Tiling groups

Some results about tilings of groups

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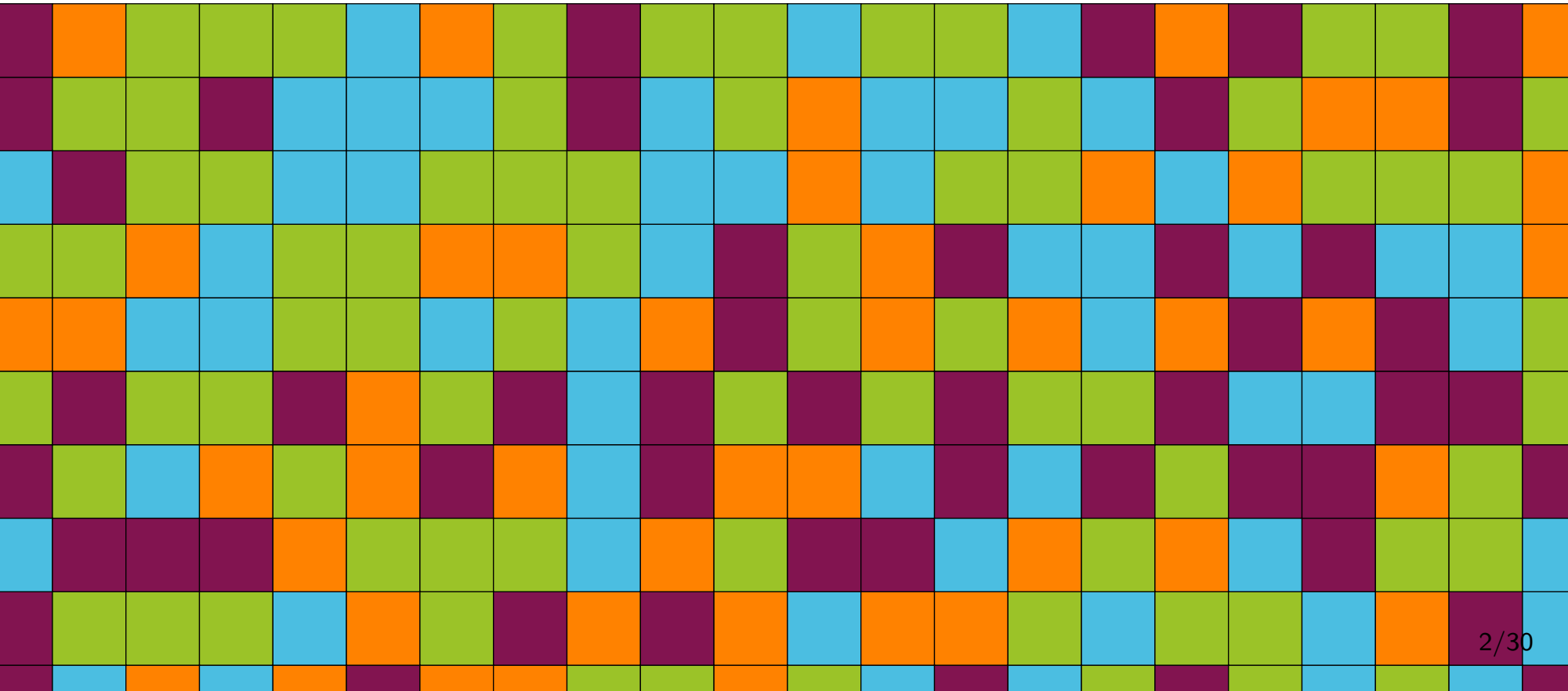
Let's start with Grids

Subshifts of Finite Type – Configurations

Finite alphabet: $\mathcal{A} = \{$  $\}$

Configuration: $x \in \mathcal{A}^{\mathbb{Z}^d}$



Subshifts of Finite Type

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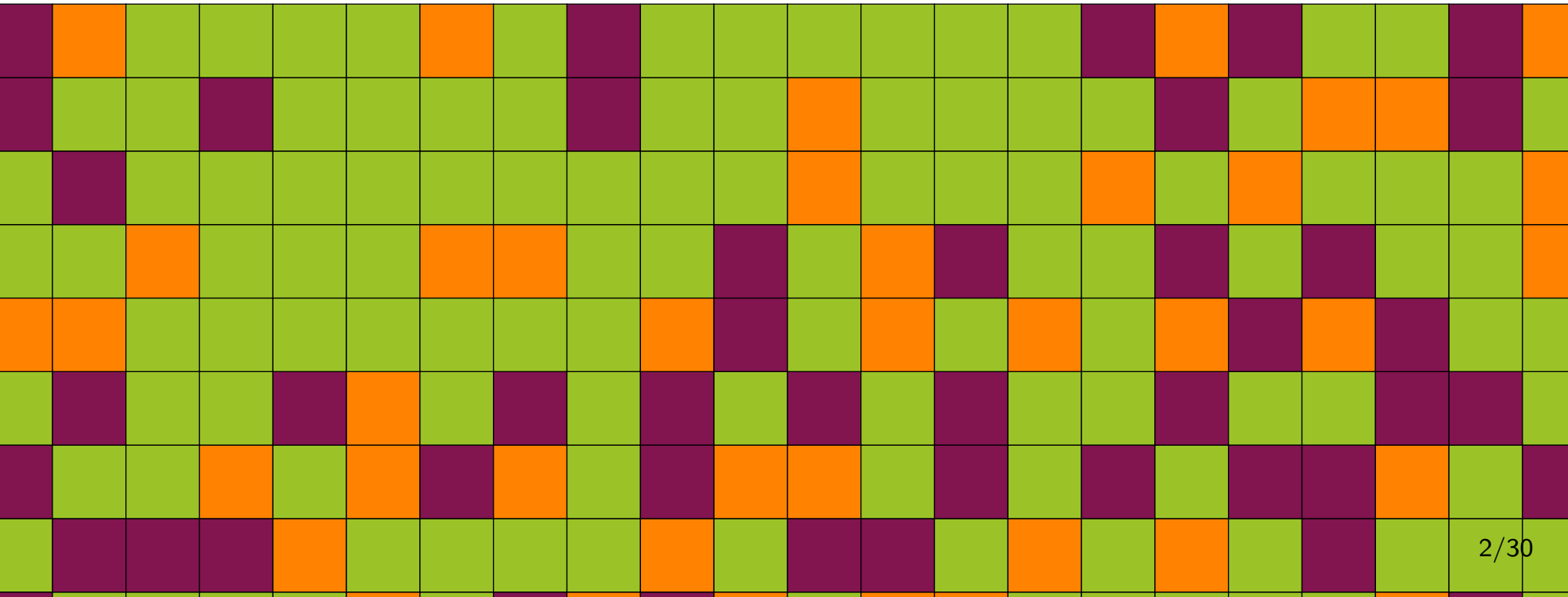

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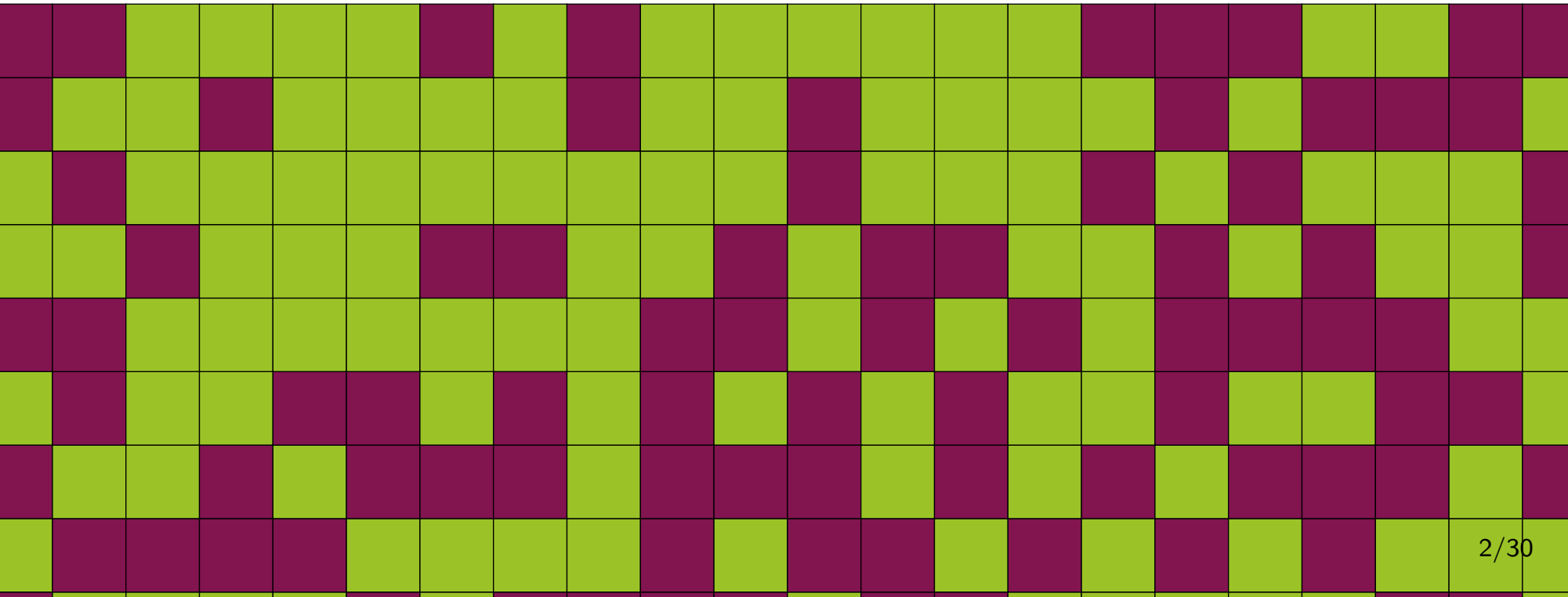

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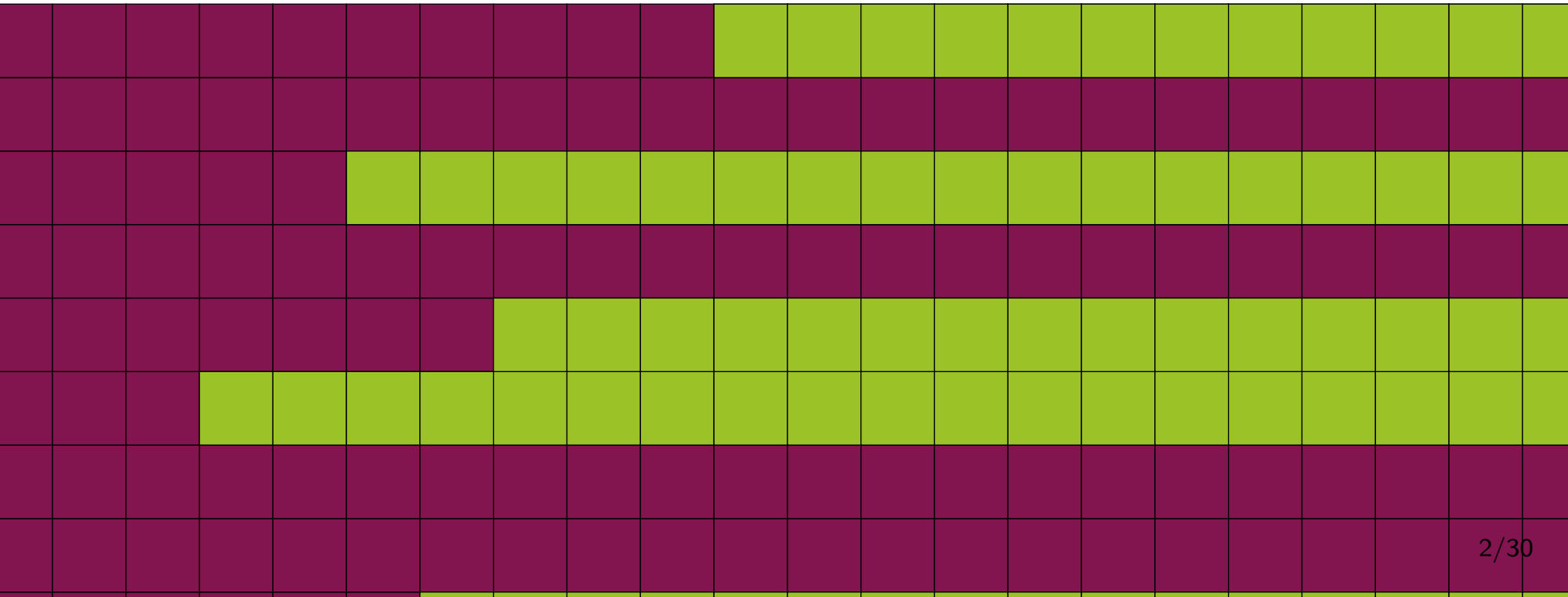

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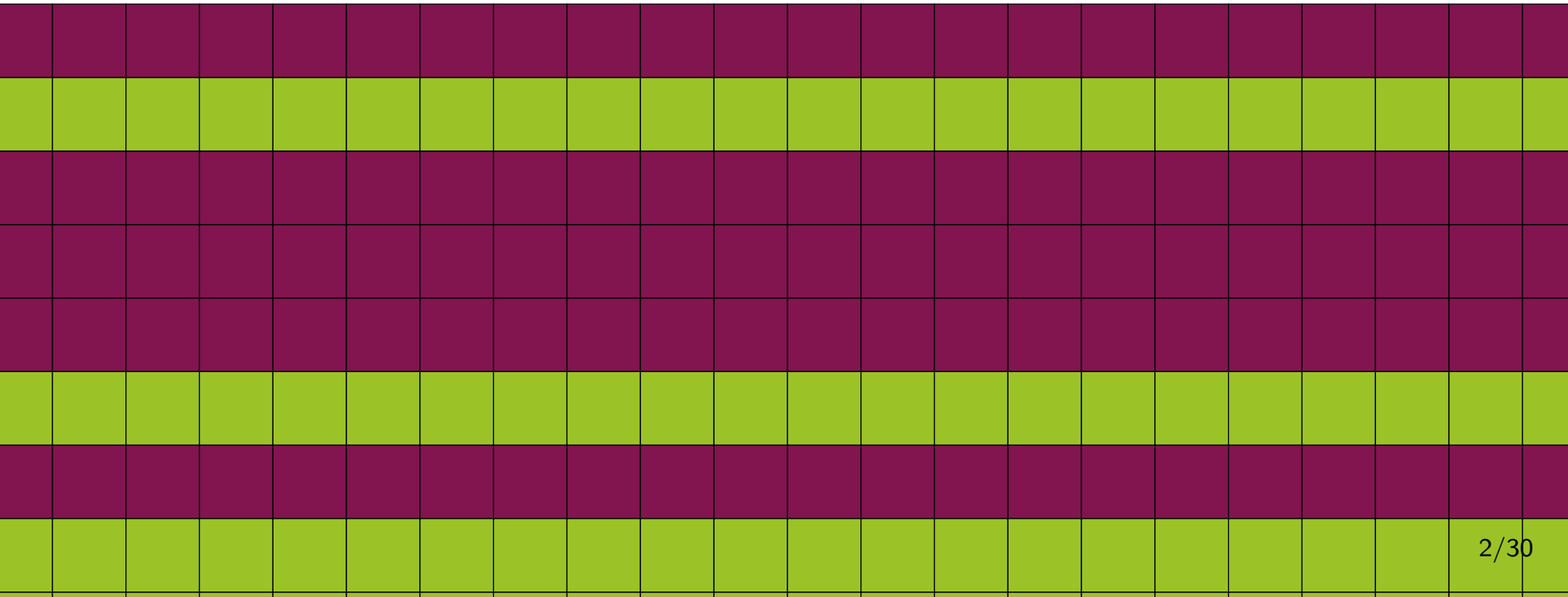

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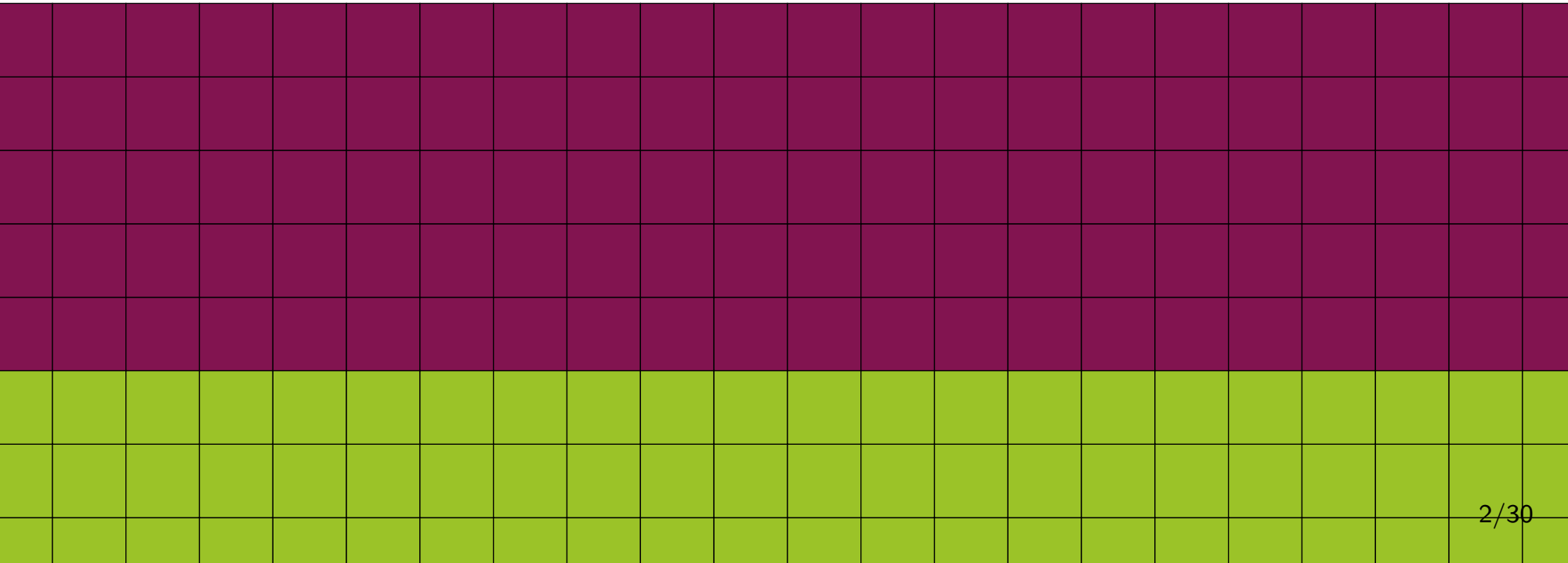

Subshifts of Finite Type

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Subshift of Finite Type (SFT):

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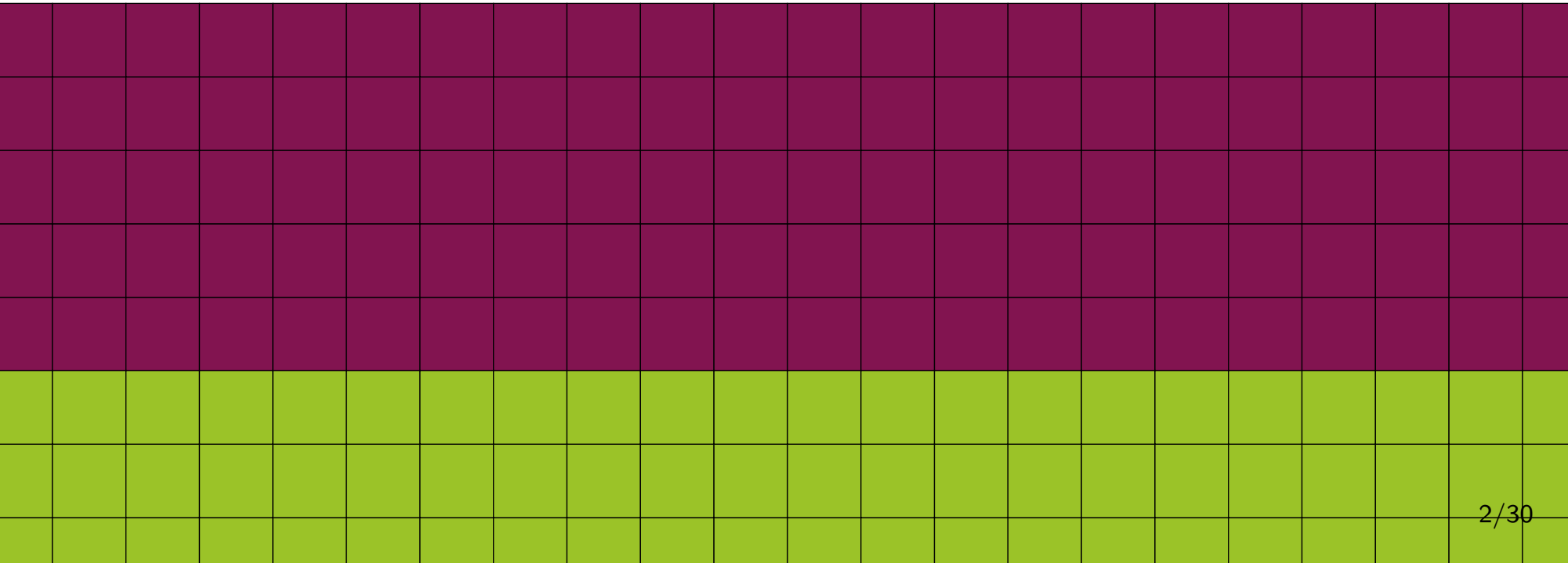

Periodicity (1D)

$x \in \mathcal{A}^{\mathbb{Z}}$ is:

- **periodic** $\exists u, \forall v, \quad x_{v-u} = x_v$

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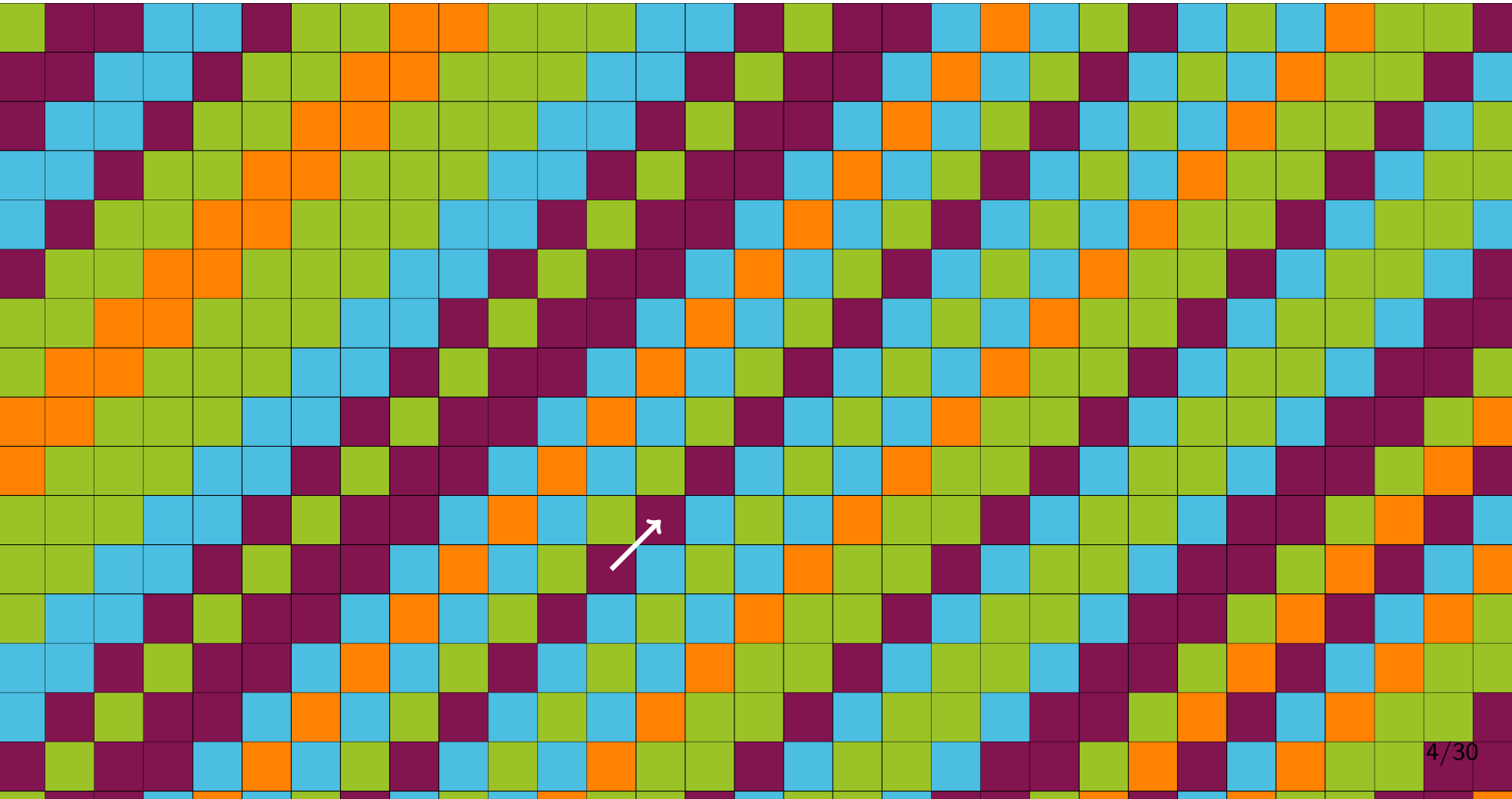
There are no **aperiodic** 1D SFT

Periodic Configuration (2D)

$x \in \mathcal{A}^{\mathbb{Z}^2}$ is:

- **1-periodic / weakly periodic**: $\exists \mathbf{u}, \forall \mathbf{v}, \quad x_{\mathbf{v}-\mathbf{u}} = x_{\mathbf{v}}$



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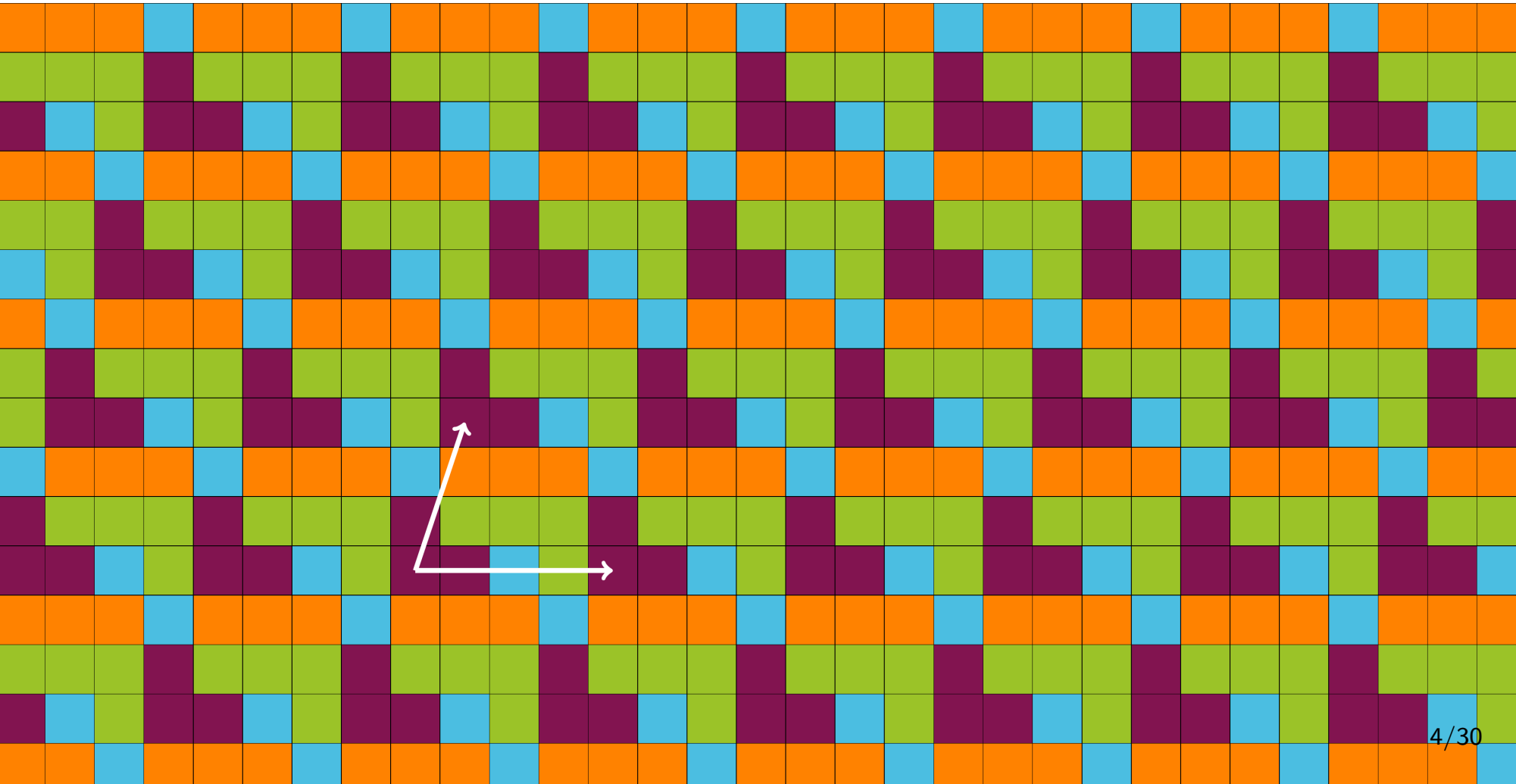
- **2-periodic** / **strongly periodic**: x is 1-periodic along $\mathbf{u}_1, \mathbf{u}_2$ not colinear



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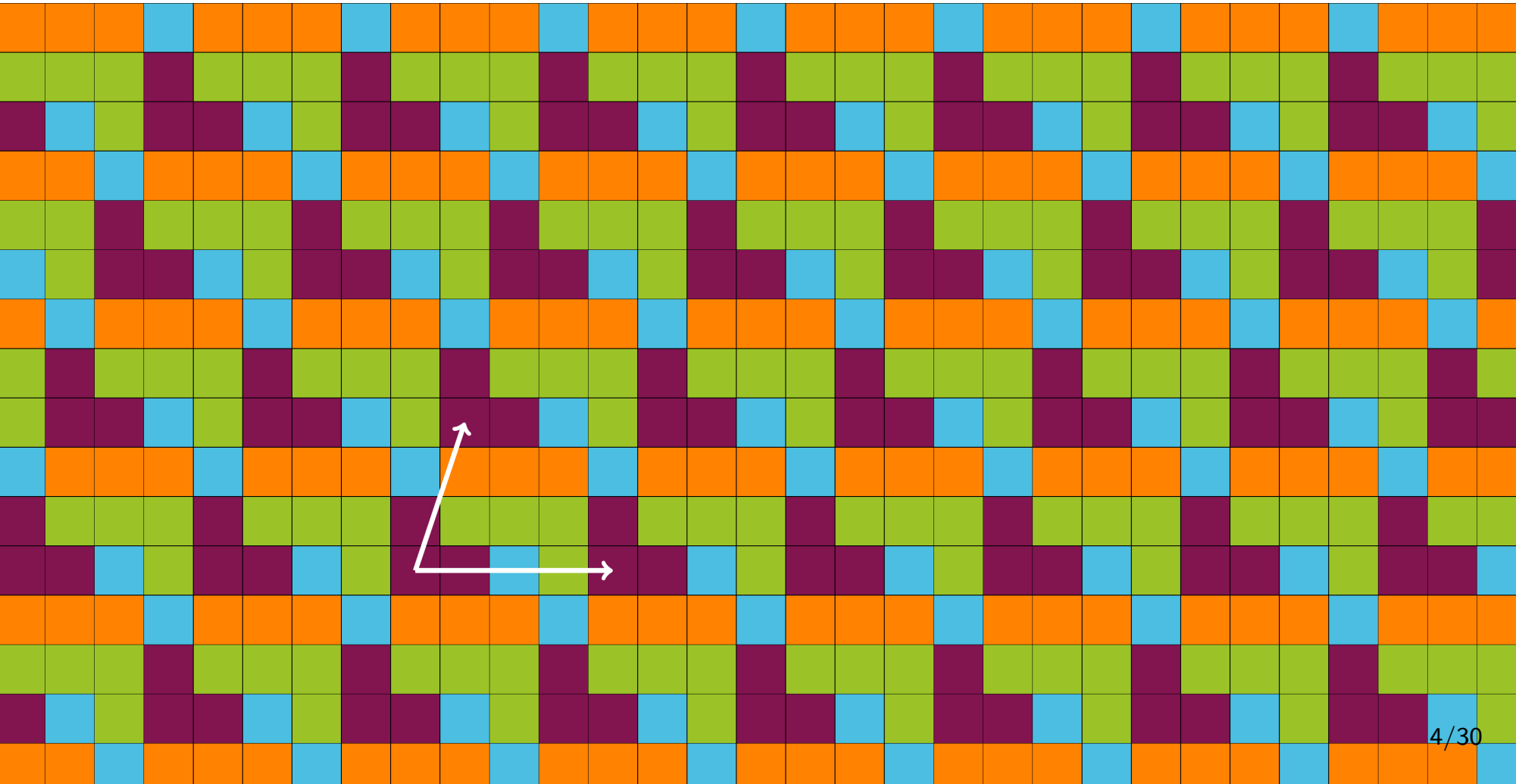
- **2-periodic / strongly periodic**: x is 1-periodic along $\mathbf{u}_1, \mathbf{u}_2$ not colinear $\Rightarrow$ Finitely many different translations of x



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Theorem (Berger, 1966)

In 2D, there are aperiodic SFTs

The Domino Problem

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Domino Problem 1D

In 1D, the domino problem is **decidable**

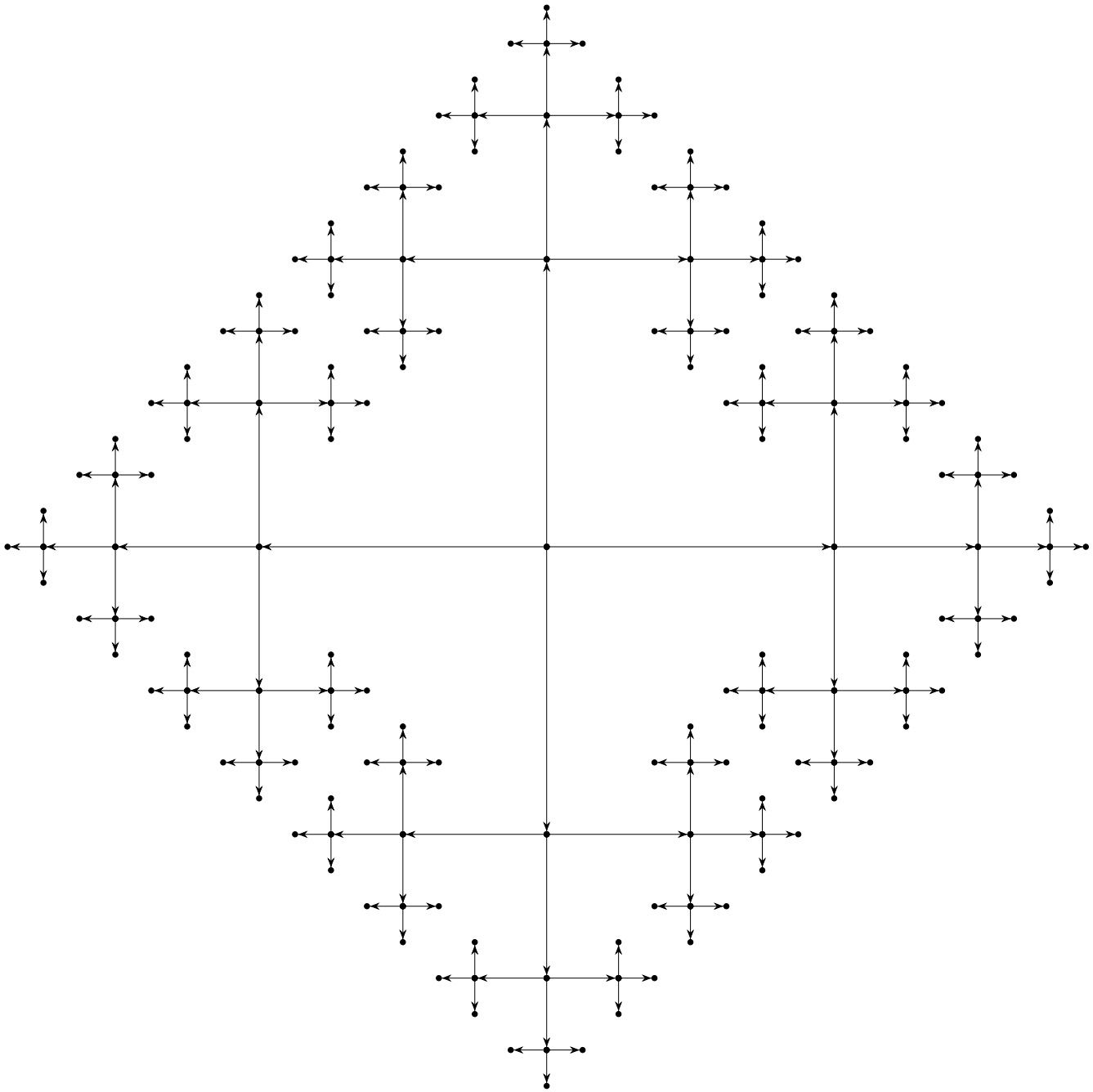
Domino Problem 2D (Berger, 1966)

In 2D, the domino problem is **undecidable**

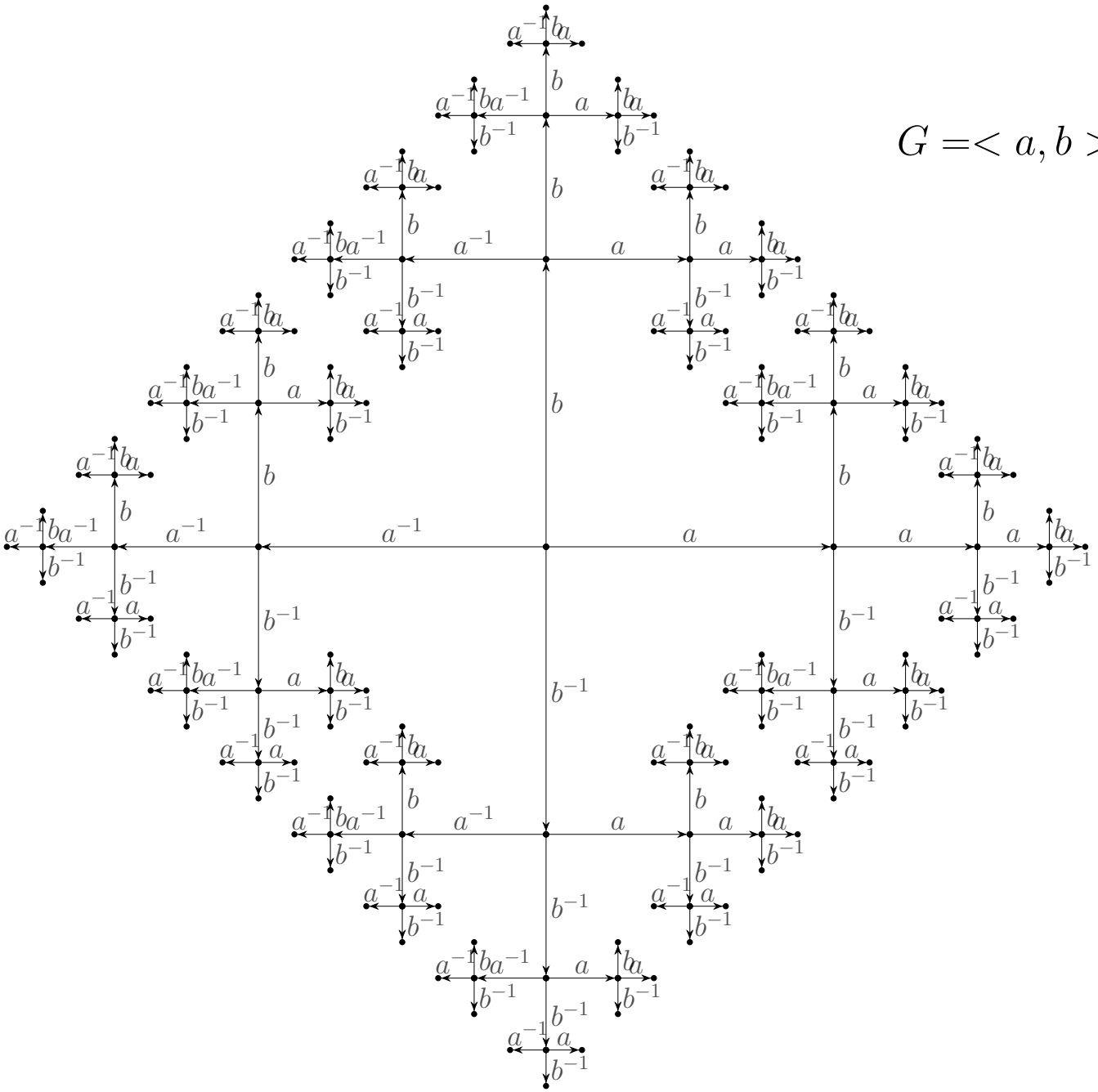
Now the interesting part:

Groups

Infinite Structures



Cayley Graph of a Group



$G =< a, b >$

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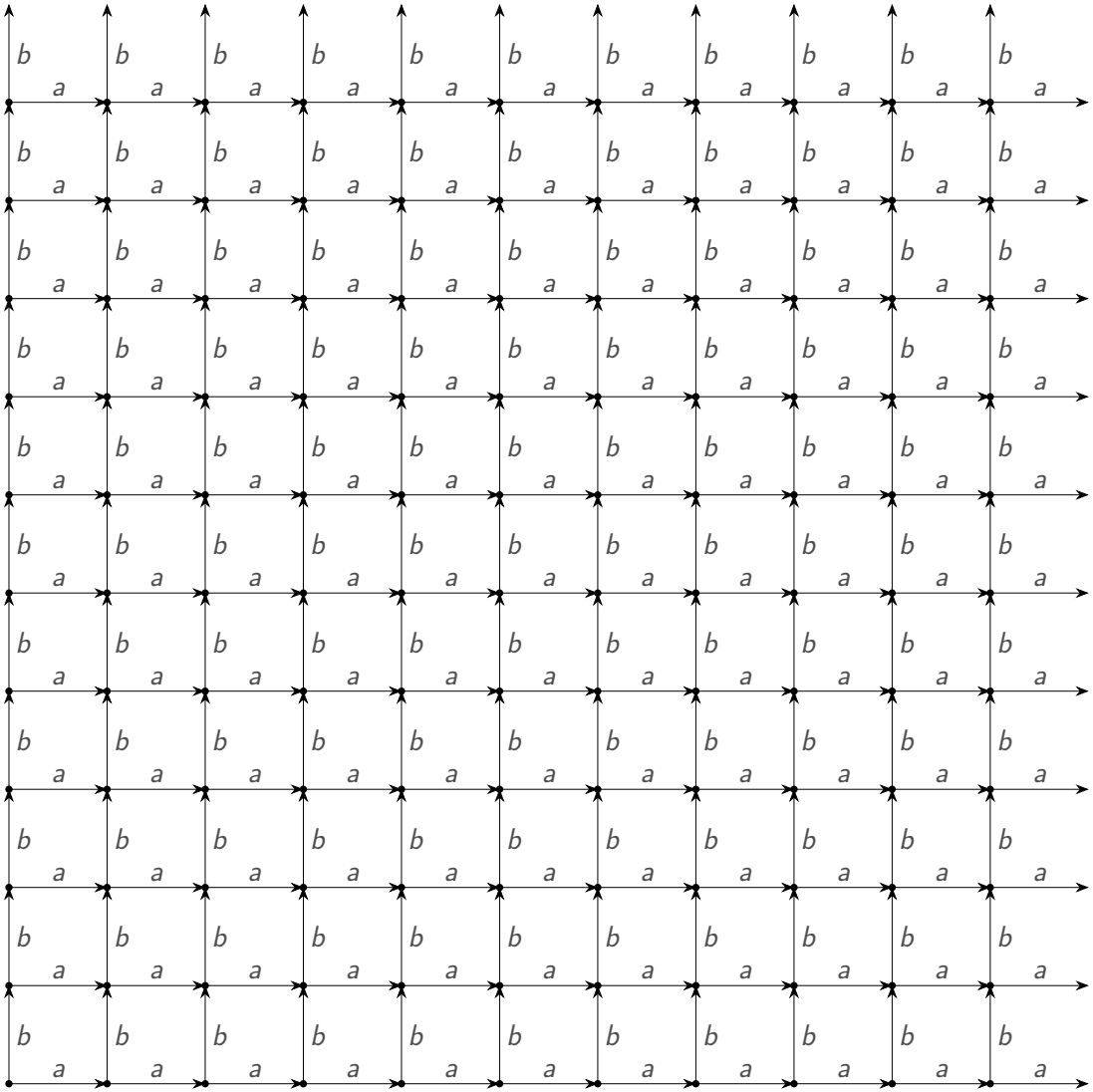
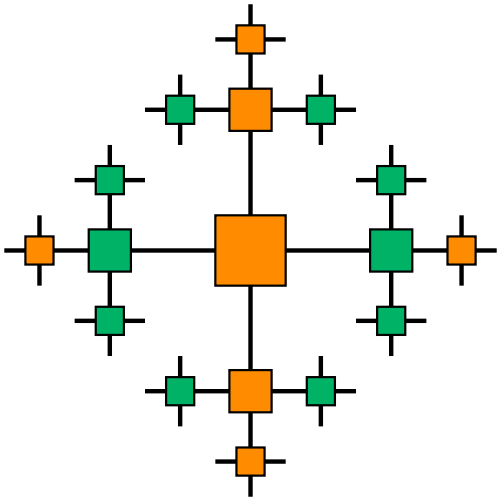


$$G =< a, b \mid aba^{-1}b^{-1} = 1_G >\cong \mathbb{Z}^2$$

Tiling the Cayley Graph of a Group

Subshifts:

- Every vertex has a color
- Forbid a set of patterns to appear in the colorings



Periodicity, Aperiodicity

- Orbit: $\mathcal{O}(x) = \{g \cdot x \mid g \in G\} \subseteq \mathcal{A}^G$

A configuration $x \in \mathcal{A}^G$ is:

- **weakly periodic**: $\exists g \in G, g \cdot x = x$
- **strongly periodic**: $|\mathcal{O}(x)| < \infty$

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Domino Problem for Groups

Domino Problem (DP)

Given a finite alphabet $\mathcal{A}$ and a finte set of forbidden patterns F, is $X_F \subseteq \mathcal{A}^G$ non-empty?

Questions

Take G your favorite (infinite) group

Periodicity:

- Is there a weakly aperiodic SFT ?
- Is there a strongly aperiodic SFT ?
- Are the two definitions equivalent ?

Domino Problem:

- Is it decidable ?

Many Questions

More generally, caractarize groups G for which

Periodicity:

- There is a weakly aperiodic SFT
- There is a strongly aperiodic SFT
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Domino Problem:

- Is (un)decidable

So Many Questions

- What is the link between weak aperiodicity and (un)decidability of DP ?
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Usually, we find an aperiodic SFT before knowing that DP is undecidable.

Sometimes weak, sometimes strong.

(un)Decidability of the domino problem

DP undecidable

- Groups with undecidable Word Problem

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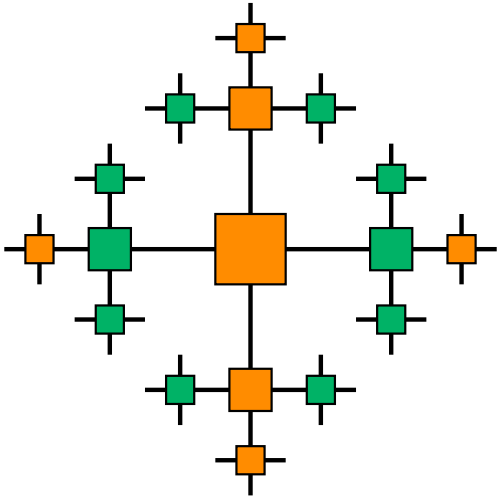
Origin-constrained DP:

- Lamplighter group [Bartholdi, Salo 2020]

DP decidable

- Virtually free groups



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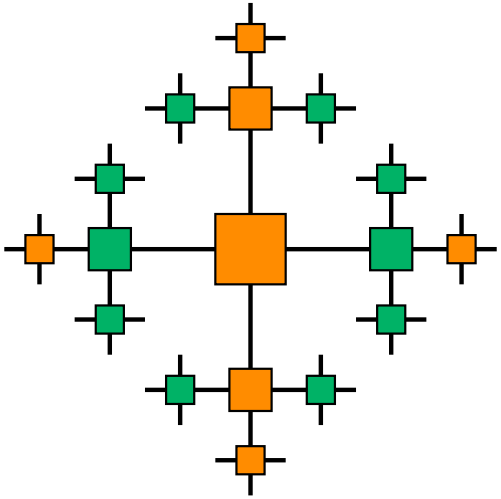

Conjecture [Ballier, Stein 2013]

G has decidable domino problem $\Leftrightarrow$ G is virtually free

Inheritance properties

(Everything is finitely generated)

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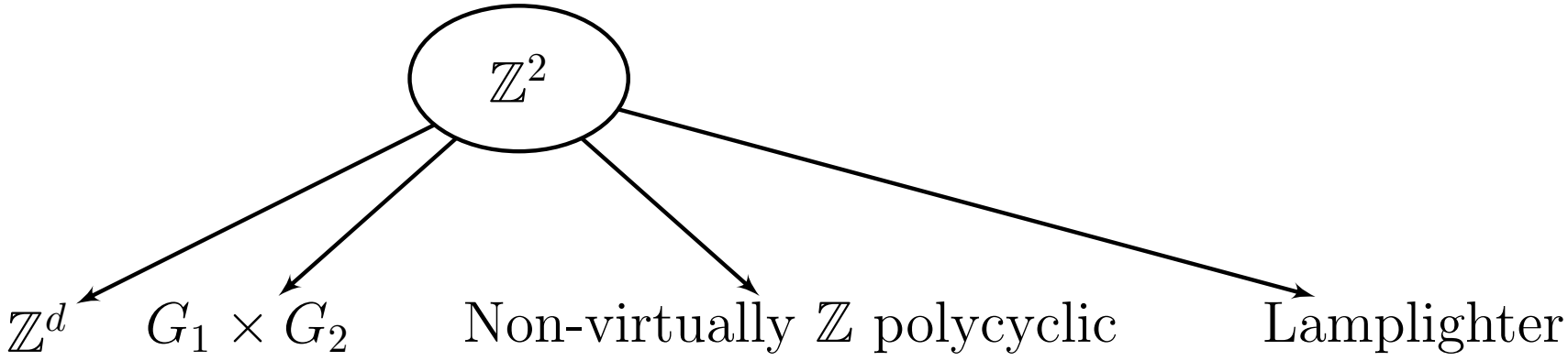
Quotient [Aubrun, Barbieri, Jeandel 2018]

N a normal subgroup of G.

G/N has undecidable DP $\Rightarrow$ G has undecidable DP

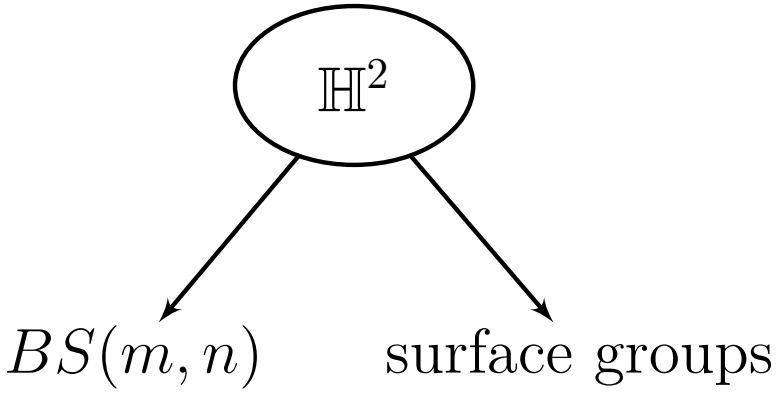
"Finding" $\mathbb{Z}^2$

Most techniques rely on "finding" $\mathbb{Z}^2$ in the group



"Finding" $\mathbb{H}^2$

For $\mathbb{H}^2$, Kari does not encode directly a Turing machine
$\rightarrow$ encodes computation in a hyperbolic space



(a)Periodicity

Where it is impossible to make aperiodic SFTs

No weakly nor strongly aperiodic SFT:

- virtually $\mathbb{Z}$

Conjecture [Carroll, Penland 2015]

A finitely generated group has no weakly aperiodic SFT

$$\Leftrightarrow$$

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No strongly but $\exists$ weakly aperiodic SFTs:

- Other virtually free groups [Piantadosi 2008]

Groups with weakly aperiodic SFTs

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We don't know if they can have **strongly** aperiodic SFTs !

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Theorem (Jeandel 2015)

If G have a strongly aperiodic SFT, then its Word Problem is decidable.

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- $BS(1, n)$, $BS(n, n)$ [Esnay, M. 2020]

Inheritance properties (1)

(Everything is finitely generated)

Subgroup [Jeandel 2015]

If H is a subgroup of G and H has a weakly aperiodic SFT, G has a weakly aperiodic SFT

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Inheritance properties (2)

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Quotient, weak [Caroll, Penland 2015]

N a normal subgroup of G.
G/N has weakly aperiodic SFT $\Rightarrow$ G has weakly aperiodic SFT

Quotient, strong [Jeandel 2015]

N a normal subgroup of G.

N and G/N have a strongly aperiodic SFT

$$\Rightarrow$$

G has a strongly aperiodic SFT

Inheritance properties (3)

(Everything is finitely generated)

Direct product [Jeandel 2015]

If G_1 and G_2 have strongly aperiodic SFT,
$G_1 \times G_2$ have strongly aperiodic SFT

Mixing the two !

A nice table to finish

	(must have one end) [Coh17]		
Aperiocity / DP	$\exists$ Strongly aperiodic SFT $\exists$ Weakly aperiodic SFT	$\nexists$ Strongly aperiodic SFT $\exists$ Weakly aperiodic SFT	$\nexists$ Strongly aperiodic SFT $\nexists$ Weakly aperiodic SFT
Decidable DP Decidable WP	?	Virtually free groups [Pia08]	Virtually $\mathbb{Z}$ groups
Undecidable DP Decidable WP	(virtually) $\mathbb{Z}^d, d \geq 2$ $\mathbb{H}^2$ [GS05][Kar08] $\mathbb{Z}^2 \rtimes H$ [BS19] $G_1 \times G_2 \times G_3$ [Jea15][Bar19] surface groups [CG17][ABM19] $BS(1,n)$ [AK13][EM20] $BS(n,n)$ [AK13][EM20]	$BS(m,n)$? Lamplighter Group ?	
Undecidable DP Undecidable WP	[Jea15]	?	?

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- DP undecidable for hyperbolic groups
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- Lamplighter do not have strongly aperiodic SFTs

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- Lamplighter do not have strongly aperiodic SFTs
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Next steps ?

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Thank you !

Especially Julien Esnay for the initial table and the nice
Master's thesis with many results already gathered