On the Longest Flip Sequence to Untangle Segments in the Plane

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2023

Section 1

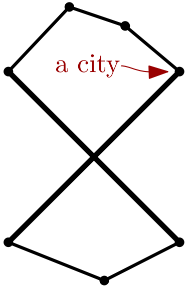
Introduction

Motivation: Untangling TSP Tours

- 2d Euclidean TSP (NP-hard):

Input: A set of n points called *cities*.
Output: The shortest *tour* (polygon whose vertices are the cities).

- Heuristics output tours with crossings.



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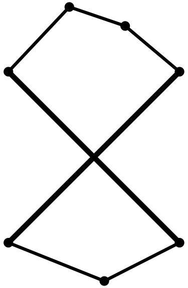

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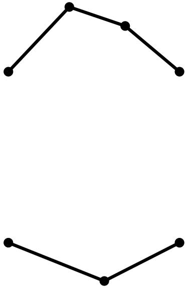

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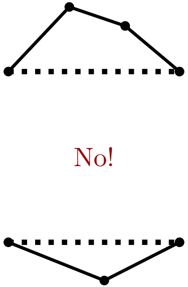

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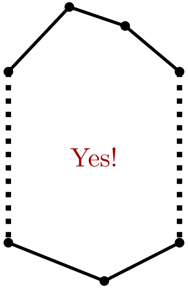

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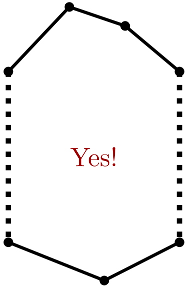

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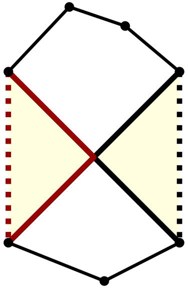

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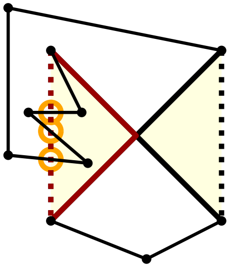

A Potential Argument

- An infinite flip sequence?
- Measuring progress with a potential,
  i.e., an integer function on tours which is:
  - bounded
  - decreasing at each step.

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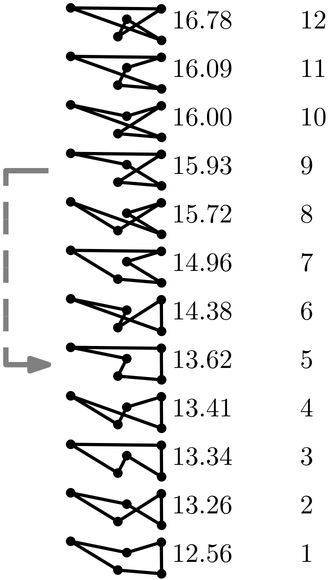
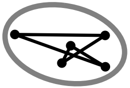
$$1 \le \underbrace{\Phi_{\text{rank when sorted by length}}(T)}_{\text{potential of the tour } T} \le n!$$



The Longest Flip Sequences

- The deletion choice may impact the number of flips.



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- $\mathbf{D}(n)$: number of flips in the longest flip sequences.
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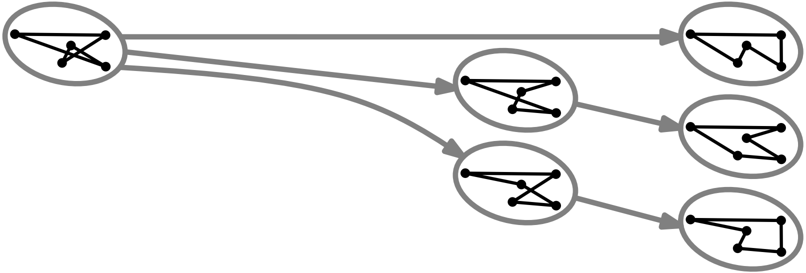

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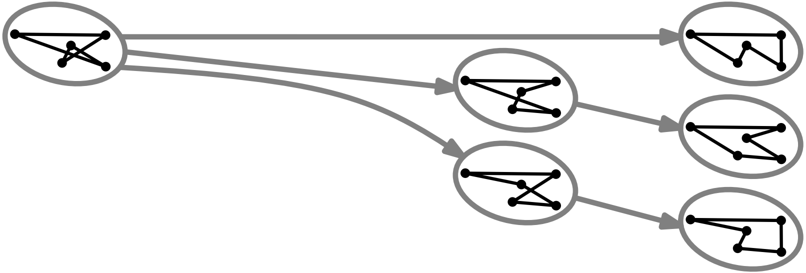

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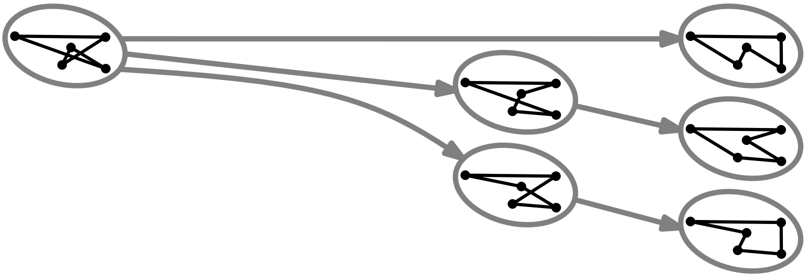

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The Flip Graph

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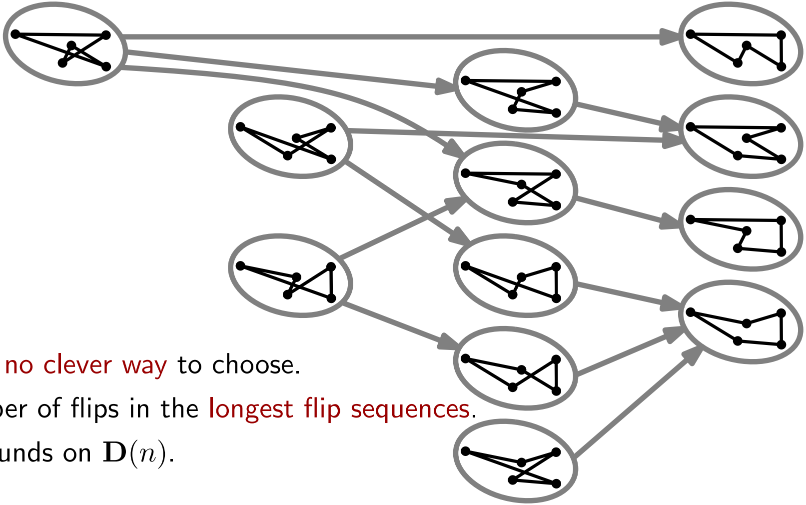

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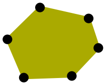
Previous Bounds on the Longest Flip Sequences



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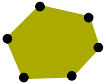

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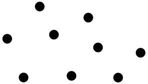
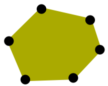


New Bound



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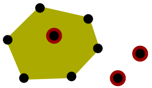

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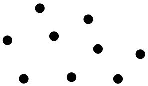
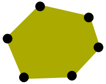


Section 2

Convex Proof



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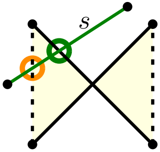
Proving $\mathbf{D}_{\text{convex}}(n) = O(n^2)$

- $\Phi_{\text{crossings}}(T)$: number of crossings in the tour T.
- $\Phi_{\text{crossings}} = O(n^2)$
- $\Phi_{\text{crossings}}$ decreases at each flip:



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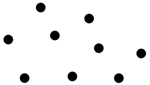

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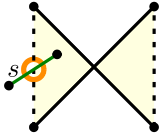
From Segments to Lines



- $\Phi_{\text{crossings}}$ may not decrease at each flip:
- Idea: consider line-segment crossings instead.
- L: lines through two cities.
- $\Phi_\ell(T)$: number of crossings with a line ℓ in the tour T.
- $\Phi_L = \sum_{\ell \in L} \Phi_\ell$
- $O(n^2)$ lines, $O(n)$ crossings per line $\implies \Phi_L = O(n^3)$.
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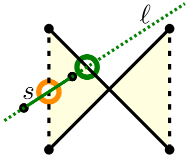

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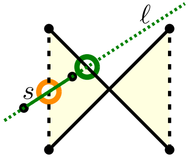

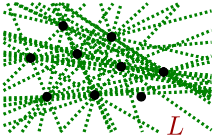

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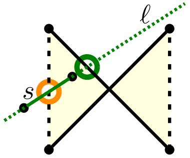

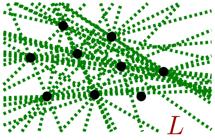

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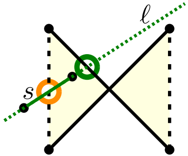
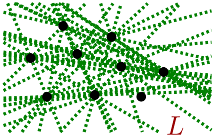
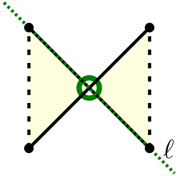
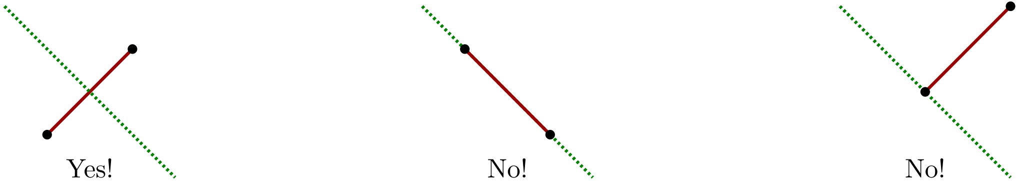






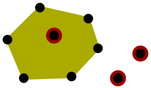
What Is a Crossing

- A single point intersection between a line and a segment is a *crossing* if it is not an endpoint of the segment.



Section 4

New Proof



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Near Convex Sets

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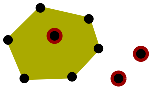
$n = 9 \qquad t = 3$



Φ: a Mixed Potential

$$\boxed{\Phi = \overbrace{\underbrace{\Phi_{\text{crossings}}}}\ + \ \overbrace{\underbrace{\Phi_{L'}}}}$$

- L': lines through at least one non-convex point.
- Case 1. If $\Phi_{\text{crossings}}$ decreases, then so does Φ (because $\Phi_{L'}$ does not increase) ✓
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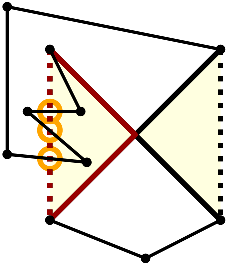

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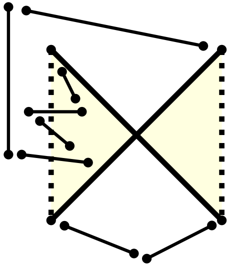

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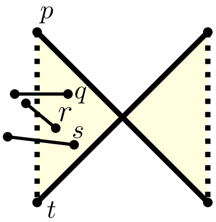

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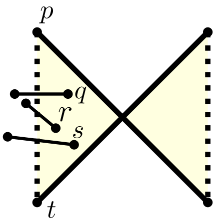

$\Phi = O(n^2 t)$

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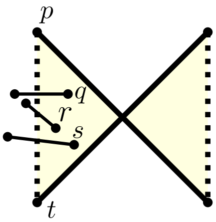

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$$\Phi = \underbrace{\overbrace{\Phi_{\text{crossings}}}^{O(n^2)}}_{\text{may not decrease!}} + \underbrace{\overbrace{\Phi_{L'}}}_{\text{compensate } \Phi_{\text{crossings}}?} = O(n^2 t)$$

- L': lines through at least one non-convex point.
- Case 1. If $\Phi_{\text{crossings}}$ decreases, then so does Φ (because $\Phi_{L'}$ does not increase) ✓
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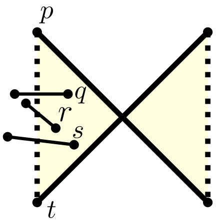

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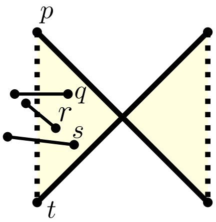

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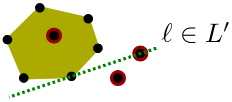
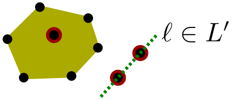
- L': lines through at least one non-convex point. $\}O(nt)$





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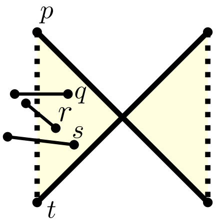

Does Φ Decreases?

$$\boxed{\Phi = \underbrace{\overbrace{\Phi_{\text{crossings}}}^{O(n^2)}}_{\text{may not decrease!}} + \underbrace{\overbrace{\Phi_{L'}}^{O(n^2 t)}}_{\text{compensate } \Phi_{\text{crossings}}?} = O(n^2 t)}$$

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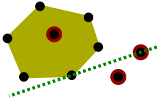

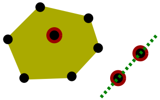

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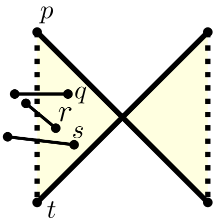

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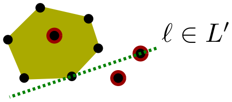

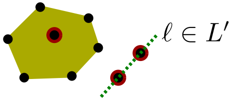

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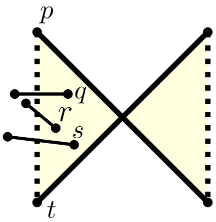

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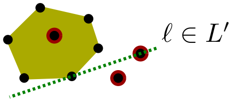
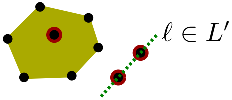
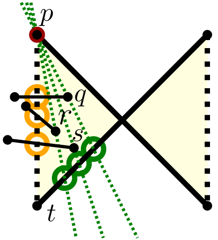




- Case 1. If $\Phi_{\text{crossings}}$ decreases, then so does Φ (because $\Phi_{L'}$ does not increase) ✓
- Case 2. If not, if p is non-convex: ✓



Does Φ Decreases?

$$\Phi = \underbrace{\overbrace{\Phi_{\text{crossings}}}^{O(n^2)}}_{\text{may not decrease!}} + \underbrace{\overbrace{\Phi_{L'}}^{O(n^2 t)}}_{\text{compensate } \Phi_{\text{crossings}}?} = O(n^2 t)$$

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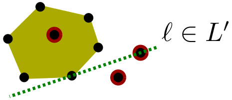

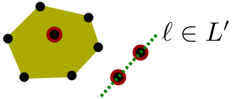

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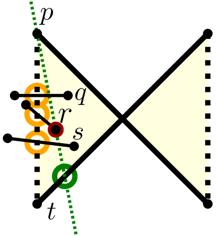

Does Φ Decreases?

$$\Phi = \overbrace{\underbrace{\Phi_{\text{crossings}}}_{\text{may not decrease!}}}^{O(n^2)} + \overbrace{\underbrace{\Phi_{L'}}_{\text{compensate } \Phi_{\text{crossings}}?}}^{O(n^2t)} = O(n^2t)$$

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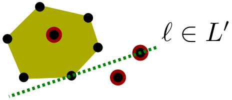

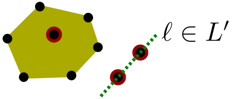

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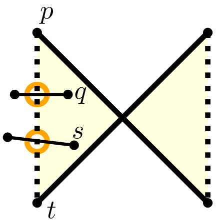

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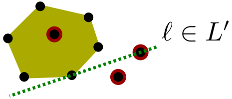

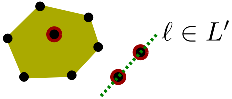

- Case 1. If $\Phi_{\text{crossings}}$ decreases, then so does Φ (because $\Phi_{L'}$ does not increase) ✓
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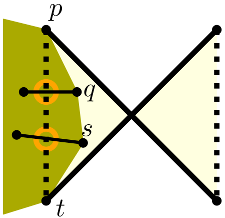

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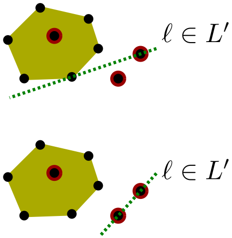
- L': lines through at least one non-convex point. $\}O(nt)$



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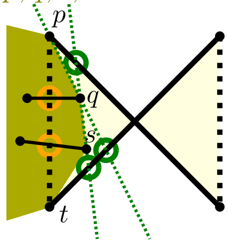

Does Φ Decreases?

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- Case 1. If $\Phi_{\text{crossings}}$ decreases, then so does Φ (because $\Phi_{L'}$ does not increase) ✓
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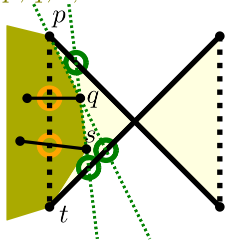

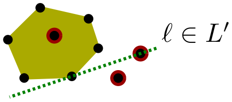

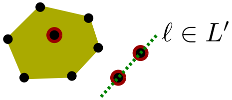

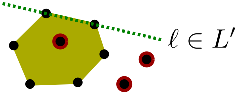

Does Φ Decreases? Yes!

$$\Phi = \underbrace{\overbrace{\Phi_{\text{crossings}}}^{O(n^2)}}_{\text{may not decrease!}} + \underbrace{\overbrace{\Phi_{L'}}^{O(n^2 t)}}_{\text{compensate } \Phi_{\text{crossings}}?} = O(n^2 t)$$

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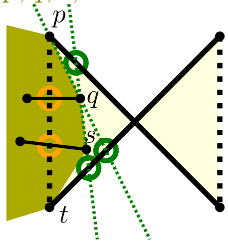
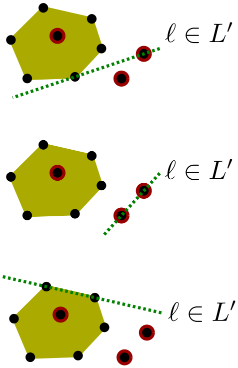




Section 5

Conclusion

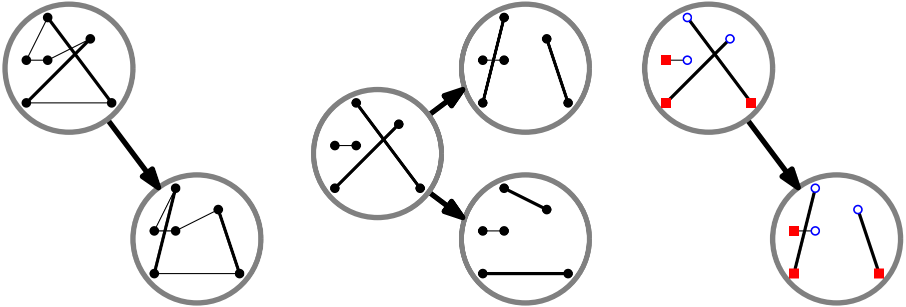
$$n^2 \preccurlyeq \mathbf{D}_{\mathsf{convex}}(n) \preccurlyeq n^2$$

$$n^2 \preccurlyeq \mathbf{D}(n,t) \preccurlyeq n^2 t$$

$$n^2 \preccurlyeq \mathbf{D}(n) \preccurlyeq \underbrace{n^2}_{\text{2016 conjecture}} \preccurlyeq n^3$$

From Tours to Segments

- Being a tour is not used in the proofs.
- A flip choice may preserve:
  - nothing special, i.e., being a set of n segments $\to \mathbf{D}$
  - being a red-blue matching $\to \mathbf{D}_{\mathrm{RB}}$
  - being a tour $\to \mathbf{D}_{\mathrm{TSP}}$
  - ...



- $\mathbf{D}, \mathbf{D}_{\mathrm{RB}}, \mathbf{D}_{\mathrm{TSP}}$ are the same asymptotically.

Conclusion

$$n^2 \preccurlyeq \mathbf{D}_{\text{convex}}(n) \preccurlyeq n^2$$

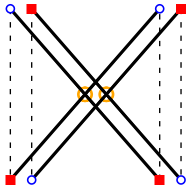
$$n^2 \preccurlyeq \mathbf{D}(n,t) \preccurlyeq n^2 t$$

$$n^2 \preccurlyeq \mathbf{D}(n) \preccurlyeq \underbrace{n^2}_{\text{2016 conjecture}} \preccurlyeq n^3$$

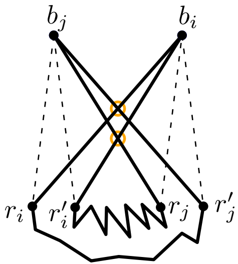
Thank you!

Reductions

- $2\mathbf{D}(n) \leq \mathbf{D}_{\mathtt{RB}}(2n) \leq \mathbf{D}(2n)$



- $2\mathbf{D}_{\mathtt{RB}}(n) \leq \mathbf{D}_{\mathtt{TSP}}(3n) \leq \mathbf{D}(3n)$



Distinct Flips

- The same pair of segments can be flipped multiple times in the same sequence.
- Counting *distinct flips* means that we do not count this multiplicity.
- A balancing argument:
  - There are $O(\frac{n^3}{k})$ flips decreasing Φ_L by at least k.
  - There are $O(n^2k^2)$ flips decreasing Φ_L by less than k:
    - We enumerate them by sweeping a line.
  - We choose $k = n^{1/3}$.