



$$\Sigma : \begin{cases} \frac{dx}{dt} &= f(x, u(t), \varphi \circ \pi(x(t))) \\ y &= h(x, u(t), \varphi \circ \pi(x(t))) \end{cases}$$

where $\varphi \circ \pi : X \rightarrow Z \rightarrow I \subset \mathbf{R}$
 $x \rightarrow z = \pi(x) \rightarrow \varphi(\pi(x))$

and $P_{\Sigma} : X \times L^{\infty}[U] \times L^{\infty}[I] \rightarrow L^{\infty}[\mathbf{R}^{d_y}]$
 $(x_0, u(\cdot), \hat{\varphi}(\cdot)) \rightarrow y(\cdot)$

φ is an unknown function of $\pi(x)$

$\hat{\varphi}$ is a function of time

P_{Σ} is the input/output function of Σ

Definition 1: Σ is identifiable at

$$(u(\cdot), y(\cdot)) \in L^\infty[U] \times L^\infty[\mathbf{R}^{d_y}]$$

if there is at most a single couple

$$(x_0, \hat{\varphi}) \in X \times L^\infty(I)$$

such that for almost all t

$$P_\Sigma(x_0, u, \hat{\varphi})(t) = y(t)$$

and $\hat{\varphi}(t) = \varphi \circ \pi(x(t))$ for some smooth function $\varphi : Z \rightarrow I$.

Σ is identifiable if it is identifiable at any admissible $(u(\cdot), y(\cdot))$.

Definition 2:

$$T\Sigma : \begin{cases} \frac{d\xi}{dt} &= T_{x,\varphi}f(x, u, \varphi; \xi, \eta) \\ \hat{y} &= d_{x,\varphi}h(x, u, \varphi; \xi, \eta) \end{cases}$$

where $(\xi, \eta) \in T_x X \times T_\varphi I$, we set

$$\begin{aligned} P_{T\Sigma}^t(\xi_0, \eta) &= d_{x,\varphi}h(x, u, \hat{\varphi}; T_{x,\varphi}\phi_t(x, u, \hat{\varphi}; \xi_0, \eta), \eta) \\ &= T_{x,\varphi}P_\Sigma^t(\xi_0, \eta) \end{aligned}$$

Σ is infinitesimally identifiable at $(x_0, u, \hat{\varphi}) \in X \times L^\infty[U] \times L^\infty[I]$ if $P_{T\Sigma}^t$ is injective $\forall t > 0$
 Σ is uniformly infinitesimally identifiable if this is true at all $(x_0, u, \hat{\varphi})$

Injectivity depends on the domain for $\hat{\varphi}$.

But for C^ω -systems:

Theorems

Σ is (infinitesimally) identifiable in the class
of **analytic functions**



Σ is (infinitesimally) identifiable in the class
of L^∞ **functions**

We prove that if Σ is not identifiable because of a pair $(x_0, \hat{\varphi}) \in X \times L^\infty$ then there exists $(\tilde{x}_0, \tilde{\varphi}) \in X \times C^\omega$ s.t. Σ is not identifiable.

The main tool is a modification of a lemma from **Gauthier–Kupka**, *Deterministic Observation Theory and Applications*, *Cambridge University Press*, 2001

Let $D_k \Phi = X \times (I \times \mathbf{R}^{k-1})$ be the space of k -jets of the system Σ , we set^(*)

$$\begin{aligned} \Phi_k^\Sigma : \quad D_k \Phi &\rightarrow \mathbf{R}^{kd_y} \\ (x_0, j^k(\hat{\varphi})) &\rightarrow j^k(y) \end{aligned}$$

Definition: Σ is differentially identifiable of order k if

$$\Phi_k^\Sigma(z_1) = \Phi_k^\Sigma(z_2) \Rightarrow (x_1, \hat{\varphi}_1(0)) = (x_2, \hat{\varphi}_2(0))$$

^(*) $j^k(u) = (u(0), u'(0), \dots, u^{(k-1)}(0))$

Proposition: Differential Identifiability \Rightarrow Identifiability

Theorem:

- If $d_y \geq 3$, differential identifiability of order $2n + 1$ is a **generic property** in the class of C^∞ systems.
- If $d_y < 3$, differential identifiability **is not** a generic property.

Proof of genericity 1/2

$$Z_i = (x_i, \varphi_i, \varphi'_i, \dots, \varphi_i^k, j_{\Sigma}^k(x_i, \varphi_i)), \quad i = 1, 2$$

$$Z = (Z_1, Z_2)$$

$$\Phi(Z) = \Phi_{\Sigma}^k(Z_1) - \Phi_{\Sigma}^k(Z_2) \in R^{k d_y},$$

$$k = 2n + 1, \quad d_y \geq 3$$

Let us suppose that Φ is a submersion

$$\text{codim} \Phi^{-1}(0) = k d_y$$

$$\text{Let } \Pi \Phi^{-1}(0) = (x_i, \varphi_i, j_{\Sigma}^k(x_i, \varphi_i))_{i=1,2}$$

$$\begin{aligned} \text{codim} \Pi \Phi^{-1}(0) &\geq k d_y - 2(k - 1) = k(d_y - 2) + 2 \\ &\geq k + 2 \geq 2n + 3 \end{aligned}$$

$$\begin{aligned} \rho_\Sigma : (X \times I)^2 \setminus \Delta &\rightarrow (J_\Sigma^k)^2 \\ (x_1, \varphi_1, x_2, \varphi_2) &\rightarrow (x_i, \varphi_i, j_\Sigma^k(x_i, \varphi_i))_{i=1,2} \end{aligned}$$

Multijet transversality theorem: the set of Σ such that ρ_Σ is transversal to $\Pi\Phi^{-1}(0)$ is residual.

$$\begin{aligned} \dim (X \times I)^2 \setminus \Delta &= 2n + 2 \\ &\Downarrow \\ \text{generically, } \rho_\Sigma &\text{ avoids } \Pi\Phi^{-1}(0) \end{aligned}$$

Theorem 2. If Σ is uniformly infinitesimally identifiable then

$$\text{i) } \frac{\partial}{\partial \varphi} \{h, L_{f_\varphi} h, \dots, (L_{f_\varphi})^{n-1} h\} \equiv 0$$

$$\text{ii) } \frac{\partial}{\partial \varphi} L_{f_\varphi}^n h \neq 0$$

$$\text{iii) } dx h \wedge \dots \wedge dx L_{f_\varphi}^{n-1} h \neq 0,$$

Therefore, locally, the system can be written

$$\left\{ \begin{array}{l} \dot{x}_1 = x_2 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = \psi(x, \varphi) \\ y = x_1 \end{array} \right. \quad \text{and} \quad \frac{\partial}{\partial \varphi} \psi(x, \varphi) \neq 0$$

Proof of the single-output case 1/2

Let $k < n$ be the first k such that $d_\varphi L_f^k h \neq 0$:

$$\Sigma \quad \begin{cases} y & = x_1 \\ \dot{x}_1 & = x_2 \cdots \\ \dot{x}_{k-1} & = x_k \\ \dot{x}_k & = L_f^k(x, \varphi) = f_k(x, \varphi) \cdots \\ \dot{x}_n & = f_n(x, \varphi) \end{cases}$$

$$T\Sigma \quad \begin{cases} \dot{x} & = f(x, \varphi) \\ \hat{y} & = \xi_1 \\ \dot{\xi}_1 & = \xi_2 \cdots \\ \dot{\xi}_{k-1} & = \xi_k \\ \dot{\xi}_k & = d_x f_k(x, \varphi) \xi + d_\varphi f_k(x, \varphi) \eta \end{cases}$$

A feedback $\eta = -\frac{d_x f_k(x, \varphi_0) \xi}{d_\varphi f_k(x, \varphi_0)}$ in φ_0 s.t. $d_\varphi f_k(x, \varphi_0) \neq$

0 gives $\frac{d\xi_k}{dt} = 0$ which contradict observability.

If $\frac{\partial}{\partial \varphi} L_{f_\varphi}^n h = 0$ at (x, φ)

$$X \times I \supset E = \{(x, \varphi); d_\varphi L_f^n h = 0\}$$

$$\downarrow \Pi$$

$$X \supset \Pi E$$

Hardt's theorem $\Rightarrow \exists \hat{\varphi}$

$$\begin{cases} y = x_1, & \dot{x}_1 = x_2, & \dots & \dot{x}_n = \psi(x, \hat{\varphi}(x)) \\ \hat{y} = \xi_1, & \dot{\xi}_1 = \xi_2, & \dots & \dot{\xi}_n = d_x \psi(x, \hat{\varphi}(x)) + 0 \end{cases}$$

Theorem 3. If Σ meets the following conditions,

i) $\frac{\partial}{\partial \varphi} \{h, L_{f_\varphi} h, \dots, (L_{f_\varphi})^{n-1} h\} \equiv 0$

ii) $\frac{\partial}{\partial \varphi} L_{f_\varphi}^n h \neq 0$

iii) $d_x h \wedge \dots \wedge d_x L_{f_\varphi}^{n-1} h \neq 0,$

then Σ is

- 1) locally identifiable,
- 2) loc. unif. infinitesimally identifiable,
- 3) loc. diff. identifiable of order $n + 1$.

Two output case: definitions of k and r

Define $E_l = \{d_x h_i, d_x L_{f_\varphi} h_i, \dots, d_x L_{f_\varphi}^{l-1} h_i, i = 1, 2\}$
 and $N(l) = \text{rank}(E_l)$ at a generic point:

k is defined by

$N(0)$	$N(1)$	\dots	$N(k-1)$	$N(k)$	$N(k+1)$	\dots	$N(k+m)$
0	2		$2k-2$	$2k$	$2k+1$		$2k+m$

$$(2k + m \leq n)$$

The **order** of the system is the first integer r such that $d_\varphi L_{f_\varphi}^r(h_1, h_2) \neq 0$.

Lemma: If Σ is uniformly infinitesimally identifiable then (1) $2k + m = n$

$$(2) \quad r \leq k + m$$

Proof:

$$(1) \quad \varphi = \varphi_0 = \text{cte} \quad \begin{cases} \dot{x} = f(x, \varphi_0) \\ \dot{\xi} = g(x, \xi, \varphi_0) \\ y = h(x, \varphi_0) \end{cases} \quad \begin{array}{l} \text{contradict} \\ \text{observability} \end{array}$$

$$(2) \quad \begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases} \quad \text{contradict identifiability}$$

Définition 5. A system Σ is regular if (1) and (2) holds.

Type 3: $r=k$ and $n=2k$

$$\left\{ \begin{array}{ll} y_1 & = x_1 \\ \dot{x}_1 & = x_3 \\ & \vdots \\ \dot{x}_{n-3} & = x_{n-1} \\ \dot{x}_{n-1} & = f_{n-1}(x, \varphi) \end{array} \quad \begin{array}{ll} y_2 & = x_2 \\ \dot{x}_2 & = x_4 \\ & \vdots \\ \dot{x}_{n-2} & = x_n \\ \dot{x}_n & = f_n(x, \varphi) \end{array} \right.$$

with $\frac{\partial}{\partial \varphi}(f_{n-1}, f_n) \neq 0$

$N(l)$ increases by steps of 2 until the last derivative and apparition of φ .

Type 1: $r > k$

$$\left\{ \begin{array}{ll} y_1 & = x_1 & y_2 & = x_2 \\ \dot{x}_1 & = x_3 & \dot{x}_2 & = x_4 \\ & \vdots & & \vdots \\ \dot{x}_{2k-3} & = x_{2k-1} & \dot{x}_{2k-2} & = x_{2k} \\ \dot{x}_{2k-1} & = f_{2k-1}(x_1, \dots, x_{2k+1}) \\ \dot{x}_{2k} & = x_{2k+1} \\ & \vdots & & \\ \dot{x}_{n-1} & = x_n \\ \dot{x}_n & = f_n(x, \varphi) \end{array} \right.$$

with $\frac{\partial f_n}{\partial \varphi} \neq 0$.

$N(l)$ increases by steps of 1 when φ appears for the first time, \simeq single-output case.

Type 2: $r < k$

$$\begin{array}{llll}
 y_1 & = & x_1 & y_2 & = & x_2 \\
 \dot{x}_1 & = & x_3 & \dot{x}_2 & = & x_4 \\
 & \vdots & & & \vdots & \\
 \dot{x}_{2r-3} & = & x_{2r-1} & \dot{x}_{2r-2} & = & x_{2r} \\
 \dot{x}_{2r-1} & = & \psi(x, \varphi) & \dot{x}_{2r} & = & F_{2r}(x_1, \dots, x_{2r+1}, \psi(x, \varphi)) \\
 & & & \dot{x}_{2r+1} & = & F_{2r+1}(x_1, \dots, x_{2r+2}, \psi(x, \varphi)) \\
 & & & \vdots & & \\
 & & & \dot{x}_{n-1} & = & F_{n-1}(x, \psi(x, \varphi)) \\
 & & & \dot{x}_n & = & F_n(x, \varphi)
 \end{array}$$

with $\frac{\partial \psi}{\partial \varphi} \neq 0, \frac{\partial F_{2r}}{\partial x_{2r+1}} \neq 0, \dots, \frac{\partial F_{n-1}}{\partial x_n} \neq 0$

φ appears when $N(l)$ increases by steps of 2.

If $r = k$ before the last derivative:

$$dxh_1 \wedge \cdots \wedge dxL_{f_\varphi}^{k-1}h_1 \wedge dxL_{f_\varphi}^{k-1}h_2 \wedge dxL_{f_\varphi}^k h_2 \neq 0$$

If $d_\varphi L_{f_\varphi}^k h_1 \neq 0$, we obtain φ using y_1 and x_{2k}, \dots, x_n using y_2

→ Type 2

If $d_\varphi L_{f_\varphi}^k h_1 \equiv 0$, we obtain φ using y_2

→ Type 1

If $d_y \geq 3$ then identifiability is a generic property (cf. theorem 3).

Let us consider for simplicity $d_y = 3$:

$$\left\{ \begin{array}{l} \frac{dx}{dt} = f(x, \varphi(x)) \\ y_1 = h_1(x, \varphi(x)) \\ y_2 = h_2(x, \varphi(x)) \\ y_3 = h_3(x, \varphi(x)) \end{array} \right.$$

$$\Phi_k^\Sigma : (x, \varphi, \dots, \varphi^{(k-1)}) \mapsto (y, \dots, y^{(k-1)})$$

Theorem The set of systems such that, in restriction to $\bar{Z} \times I$, Φ_{2n+1}^Σ is an injective immersion, is residual.

where $\bar{\Sigma} = \Sigma|_{\bar{Z}}$, \bar{Z} a large set.

Theorem: Global normal form ($d_y=3$)

Outside a bad small subset, Σ may be written

$$\begin{aligned}
 y &= z^1 = (z_1^1, z_2^1, z_3^1) \\
 \dot{z}^1 &= z^2 \\
 &\vdots \\
 \dot{z}^{k-1} &= z^k \\
 \dot{z}^k &= F(z^1, \dots, z^k, \varphi^{(k)})
 \end{aligned}$$

Moreover

$$(x, j^k \varphi) = H(z^1, \dots, z^k)$$

and

$$y^{(k)} = G(x, \varphi, \dots, \varphi^{(k)})$$

Theorem: Local normal form ($d_y=3$)

Outside a bad small subset, Σ may be **locally** (around (x, φ)) rewritten

$$\begin{aligned} y &= z^1 = (z_1^1, z_2^1) \\ \dot{z}^1 &= z^2 \\ &\vdots \\ \dot{z}^{k-1} &= z^k \\ \dot{z}^k &= F(z^1, \dots, z^k, y_3, \dots, y_3^{(k)}) \end{aligned}$$

and

$$\begin{aligned} x &= \Phi(z^1, \dots, z^k, y_3, \dots, y_3^{(k-1)}) \\ \varphi &= \Psi(x, y_3) \end{aligned}$$