Time optimal control problem

\[
\begin{cases}
\frac{dx(t)}{dt} = f(x(t), u(t)) \\
y(t) = h(x(t))
\end{cases}
\]

where \( u(t) \in [-1, 1]^p \).

Time optimal control: Minimize \( T \) such that \( x(T) \in N \), \( N \) being the target.

Following Hermes ("discontinuous vector fields and feedback control", SIAM J. Control, 1967), we assume the Brunovsky stability.
Brunovski stability

**Definition**

A synthesis \( u(t) \) is a Filippov synthesis iff any solution \( x(t) \) of

\[
\begin{align*}
\frac{dx(t)}{dt} &= f(x(t), u^*(x(t))) \\
x(0) &= \eta
\end{align*}
\]

in Caratheodory sense (absolutely continuous for a.e. \( t \)) is a Filippov solution for a.e. \( t \) i.e.

\[
\frac{dx(t)}{dt} \in \bigcap_{\delta > 0} \bigcap_{\lambda N = 0} \overline{\text{conv}} f(B(x(t), \delta) \setminus N)
\]
Exemple: chemical reaction

Chemical reaction of type

\[ A \rightarrow B \rightarrow C \]

\[
\begin{align*}
\frac{d}{dt} [A]_t &= -k_1 (T_t) [A]_t \\
\frac{d}{dt} [B]_t &= k_1 (T_t) [A]_t - k_2 (T_t) [B]_t 
\end{align*}
\]

where \( k_i (T) = A_i e^{-\frac{E_i}{RT}} \)

Problem: Maximize \([B]\) in minimal time.
The synthesis has only regular trajectories. It is obviously a Filippov synthesis.
The temperature is a state variable, the control is the heat duty:
\[
\frac{dT_t}{dt} = -h(T_t) Q_t
\]

Singular trajectory
Regular trajectories
A bang-bang trajectory followed by a singular arc.
We should verify

\[ f(x(t), u^*(x(t))) \in \bigcap_{\delta > 0} \bigcap_{\lambda N = 0} \text{conv} f(B(x(t), \delta) \setminus N) \]

The main problem concern the Caratheodory solution on the singular arc

\[ \gamma_s = \{(y, v) ; \quad y(\alpha \beta v^{\alpha-1} - 1) = 1\} \]

with singular control

\[ u_s = -\frac{v^2}{h(v) \alpha y} \]
Let $x$ being a point on $\gamma_s$ and $\delta > 0$ small enough. $\gamma_x$ cross $B(x, \delta)$ on two open sets $B(x, \delta) \cap \gamma_+$ et $B(x, \delta) \cap \gamma_-$. Hence, any Filippov solution satisfy

\[
\begin{align*}
\dot{y} &= v - \beta v^\alpha y + vy \\
\dot{v} &= ah(v) u_+ + (1 - a) h(v) u_- = h(v) \left( au_+ - (1 - a) u_- \right) \\
(\dot{y}, \dot{v}) &\in T_x \gamma_s
\end{align*}
\]

The unique solution is

\[
(v - \beta v^\alpha y + vy) \frac{d}{dy} + h(v) u_s \frac{d}{dv}
\]
Closed loop system

\[
\begin{aligned}
\frac{dx(t)}{dt} &= f(x(t), u^*(\hat{x}(t))) \\
\frac{dx(t)}{dt} &= f(x(t), u^*(\hat{x}(t))) + K_\theta(t)(x(t) - \hat{x}(t)) \\
x(0) &= \eta \\
\hat{x}(0) &= \hat{\eta}
\end{aligned}
\]

**Theorem** Let $u^*$ be a Filippov synthesis. Let $\varepsilon > 0$ and $N_\varepsilon = \{x \in M : d(x, M) < \varepsilon\}$. For $\theta$ large enough, the closed loop system reach $N_\varepsilon$ at time $T_\varepsilon$ where $|T - T_\varepsilon| < \varepsilon$. 
Proof Set \( y(t) = x^*(t) - \hat{x}(t) \) hence

\[
|y| \cdot \frac{d|y|}{dt} = \frac{1}{2} \frac{d|y|^2}{dt} = y \cdot \frac{d|y|}{dt} = y \cdot \frac{d|x^*|}{dt} - y \cdot \frac{d|x|}{dt} \\
\leq \text{essmax}_{\xi^* \in B(x^*,\delta)} (y \cdot f_u(\xi^*)) \\
- \text{essmin}_{\xi \in B(\hat{x},\delta)} (y \cdot (f_u(\hat{\xi}) + K(t)(x - \hat{\xi}))) \\
\leq \cdots
\]

until

\[
\frac{d|y|}{dt} \leq M(L(t,|y|)) + \|K\|(x - \hat{x})
\]
Closed loop behaviour

Closed loop trajectory up to a change of coordinates (rectification):

exponential convergence
Open loop system

\[ \begin{align*}
\frac{dx}{dt} &= Ax + b(x, u) \\
y &= Cx
\end{align*} \]

High-gain observer

\[ \begin{align*}
\frac{dz}{dt} &= Az + b(z, u(z)) - S(t)^{-1}C'r^{-1}(Cz - y(t)) \\
\frac{dS}{dt} &= -(A + b^*(z, u(z)))'S - S(A + b^*(z, u(z))) \\
&\quad + C'r^{-1}C - SQ_\theta S
\end{align*} \]

Globally asymptotically stable control \( u \)
Main result

(weakly) g.a.s. closed loop system

\[
\begin{align*}
\frac{dx}{dt} &= Ax + b(x, u(z)) \\
\frac{dz}{dt} &= Az + b(z, u(z)) - S(t)^{-1}C'r^{-1}(Cz - y(t)) \\
\frac{dS}{dt} &= -(A + b^*(z, u(z)))'S - S(A + b^*(z, u(z))) + C'r^{-1}C - SQ_\theta S
\end{align*}
\]

in the sense that bounded trajectories are contained in the attracting set of the unique equilibrium point \((x^*, z^*; S^*)\)
**First step:** On the invariant set

\[ N = \{(\varepsilon, z, S); \varepsilon = x - z = 0\} \]

the system has a triangular form, hence (Vidyasagar, Sontag) we can deduce the local asymptotic stability on \( N \).

Using a Byrnes–Isidori lemma, we can deduce local asymptotic stability on the whole space (from the observer exponential convergence).
**Second step:** Any positive trajectory $\Lambda^+ (x_0, z_0, S_0)$ such that $x(t)$ is bounded, is bounded.

Let $\Omega$ be the $\omega$–limit set of $\Lambda^+ (x_0, z_0, S_0)$ (Lasalle–Lefschetz)
- $\Omega$ is not empty
- it is a closed set
- It is a positively invariant set
- $(x^*, z^*; S^*) \in \Omega$

and the first step implies $\Omega = (x^*, z^*; S^*)$
Exemple: a polymerization reactor

CSTR model: six highly nonlinear differential equations describing weight fraction of monomer $W_M$, of solvent $W_S$ and of initiator $W_I$, the temperature $T$ and the leading moments of the molecular weight distribution $\lambda_i$, $i = 0, 1, 2$ (but $\lambda_1 \approx W_P = 1 - W_M - W_S$).

<table>
<thead>
<tr>
<th>Steady state</th>
<th>Conversion rate</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>stable</td>
<td>poor</td>
<td>Usually choised as operating point</td>
</tr>
<tr>
<td>unstable</td>
<td>medium</td>
<td></td>
</tr>
<tr>
<td>stable</td>
<td>high</td>
<td>risk of solidification due to high viscosity</td>
</tr>
</tbody>
</table>
1. The model can be put globally in an MIMO observability canonical form.

2. Characteristic index (Isidori) are computed and a partially linearizing state feedback is designed.

3. Global asymptotic stability of the zero-dynamics is proved

4. The nonlinear separation principle is applied

5. The controller has never been actually applied