Near-Optimal ε-Kernel Construction and Related Problems

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SoCG 2017

ε-Kernel

Input

S: Set of n points in $\mathbb{R}^d$
$\varepsilon > 0$: Approximation parameter
d: Constant dimension

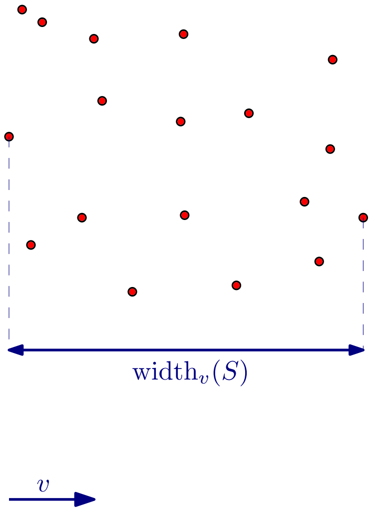
Output

$Q \subseteq S$ such that for all vector v,

$$\text{width}_v(Q) \geq (1-\varepsilon)\,\text{width}_v(S)$$

and $|Q| = O(1/\varepsilon^{(d-1)/2})$

- Approximation of the convex hull
- Connected to diameter, width...



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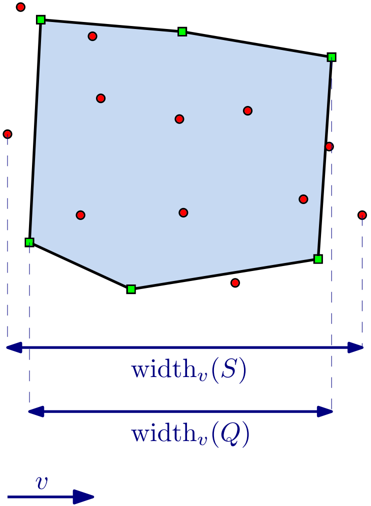
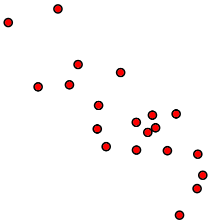


Simple ε-Kernel Construction [AHV04]



n points

1. Fatten the point set inside a unit hypercube
   time $O(n)$
2. Keep one point from top and bottom non-empty ε-grid cells in each column
   time $O(n + 1/\varepsilon^{d-1})$
3. Place hypercube of size 2 around and $\sqrt{\varepsilon}$-grid on facets [Brl76]
   time $O(1/\varepsilon^{(d-1)/2})$
4. Find nearest neighbor of each grid vertex
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Total time: $O(n + 1/\varepsilon^{3(d-1)/2})$

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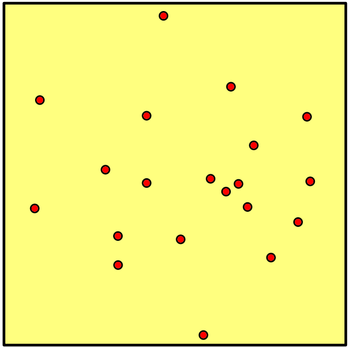

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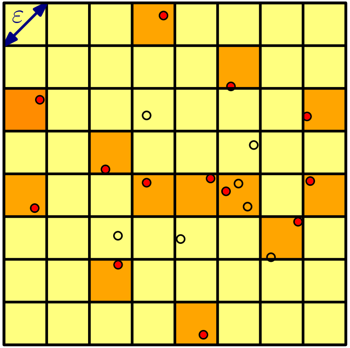

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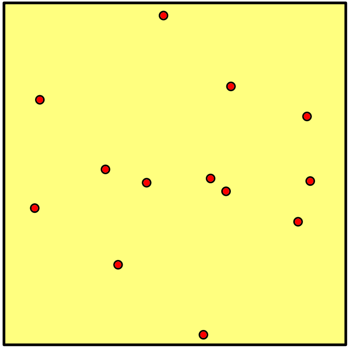

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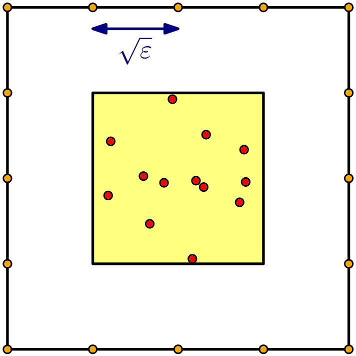

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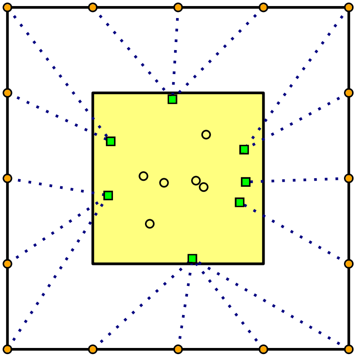

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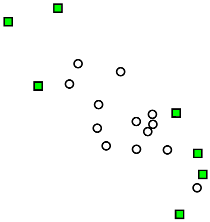

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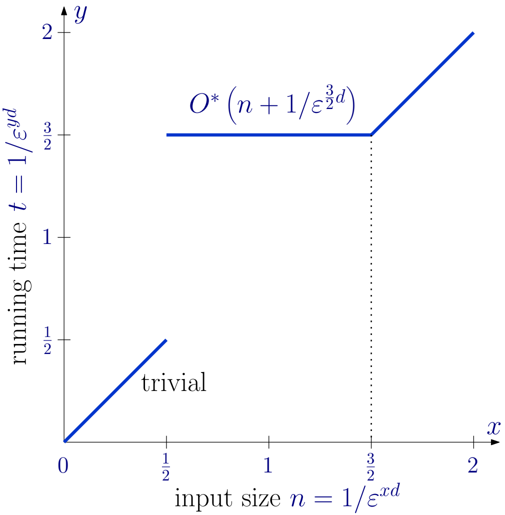
- [AHV04] $O\left(n + 1/\varepsilon^{\frac{3(d-1)}{2}}\right)$
- [Cha06] $O\left(n \log \frac{1}{\varepsilon} + 1/\varepsilon^{d-2}\right)$
- [ArC14] $O\left(n + \sqrt{n}/\varepsilon^{\frac{d}{2}}\right)$
- [Cha17] $\widetilde{O}\left(n\sqrt{\frac{1}{\varepsilon}} + 1/\varepsilon^{\frac{d-1}{2}+\frac{3}{2}}\right)$

Our near-optimal construction

- $O\left(n \log \frac{1}{\varepsilon} + 1/\varepsilon^{\frac{d-1}{2}+\alpha}\right)$
- $\alpha > 0$ arbitrarily small
- Independent of [Cha17] and completely different technique



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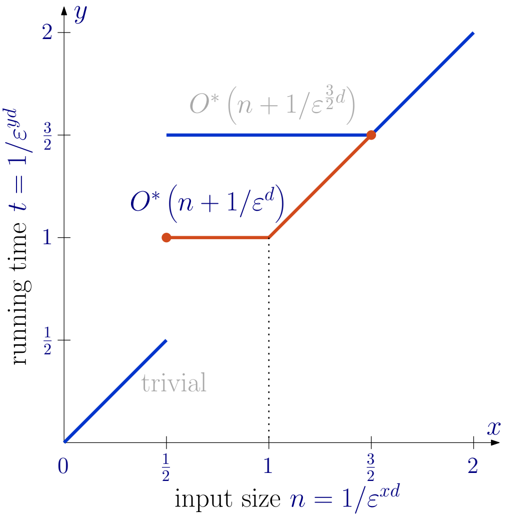

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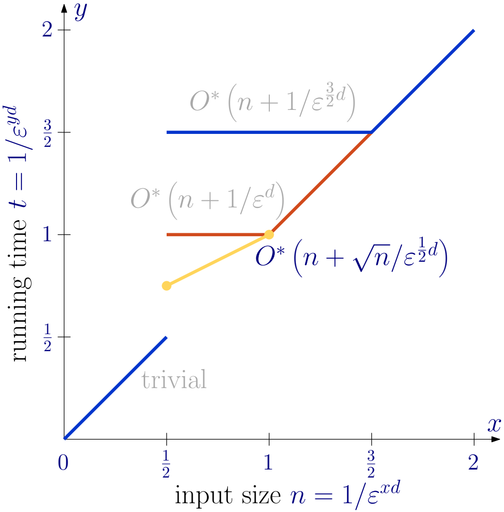

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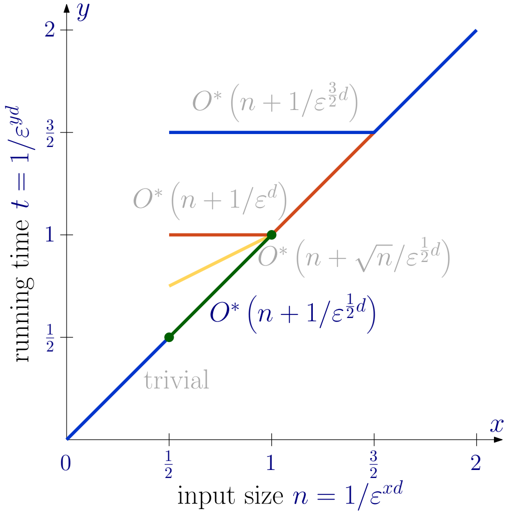
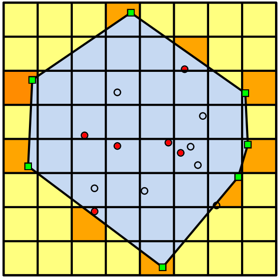


Results

Main results

- Near-optimal ε-kernel construction in $O\left(n \log \frac{1}{\varepsilon} + 1/\varepsilon^{\frac{d-1}{2}+\alpha}\right)$ time
- Diameter approximation in $O\left(n \log \frac{1}{\varepsilon} + 1/\varepsilon^{\frac{d-1}{2}+\alpha}\right)$ time
- Bichromatic closest pair approximation in $O\left(n/\varepsilon^{\frac{d}{4}+\alpha}\right)$ expected time
- Euclidean minimum spanning/bottleneck tree approximation in $O\left((n \log n)/\varepsilon^{\frac{d}{4}+\alpha}\right)$ expected time
- Near-optimal preprocessing time for approximate polytope membership data structure
- Near-optimal preprocessing time for approximate nearest neighbor data structure
- $\alpha > 0$ arbitrarily small

Techniques

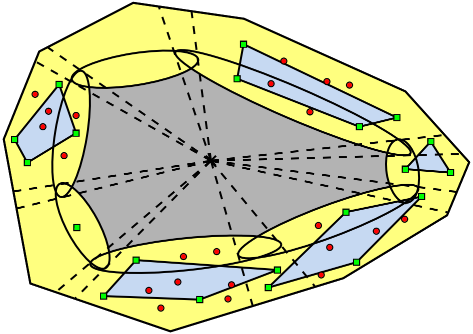


Previous solutions use grids

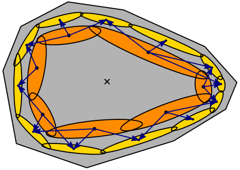
- Similar width in all directions
- Ignores curvature

We use a hierarchy of Macbeath regions

- Adapt to the curvature of the body
- Narrow in directions of high curvature
- Wide in directions of low curvature

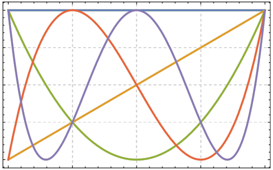


Comparison to Chan 2017



Our solution:

- Geometric: Macbeath regions
- Sligthly faster: $O\left(n \log \frac{1}{\varepsilon} + 1/\varepsilon^{\frac{d-1}{2}+\alpha}\right)$
- Euclidean metric only



Timothy Chan, 2017:

- Algebraic: Chebyshev polynomials
- $\widetilde{O}\left(n\sqrt{\frac{1}{\varepsilon}} + 1/\varepsilon^{\frac{d-1}{2}+\frac{3}{2}}\right)$
- Generalizes to other metrics

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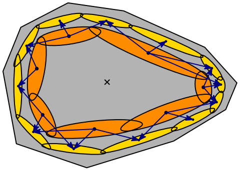

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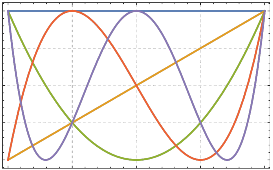

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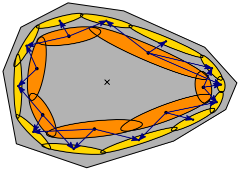

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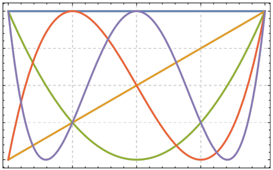

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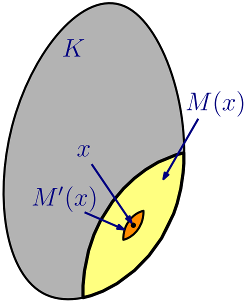

Given a convex body K, $x \in K$, and $\lambda > 0$:

- $M^\lambda(x) = x + \lambda((K - x) \cap (x - K))$
- $M(x) = M^1(x)$: intersection of K and K reflected around x
- $M'(x) = M^{1/5}(x)$

Properties

- $M'(x) \cap M'(y) \neq \emptyset \;\Rightarrow\; M'(x) \subseteq M(y)$
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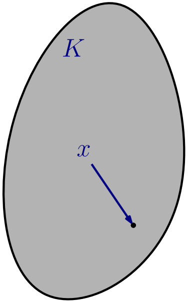

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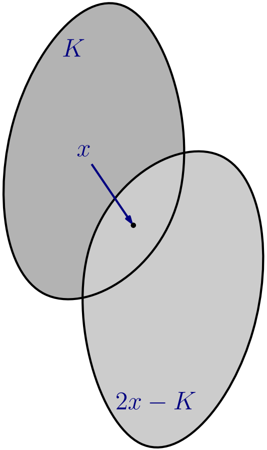

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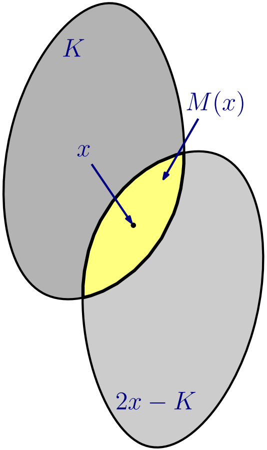

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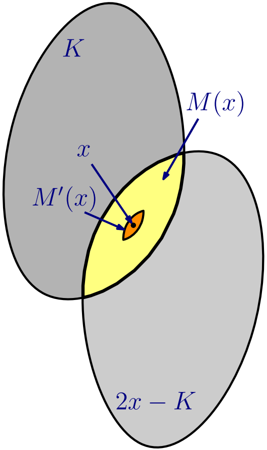

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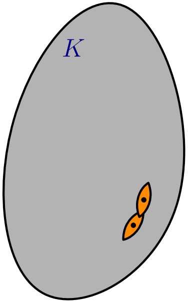

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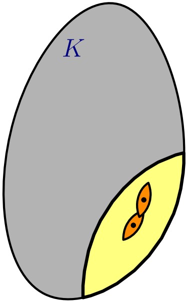

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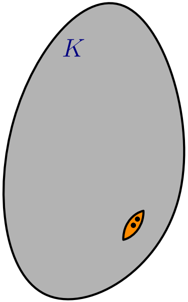

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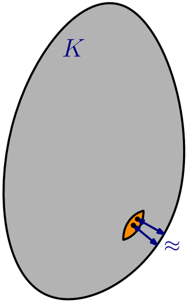

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John Ellipsoid [Joh48]

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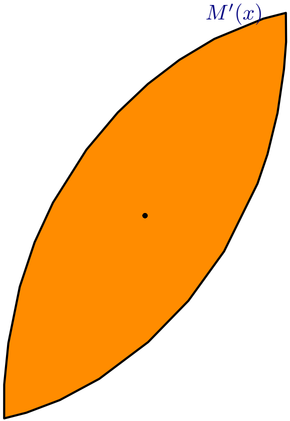
Macbeath Ellipsoid

- $E(x)$: enclosed John ellipsoid of $M'(x)$
- Constant approximation of $M'(x)$



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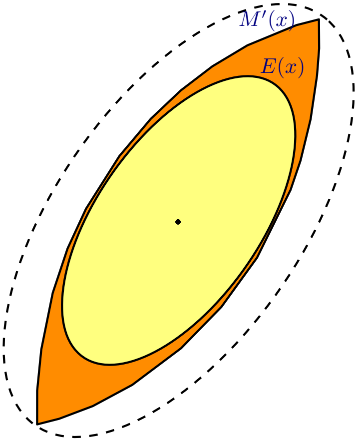
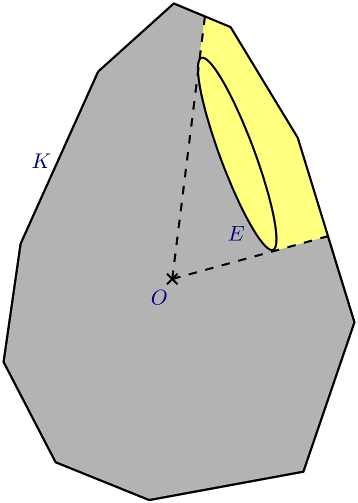
Macbeath Ellipsoid

- $E(x)$: enclosed John ellipsoid of $M'(x)$
- Constant approximation of $M'(x)$



Shadow of Macbeath Ellipsoids



Shadow of ellipsoid E

Points $p \in K$ such that ray Op intersects E

- Reaches the boundary
- Constant approximation of E

Covering with Macbeath Ellipsoids

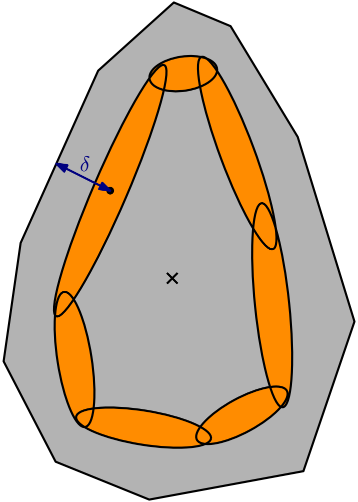
Covering (see [Bar07])

Given:

- K: convex body
- δ: small positive parameter

There exist ellipsoids $E(x_1), \ldots, E(x_k)$

- $\delta(x_1) = \cdots = \delta(x_k) = \delta$
- Cover: Shadows cover the boundary
- $k = O(1/\delta^{\frac{d-1}{2}})$ [AFM16]



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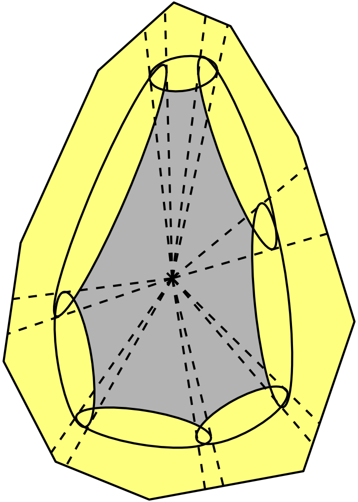

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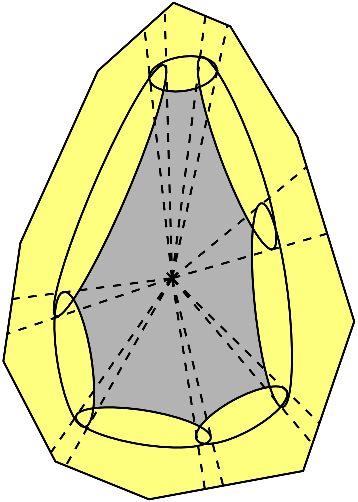
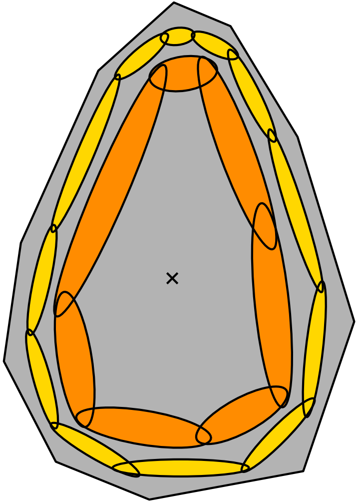


Hierarchy of Macbeath Ellipsoids



Hierarchy [AFM17a]

- Each level i a δ_i-covering
- $\ell = \Theta(\log \frac{1}{\varepsilon})$ levels
- $\delta_0 = \Theta(1)$, $\delta_\ell = \Theta(\varepsilon)$
- $\delta_{i+1} = \delta_i / 2$
- E is parent of E' if
  - Levels are consecutive
  - Shadow of E intersects E'
- Each node has $O(1)$ children
- Total size: $O\left(1/\varepsilon^{\frac{d-1}{2}}\right)$

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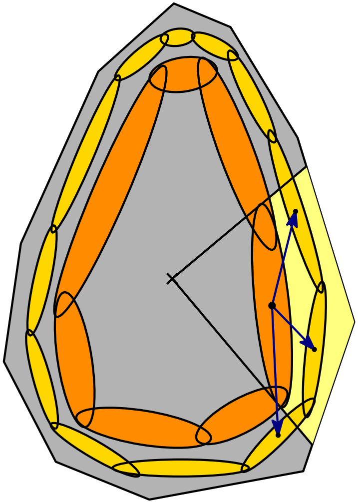

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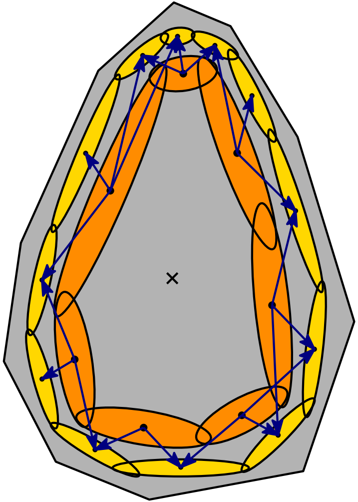

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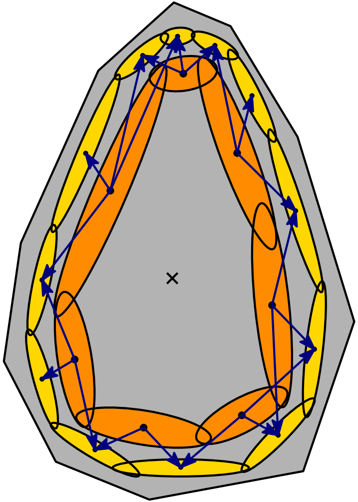

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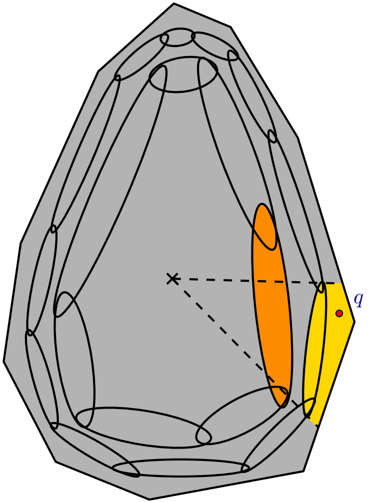
Hierarchy Properties

- Query point $q \in K$:
  - Find leaf shadow that contains q
  - Or report q as far from the boundary
  - $O(\log \frac{1}{\varepsilon})$ time
- Hierarchy $\longrightarrow$ Kernel
  - Split points among leaf shadows
  - Pick one point per leaf shadow (if it exists)
  - $O(n \log \frac{1}{\varepsilon})$ time
- Hierarchy construction takes: $O\left(n + 1/\varepsilon^{\frac{3(d-1)}{2}}\right)$ time (too slow!)



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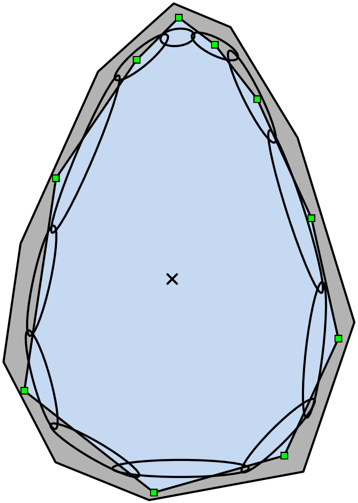

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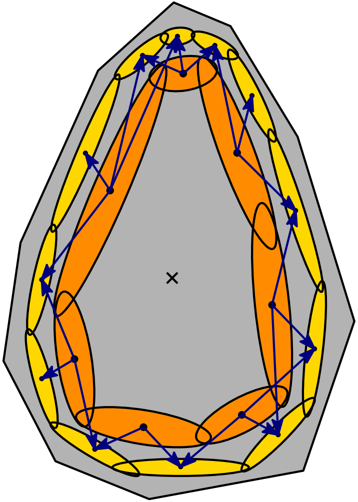
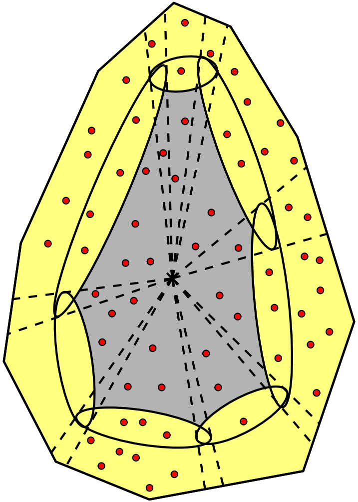


Kernel Construction



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4. Return union of kernels

Time: $O\left(n \log \frac{1}{\varepsilon} + 1/\varepsilon^{\frac{5(d-1)}{6}}\right)$

Kernel size: $O\left(\left(\frac{1}{\delta}\right)^{\frac{d-1}{2}} \left(\frac{\delta}{\varepsilon}\right)^{\frac{d-1}{2}}\right) = O\left(1/\varepsilon^{\frac{d-1}{2}}\right)$

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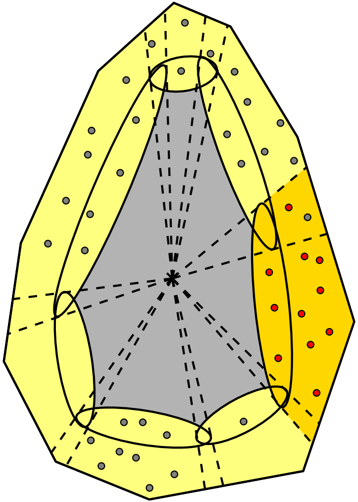

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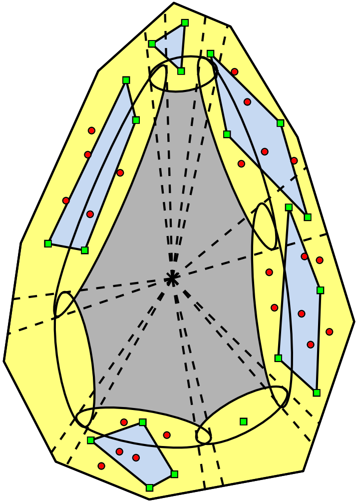

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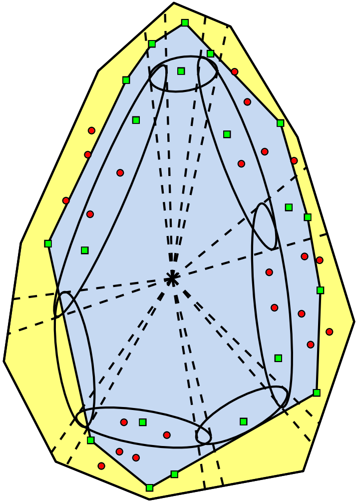

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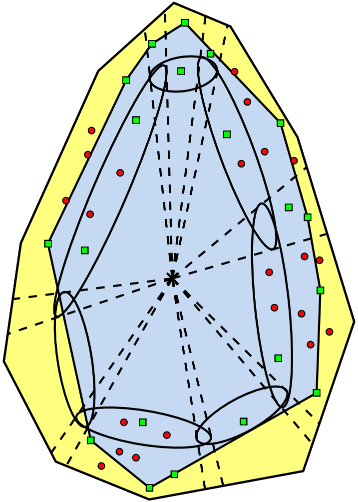

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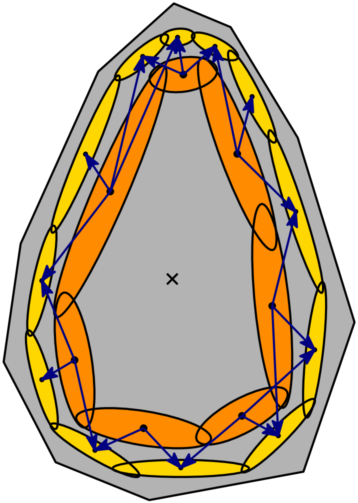
Bootstrapping

Bootstrap using improved ε-kernel construction:

- $\widetilde{O}\left(n + \left(\frac{1}{\varepsilon}\right)^{t(d-1)}\right)$ time $\longrightarrow \widetilde{O}\left(n + \left(\frac{1}{\varepsilon}\right)^{\frac{4t+1}{6}(d-1)}\right)$ time
- $t: \; 1 \longrightarrow \frac{5}{6} \longrightarrow \frac{13}{18} \longrightarrow \frac{35}{54} \longrightarrow \cdots \longrightarrow \frac{1}{2} + \alpha$
- Exponent t arbitrarily close to $\frac{1}{2}$: $\widetilde{O}\left(n + 1/\varepsilon^{\frac{d-1}{2}+\alpha}\right)$ time

Open Problems

- Other applications of the hierarchy
- Remove $1/\varepsilon^{\alpha}$ factor
- Minimum width approximation
- Lower bound for approximate diameter (or improved upper bound)



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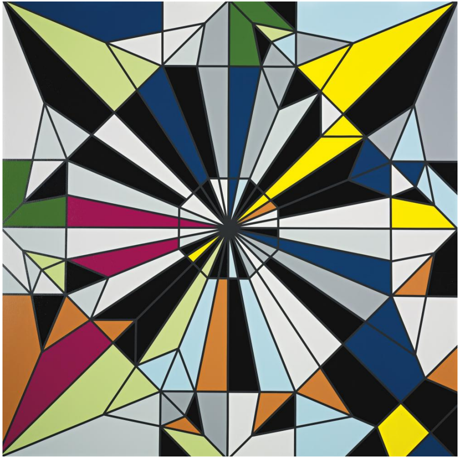

Painting by Sarah Morris

Thank you!