Observers

$$\begin{cases} \dfrac{dx(t)}{dt} = f(x(t), u(t)) \\ x(0) = x_0 \text{ unknown} \\ y(t) = h(x(t), u(t)) \end{cases}$$

$x(t) \in X$ state, $\dim(X) = n$

$y(t) \in \mathbb{R}^{d_y}$ outputs (measurements)

$u(t) \in \mathbb{R}^{d_u}$ inputs (control)

Control $P_{x_0} : u(\cdot) \mapsto y(\cdot)$

Observation $P_{u(\cdot)} : x_0 \mapsto y(\cdot)$

Observability $=$ $P_{u(\cdot)}$ injective

(uniform) (for any $u(\cdot)$)

Observer: définition

Observer = algorithm wich calculate $P_{u(\cdot)}^{-1}$

$$\begin{cases} \dfrac{d\xi(t)}{dt} & = & g(\xi(t), u(t), y(t)) \\ \xi(0) & = & \xi_0 \text{ fixed} \\ \widehat{x}(t) & = & \psi(\xi(t), u(t), y(t)) \end{cases}$$

s.t. $d(x(t), \widehat{x}(t)) \overset{t \to \infty}{\longrightarrow} 0$

Some very classical « observers »

Luenberger (linear systems)

Linear Kalman observer (from Kalman filter)

If $\mathcal{L}(x(0)) = \mathcal{N}(\widehat{x}_0, P_0)$
then $\mathcal{L}(x(t)/y(0\ldots t)) = \mathcal{N}(\widehat{x}(t), P(t))$

Extended Kalman filter (linearization)

$$A_t = \frac{\partial f}{\partial x}(z(t), u(t)) \text{ and } C_t = \frac{\partial h}{\partial x}(z(t), u(t))$$

(Linear) Kalman filter: $f(z,u) = Az + Bu$ and $h(z,u) = Cz$

$$\begin{cases} \frac{dz}{dt} & = & Az + Bu + PC'R^{-1}(y(t) - Cz) \\ \frac{dP}{dt} & = & AP + PA' + Q - PC'R^{-1}CP \end{cases}$$

Extended Kalman filter

$$\begin{cases} \frac{dz}{dt} & = & f(z,u) + PC_t'R^{-1}(y(t) - h(z,u)) \\ \frac{dP}{dt} & = & A_tP + PA_t' + Q - PC_t'R^{-1}C_tP \end{cases}$$

$\left(A_t = \frac{\partial f}{\partial x}(z(t), u(t))\right.$ and $\left.C_t = \frac{\partial h}{\partial x}(z(t), u(t))\right)$

Interesting properties

+ It is an optimal local observer

+ It works well

- It is not an observer

+ It is not an intrinsic tool

+ It can be used for identification purpose

Observability and normal forms

$$\begin{cases} \dfrac{dx(t)}{dt} &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)) \end{cases}$$

We consider the linearized system

$$\begin{cases} \dfrac{d\xi}{dt} &= T_X f(\xi, u) \\ \eta &= d_X h(\xi, u) \end{cases}$$

and its input-output mapping

$$\begin{array}{llll} dP_{(x_0,u}:(\xi_0) & : & T_{X_0}X & \longrightarrow & L^\infty_{\mathrm{loc}}([0,\tau); R^{d_y} \\ & & \xi_0 & \longrightarrow & dP_{(x_0,u}(\xi_0)(t) \end{array}$$

Infinitesimal observability

Definition

A system is called **infinitesimally observable at** (x_0, u) if the linear mapping $dP_{x_0,u}$ is injective.

It is **infinitesimally observable at** u if the linear mapping $dP_{x_0,u}$ is injective for any initial condition x_0.

It is **uniformly infinitesimally observable** if it is infinitesimally observable for any input u.

Preliminay definitions

Let us consider the set S of systems Σ, and define

$$j^k u = (u, \frac{du}{dt}, \frac{d^2u}{dt^2}, \cdots, \frac{d^{k-1}u}{dt^{k-1}})$$

Then

$$S\Phi_N : X \times U \times \mathbb{R}^{(N-1)d_u} \times S \longrightarrow \mathbb{R}^{Nd_y} \times \mathbb{R}^{Nd_u}$$

$$\begin{aligned} x, j^N u, \Sigma \quad \longrightarrow & \quad (h(x,u), L_f h(x, j^2 u), \\ & \quad \cdots L_f^{N-1} h(x, j^N u), j^N u) \\ = & \quad (j^N y, j^N u) \end{aligned}$$

Differential observability

Definition

A system Σ is differentially observable of order N if $S\Phi_N^\Sigma = S\Phi_N(\cdot,\cdot,\Sigma)$ is an injective mapping and strongly differentially observable if $S\Phi_N^\Sigma$ is an injective immersion.

Proposition Differential observability at any order implies observability.

This proposition is used to verify observability, not to build observers.

Gauthier-Kupka theory I

$d_y > d_u$:

The set of systems that are strongly differentially observable of order $2n + 1$ is residual in S;

The set of **analytic** strongly differentially observable systems (of order $2n + 1$) that are moreover L^∞-observable is dense in S.

Observability is a generic property.

Phase variable representation

The following is a generic (residual) property on S : Set $k = 2n + 1$.

On any $\Gamma \subset X$ compact and $u, \dot{u}, ..., u^{(k)}$ bounded, the mappings $\Phi^{\Sigma}_{k,j^k u} : x(t) \mapsto j^k y$ are smooth injective immersions that map the trajectories of the system (restricted to Γ) to

$$
\begin{aligned}
y &= z_1 \\
\dot{z}_1 &= z_2, \\
&\vdots \\
\dot{z}_{k-1} &= z_k \\
\dot{z}_k &= \varphi_K(z_1, \dots, z_k, u, \dot{u}, \dots, u^{(k)}).
\end{aligned}
$$

Gauthier-Kupka theory II

$d_y \leq d_u$:

Observability is a property of infinite codimension.

Observable systems admits a (local) normal form of observability.

If $d_y = 1$, if the system is uniformly infinitesimally observable, then, (outside a subanalytic subset of X of codimension 1), for all $x^0 \in X$, there is a coordinate neighborhood of x^0, (V_{x^0}, x), such that, in these coordinates, the system $\Sigma_{|V_{x^0}}$ (Σ restricted to V_{x^0}) can be written as:

Canonical form of observability

$$\begin{cases} y &= h(x_1, u) \\ \dot{x}_1 &= f_1(x_1, x_2, u) \\ \dot{x}_2 &= f_2(x_1, x_2, x_3, u) \\ &\vdots \\ \dot{x}_{n-1} &= f_{n-1}(x_1, x_2, .., x_n, u) \\ \dot{x}_n &= f_n(x_1, x_2, ..., x_n, u) \end{cases}$$

where moreover

$$\frac{\partial h}{\partial x_1}$$

and

$$\frac{\partial f_i}{\partial x_{i+1}}, i = 1, \ldots, n-1$$

are never zero on $V_{x_0} \times U$.

Affine analytic SISO case I

$$\begin{cases} \frac{dx}{dt} & = & f(x) + g(x)u \\ y & = & h(x) \end{cases}$$

$u \in \mathbb{R}$

$$\begin{array}{cccc} \Phi & : & X & \longmapsto & \mathbb{R}^n \\ & & x & \longrightarrow & (h(x), L_f h(x), \ldots, L_f^{n-1} h(x)) \end{array}$$

is a local diffeomorphism outside M, analytic closed subset of X.

Infinitesimal observability $\Rightarrow$ M has codimension ≥ 1 and if $Y \subset X \setminus M$ open, $\Phi_{|Y}$ being a diffeomorphism, then $\Phi_{|Y}$ maps the system into canonical form of observability:

Affine analytic SISO case II

$$\begin{cases} \dot{x}_1 = x_2 + u\, g_1(x_1) \\ \dot{x}_2 = x_3 + u\, g_2(x_1, x_2) \\ \quad \vdots \\ \dot{x}_{n-1} = x_n + u\, g_{n-1}(x_1, \ldots, x_{n-1}) \\ \dot{x}_n = \varphi(x) + u\, g_n(x) \end{cases}$$

Moreover, if the system can be written in this form on $\Omega \in \mathbb{R}^n$ then is is observable on Ω.

This result is much simple than previous ones

Proof

If $g_k = g_k\left(x_1, x_2, \ldots, x_k, x_{k+1}\right)$ then

$$\dot{x}_k = x_{k+1} + u\, g_k\left(x_1, \ldots, x_k, x_{k+1}\right)$$

hence

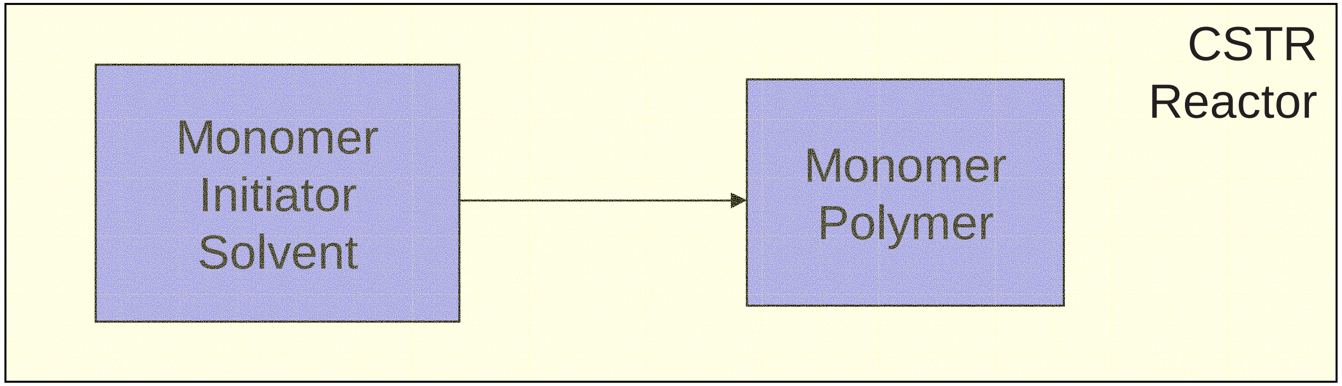
$$\dot{\xi}_k = \xi_{k+1} + u\left(\frac{\partial g}{\partial x_1}\xi_1 + \cdots + \frac{\partial g}{\partial x_{k+1}}\xi_{k+1}\right)$$

therefore

$$u\left(\xi_1, \ldots \xi_{k+1}\right) = -\frac{\xi_{k+1}}{\frac{\partial g}{\partial x_1}\xi_1 + \cdots + \frac{\partial g}{\partial x_k}\xi_k}$$

make the system not infinitesimally observable!

Polymerization reactor



Initiator decomposition	$I \xrightarrow{k_d} 2R^{\cdot}$
Initiation	$R^{\cdot} + M \xrightarrow{k_i} P_1^{\cdot}$
Propagation	$P_x^{\cdot} + M \xrightarrow{k_p} P_{x+1}^{\cdot}$
Chain transfer to monomer	$P_x^{\cdot} + M \xrightarrow{k_{ttM}} P_x^{\cdot} + P_1^{\cdot}$
Chain transfer to solvent	$P_x^{\cdot} + S \xrightarrow{k_{ttS}} P_x^{\cdot} + P_1^{\cdot}$
Chain termination by combination	$P_x^{\cdot} + P_y^{\cdot} \xrightarrow{k_{tc}} P_{x+y}$
Chain termination by disproportionation	$P_x^{\cdot} + P_y^{\cdot} \xrightarrow{k_{td}} P_x + P_y$

Polymerization reactor model I

$$
\begin{cases}
\frac{dW_M}{dt} = \frac{Q_{mF}}{\rho V}\left(W_{MF} - W_M\right) - \left(k_p + k_{ttM}\right) W_M C_R \\
\qquad - 2 f k_d W_I \frac{M_M}{M_I} \\
\frac{dW_S}{dt} = \frac{Q_{mF}}{\rho V}\left(W_{SF} - W_S\right) - k_{ttS} W_S C_R \\
\frac{dW_I}{dt} = \frac{Q_{mF}}{\rho V}\left(W_{IF} - W_I\right) - k_d W_I \\
\frac{dT}{dt} = \frac{Q_{mF}}{\rho V}\left(\frac{c_{pF}}{c_p} T_F - T\right) + \frac{UA}{\rho V c_p}\left(T_j - T\right) - \frac{k_p C_R W_M \Delta H}{M_M c_p} \\
\frac{d\lambda_0}{dt} = \frac{Q_{mF}}{\rho V}\left(\lambda_{0F} - \lambda_0\right) + \frac{\Phi_0}{\rho} \\
\frac{d\lambda_2}{dt} = \frac{Q_{mF}}{\rho V}\left(\lambda_{2F} - \lambda_2\right) + \frac{\Phi_2}{\rho}
\end{cases}
$$

$$
\frac{1}{\rho} = \frac{W_M}{\rho_M} + \frac{W_S}{\rho_S} + \frac{1 - W_M - W_S}{\rho_P}
$$

Polymerization reactor model II

$$C_R = \sqrt{\frac{f\,k_d C_I}{k_{tc} + k_{td}}}$$

$$C_x = \frac{W_x \rho}{M_x} x = M, I, S$$

$$\Phi_i = \Phi_i\left(W_M, W_S, W_I\right)$$

k are given by Arrhenius law

Measured outputs are temperature T and density ρ.

Observability of the CSTR model

$$\left\{\begin{array}{lcl}
\frac{dW_M}{dt} & = & \frac{Q_{mF}}{\rho V}(W_{MF}-W_M)-(k_p+k_{ttM})W_M C_R \\
 & & -2fk_d W_I \frac{M_M}{M_I} \\
\frac{dW_S}{dt} & = & \frac{Q_{mF}}{\rho V}(W_{SF}-W_S)-k_{ttS}W_S C_R \\
\frac{dW_I}{dt} & = & \frac{Q_{mF}}{\rho V}(W_{IF}-W_I)-k_d W_I \\
\frac{dT}{dt} & = & \frac{Q_{mF}}{\rho V}\left(\frac{c_{pF}}{c_p}T_F-T\right)+\frac{UA}{\rho V c_p}\left(T_j-T\right)-\frac{k_p C_R W_M \Delta H}{M_M c_p} \\
\frac{d\lambda_0}{dt} & = & \frac{Q_{mF}}{\rho V}(\lambda_{0F}-\lambda_0)+\frac{\Phi_0}{\rho} \\
\frac{d\lambda_2}{dt} & = & \frac{Q_{mF}}{\rho V}(\lambda_{2F}-\lambda_2)+\frac{\Phi_2}{\rho}
\end{array}\right.$$

$\sqrt{W_I}W_M$

$$\frac{d(\sqrt{W_I}W_M)}{dt}=\varphi(T,W_M,W_I)$$

$$\frac{d\rho^{-1}}{dt}=\psi(T,\rho,W_M,W_S)$$

- ☑ T
- ☑ W_M
- ☑ W_I
- ☑ W_S

More general canonical form of observability

$$\begin{cases} \dfrac{dx}{dt} & = A(t)x + b(x,u) \\ y & = C(t)x \end{cases}$$

$$A(t) = \begin{pmatrix} 0 & a_2(t) & 0 & \cdots & 0 \\ & & a_3(t) & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & & a_n(t) \\ 0 & & \cdots & & 0 \end{pmatrix}$$

$$C(t) = \begin{pmatrix} a_1(t) & 0 & \cdots & 0 \end{pmatrix}$$

$$0 < a_m \leq a_i(t) \leq a_M$$

$$b(x,u) = b_1(x_1,u)\frac{\partial}{\partial x_1} + b_2(x_1,x_2,u)\frac{\partial}{\partial x_2} + b_n(x_1,\ldots,x_n,u)\frac{\partial}{\partial x_n}$$

Multi-output Canonical form of observability (rare !) I

$$\begin{cases} \dot{X} &= AX + B(X,u) \\ y &= CX \end{cases}$$

$$X = \begin{pmatrix} X^1 \\ X^2 \\ \vdots \\ X^{d_y} \end{pmatrix} \qquad X^i = \begin{pmatrix} X^i_1 \\ X^i_2 \\ \vdots \\ X^i_{\rho_i} \end{pmatrix}$$

$$C = \begin{pmatrix} C^1 & & & \\ & C^2 & & \\ & & \ddots & \\ & & & C^{d_y} \end{pmatrix} \qquad C^i = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$$

Multi-output Canonical form of observability (rare !) II

$$A = \begin{pmatrix} A^1 & & & \\ & A^2 & & \\ & & \ddots & \\ & & & A^{d_y} \end{pmatrix} \qquad A^i = \begin{pmatrix} 0 & 1 & 0 & & \\ & 0 & 1 & \ddots & \\ & & \ddots & \ddots & 0 \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} B^1(X,u) \\ B^2(X,u) \\ \vdots \\ B^{d_y}(X,u) \end{pmatrix} \qquad B^i = \begin{pmatrix} B_1^i\left(X_1^i, u\right) \\ B_2^i\left(X_1^i, X_2^i, u\right) \\ \\ B_{\rho_i-1}^i\left(X_1^i, \ldots, X_{\rho_i-1}^i, u\right) \\ B_{\rho_i}^i(X,u) \end{pmatrix}$$

UNIVERSITÉ DE BOURGOGNE

Multi-output high-gain EKF (rare !)

$$\begin{cases} \frac{dz}{dt} & = & Az + B(z,u) + PC'R_\theta^{-1}(y(t) - Cz) \\ \frac{dP}{dt} & = & (A + B^*(z,u))P + P(A + B^*(z,u))' \\ & & +Q_\theta - PC'R_\theta^{-1}CP \end{cases}$$

$$\Delta = \begin{pmatrix} \Delta^1 & & & \\ & \Delta^2 & & \\ & & \ddots & \\ & & & \Delta^{d_y} \end{pmatrix} \quad \Delta^i = \begin{pmatrix} \theta^{\rho_i - 1} & & & \\ & \theta^{\rho_i - 2} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$Q_\theta = \theta \Delta^{-1} Q \Delta^{-1}$$

and

$$R_\theta \text{ s.t. } \Delta^{-1} C' R_\theta^{-1} C \Delta^{-1} = \theta C' R^{-1} C$$

Single-output high-gain EKF

$$\left\{\begin{array}{lcl} \frac{dz}{dt} & = & Az + B(z,u) + PC'R^{-1}(y(t) - Cz) \\ \frac{dP}{dt} & = & (A + B^*(z,u))P + P(A + B^*(z,u))' \\ & & +Q_\theta - PC'R^{-1}CP \end{array}\right.$$

$$\Delta = \begin{pmatrix} 1 & & & \\ & \frac{1}{\theta} & & \\ & & \ddots & \\ & & & (\frac{1}{\theta})^{n-1} \end{pmatrix}$$

$$Q_\theta = \theta^2 \Delta^{-1} Q \Delta^{-1}$$

High-gain extended Kalman filter

Theorem

For θ large enough and for all $T > 0$, the high-gain extended Kalman filter satisfies for $t > \frac{T}{\theta}$

$$\|z(t) - x(t)\| \leq \theta^{n-1} k(T) \left\| z\left(\frac{T}{\theta}\right) - x\left(\frac{T}{\theta}\right) \right\| e^{-(\theta\omega(T) - \mu(T))\left(t - \frac{T}{\theta}\right)}$$

for some positive continuous functions $k(T)$, $\omega(T)$ and $\mu(T)$.

Canonical form for observer construction

$$\begin{cases} \dot{x}_1 &= F_1(x_1, x_2, u) & \dfrac{\partial F_1}{\partial x_2} \neq 0 \\ \dot{x}_2 &= F_2(x_1, x_2, x_3, u) & \dfrac{\partial F_2}{\partial x_3} \neq 0 \\ &\vdots & \\ \dot{x}_n &= F_n(x, u) & \end{cases}$$

$$\xi_1 = y = x_1, \ \xi_2 = F_1(x_1, x_2, u)$$

$$\xi_3 = \frac{\partial F_1}{\partial x_2} F_2(x_1, x_2, u), \cdots$$

$$\xi_{i+1} = \frac{\partial F_1}{\partial x_2} \cdots \frac{\partial F_{i-1}}{\partial x_i} F_i\left(x_1, \ldots, x_{i+1}, u\right)$$

$$\downarrow$$

$$\begin{cases} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 + \dfrac{\partial F_1}{\partial x_1}\dot{x}_1 + \dfrac{\partial F_1}{\partial u}\dot{u} \\ &\vdots \\ \dot{\xi}_n &= G(x, u, \dot{u}) \end{cases}$$

Ref. H. **Hammouri, M. Farza,** *Nonlinear observers for local uniform observable systems*

Continuous-discrete time version

$$\begin{cases} \frac{dx}{dt} &= A(u)x + b(x,u) \\ y_k &= C(u)\,x(t_k) \end{cases}$$

Correction step

$$\begin{cases} z\left(t_k^+\right) &= z(t_k) + G(k)\left(y_k - C(u)\,z(t_k)\right) \\ G(k) &= P(t_k)\,C(u)'\left(C(u)\,P(t_k)\,C(u)' + \frac{1}{\Delta t}R\right)^{-1} \\ P\left(t_k^+\right) &= \left(I - G(k)\,C(u)\right)P(t_k) \end{cases}$$

Prediction step

$$\begin{cases} \frac{dz}{dt} &= A(u)x + b(x,u) \\ \frac{dP}{dt} &= (A(u) + b^*(z,u))P + P(A(u) + b^*(z,u))' + Q_\theta \end{cases}$$

Cascaded high-gain observers

$$\begin{cases} \dot{X}_1 &= f_1(X_1, u) \\ \dot{X}_2 &= f_2(X_1, X_2, u) \end{cases}$$

Let Z_1 be the state of a high-gain observer for Σ_1 with parameter θ_1 and Z_2 be the state of a high-gain observer for Σ_2 with parameter θ_2, X_1 being an input.

For θ_1 and θ_2 large enough, (Z_1, Z_2) is an exponential observer.

Modified Extended Kalman filter

$$
\begin{aligned}
\frac{dz}{dt} &= A(t)z + b(z,u) - S(t)^{-1}C(t)' r^{-1}(C(t)z - y(t)) \\
\frac{dS}{dt} &= -(A(t) + b^*(z,u))'S - S(A(t) + b^*(z,u)) \\
&\quad + C(t)' r^{-1} C(t) - SQ_\theta S
\end{aligned}
$$

$$
\Delta = \begin{pmatrix} 1 & & & \\ & \frac{1}{\theta} & & \\ & & \ddots & \\ & & & (\frac{1}{\theta})^{n-1} \end{pmatrix}
\qquad
Q_\theta = \theta^2 \Delta^{-1} Q \Delta^{-1}
$$

If θ is large, high-gain observer (HGEKF)

If $\theta \approx 1$, Classical Extended Kalman filter (EKF)

Theorem

There exist $\lambda_0 > 0$ such that for any $0 \leq \lambda \leq \lambda_0$, there exist θ_0 such that for any $\theta(0) > \theta_0$, for any $S(0) \geq c\, Id$, for any compact $K \subset \mathbf{R}^n$, for any $z(0) \in K$ then if we set $\varepsilon(t) = z(t) - x(t)$ for any $t \geq 0$

$$||\varepsilon(t)||^2 \leq R(\lambda, c) e^{-at} \Lambda(\theta(0), t, \lambda) ||\varepsilon(0)||^2 \quad (1)$$

where

$$\Lambda(\theta(0), t, \lambda) = \theta(0)^{2(n-1)+\frac{a}{\lambda}} e^{-\frac{a}{\lambda}\theta(0)(1-e^{-\lambda t})}$$

and a is a positive constant and $R(\lambda, c)$ is a decreasing function of c.

Passer la preuve

Proof

Change of variables $\begin{cases} \widetilde{x} & = \Delta x \\ \widetilde{P} & = \frac{1}{\theta} \Delta P \Delta \end{cases} \qquad \left(P = S^{-1}\right)$

$+$ time change $d\tau = \theta(t)\, dt$

We set $\varepsilon = z - x = error$ then we calculate $\varepsilon'(\tau) S(\tau) \varepsilon(\tau)$.
Observability give us $\alpha I \leq S(\tau) \leq \beta I$ then

$$\varepsilon'(\tau) S(\tau) \varepsilon(\tau) \longrightarrow 0 \Longleftrightarrow \varepsilon(\tau) \longrightarrow 0$$

When $\tau \leq T$

$$\|\varepsilon(\tau)\|^2 \leq \theta(\tau)^{2(n-1)} H(c)\, e^{-(a_1\theta(T)-a_2)\tau} \|\varepsilon(0)\|^2$$

Parallel high-gain and non-high-gain EKF

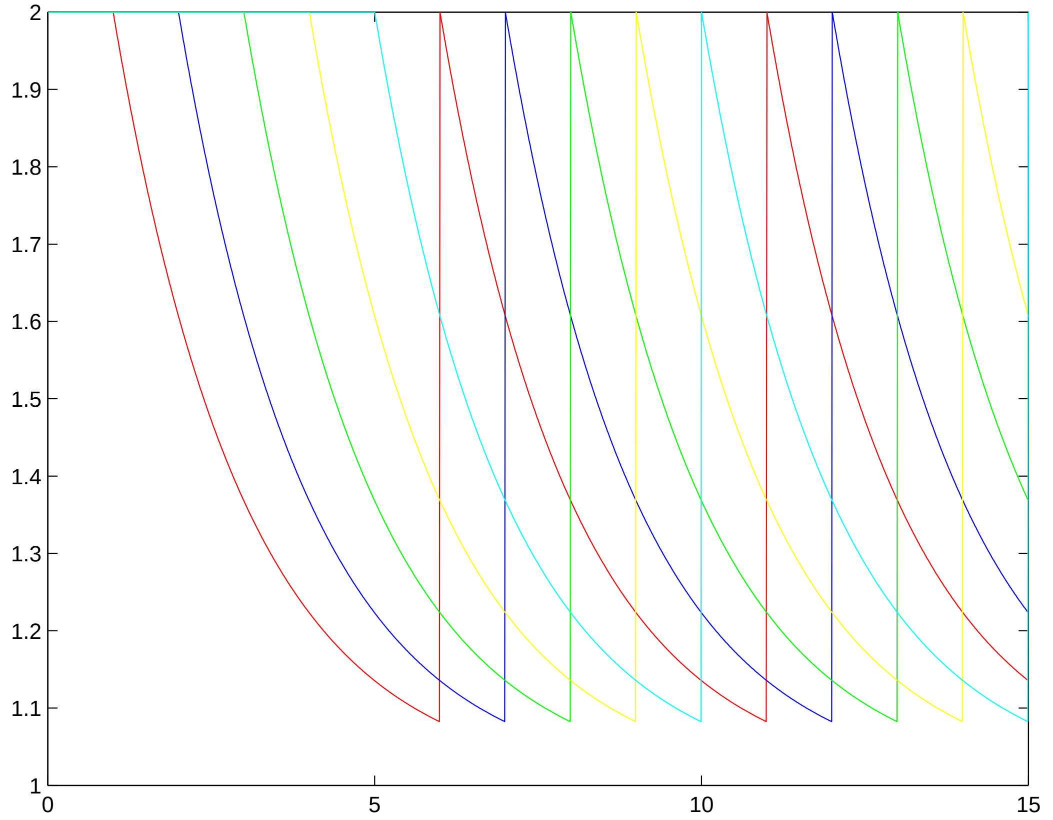
We use N observers in parallel. At times kT :

- a new observer is initialized with $\theta(kT) = \theta_0$,
- the older observer is killed.

Therefore, at any time t, we have N observers initialized at times kT, $(k-1)T \ldots (k-N+1)T$ where $k = \left\lfloor \frac{t}{T} \right\rfloor$.

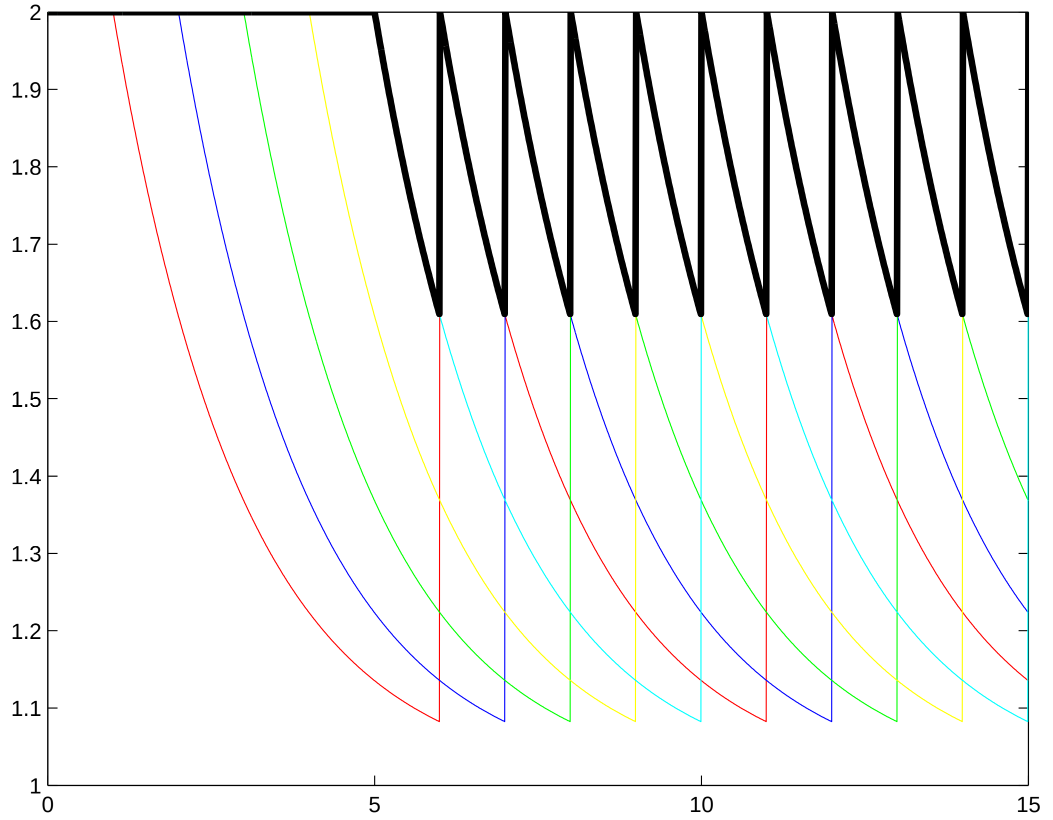
State estimation: the estimation given by the observer with smallest innovation $\|y - C\widehat{x}\|$.

Parallel Extended Kalman Filter



θ for 5 observers

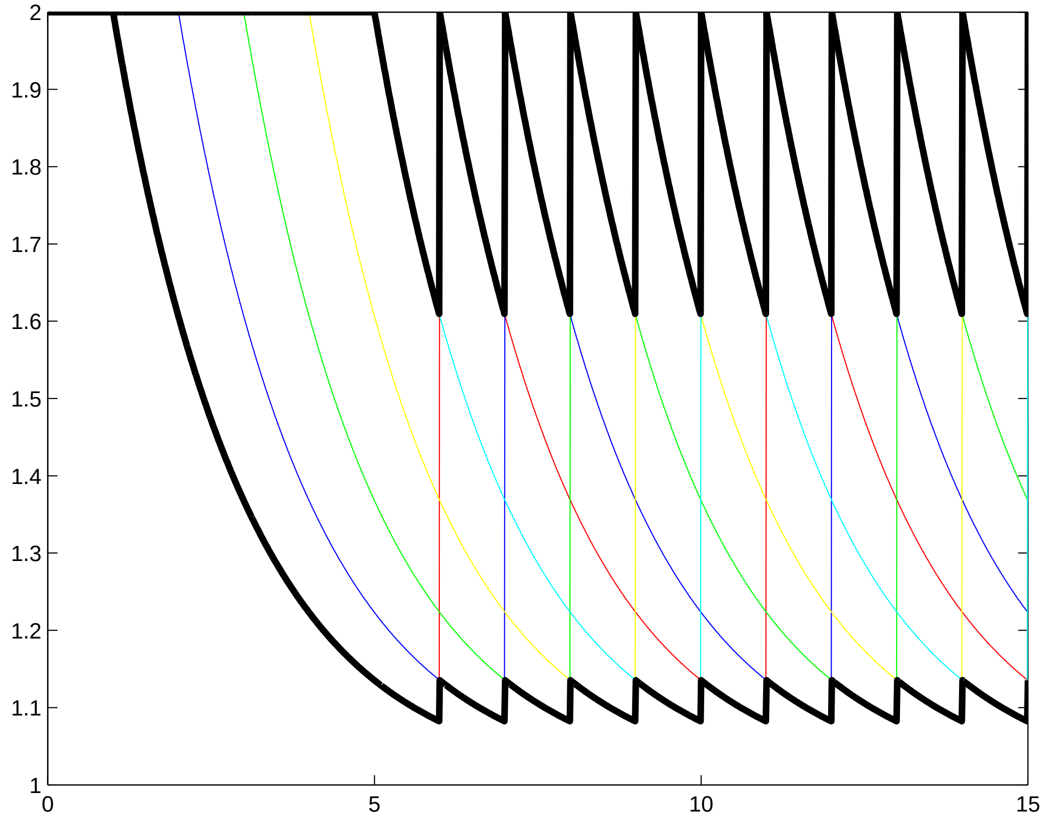
High-gain Extended Kalman Filter



θ for the youngest observer is

$$1 + e^{-\lambda(t-kT)} (\theta_0 - 1) \geq 1 + e^{-\lambda T} (\theta_0 - 1)$$

Standard Extended Kalman Filter



θ for the oldest observer is

$$1 + e^{-\lambda(t-kT+(N+1)T)}(\theta_0 - 1) \approx 1$$

Adaptive-gain extended Kalman filter

Main idea: Modify θ s.t.

- θ exponentially decrease if everything seems ok (previous result)

- θ exponentially increase when something goes wrong

$$\frac{d\theta}{dt} = F(t, \theta)$$

where

- $F(t, \theta) \simeq \lambda(1 - \theta)$

  if estimation Z is closed to state x

- $F(t, \theta) \simeq (\theta_{\text{max}} - \theta)$

  if Z is far from state x

Persistant observability

Lemma

We consider a system in its canonical form of observability.

Let $x_1^0,\ x_2^0 \in \mathbf{R}^n$.

Let us consider the outputs $y_i\left(t\right)$ with initial conditions $x_i^0,\ i = 1, 2$.

$$\forall T > 0 \quad \forall u \in L_b^1\left(\mathcal{U}_{\text{adm}}\right) \quad \exists \lambda_T > 0$$

$$\left\| x_1^0 - x_2^0 \right\| \leq \frac{1}{\lambda_T} \int_0^T \left\| y_1\left(\tau\right) - y_2\left(\tau\right) \right\| d\tau$$

Innovation process

$$
\begin{aligned}
\frac{d\theta}{dt} &= F(\theta, \mathcal{I}) \\
&= \lambda\ (1-\theta) + K\ (\theta_{\mathsf{max}} - \theta)\ \mathcal{I}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I} &= \int_{t-T}^{t} \left\| y(s) - \bar{y}_{t-T}(s) \right\|^2 ds \\
&= \left\| y - \bar{y}_{t-T} \right\|^2_{L^2(t-T,t)}
\end{aligned}
$$

and

$$
\left\{
\begin{aligned}
\frac{d\xi}{d\tau} &= A(u)\xi\,(\tau) + b(\xi\,(\tau)\,, u) \\
\xi\,(t-T) &= Z\,(t-T) \\
\bar{y}_{t-T}\,(\tau) &= C\,(u)\,\xi\,(\tau)
\end{aligned}
\right.
$$

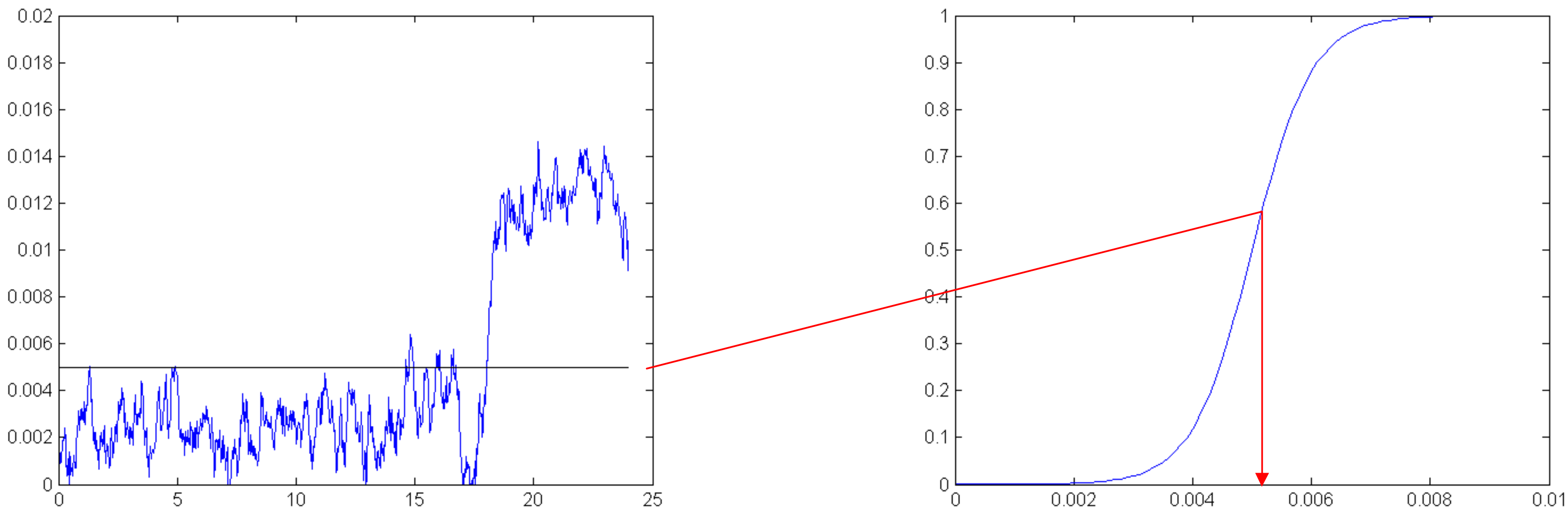
UNIVERSITÉ DE BOURGOGNE

A more practical solution

$$\frac{d\theta}{dt} = \lambda(1 - s(\mathcal{I}))\,(1 - \theta) + K\, s(\mathcal{I})\,(\theta_{max} - \theta)$$

where s is a sigmoid function:

$$s(\mathcal{I}) = \frac{1}{1 + \mathrm{e}^{-\beta\,(\mathcal{I} - m)}}$$



Theorem

$$\begin{cases} \dfrac{dZ}{dt} &= A(u)Z + b(Z,u) + S^{-1}C'R_\theta^{-1}(CZ - y(t)) \\ \dfrac{dS}{dt} &= -(A(u) + b(Z,u))'S - S(A(u) + b^*(Z,u)) \\ & \quad +C'R_\theta^{-1}C - SQ_\theta S \\ \dfrac{d\theta}{dt} &= \lambda(1-\theta) + K(\theta_{\mathrm{max}} - \theta)\,\mathcal{I} \end{cases}$$

with $Q_\theta = \theta\Delta^{-1}Q\Delta^{-1}$ and $R_\theta = \theta^{-1}R$.

Theorem For λ small enough, K large enough, and θ_{max} large enough, this observer is an exponentially converging *persistant* observer.

Application on a DC motor