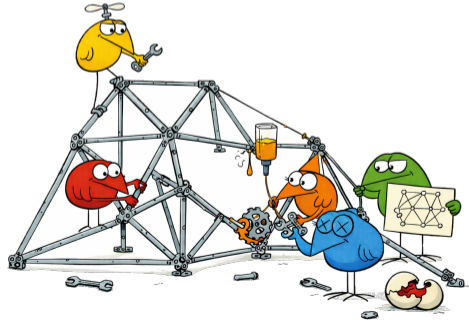


Parallel Reconfiguration of Triangulations

Guilherme D. da Fonseca – LIS, Aix-Marseille Université
Fabien Feschet – LIMOS, Université Clermont Auvergne
Yan Gerard – LIMOS, Université Clermont Auvergne



Reconfiguration Path

Shadoks

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Fewer
Variables

Happy Edges

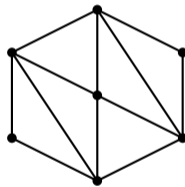
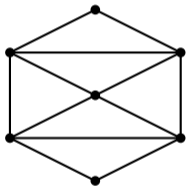
Center

Initial

Improve

Thanks

- ▶ **Challenge problem:** find a **central triangulation** to several triangulations
- ▶ **Unit flip:** remove **one edge** and insert one edge
- ▶ **Parallel flip:** remove k edges and insert k edges in disjoint quadrilaterals
- ▶ **This talk:** find a **short reconfiguration path** connecting two triangulations



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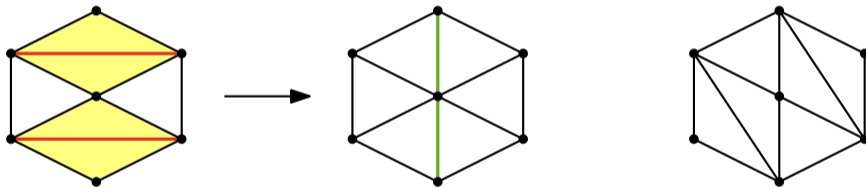
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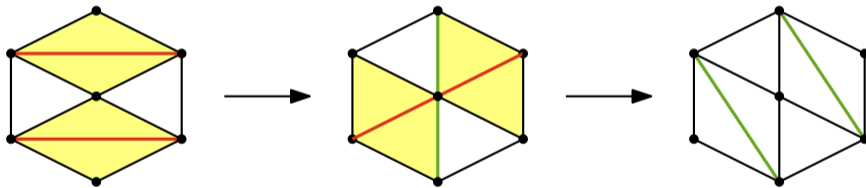
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Greedy Heuristic

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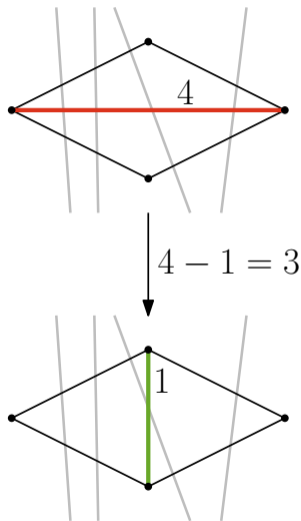
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Thanks

- ▶ **Crossings:** Number of segments of the target triangulation that are crossed
- ▶ Build a **parallel flip** greedily:
Add the unit flip that **decreases the crossings** the most
- ▶ Perform the parallel flip and repeat
- ▶ **Always finds a path**, but may not be the shortest
- ▶ **Fast** and often finds optimal paths
- ▶ Improvements: Run on both directions and use squeaky wheel



Greedy Heuristic

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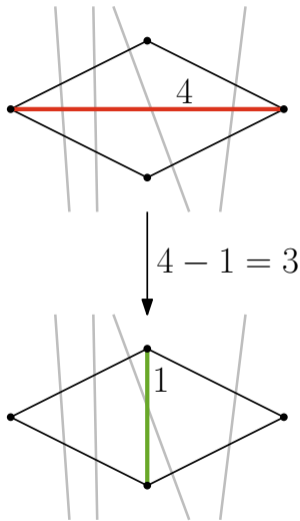
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SAT Formulation

Shadoks

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- ▶ **Edge variable** $e(u, v, i)$: Edge uv exists at time i
- ▶ **Flip variable** $f(u, v, u_2, v_2, i)$: Edge uv is replaced by u_2v_2 at time i

- ▶ Consider n points and a path of length t
- ▶ $O(n^2t)$ edge variables and $O(n^4t)$ flip variables
- ▶ Number of clauses is linear in the number of variables
- ▶ Most clauses are binary:
 - $f(u, v, u_2, v_2, i) \implies e(u, v, i)$
 - $f(u, v, u_2, v_2, i) \implies \neg e(u, v, i + 1)$
 - ...

SAT Formulation

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Eliminating Flip Variables

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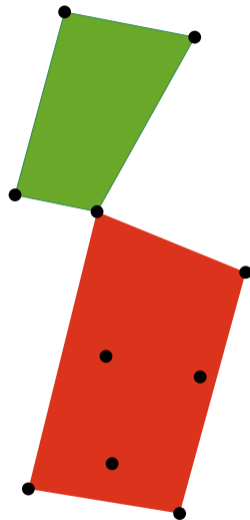
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Thanks

- ▶ A flip can only be performed on an **empty convex quadrilateral**
- ▶ For n random points, there are $O(n^2)$ such quadrilaterals
- ▶ For convex position, all $\Theta(n^4)$ quadrilaterals are convex and empty, but the challenge instances are not in convex position



Eliminating Edge Variables

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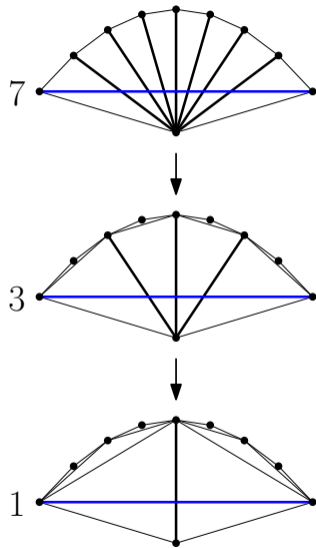
Center

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Thanks

- ▶ $\chi(T, s)$: number of edges of triangulation T crossed by a segment s
- ▶ Consider two triangulations T, T' related by a parallel flip
- ▶ Easy to show that $\chi(T', s) \geq \left\lfloor \frac{\chi(T, s)}{2} \right\rfloor$
- ▶ Hence, edge s cannot appear in the first $1 + \lfloor \log_2(\chi(T, s)) \rfloor$ triangulations
- ▶ Eliminate edge variables that **cross too many edges** of the source or target triangulations
- ▶ **Open problem:** better bounds based on the sequence of triangles crossed by s



Eliminating Edge Variables

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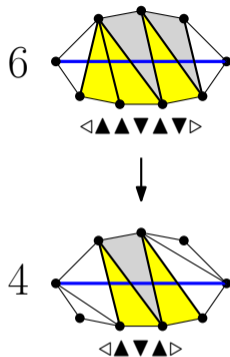
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Happy Edges Conjecture

Shadoks

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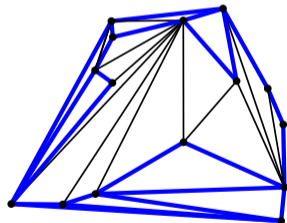
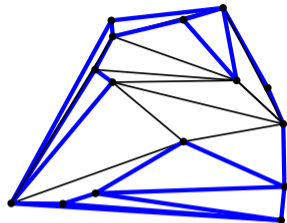
Center

Initial

Improve

Thanks

- ▶ Many other variables may be eliminated if we do not require exact solutions
- ▶ **Happy edge**: Edge common to the source and target triangulations
- ▶ Conjecture: There is a shortest path that never removes happy edges
- ▶ The conjecture is **false**, but...
- ▶ Among 6000 tested pairs of triangulations, only 2 pairs falsify the conjecture: paths that keep the happy edges are 1 unit longer
- ▶ **Eliminate all edge variables that cross the happy edges**



random_instance_111_15_2

Happy Edges Conjecture

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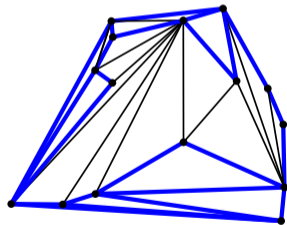
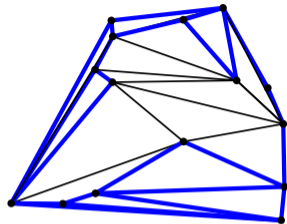
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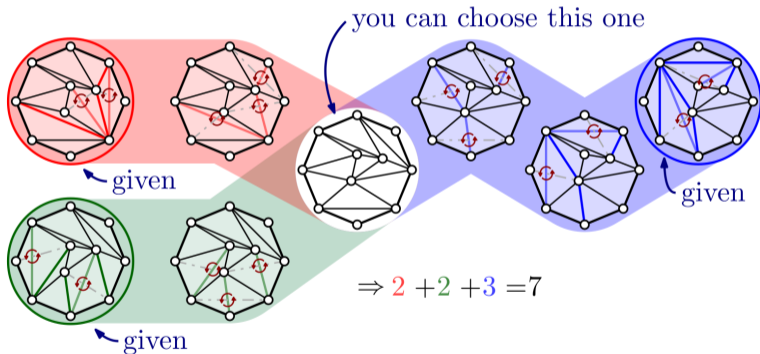


random_instance_111_15_2

Actual Challenge Problem: Find a Good Center

Shadoks

- ▶ We now briefly look at the actual challenge problem
- ▶ The goal is to find a **central triangulation**



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Find an Initial Solution

Shadoks

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Simple but effective heuristic:

- ▶ Initialize a set \mathcal{C} with de Delaunay triangulation
- ▶ While \mathcal{C} is not empty (and time limit not reached):
 - ▶ Extract a center C from \mathcal{C}
 - ▶ Calculate paths from C to the given triangulations
 - ▶ Add to \mathcal{C} the triangulations that are adjacent to C
- ▶ Return the best solution found

Improve a Solution with SAT and MaxSAT

Shadoks

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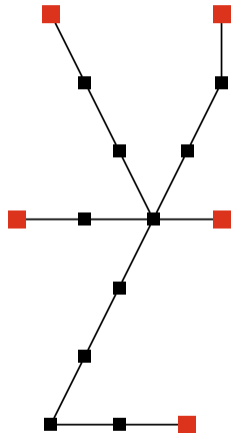
Center

Initial

Improve

Thanks

- ▶ The **SAT** model can be extended to **multiple paths of fixed lengths**
- ▶ Problem may be too large for the SAT solver
- ▶ **MaxSAT** is used to concentrate unit flips far from the center
- ▶ The **central portion** forms a new small instance that we **solve exactly using SAT**



Improve a Solution with SAT and MaxSAT

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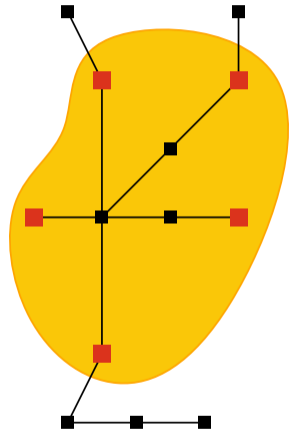
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Thank You!

Shadoks

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Full paper: <https://arxiv.org/abs/2603.21293>