Linear-Time Approximation Algorithms for Geometric Intersection Graphs

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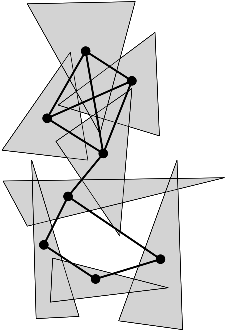
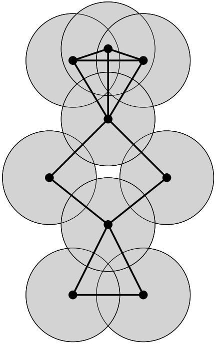
2015

Geometric Intersection Graphs



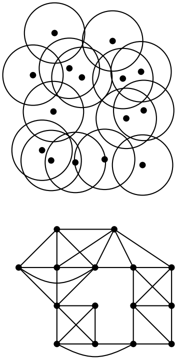
- *Geometric intersection graph*: Intersection graph of geometric objects
- Objects may be disks, unit disks, squares, unit squares, rectangles, etc
- Different graph classes depending on the object type
- Recognition is NP-hard for all objects listed above (and may not be in NP)
- Generalize interval graphs to higher dimensions

Unit Disk Graphs



- *Unit disk graph (UDG)*: Intersection graph of unit-disks in the plane
- Applications in wireless networks
- Neither planar nor perfect: K_i and C_i are UDGs for all i
- Vertex coordinates (disk centers) are real numbers

Unit Disk Graph Algorithms



- Two types of algorithms:
  - Geometric: vertex coordinates
  - Graph-based: adjacency information only
- PTASs for several problems:
  - Minimum Dominating Set
  - Maximum (Weight) Independent Set
  - Minimum (Weight) Vertex Cover
  - Minimum Connected Dominating Set
  - ...

Our assumptions

- Vertex coordinates as input (geometric algorithm)
- Floor function and $O(1)$-time hashing

PTAS vs Constant Approximations

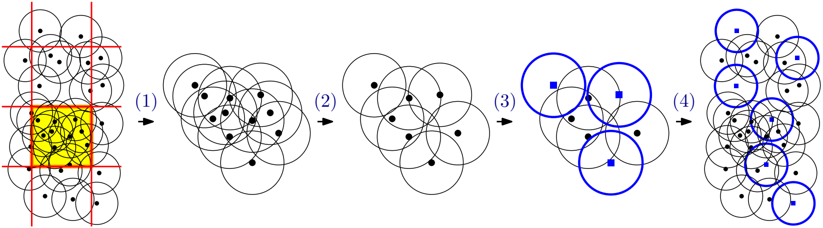
- PTASs for UDGs have high complexity:
  $O(n^{10})$ to 4-approximate the *minimum dominating set*
- Faster constant-factor approximations exist:
  - 5-approximation in $O(n)$ time
  - 4.89-approximation in $O(n \log n)$ time
  - 4.78-approximation in $O(n^4)$ time
  - 4-approximation in $O(n^6 \log n)$ time
  - 3-approximation in $O(n^{11} \log n)$ time

Our Results for UDGs

New method to obtain $O(n)$-time approximations:

- Min Dominating Set for UDGs: $(4 + \varepsilon)$-approximation
- Works for several geometric intersection graphs
- Works for several graph problems

Overview of Our Method



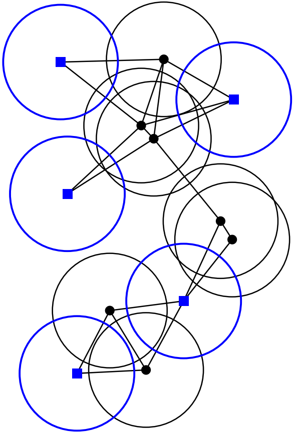
(1) Break the original problem into subproblems of $O(1)$ diameter (shifting strategy)

(2) Build a coreset with $O(1)$ objects for each subproblem, which gives an α-approximation to the subproblem

(3) Solve the coreset optimally

(4) Combine the solutions into an $(\alpha + \varepsilon)$-approximation

Maximum-Weight Independent Set for UDGs



- *Independent Set*: Subset of points with minimum distance > 2
- *Maximum-Weight Independent Set*:
  - Points have real weights

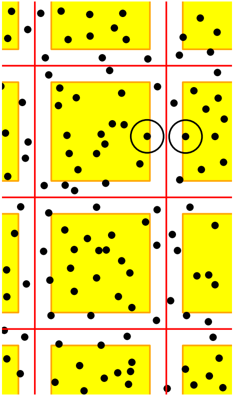
Previous results:

- $(1+\varepsilon)$-approx in $O(n^{4\lceil 2/\varepsilon\sqrt{3}\rceil})$ time: 4-approximation in $O(n^4)$ time
- 5-approximation in $O(n \log n)$ time

Our result:

- $(4+\varepsilon)$-approximation in $O(n)$ time

Breaking the Problem into Subproblems

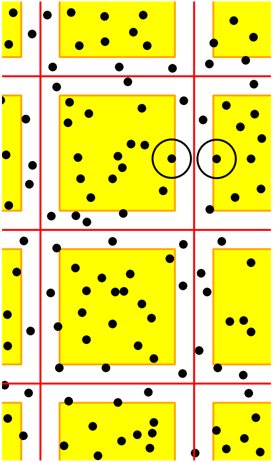


Break problem into $O(1)$-diameter subproblems (shifting strategy):

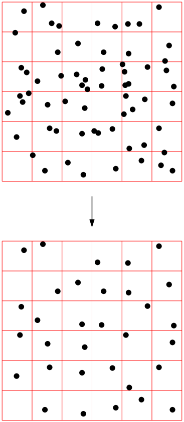
- Set k to smallest integer with $\left(\frac{k-2}{k}\right)^2 \geq \frac{4}{4+\varepsilon}$
- Use grids of size $2k$
- Create k^2 shifted grids with even origins
- Contract grid cells by 1 in all directions
- Each contracted cell is a subproblem

Analysis of Shifting Strategy

- Contracted cells are distance 2 apart: union preserves independence
- 4-approximation in yellow area
- Yellow area gets much bigger than white area as $k \to \infty$
- Expected number of OPT points in white area is small
- Maximum is larger than expectation

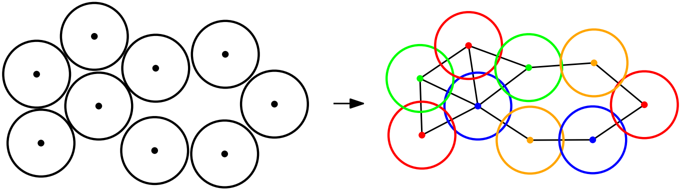


Constant-Diameter Coreset



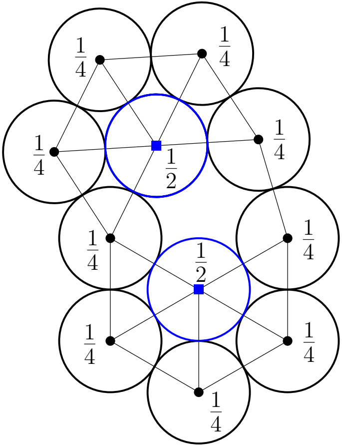
- *Coreset*: Subset with $O(1)$ points that approximates the original solution
- Algorithm:
  - Create grid with cells of diameter $0.29 < (2 - \sqrt{2})/2$
  - Select a point of maximum weight inside each cell (coreset)
  - Find the optimal independent set among the selected points
- We need to prove it gives a 4-approximation!

Proof of 4-Approximation



- Consider the optimal independent set
- Moving points by at most 0.29, we obtain a planar graph
- Planar graphs are 4-colorable
- The color of maximum weight is a 4-approximation

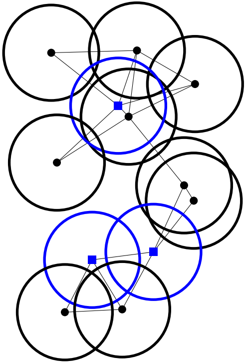
Lower Bound of 3.25



- P_1: Set of points from the figure
- P_2: Multiply coordinates from P_1 by $(1+\varepsilon)$ and weights by $(1-\varepsilon)$
- $P_1 \cup P_2$ gives a lowerbound of 3.25
  - P_2 is independent
  - MWIS: P_2, with $w(P_2) \approx 3.25$
  - Coreset: P_1
  - P_1 has MWIS with weight 1

Minimum Dominating Set for UDGs

Dominating Set: Subset of points D such that all input points are within distance at most 2 from a point in D



- 5-approximation in $O(n)$ time
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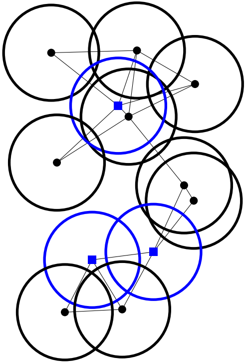
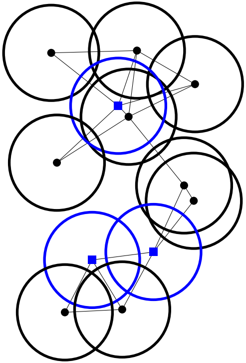


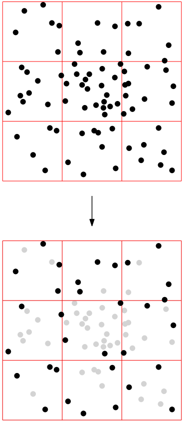
- 5-approximation in $O(n)$ time
- 4.89-approximation in $O(n \log n)$ time
- 4.78-approximation in $O(n^4)$ time
- new $(4 + \varepsilon)$-approximation in $O(n)$ time
- 4-approximation in $O(n^6 \log n)$ time
- 3-approximation in $O(n^{11} \log n)$ time

Minimum Dominating Set Algorithm



- Break the problem into subproblems of $O(1)$ diameter using the shifting strategy
- Cells need to be expanded rather than contracted
- We'll present only the coreset

Constant-Diameter Coreset



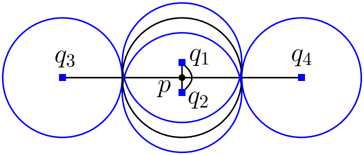
- Algorithm:
  - Create grid with cells of diameter $\gamma = 0.24$ (any positive γ satisfying

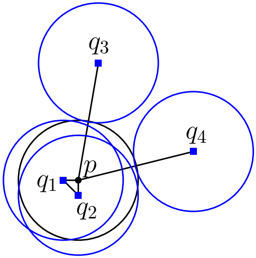
$$\sqrt{8 - 8\cos\left(\frac{\frac{\pi}{2} + 2\arcsin(\frac{\gamma}{2})}{2}\right)} + \gamma < 2$$

    suffices)
  - Select the points of min and max x and y coordinates
  - Find the optimal dominating set among the coreset points, but dominating all points
- We need to prove it's a 4-approximation!

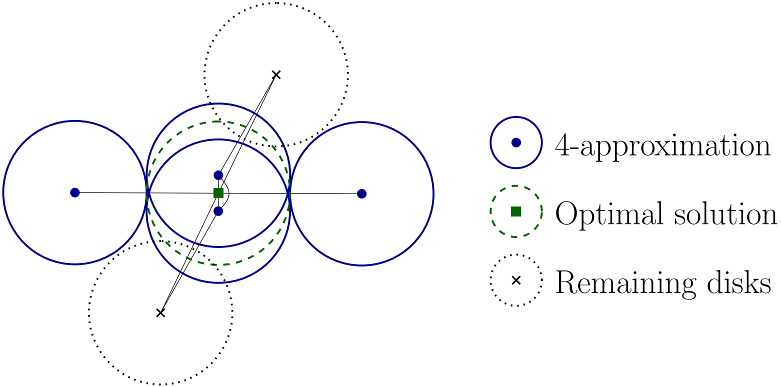
Proof of 4-Approximation

- For each point p in OPT,
  - either p is in the coreset (great!)
  - or there are points q_1, q_2 near p with angle $\geq 90°$
- We dominate all points dominated by p using at most 4 points q_1, q_2, q_3, q_4

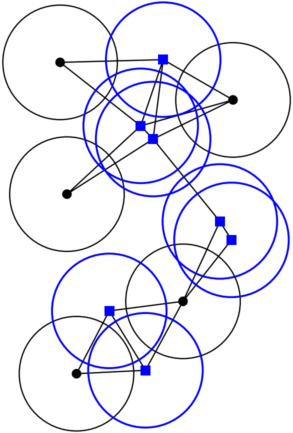




Lower Bound of 4

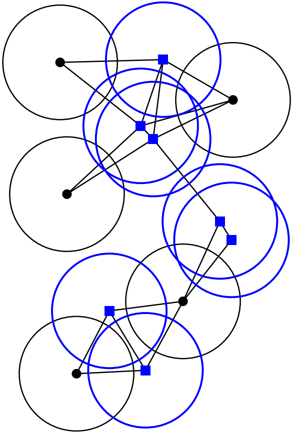


Minimum Vertex Cover for UDGs



- *Vertex Cover*: Complement of independent set
- Linear-time PTAS already known
- Minimum vertex cover corresponds to maximum independent set
- C: Vertex cover, I: Independent set, $|C| = n - |I|$
- Approximation ratio is not preserved

Minimum Vertex Cover for UDGs

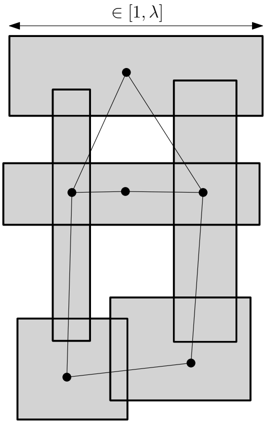


- *Vertex Cover*: Complement of independent set
- Linear-time PTAS already known
- Minimum vertex cover corresponds to maximum independent set
- C: Vertex cover, I: Independent set, $|C| = n - |I|$
- Approximation ratio is not preserved
  - Bad when $|C| \ll n$
  - Great when $|I| \ll n$

Linear-Time Approximation Scheme

- Break the problem into subproblems of $O(1)$ diameter using the shifting strategy
- A set of diameter d has at most $(d+2)^2/4$ independent vertices
- If n is sufficiently small (constant), solve the problem optimally $\left(n < \left(1 + \frac{3}{4\varepsilon}\right)\frac{(d+2)^2}{4}\right)$
- Otherwise, compute the 4-approximate maximum independent set and use its complement

A Subclass of Rectangle Graphs

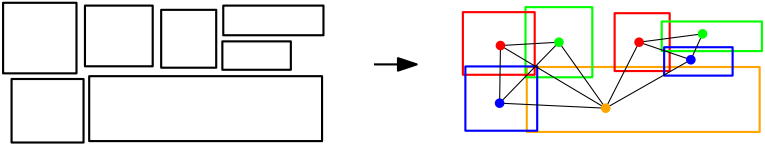


- *Rectangle graph*: Intersection graph of axis-aligned rectangles in the plane
- *Independent set*: No constant factor approximation known
- *Our subclass*: Width and height between 1 and λ for *constant* λ
- PTASs exist for this subclass (very high complexity)
- *Our result*: Linear-time $(6 + \varepsilon)$-approximation to the maximum-weight independent set

Constant-Diameter Coreset

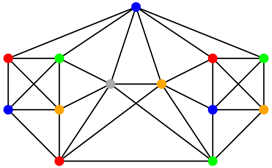
- A rectangle q centered at (x_q, y_q) with width w_q and height h_q is a point $(x_q, y_q, w_q, h_q) \in \mathbb{R}^4$
- *Coreset*: Subset with $O(1)$ points that approximates the original solution
- Algorithm:
  - Create 4-dimensional grid with cells of diameter $0.16 < 1/6$
  - Select a point (rectangle) of maximum weight inside each cell (coreset)
  - Find the optimal independent set among the selected points
- We need to prove it gives a 6-approximation!

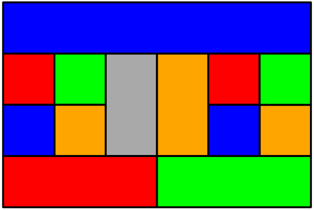
Proof of 6-Approximation



- Consider the optimal independent set
- Moving and resizing rectangles by less than $1/6$, we obtain a 1-planar graph (each edge crosses at most one other edge)
- 1-planar graphs are 6-colorable
- The color of maximum weight is a 6-approximation

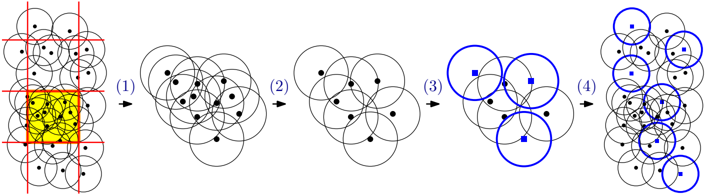
Lower Bound of $13/3 = 4.333\ldots$





- Contact graph of rectangles
- Vertices: 13
- Maximum independent set: 3
- Lower bound: $13/3$
- Are all graphs in the class 5-colorable?

Conclusion



New method to obtain $O(n)$-time algorithms for problems on geometric intersection graphs, yielding:

- $(4+\varepsilon)$-approx to max-weight independent set for UDGs
- $(4+\varepsilon)$-approx to minimum dominating set for UDGs
- $(1+\varepsilon)$-approx to minimum vertex cover for UDGs
- $(6+\varepsilon)$-approx to max-weight independent set for certain rectangle graphs

Open Problems

- Tight analysis for both max-weight independent set algorithms?
- Improvement for the unweighted version (by considering extreme points in several directions)?
- Similar method without geometric information?
- Solve other problems:
  - Minimum-weight dominating set?
  - Minimum connected dominating set?
  - Minimum independent dominating set?
- Other geometric intersection graphs?

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Thank you!

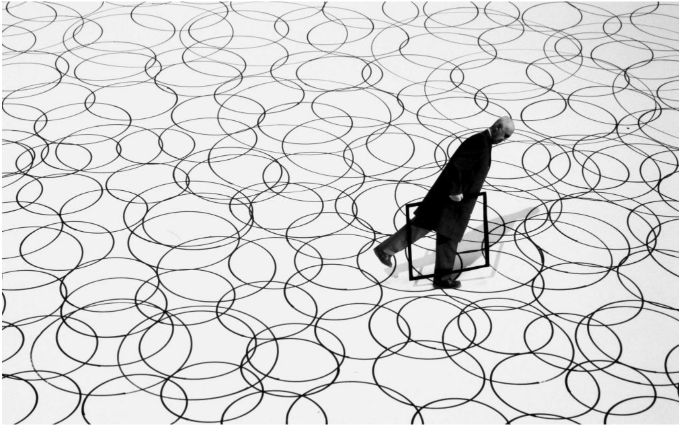

Photo by Gilbert Garcin