Introduction

$$\Sigma : \begin{cases} \frac{dx}{dt} & = & f(x, u(t), \varphi \circ \pi(x(t))) \\ y & = & h(x, u(t), \varphi \circ \pi(x(t))) \end{cases}$$

where $\varphi \circ \pi : \begin{array}{ccccc} X & \longrightarrow & Z & \longrightarrow & I \subset \mathbf{R} \\ x & \longrightarrow & z = \pi(x) & \longrightarrow & \varphi(\pi(x)) \end{array}$

and $P_\Sigma : \begin{array}{ccccccc} X & \times & L^\infty[U] & \times & L^\infty[I] & \longrightarrow & L^\infty[\mathbf{R}^{d_y}] \\ (x_0 & , & u(.) & , & \widehat{\varphi}(.)) & \longrightarrow & y(.) \end{array}$

φ is an unknown function of $\pi(x)$

$\widehat{\varphi}$ is a function of time

P_Σ is the input/output function of Σ

Identifiability

Definition 1: Σ is identifiable at

$$(u(.), y(.)) \in L^\infty[U] \times L^\infty[\mathbf{R}^{d_y}]$$

if there is at most a single couple

$$(x_0, \hat{\varphi}) \in X \times L^\infty(I)$$

such that for almost all t

$$P_\Sigma(x_0, u, \hat{\varphi})(t) = y(t)$$

and $\hat{\varphi}(t) = \varphi \circ \pi(x(t))$ for some smooth function $\varphi : Z \to I$.
Σ is identifiable if it is identifiable at any admissible $(u(.), y(.))$.

Infinitesimal Identifiability

Definition 2:

$$T\Sigma : \begin{cases} \frac{d\xi}{dt} & = & T_{x,\varphi} f(x,u,\varphi;\xi,\eta) \\ \widehat{y} & = & d_{x,\varphi} h(x,u,\varphi;\xi,\eta) \end{cases}$$

where $(\xi,\eta) \in T_x X \times T_\varphi I$, we set

$$\begin{aligned} P^t_{T\Sigma}(\xi_0,\eta) &= d_{x,\varphi} h(x,u,\widehat{\varphi}; T_{x,\varphi}\phi_t(x,u,\widehat{\varphi};\xi_0,\eta),\eta) \\ &= T_{x,\varphi} P^t_\Sigma(\xi_0,\eta) \end{aligned}$$

Σ is infinitesimally identifiable at $(x_0,\ u,\ \widehat{\varphi}) \in X \times L^\infty[U] \times L^\infty[I]$ if $P^t_{T\Sigma}$ is injective $\forall t > 0$

Σ is uniformly infinitesimally identifiable if this is true at all $(x_0,\ u,\ \widehat{\varphi})$

Identifiability for smooth functions

Injectivity depends on the domain for $\hat{\varphi}$.

But for C^ω–systems:

Theorems

Σ is (infinitesimally) identifiable in the class of **analytic functions**

$$\Updownarrow$$

Σ is (infinitesimally) identifiable in the class of L^∞ **functions**

Main idea of the proof

We prove that if Σ is not identifiable because of a pair $(x_0, \widehat{\varphi}) \in X \times L^\infty$ then there exists $\left(\widetilde{x_0}, \widetilde{\widehat{\varphi}}\right) \in X \times C^\omega$ s.t. Σ is not identifiable.

The main tool is a modification of a lemma from **Gauthier–Kupka**, Deterministic Observation Theory and Applications, *Cambridge University Press*, 2001

Differential Identifiability

Let $D_k\Phi = X \times (I \times \mathbf{R}^{k-1})$ be the space of k-jets of the system Σ, we set$^{(*)}$

$$\begin{array}{rccc} \Phi_k^\Sigma : & D_k\Phi & \rightarrow & \mathbf{R}^{kd_y} \\ & (x_0, j^k(\widehat{\varphi})) & \rightarrow & j^k(y) \end{array}$$

Definition: Σ is differentially identifiable of order k if

$\Phi_k^\Sigma(z_1) = \Phi_k^\Sigma(z_2) \Rightarrow (x_1, \widehat{\varphi}_1(0)) = (x_2, \widehat{\varphi}_2(0))$

$^{(*)}$ $j^k(u) = \left(u(0), u'(0), \ldots, u^{(k-1)}(0)\right)$

Genericity

Proposition: Differential Identifiability $\Rightarrow$ Identifiability

Theorem:

- If $d_y \geq 3$, differential identifiability of order $2n+1$ is a **generic property** in the class of C^∞ systems.
- If $d_y < 3$, differential identifiability **is not** a generic property.

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Proof of genericity 1/2

$$Z_i = \left(x_i, \varphi_i, \varphi_i', \ldots, \varphi_i^k, j_\Sigma^k(x_i, \varphi_i)\right),\ i = 1, 2$$

$$Z = (Z_1, Z_2)$$

$$\Phi(Z) = \Phi_k^\Sigma(Z_1) - \Phi_k^\Sigma(Z_2) \in R^{k\,d_y},$$

$$k = 2n + 1,\ d_y \geq 3$$

Let us suppose that Φ is a submersion

$\mathrm{codim}\Phi^{-1}(0) = k\,d_y$

Let $\Pi\Phi^{-1}(0) = \left(x_i, \varphi_i, j_\Sigma^k(x_i, \varphi_i)\right)_{i=1,2}$

$$\begin{aligned} \mathrm{codim}\Pi\Phi^{-1}(0) &\geq k\,d_y - 2(k-1) = k(d_y - 2) + 2 \\ &\geq k + 2 \geq 2n + 3 \end{aligned}$$

Proof of genericity 2/2

$$\begin{array}{rccc} \rho_\Sigma : & (X \times I)^2 \setminus \Delta & \to & \left(J^k_\Sigma\right)^2 \\ & (x_1, \varphi_1, x_2, \varphi_2) & \to & \left(x_i, \varphi_i, j^k_\Sigma(x_i, \varphi_i)\right)_{i=1,2} \end{array}$$

Multijet transversality theorem: the set of Σ such that ρ_Σ is transversal to $\Pi\Phi^{-1}(0)$ is residual.

$$\dim (X \times I)^2 \setminus \Delta = 2n + 2$$

$$\Downarrow$$

generically, ρ_Σ avoids $\Pi\Phi^{-1}(0)$

Single-output case

Theorem 2. If Σ is uniformly infinitesimally identifiable then

i) $\frac{\partial}{\partial\varphi}\left\{h, L_{f_\varphi}h, \ldots, (L_{f_\varphi})^{n-1}h\right\} \equiv 0$

ii) $\frac{\partial}{\partial\varphi}L_{f_\varphi}^{n}h \neq 0$

iii) $d_x h \wedge \ldots \wedge d_x L_{f_\varphi}^{n-1}h \neq 0$,

Therefore, locally, the system can be written

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & x_2 \\ & \vdots & \\ \dot{x}_{n-1} & = & x_n \\ \dot{x}_n & = & \psi(x,\varphi) \\ y & = & x_1 \end{array}\right. \quad \text{and } \frac{\partial}{\partial\varphi}\psi(x,\varphi) \neq 0$$

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Single-output case, pseudo-converse

Theorem 3. If Σ meets the following conditions,

i) $\frac{\partial}{\partial\varphi}\left\{h, L_{f_\varphi}h, \ldots, (L_{f_\varphi})^{n-1}h\right\} \equiv 0$

ii) $\frac{\partial}{\partial\varphi}L^n_{f_\varphi}h \neq 0$

iii) $d_x h \wedge \ldots \wedge d_x L^{n-1}_{f_\varphi} h \neq 0$,

then Σ is

1) locally identifiable,

2) loc. unif. infinitesimally identifiable,

3) loc. diff. identifiable of order $n+1$.

Passer la preuve

Proof of the single-output case 1/2

Let $k < n$ be the first k such that $d_\varphi L_f^k h \neq 0$:

$$\Sigma \quad \begin{cases} y & = & x_1 \\ \dot{x}_1 & = & x_2 \cdots \\ \dot{x}_{k-1} & = & x_k \\ \dot{x}_k & = & L_f^k(x,\varphi) = f_k(x,\varphi) \cdots \\ \dot{x}_n & = & f_n(x,\varphi) \end{cases}$$

$$T\Sigma \quad \begin{cases} \dot{x} & = & f(x,\varphi) \\ \widehat{y} & = & \xi_1 \\ \dot{\xi}_1 & = & \xi_2 \cdots \\ \dot{\xi}_{k-1} & = & \xi_k \\ \dot{\xi}_k & = & d_x f_k(x,\varphi)\,\xi + d_\varphi f_k(x,\varphi)\,\eta \end{cases}$$

Proof of the single-output case 2/2

A feedback $\eta = -\dfrac{d_x f_k(x,\varphi_0)\,\xi}{d_\varphi f_k(x,\varphi_0)}$ in φ_0 s.t. $d_\varphi f_k(x,\varphi_0) \neq 0$ gives $\frac{d\xi_k}{dt} = 0$ which contradict observability.

If $\frac{\partial}{\partial\varphi} L^n_{f_\varphi} h = 0$ at (x,φ)

$$
\begin{array}{ccc}
X \times I & \supset & E = \left\{ (x,\varphi)\,;\, d_\varphi L^n_f h = 0 \right\} \\
 & & \downarrow \Pi \\
X & \supset & \Pi E
\end{array}
$$

Hardt's theorem $\Rightarrow \exists \widehat{\varphi}$

$$
\left\{
\begin{array}{lllllllll}
y & = & x_1, & \dot{x}_1 & = & x_2, & \ldots & \dot{x}_n & = \;\; \psi(x,\widehat{\varphi}(x)) \\
\widehat{y} & = & \xi_1, & \dot{\xi}_1 & = & \xi_2, & \ldots & \dot{\xi}_n & = \;\; d_x\psi(x,\widehat{\varphi}(x)) + 0
\end{array}
\right.
$$

Two output case: definitions of *k* and *r*

Define $E_l = \left\{ d_x h_i, d_x L_{f_\varphi} h_i, \ldots, d_x L_{f_\varphi}^{l-1} h_i \,,\, i = 1, 2 \right\}$ and $N(l) = \text{rank}(E_l)$ at a generic point:

k is defined by

$N(0)$	$N(1)$	$\cdots$	$N(k-1)$	$N(k)$	$N(k+1)$	$\cdots$	$N(k+m)$
0	2		$2k-2$	$2k$	$2k+1$		$2k+m$

$(2k + m \le n)$

The **order** of the system is the first integer r such that $d_\varphi L_{f_\varphi}^r (h_1, h_2) \neq 0$.

Classification

Lemma: If Σ is uniformly infinitesimally identifiable then (1) $2k + m = n$

(2) $r \leq k + m$

Proof:

(1) $\varphi = \varphi_0 =$cte $\left\{ \begin{array}{rcl} \dot{x} & = & f(x, \varphi_0) \\ \dot{\xi} & = & g(x, \xi, \varphi_0) \\ y & = & h(x, \varphi_0) \end{array} \right.$ contradict observability

(2) $\left\{ \begin{array}{rcl} \dot{x} & = & f(x) \\ y & = & h(x) \end{array} \right.$ contradict identifiability

Définition 5. A system Σ is regular if (1) and (2) holds.

Type 3: $r=k$ and $n=2k$

$$\left\{\begin{array}{lllll} y_1 & = & x_1 & y_2 & = & x_2 \\ \dot{x}_1 & = & x_3 & \dot{x}_2 & = & x_4 \\ & \vdots & & & \vdots & \\ \dot{x}_{n-3} & = & x_{n-1} & \dot{x}_{n-2} & = & x_n \\ \dot{x}_{n-1} & = & f_{n-1}(x,\varphi) & \dot{x}_n & = & f_n(x,\varphi) \end{array}\right.$$

with $\frac{\partial}{\partial \varphi}(f_{n-1}, f_n) \neq 0$

$N(l)$ increases by steps of 2 until the last derivative and apparition of φ.

Type 1: *r>k*

$$\begin{cases} y_1 = x_1 & y_2 = x_2 \\ \dot{x}_1 = x_3 & \dot{x}_2 = x_4 \\ \vdots & \vdots \\ \dot{x}_{2k-3} = x_{2k-1} & \dot{x}_{2k-2} = x_{2k} \\ \dot{x}_{2k-1} = f_{2k-1}(x_1, ..., x_{2k+1}) & \\ \dot{x}_{2k} = x_{2k+1} & \\ \vdots & \\ \dot{x}_{n-1} = x_n & \\ \dot{x}_n = f_n(x, \varphi) & \end{cases}$$

with $\frac{\partial f_n}{\partial \varphi} \neq 0$.

$N(l)$ increases by steps of 1 when φ appears for the first time, $\simeq$ single-output case.

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Type 2: *r*<*k*

$$
\begin{array}{lllllll}
y_1 & = & x_1 & y_2 & = & x_2 \\
\dot{x}_1 & = & x_3 & \dot{x}_2 & = & x_4 \\
 & \vdots & & & \vdots & \\
\dot{x}_{2r-3} & = & x_{2r-1} & \dot{x}_{2r-2} & = & x_{2r} \\
\dot{x}_{2r-1} & = & \psi(x,\varphi) & \dot{x}_{2r} & = & F_{2r}(x_1,\ldots,x_{2r+1},\psi(x,\varphi)) \\
 & & & \dot{x}_{2r+1} & = & F_{2r+1}(x_1,\ldots,x_{2r+2},\psi(x,\varphi)) \\
 & & & & \vdots & \\
 & & & \dot{x}_{n-1} & = & F_{n-1}(x,\psi(x,\varphi)) \\
 & & & \dot{x}_n & = & F_n(x,\varphi)
\end{array}
$$

with $\frac{\partial\psi}{\partial\varphi} \neq 0, \frac{\partial F_{2r}}{\partial x_{2r+1}} \neq 0, \ldots., \frac{\partial F_{n-1}}{\partial x_n} \neq 0$

φ appears when $N(l)$ increases by steps of 2.

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$r=k$ and $2k<n$

If $r = k$ before the last derivative:

$$d_x h_1 \wedge \cdots \wedge d_x L_{f_\varphi}^{k-1} h_1 \wedge d_x L_{f_\varphi}^{k-1} h_2 \wedge d_x L_{f_\varphi}^{k} h_2 \not\equiv 0$$

If $d_\varphi L_{f_\varphi}^{k} h_1 \neq 0$, we obtain φ using y_1 and $x_{2k},\ldots,x_n$ using y_2

→ Type 2

If $d_\varphi L_{f_\varphi}^{k} h_1 \equiv 0$, we obtain φ using y_2

→ Type 1

Normal form in generic case ($d_y>2$)

If $d_y \geq 3$ then identifiability is a generic property (cf. theorem 3).

Let us consider for simplicity $d_y = 3$:

$$\begin{cases} \dfrac{dx}{dt} & = & f\left(x, \varphi(x)\right) \\ y_1 & = & h_1\left(x, \varphi(x)\right) \\ y_2 & = & h_2\left(x, \varphi(x)\right) \\ y_3 & = & h_3\left(x, \varphi(x)\right) \end{cases}$$

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Main result

$$\Phi_k^{\Sigma} : \left(x, \varphi, \ldots, \varphi^{(k-1)}\right) \mapsto \left(y, \ldots, y^{(k-1)}\right)$$

Theorem The set of systems such that, in restriction to $\bar{Z} \times I$, $\Phi_{2n+1}^{\bar{\Sigma}}$ is an injective immersion, is residual.

where $\bar{\Sigma} = \Sigma|_{\bar{Z}}$, $\bar{Z}$ a large set.

Theorem: Global normal form (d_y=3)

Outside a bad small subset, Σ may be written

$$\begin{aligned} y &= z^1 = \left(z_1^1, z_2^1, z_3^1\right) \\ \dot{z}^1 &= z^2 \\ &\vdots \\ \dot{z}^{k-1} &= z^k \\ \dot{z}^k &= F\left(z^1, \ldots, z^k, \varphi^{(k)}\right) \end{aligned}$$

Moreover

$$\left(x, j^k\varphi\right) = H\left(z^1, \ldots, z^k\right)$$

and

$$y^{(k)} = G\left(x, \varphi, \ldots, \varphi^{(k)}\right)$$

Theorem: Local normal form (d_y=3)

Outside a bad small subset, Σ may be **locally** (around (x, φ)) rewritten

$$\begin{aligned}
y &= z^1 = \left(z^1_1, z^1_2\right) \\
\dot{z}^1 &= z^2 \\
&\vdots \\
\dot{z}^{k-1} &= z^k \\
\dot{z}^k &= F\left(z^1, \ldots, z^k, y_3, \ldots, y_3^{(k)}\right)
\end{aligned}$$

and

$$\begin{aligned}
x &= \Phi\left(z^1, \ldots, z^k, y_3, \ldots, y_3^{(k-1)}\right) \\
\varphi &= \Psi\left(x, y_3\right)
\end{aligned}$$

Canonical form for observer construction

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$$\begin{cases} \dot{x}_1 &= F_1(x_1, x_2, u) \\ \dot{x}_2 &= F_2(x_1, x_2, x_3, u) \\ &\vdots \\ \dot{x}_n &= F_n(x, u) \end{cases} \qquad \frac{\partial F_1}{\partial x_2} \neq 0, \quad \frac{\partial F_2}{\partial x_3} \neq 0$$

$$\downarrow$$

$$\xi_1 = y = x_1, \; \xi_2 = F_1(x_1, x_2, u)$$

$$\xi_3 = \frac{\partial F_1}{\partial x_2} F_2(x_1, x_2, u), \cdots$$

$$\xi_{i+1} = \frac{\partial F_1}{\partial x_2} \cdots \frac{\partial F_{i-1}}{\partial x_i} F_i\left(x_1, \ldots, x_{i+1}, u\right)$$

$$\begin{cases} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 + \frac{\partial F_1}{\partial x_1}\dot{x}_1 + \frac{\partial F_1}{\partial u}\dot{u} \\ &\vdots \\ \dot{\xi}_n &= G(x, u, \dot{u}) \end{cases}$$

Ref. H. **Hammouri, M. Farza,** *Nonlinear observers for local uniform observable systems*

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