Time optimal control problem

$$\begin{cases} \dfrac{dx(t)}{dt} & = & f\left(x\left(t\right), u\left(t\right)\right) \\ y(t) & = & h(x(t)) \end{cases}$$

where $u(t) \in [-1, 1]^p$.

Time optimal control: Minimize T such that $x(T) \in N$, N being the target.

Following Hermes ("discontinuous vector fields and feedback control", SIAM J. Control, 1967), we assume the Brunovsky stability.

Brunovski stability

Definition

A synthesis $u(t)$ is a Filippov synthesis iff any solution $x(t)$ of

$$\begin{cases} \frac{dx(t)}{dt} & = & f\left(x\left(t\right), u^{*}\left(x\left(t\right)\right)\right) \\ x\left(0\right) & = & \eta \end{cases}$$

in Caratheodory sense (absolutely continuous for a.e. t) is a Filippov solution for a.e. t i.e.

$$\frac{dx(t)}{dt} \in \bigcap_{\delta>0} \bigcap_{\lambda N=0} \overline{\text{conv}}\, f\left(B\left(x\left(t\right), \delta\right) \setminus N\right)$$

Exemple: chemical reaction

Chemical reaction of type

$$A \to B \to C$$

$$\begin{cases} \frac{d}{dt}[A]_t & = & -k_1(T_t)[A]_t \\ \frac{d}{dt}[B]_t & = & k_1(T_t)[A]_t - k_2(T_t)[B]_t \end{cases}$$

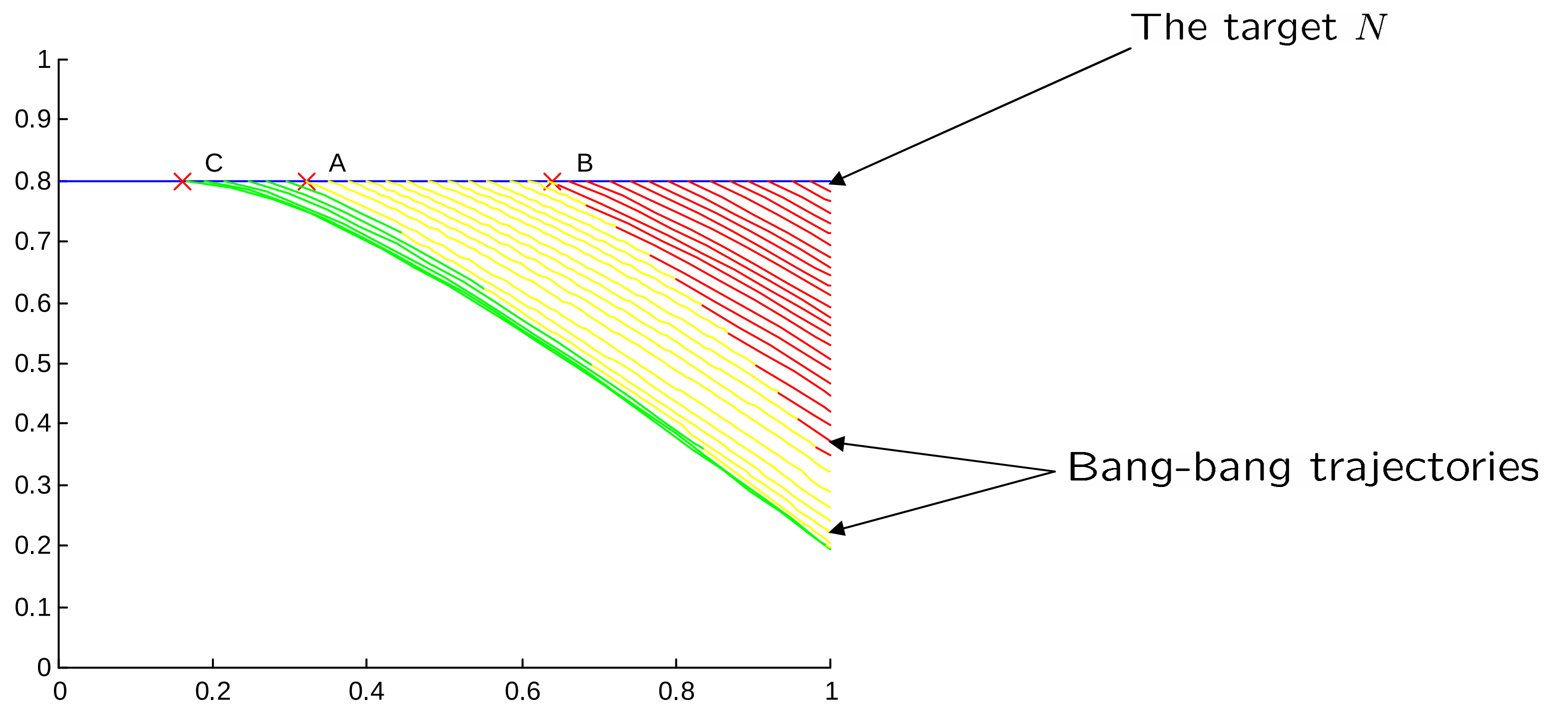
where $k_i(T) = A_i \mathrm{e}^{-\frac{E_i}{RT}}$

Problem: Maximize $[B]$ in minimal time.

Temperature control

The synthesis has only regular trajectories.

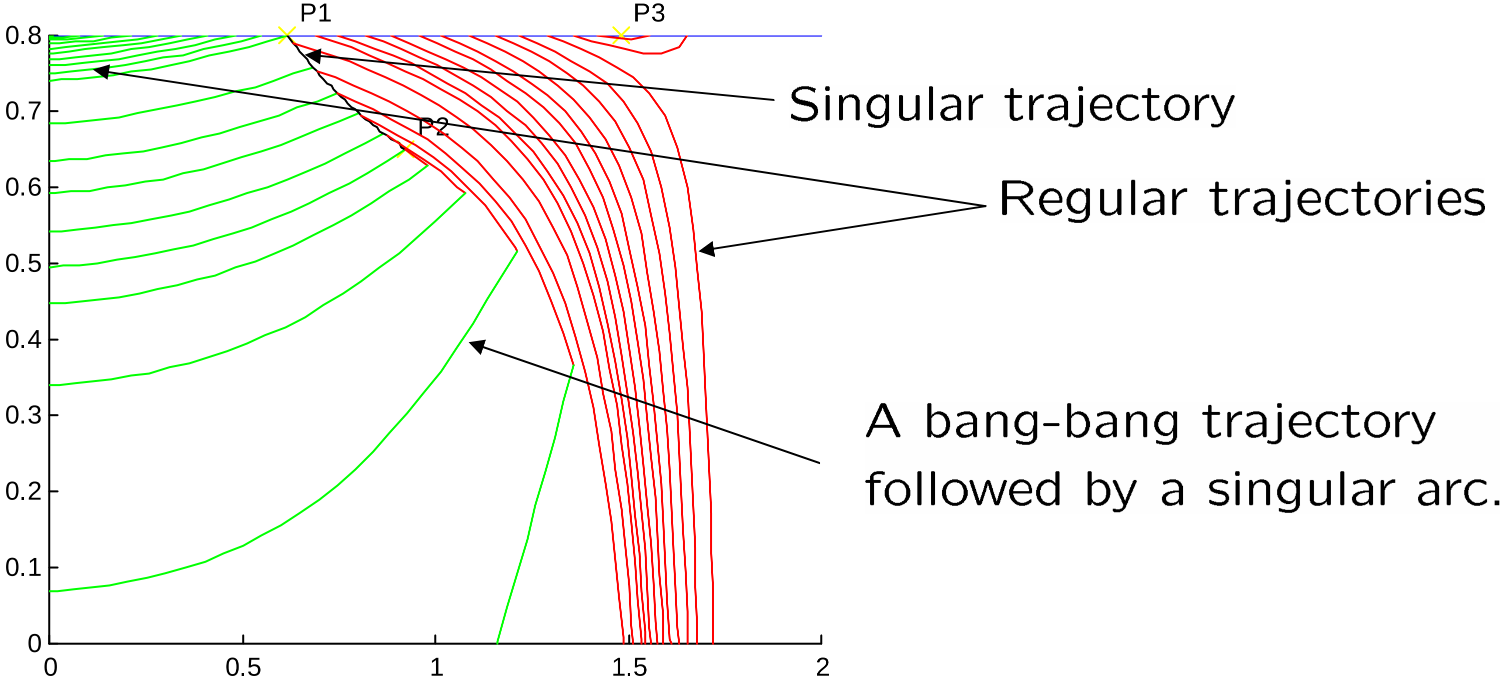
It is obviously a Filippov synthesis.



Heat control

The temperature is a state variable, the control is the heat duty:

$$\frac{dT_t}{dt} = -h(T_t)\,Q_t$$



Regular trajectory I

We should verify

$$f\left(x\left(t\right),u^{*}\left(x\left(t\right)\right)\right)\in\bigcap_{\delta>0}\bigcap_{\lambda N=0}\overline{\mathrm{conv}}\, f\left(B\left(x\left(t\right),\delta\right)\setminus N\right)$$

The main problem concern the Caratheodory solution on the singular arc

$$\gamma_s=\left\{\left(y,v\right);\quad y\left(\alpha\beta v^{\alpha-1}-1\right)=1\right\}$$

with singular control

$$u_s=-\frac{v^2}{h\left(v\right)\alpha y}$$

Regular trajectory II

Let x being a point on γ_s and $\delta > 0$ small enough. γ_x cross $B(x, \delta)$ on two open sets $B(x, \delta) \cap \gamma_+$ et $B(x, \delta) \cap \gamma_-$. Hence, any Filippov solution satisfy

$$\begin{cases} \dot{y} = v - \beta v^{\alpha} y + v y \\ \dot{v} = a h(v) u_+ + (1-a) h(v) u_- = h(v) \left(a u_+ - (1-a) u_- \right) \\ (\dot{y}, \dot{v}) \in T_x \gamma_s \end{cases}$$

The unique solution is

$$(v - \beta v^{\alpha} y + v y) \frac{d}{dy} + h(v) u_s \frac{d}{dv}$$

Closed loop system

$$\begin{cases} \dfrac{dx(t)}{dt} &= f\left(x\left(t\right), u^{*}\left(\widehat{x}\left(t\right)\right)\right) \\ \dfrac{d\widehat{x}(t)}{dt} &= f\left(x\left(t\right), u^{*}\left(\widehat{x}\left(t\right)\right)\right) + K_{\theta}\left(t\right)\left(x\left(t\right) - \widehat{x}\left(t\right)\right) \\ x\left(0\right) &= \eta \\ \widehat{x}\left(0\right) &= \widehat{\eta} \end{cases}$$

Theorem Let u^{*} be a Filippov synthesis. Let $\varepsilon > 0$ and $N_{\varepsilon} = \{x \in M \; ; \; d\left(x, M\right) < \varepsilon\}$.
For θ large enough, the closed loop system reach N_{ε} at time T_{ε} where $|T - T_{\varepsilon}| < \varepsilon$.

Sketch of proof

Proof Set $y(t) = x^*(t) - \widehat{x}(t)$ hence

$$\begin{aligned}
|y| \cdot \frac{d|y|}{dt} &= \frac{1}{2}\frac{d|y|^2}{dt} = y \cdot \frac{d|y|}{dt} = y \cdot \frac{d|x^*|}{dt} - y \cdot \frac{d|\widehat{x}|}{dt} \\
&\leq essmax_{\xi^* \in B(x^*,\delta)} \left(y \cdot f_u\left(\xi^*\right)\right) \\
&\quad - essmin_{\widehat{\xi} \in B(\widehat{x},\delta)} \left(y \cdot \left(f_u\left(\widehat{\xi}\right) + K(t)\left(x - \widehat{\xi}\right)\right)\right) \\
&\leq \cdots
\end{aligned}$$

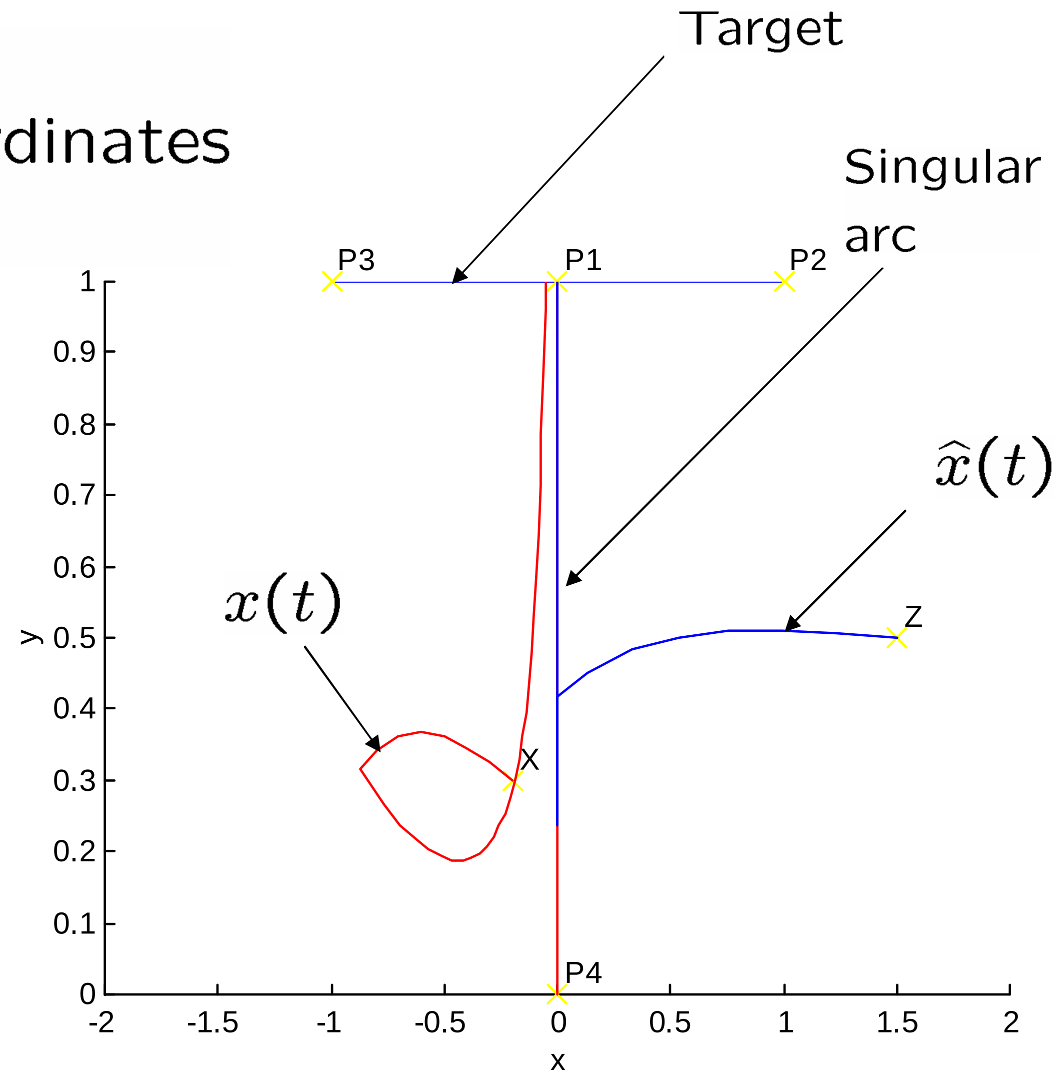
until

$$\frac{d|y|}{dt} \leq M\left(L\left(t, |y|\right)\right) + \|K\|\left(x - \widehat{x}\right)$$

Closed loop behaviour

Closed loop trajectory
up to a change of coordinates
(rectification):

exponential
convergence



« Nonlinear separation principle »

Open loop system

$$\begin{cases} \dfrac{dx}{dt} &= Ax + b(x, u) \\ y &= Cx \end{cases}$$

High-gain observer

$$\frac{dz}{dt} = Az + b(z, u(z)) - S(t)^{-1}C'r^{-1}(Cz - y(t))$$

$$\frac{dS}{dt} = -(A + b^*(z, u(z)))'S - S(A + b^*(z, u(z))) + C'r^{-1}C - SQ_\theta S$$

+

Globally asymptotically stable control u

Main result

= (weakly) g.a.s. closed loop system

$$\frac{dx}{dt} = Ax + b(x, u(z))$$
$$\frac{dz}{dt} = Az + b(z, u(z)) - S(t)^{-1}C'r^{-1}(Cz - y(t))$$
$$\frac{dS}{dt} = -(A + b^*(z, u(z)))'S - S(A + b^*(z, u(z))) + C'r^{-1}C - SQ_\theta S$$

in the sense that bounded trajectories are contained in the attracting set of the unique equilibrium point $(x^*, z^*; S^*)$

Sketch of proof I

First step: On the invariant set

$$N = \{(\varepsilon, z, S)\,;\, \varepsilon = x - z = 0\}$$

the system has a triangular form, hense (Vidyasagar, Sontag) we can deduce the local asymptotic stability on N.

Using a Byrnes–Isidori lemma, we can deduce local asymptotic stability on the whole space (from the observer exponential convergence).

Sketch of proof II

Second step: Any positive trajectory $\Lambda^{+}(x_0, z_0, S_0)$ such that $x(t)$ is bounded, is bounded.

Let Ω be the ω–limit set of $\Lambda^{+}(x_0, z_0, S_0)$ (Lasalle–Lefschetz)

- Ω is not empty
- it is a closed set
- It is a positively invariant set
- $(x^*, z^*; S^*) \in \Omega$

and the first step implies $\Omega = (x^*, z^*; S^*)$

Exemple: a polymerization reactor

CSTR model: six highly nonlinear differential equations describing weight fraction of monomer W_M, of solvent W_S and of initiator W_I, the temperature T and the leading moments of the molecular weight distribution λ_i, $i = 0, 1, 2$ (but $\lambda_1 \approx W_P = 1 - W_M - W_S$).

Steady state	Conversion rate	Comments
stable	poor	Usually chosed as operating point
unstable	medium	
stable	high	risk of solidification due to high viscosity

Application story (Royal Dutch Shell)

1. The model can be put globally in an MIMO observability canonical form.
2. Characteristic index (Isidori) are computed and a partially linearizing state feedback is designed.
3. Global asymptotic stability of the zero-dynamics is proved
4. The nonlinear separation principle is applied
5. The controller has never been actually applied