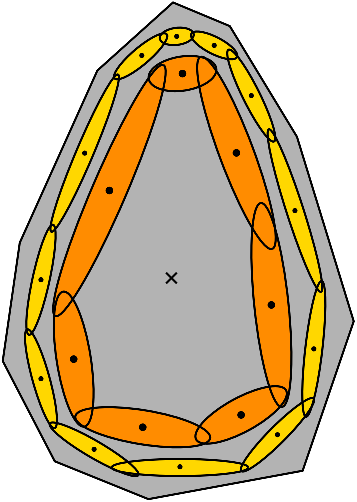


Hierarchy of Macbeath Ellipsoids



Hierarchy

Given:

- K: convex body
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Hierarchy:

- Each level i a δ_i-covering
- $\ell = \Theta(\log \frac{1}{\varepsilon})$ levels
- $\delta_0 = \Theta(1)$, $\delta_\ell = \Theta(\varepsilon)$
- $\delta_{i+1} = \delta_i / 2$
- E, E' are parent/child if
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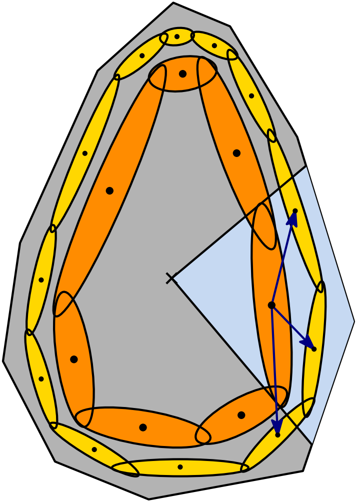

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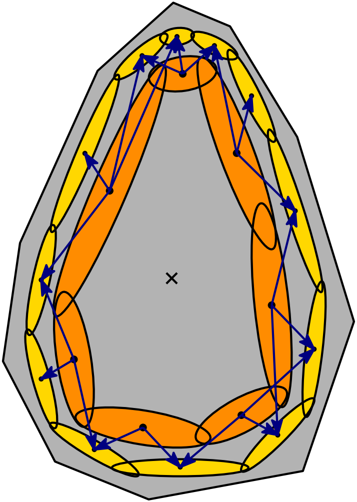

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Ray Shooting from the Origin

Ray Shooting from the Origin (generalizes polytope membership)

Preprocess:

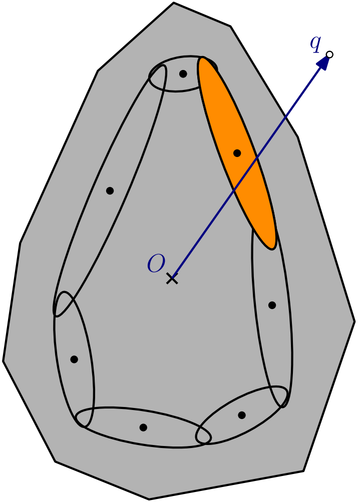
- K: convex body
- ε: small positive parameter

Query:

- Oq: ray from the origin towards q

Query algorithm:

- Find an ellipsoid intersecting Oq at level 0
- Repeat among children at next level
- Stop at leaf node
- Leaf ellipsoid ε-approximates boundary



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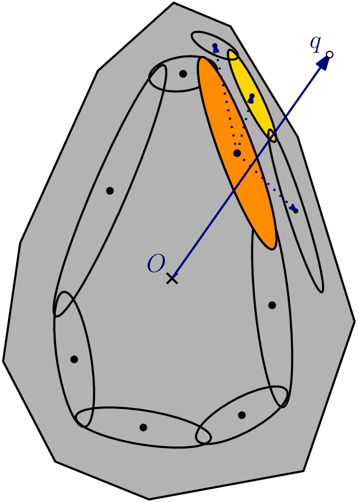


Ray Shooting from the Origin

Ray Shooting from the Origin (generalizes polytope membership)

Preprocess:

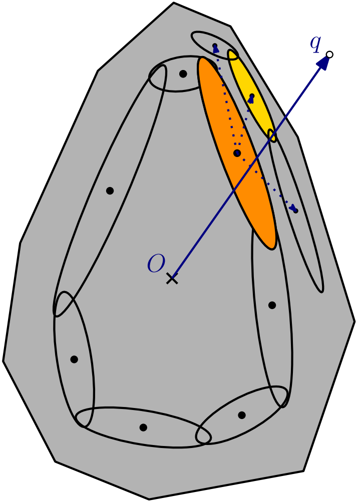
- K: convex body
- ε: small positive parameter

Query:

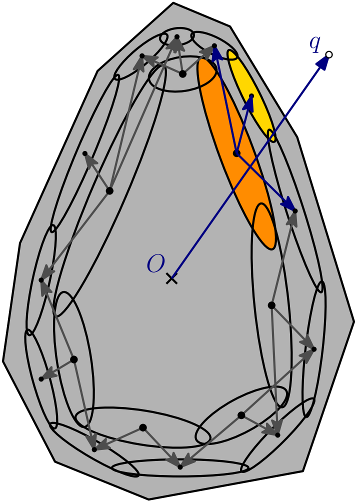
- Oq: ray from the origin towards q

Query algorithm:

- Find an ellipsoid intersecting Oq at level 0
- Repeat among children at next level
- Stop at leaf node
- Leaf ellipsoid ε-approximates boundary



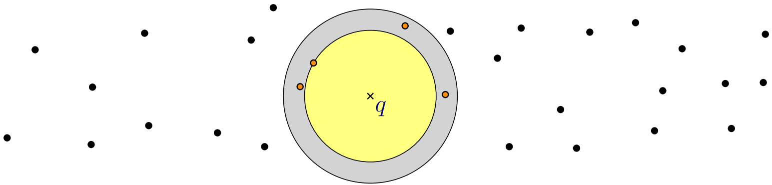
Analysis



- Out-degree: $O(1)$
- Query time per level: $O(1)$
- Number of levels: $O(\log \frac{1}{\varepsilon})$
- Query time: $O(\log \frac{1}{\varepsilon})$

- Storage for bottom level: $O(1/\varepsilon^{(d-1)/2})$
- Geometric progression of storage per level
- Total storage: $O(1/\varepsilon^{(d-1)/2})$

Impact



Approximate Nearest Neighbor

Preprocess n points such that, given a query point q, we can find a point within at most $1 + \varepsilon$ times the distance to q's nearest neighbor

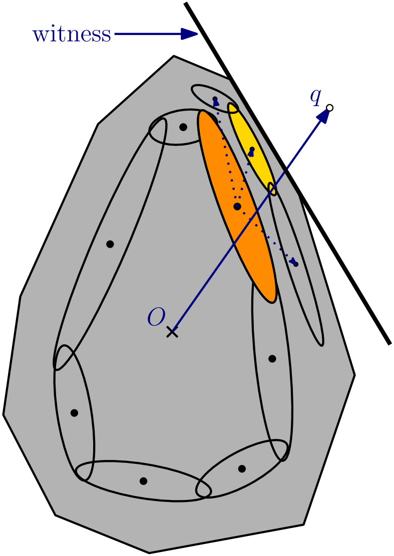
- For $\log \frac{1}{\varepsilon} \leq m \leq 1/\varepsilon^{d/2}$
  Query time: $O(\log n + 1/(m\, \varepsilon^{d/2}))$ Storage: $O(n\, m)$
- If $m = 1/\varepsilon^{d/2}$
  Query time: $O(\log n)$ Storage: $O(n/\varepsilon^{d/2})$

What else is in the paper?

- Proofs
- Witness (important to find the approximate nearest neighbor)
- Reduction from ANN to approximate ray shooting

Full Paper

arxiv.org/abs/1612.01696



Conclusions and Open Problems

Our approximate polytope membership data structure is optimal

- Query time: $O(\log \frac{1}{\varepsilon})$
- Storage: $O(1/\varepsilon^{(d-1)/2})$

Still, several open problems remain

- Further improvements to approximate nearest neighbor searching?
- Generalization to k-nearest neighbors?
- Other applications of the hierarchy?

Recent applications of the hierarchy

- Near-optimal ε-kernel computation
- Approximate diameter
- Approximate bichromatic closest pair

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Sculpture by José Mérino

Thank you!