Economical Convex Coverings and Applications

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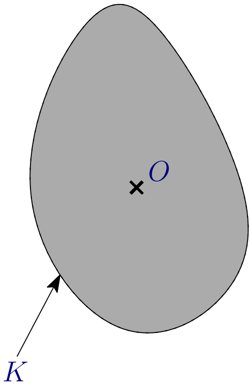
(c,ε)-covering:

- Given c, ε, and a convex body $K \subset \mathbb{R}^n$ (with a central origin)
- Collection $\mathcal{Q}$ of convex bodies
- Union covers K
- Factor-c expansion of each $Q \in \mathcal{Q}$ about its centroid lies inside $(1+\varepsilon)K$
- Usually $c = 2$

In our case:

- A constant contraction forms a packing
- Bodies are centrally symmetric



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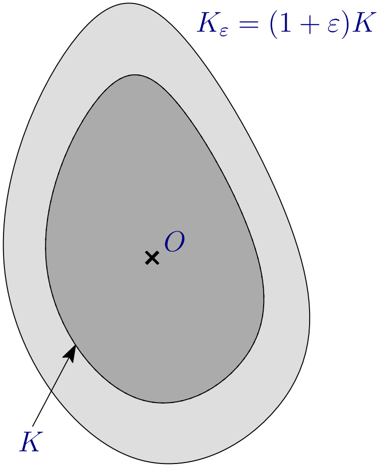

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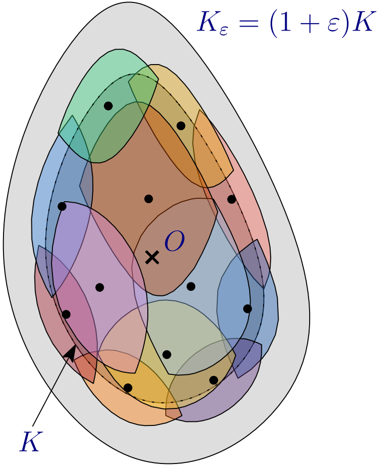

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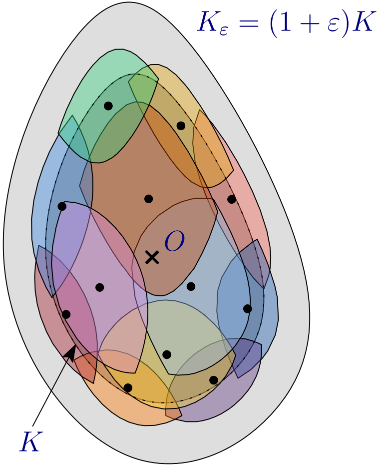

Introduction

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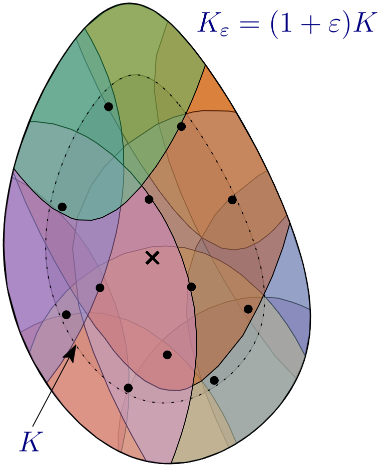

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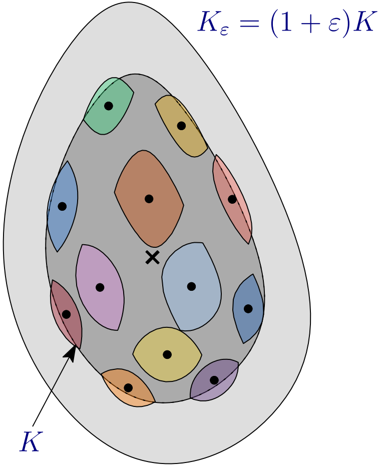
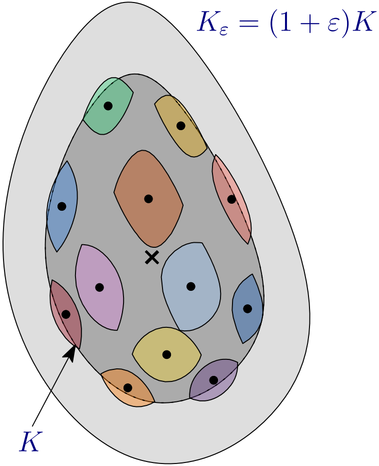


Previous and New Cover Sizes



Previous (c, ε) cover sizes (for constant c):

- $2^{O(n)}/\log^n(1/\varepsilon)$ for ℓ_∞ balls [ENN11]
- $n^{O(n)}/\varepsilon^{(n-1)/2}$ for any convex body [AM18]
- $2^{O(n)}/\varepsilon^{n/2}$ for ℓ_p balls [NV22]
- Lower bound for ℓ_2 balls: $2^{-O(n)}/\varepsilon^{(n-1)/2}$ [NV22]

Our cover size:

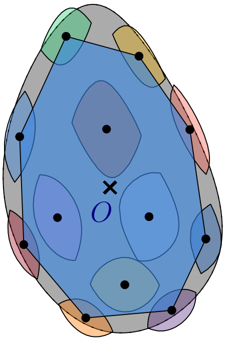
- $2^{O(n)}/\varepsilon^{(n-1)/2}$ for any convex body

- We want an approximation P of K such that:
  $K \subseteq P \subseteq (1+\varepsilon)K$
- Implies Banach-Mazur metric
- Compared to Hausdorff:
  Finer approximation in narrow directions
- Goal: small number of vertices

From (c,ε)-covering to polytope approximation:

Let X be the set of centers of any (c, ε')-covering of $K(1+\varepsilon/c)$. Then $K \subset \mathrm{conv}(X) \subset K(1+\varepsilon)$.

- Number of vertices: $|\mathcal{Q}| = 2^{O(n)}/\varepsilon^{(n-1)/2}$
- Matches best previous bound [NNR20]



Application 2: Approximate Closest Vector Problem (CVP)

Closest Vector Problem (CVP) :

- Given:
  - n-dimensional lattice L in $\mathbb{R}^n$
  - target vector $t \in \mathbb{R}^n$
  - convex body K representing a "norm" $\| \cdot \|_K$
- Find:
  vector x minimizing $\|tx\|_K$
- Approximation:
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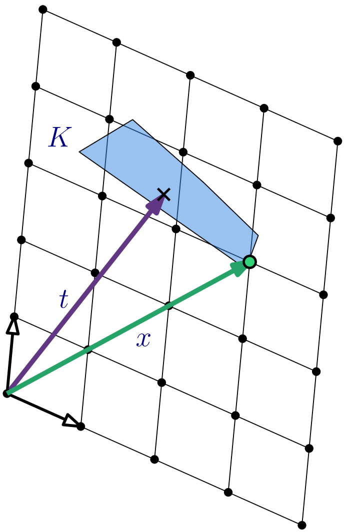

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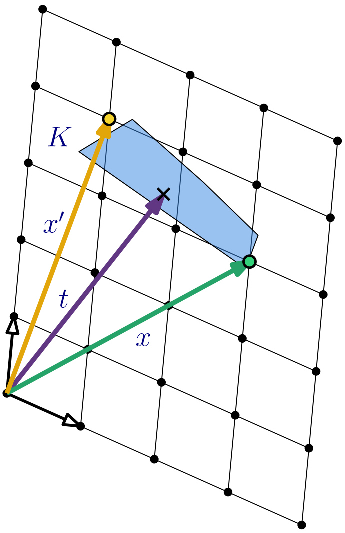
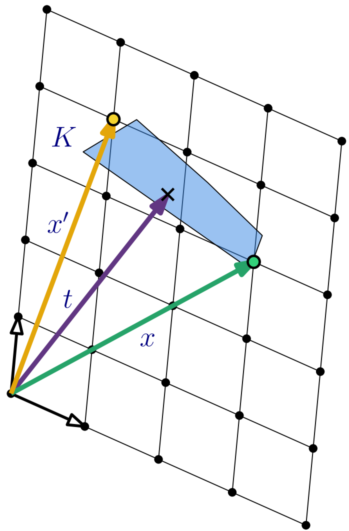


Application 2: Approximate Closest Vector Problem (CVP)

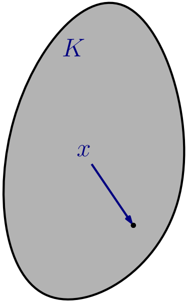
From (c,ε)-covering to approximate CVP [NV22] :

Given a $(2, \varepsilon)$-covering of K consisting of N centrally symmetric convex bodies, we can solve $(1 + 7\varepsilon)$-CVP under $\| \cdot \|_K$ with $\widetilde{O}(N)$ calls to a 2-CVP solver.

- Previous solution in $2^{O(n)}/\varepsilon^n$ time [DK16]
- We use it to get $2^{O(n)}/\varepsilon^{(n-1)/2}$ time
- Same time for approximate integer programming



Main Tool: Macbeath Regions



Macbeath region [Mac52]:

- Given a convex body K, $x \in K$, and $\lambda > 0$:
- $M^\lambda(x) = x + \lambda((K - x) \cap (x - K))$
- $M(x) = M^1(x)$: intersection of K and K reflected around x

Equivalently:

- $M(x)$: largest centrally symmetric convex body centered on x
- $M^\lambda(x)$: $M(x)$ scaled by λ around x

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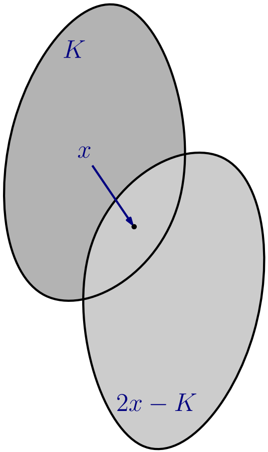

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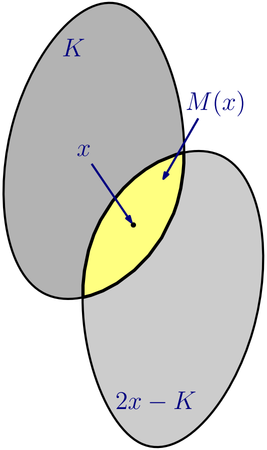

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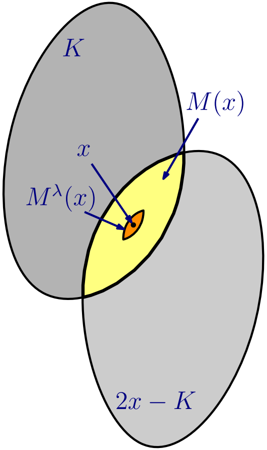

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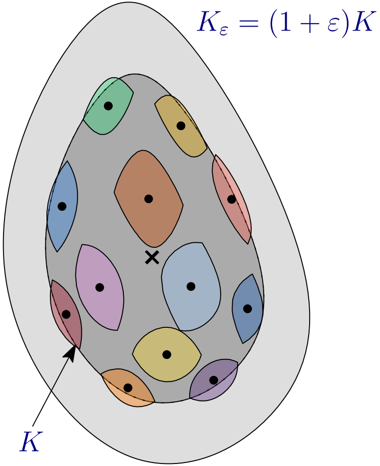
Maximal Packing of Macbeath Regions

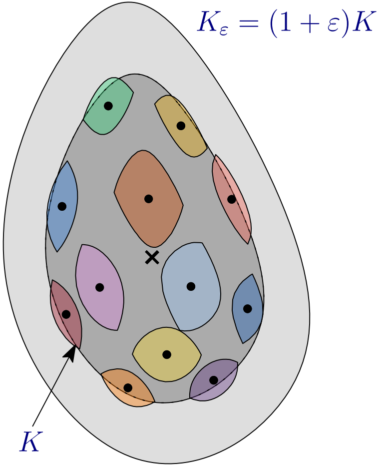
- Our covering is defined by a maximal set of disjoint Macbeath regions for K_ε with $\lambda = 1/4c$
- Scaling them by 4 to $\lambda = 1/c$ gives our (c, ε)-cover $\mathcal{Q}$ of K
- Scaling $\mathcal{Q}$ by c to $\lambda = 1$ stays inside K_ε

- Previous bound was $|\mathcal{Q}| = n^{O(n)}/\varepsilon^{(n-1)/2}$ [AAFM22]
- We show that $|\mathcal{Q}| = 2^{O(n)}/\varepsilon^{(n-1)/2}$
- New techniques are needed



Large Macbeath Regions

- Assume $\mathrm{vol}(K_\varepsilon) = 1$
- $\mathcal{Q}_{\geq t}$: Subset of regions of volume at least t
- Shrinking the regions by 4 produces a packing
- Hence, $|\mathcal{Q}_{\geq t}| = O(4^n/t) = 2^{O(n)}/t$
- For $t = \varepsilon^{(n+1)/2}$: $|\mathcal{Q}_{\geq t}| = 2^{O(n)}/\varepsilon^{(n+1)/2}$
- Bounds with $n-1$ instead of $n+1$ come from splitting K into layers
- Roughly, Macbeath regions with center x at distance α from the boundary are in a layer of volume $O(\alpha)$
- As α increases, Macbeath regions get larger
- Forms a geometric progression



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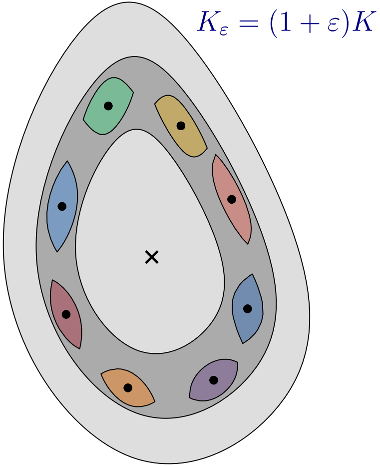

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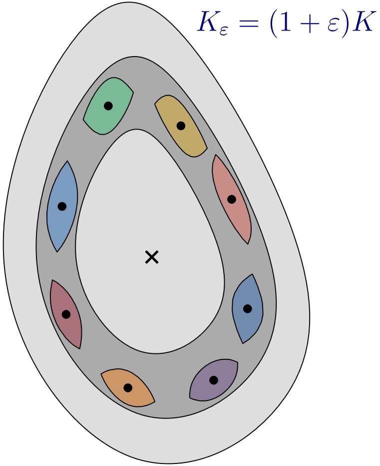
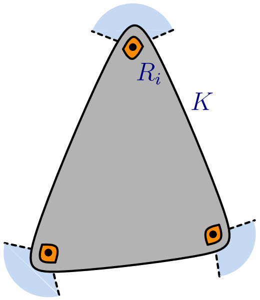


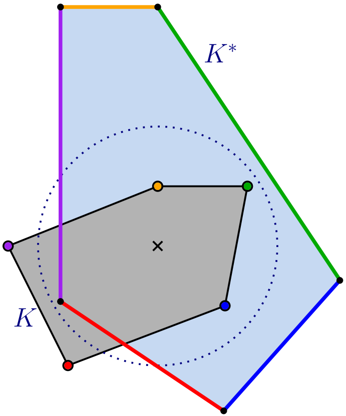
Small Macbeath Regions

- Consider a tiny Macbeath region of volume $O(\varepsilon^n)$
- Such Macbeath region must be close to a portion of K's boundary with high curvature
- By convexity, K's boundary curvature is bounded
- Therefore, the number of such Macbeath regions is $2^{O(n)}$

- How to make this formal for all small regions?



Polar Body



- q: point
- Polar hyperplane $q^* = \{p : p \cdot q = 1\}$

- K: convex body
- Polar convex body $K^* = \{p : p \cdot q \leq 1 \text{ for all } q \in K\}$
- High curvature maps to low curvature

- Mahler volume $\text{vol}(K) \cdot \text{vol}(K^*) \geq 2^{-O(n)} \cdot \omega_n^2$
- If the origin is well-centered:
  $\text{vol}(K) \cdot \text{vol}(K^*) \leq 2^{O(n)} \cdot \omega_n^2$
- ω_n: volume of the n-dimensional unit Euclidean ball

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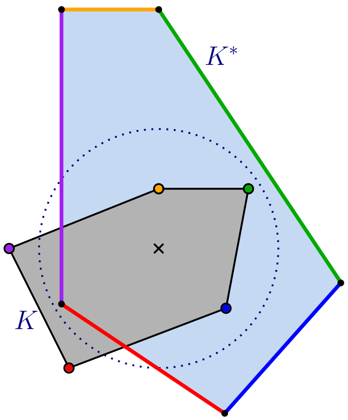

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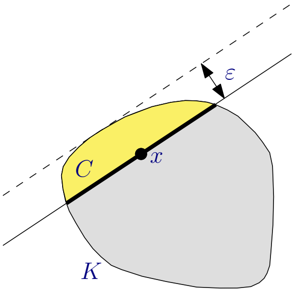
Cap

- Cap:
  intersection of K and a halfspace
- Base of a cap:
  intersection of K and a hyperplane
- Width of a cap:
  maximum orthogonal distance from the base (often ε)
- x: centroid of the base of a cap C
- Cap and Macbeath region have similar volumes:
  $2^{-O(n)} \cdot \mathrm{vol}(C) \ \le \ \mathrm{vol}(M(x)) \ \le \ 2 \cdot \mathrm{vol}(C)$



Caps in the Primal and Polar

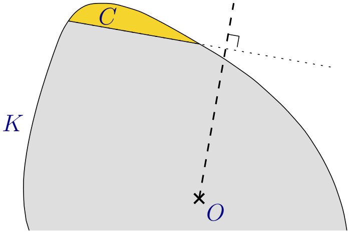
Key Lemma:

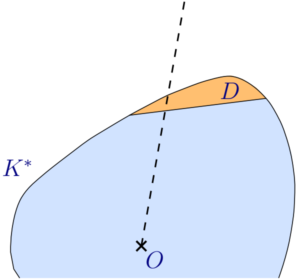
For a cap C of K and a related cap D of the polar K^*, both of width at least ε:

$$\mathrm{vol}_K(C) \cdot \mathrm{vol}_{K^*}(D) \ \geq \ 2^{-O(n)} \varepsilon^{n+1}.$$

Relationship: ray from the origin orthogonal to the base of C intersects D.

- We can bound the number of Macbeath regions: small caps in the primal are large in the polar
- How do we prove the lemma?





Dual Cap and Inner Cone

- Dual cap:
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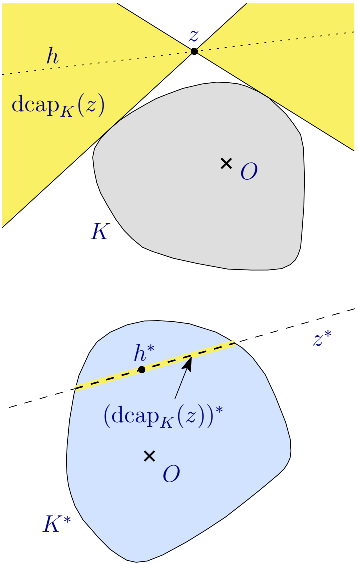
- Inner cone:
  points in all rays from a point z towards a point in K



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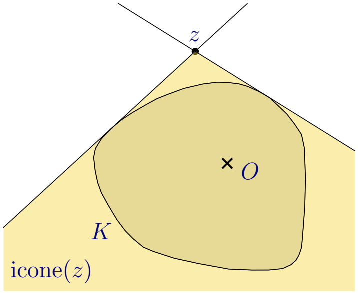
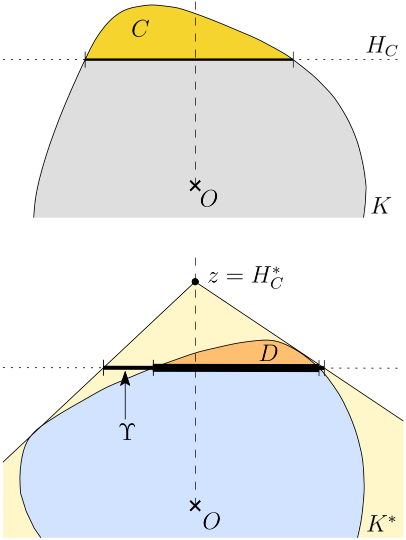


First Attempt to Prove the Key Lemma

- Region Υ: Intersection of the inner cone and the base hyperplane of D
- We show: Υ is the polar of the base of C scaled by $\Theta(\varepsilon)$
- Remember: Mahler volume $\text{vol}(K) \cdot \text{vol}(K^*) = 2^{O(n)}$
- Problem 1: Υ is larger than the base of D
- Easy fix: Scale up D by $O(n)$
- Problem 2: Increases the volume by $n^{O(n)}$



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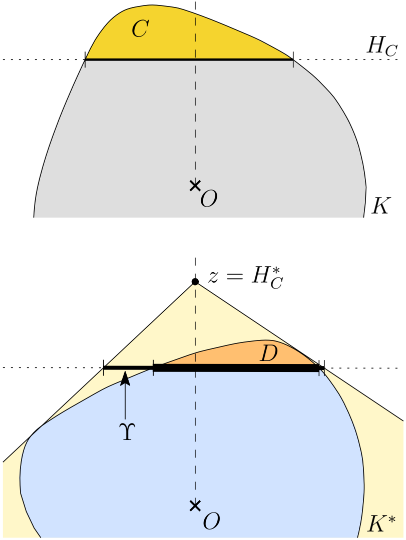

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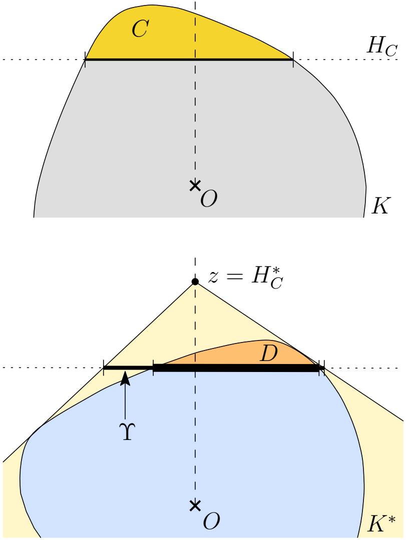
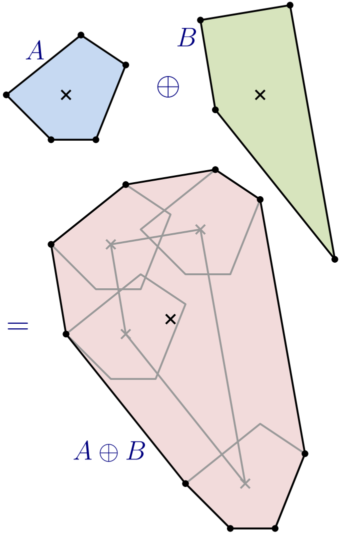


Difference Body

- Minkowski sum:
  $A \oplus B = \{p + q : p \in A,\ q \in B\}$
- Difference body:
  $\Delta(K) = K \oplus (-K)$
- $\text{vol}(\Delta(K)) \leq 4^n \, \text{vol}(K)$ [RS59]
- No $n^{O(n)}$ factor



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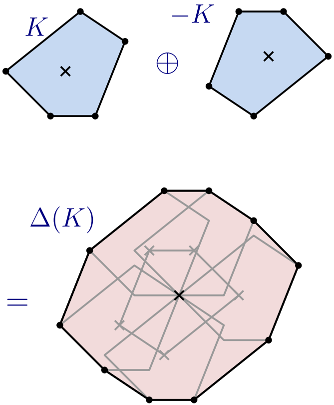
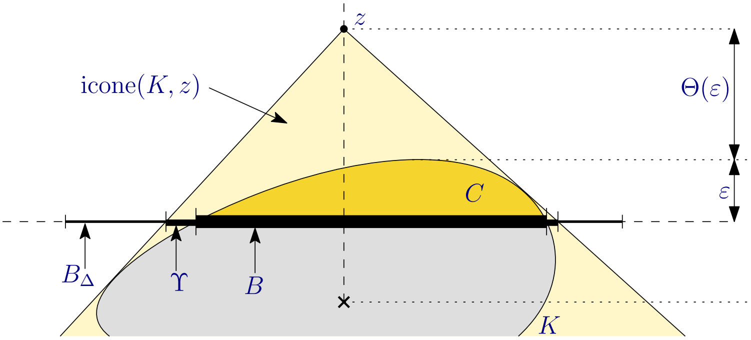


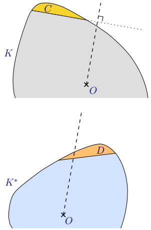
Difference Body and Inner Cone

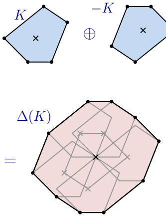
- B_Δ: difference body of the base of a cap scaled by 5 (instead of $O(n)$)
- B_Δ contains Υ

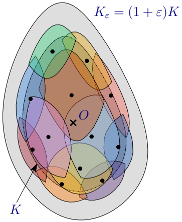


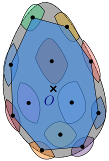
Conclusion

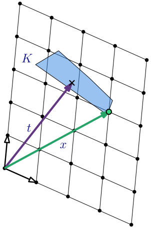
- We show that given a cap C of K there is a cap D of the polar K^* with $\mathrm{vol}(C) \cdot \mathrm{vol}(D) \geq 2^{-O(n)} \varepsilon^{n+1}$
- Key tools: Mahler volume and difference body
- Small caps in the primal take a large volume in the polar
- We get a (c, ε)-covering $\mathcal{Q}$ with $|\mathcal{Q}| = 2^{O(n)} / \varepsilon^{n/2}$
- Implies polytope approximation in the Banach-Mazur metric
- Implies the same running time for ε-approximate CVP and integer programming











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Thank you!