# On the Longest Flip Sequence to Untangle Segments in the Plane

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2023

# Section 1

# Introduction

# Motivation: Untangling TSP Tours

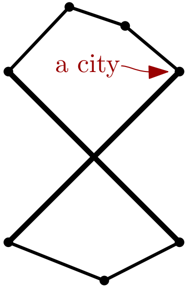


- 2d Euclidean TSP (NP-hard):

**Input:** A set of $n$ points called *cities*.

**Output:** The shortest *tour* (polygon whose vertices are the cities).

- Heuristics output tours with crossings.

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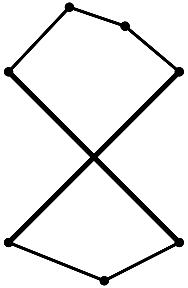

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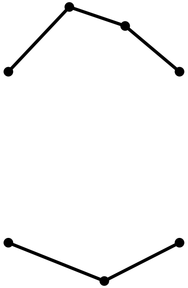

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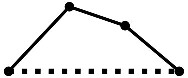
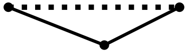


No!



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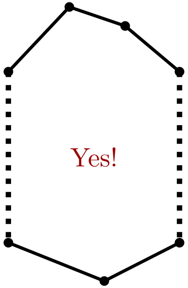

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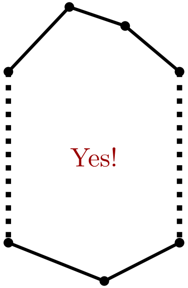

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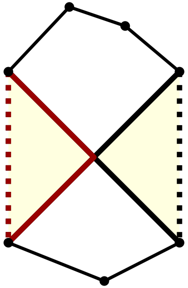

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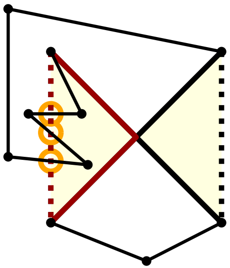

# A Potential Argument

- An infinite flip sequence?
- Measuring progress with a potential,
  i.e., an integer function on tours which is:
  - bounded
  - decreasing at each step.

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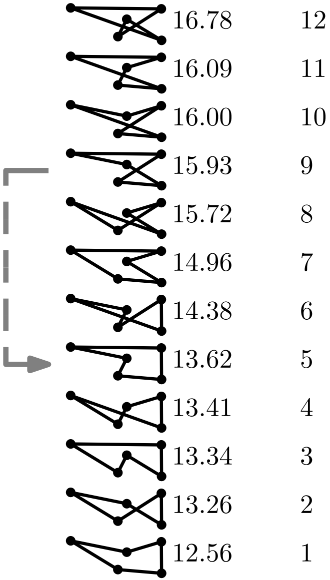
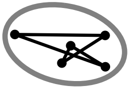
$$1 \quad \underbrace{\text{rank when sorted by length}(T)}_{\text{potential of the tour } T} \quad n!$$



# The Longest Flip Sequences



- The deletion choice may impact the number of flips.



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- $\mathrm{D}(n)$: number of flips in the longest flip sequences.
- We want bounds on $\mathrm{D}(n)$.

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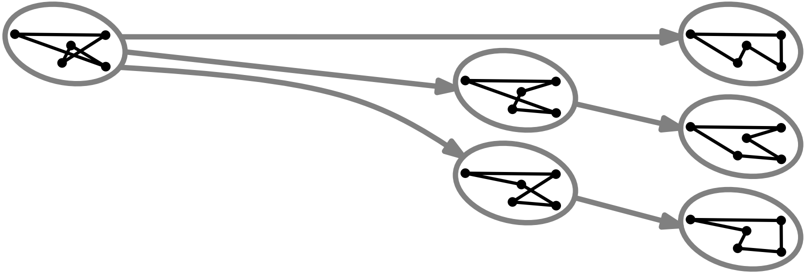

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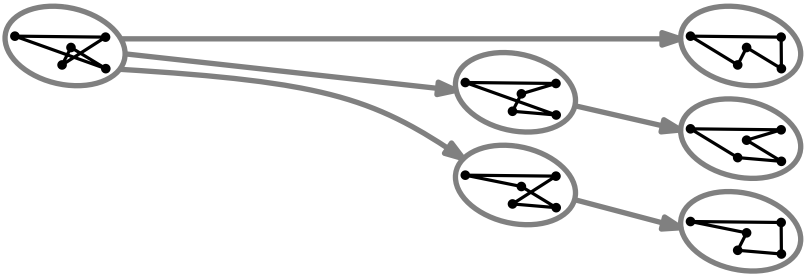

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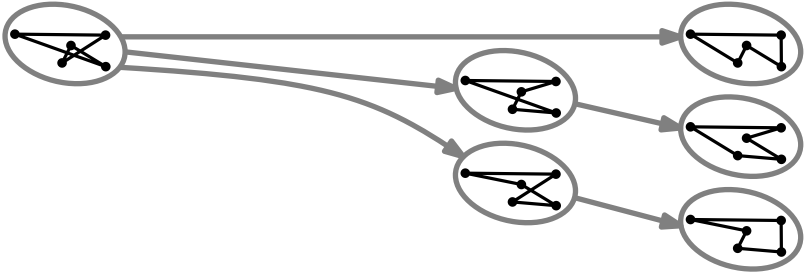

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# The Flip Graph

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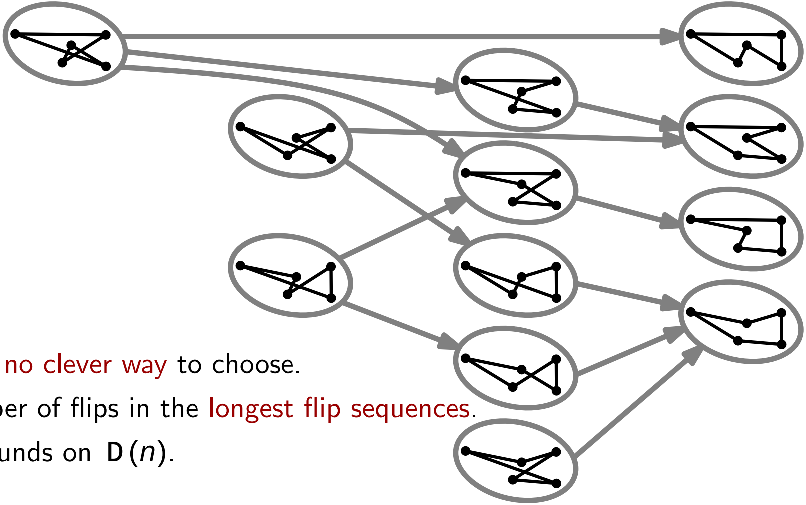

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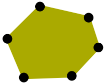
# Previous Bounds on the Longest Flip Sequences



$$n^2 \quad 4 \quad \overbrace{\text{cities in convex position}}^{\mathbf{D}_{\text{convex}}(n)} \quad 4 \quad \underbrace{n^2}_{\text{next slide}}$$

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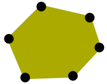
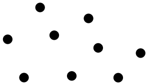


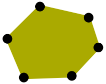
$$n^2 \le \overbrace{\text{cities in convex position}}^{\mathbf{D}_{\text{convex}}(n)} \le \underbrace{n^2}_{\text{next slide}}$$

$$n^2 \le \overbrace{\text{cities in general position}}^{\mathbf{D}(n)} \le \underbrace{n^3}_{1980} \le \underbrace{n!}_{\text{previous slide}}$$



# New Bound





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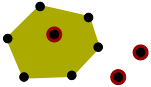

$$n^2 \leq \overbrace{t \text{ non-convex cities}}^{\mathbf{D}(n;t)} \leq \underbrace{n^2 t}_{\text{new}}$$

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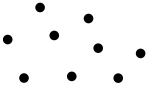
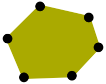


# Section 2

## Convex Proof



$$n^2 \quad 4 \quad \mathtt{D}_{\text{convex}}(n) \quad 4 \quad n^2$$

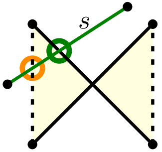
$$n^2 \quad 4 \quad \mathtt{D}(n; t) \quad 4 \quad n^2 t$$

$$n^2 \quad 4 \quad \mathtt{D}(n) \quad 4 \quad n^3$$

# Proving $\mathbf{D}_{\text{convex}}(n) = O(n^2)$



- $\text{crossings}(T)$: number of crossings in the tour $T$.
- $\text{crossings} = O(n^2)$
- crossings decreases at each flip:



# Section 3

## 1980 Proof

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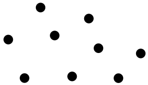

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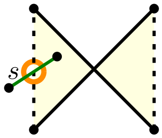
# From Segments to Lines






- crossings may not decrease at each flip:
- Idea: consider line-segment crossings instead.
- $L$: lines through two cities.
- `$(T)$: number of crossings with a line ` in the tour $T$.
- $_L$ = P $_{`2L}$ `
- $O(n^2)$ lines, $O(n)$ crossings per line =) $_L = O(n^3)$.
- ` does not increase at a flip.
- $_L$ decreases at each flip.

# From Segments to Lines



- crossings may not decrease at each ip:
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- $L$: lines through two cities.
- $\ell(T)$: number of crossings with a line $\ell$ in the tour $T$.
- $_L = \sum_{\ell \in L} \ell$
- $O(n^2)$ lines, $O(n)$ crossings per line $\Rightarrow$ $_L = O(n^3)$.
- $\ell$ does not increase at a ip.
- $_L$ decreases at each ip.

# Line Potentials



- crossings may not decrease at each flip:
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- $\ell(T)$: number of crossings with a line $\ell$ in the tour $T$.
- $\Phi_L = \sum_{\ell \in L} \ell$
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- $\ell$ does not increase at a flip.
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# $\Phi_L$ Decreases



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# What Is a Crossing

- A single point intersection between a line and a segment is a crossing if it is not an endpoint of the segment.

# Section 4

## New Proof

$$n^2 \quad 4 \quad D_{convex}(n) \quad 4 \quad n^2$$

$$n^2 \quad 4 \quad D(n;t) \quad 4 \quad n^2 t$$

$$n^2 \quad 4 \quad D(n) \quad 4 \quad n^3$$

# Near Convex Sets



- Near Convex sets: the $n$ points are convex except $t$ of them.

$n = 9$ $t = 3$

# : a Mixed Potential



$$= \underbrace{\overbrace{\qquad}}_{\text{crossings}} + \underbrace{\overbrace{\qquad}}_{L^0}$$

- $L^0$: lines through at least one non-convex point.
- Case 1. If crossings decreases, then so does (because $L^0$ does not increase)
- Case 2. If not,

= z}|{ crossings |{z} + z}|{ $L^0$ |{z}

may not decrease!

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$O(n^2)$

= crossings + $L^0$ = $O(n^2t)$

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- $L^0$: lines through at least one non-convex point. $O(nt)$
- Case 1. If crossings decreases, then so does (because $L^0$ does not increase) ✗
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$= \underbrace{\ldots}_{\text{crossings}}^{O(n^2)} + \underbrace{\ldots}_{L^0}^{O(n^2 t)} = O(n^2 t)$

may not decrease! compensate crossings?

- $L^0$: lines through at least one non-convex point. $O(nt)$
- Case 1. If crossings decreases, then so does (because $L^0$ does not increase) X
- Case 2. If not,

# Does Decreases?

$$= \underbrace{\overbrace{\quad}^{O(n^2)}}_{|\{\quad\}|} \text{ crossings } + \underbrace{\overbrace{\quad}^{O(n^2 t)}}_{|\{L^0\}|} = O(n^2 t)$$

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Introduction
Convex Proof
1980 Proof
New Proof
Near Convex
Mixed Potential
Bounded
Decreasing
Conclusion

# Does    Decreases?



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may not decrease!    compensate crossings?

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# Does Decreases?



$= \overbrace{\qquad}^{O(n^2)}_{\text{crossings}}$ may not decrease! $+ \overbrace{\qquad}^{O(n^2 t)}_{L^0}$ compensate crossings? $= O(n^2 t)$

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# Does Decreases?



= z}|{ O($n^2$) crossings |{z} may not decrease! + z}|{ O($n^2t$) |{z} $L^0$ compensate crossings? = O($n^2t$)

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# Does Decreases?

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- $L^0$: lines through at least one non-convex point: $O(nt)$
  $\cup$ lines through two consecutive convex points: $O(n)$
- Case 1. If $_{\text{crossings}}$ decreases, then so does
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# Does Decreases? Yes!

$$= \underbrace{\text{crossings}}_{\text{may not decrease!}}^{O(n^2)} + \underbrace{L^0}_{\text{compensate crossings?}}^{O(n^2 t)} = O(n^2 t)$$

- $L^0$: lines through at least one non-convex point. $O(nt)$
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# Section 5

## Conclusion

$$n^2 \quad 4 \quad D_{convex}(n) \quad 4 \quad n^2$$

$$n^2 \quad 4 \quad D(n; t) \quad 4 \quad n^2 t$$

$$n^2 \quad 4 \quad D(n) \quad 4 \quad \underbrace{n^2}_{\text{2016 conjecture}} \quad 4 \quad n^3$$

# From Tours to Segments

- Being a tour is not used in the proofs.
- A  ip choice may preserve:
  - nothing special, i.e., being a set of segments ! $D$
  - being a red-blue matching ! $D_{RB}$
  - being a tour ! $D_{TSP}$
  - ...

- $D; D_{RB}; D_{TSP}$ are the same asymptotically.

# Conclusion

$$n^2 \quad 4 \quad D_{convex}(n) \quad 4 \quad n^2$$

$$n^2 \quad 4 \quad D(n; t) \quad 4 \quad n^2 t$$

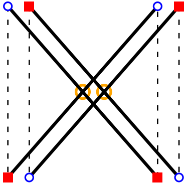
$$n^2 \quad 4 \quad D(n) \quad 4 \quad \underbrace{n^2}_{\text{2016 conjecture}} \quad 4 \quad n^3$$

|{z}
2016 conjecture

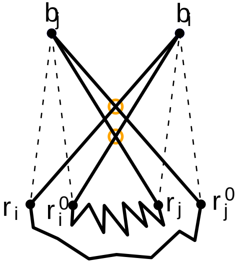
Thank you!

# Reductions

- $2D(n)$ $D_{RB}(2n)$ $D(2n)$



- $2D_{RB}(n)$ $D_{TSP}(3n)$ $D(3n)$



# Distinct Flips

- The same pair of segments can be flipped multiple times in the same sequence.
- Counting *distinct  ips* means that we do not count this multiplicity.
- A balancing argument:
  - There are $O(\frac{n^3}{k})$ flips decreasing $_L$ by at least $k$.
  - There are $O(n^2k^2)$ flips decreasing $_L$ by less than $k$:
    - We enumerate them by sweeping a line.
  - We choose $k = n^{1=3}$.