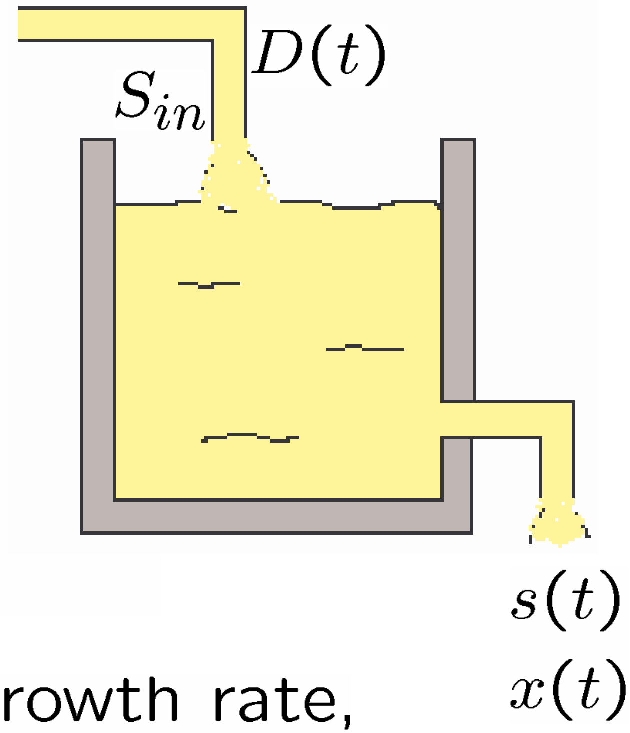
Biological reactor

$$\begin{cases} \dfrac{ds(t)}{dt} = -\mu(s(t))\,x(t) + D(t)(S_{in} - s(t)) \\ \dfrac{dx(t)}{dt} = (\mu(s(t)) - D(t))\,x(t) \end{cases}$$



$s(t)$: substrates
$x(t)$: biomass
$D(t)$: influent flow rate
S_{in} : substrate concentration in the influent

$\mu(s) > 0$, $\mu(0) = 0$ specific growth rate,

Monod $\mu(s) = \frac{\mu_0 s}{k_m + s}$ or

Haldane $\mu(s) = \frac{\mu_0 s}{k_m + s + \frac{s^2}{k_i}}$ ...

Observability

Only $s(t)$ is measured

$$\frac{ds(t)}{dt} = -\mu(s(t))\,x(t) + D(S_{in} - s(t))$$

The system is observable.

Only $x(t)$ is measured

$$\begin{aligned} \frac{dx(t)}{dt} &= (\mu(s(t)) - D(t))x(t) \\ \frac{d\mu(s(t))}{dt} &= \mu'(s(t))\,(-\mu(s(t))\,x(t) + D(t)\,(S_{in} - s(t))) \end{aligned}$$

The system is observable (depending on μ).

Identifiability

Same questions but $\mu(s)$ is unknown

$$\begin{cases} \frac{ds(t)}{dt} = -\mu(s(t))x(t) + D(S_{in} - s(t)) \\ \frac{dx(t)}{dt} = (\mu(s(t)) - D)x(t) \end{cases}$$

Only $x(t)$ is measured,
can we reconstruct $\mu(s)$ and $s(t)$? **no**

Only $s(t)$ is measured,
can we reconstruct $\mu(s)$ and $x(t)$? **yes, if ...**

Both $x(t)$ and $s(t)$ are measured,
can we reconstruct $\mu(s)$? **yes**

$x(t)$ is observed

$$\begin{cases} \frac{dx(t)}{dt} = (\mu(s) - D)x \\ \frac{ds(t)}{dt} = -\mu(s)\, x + D(S_{in} - s) \\ y = x \end{cases}$$

$$s(t) = e^{-Dt}s(0) + \int_0^t e^{-D(t-\tau)}(-(\mu x)(\tau) + DS_{in})d\tau$$

$$s(0) = s_0$$

$$\tilde{s}(t) = e^{-Dt}\tilde{s}_0 + \int_0^t e^{-D(t-\tau)}(-(\mu x)(\tau) + DS_{in})d\tau$$

$$\tilde{s}(0) = \tilde{s}_0 \approx s_0$$

$$\tilde{\mu}(\tilde{s}) = \mu(s) \Rightarrow \frac{dx(t)}{dt} = (\tilde{\mu}(\tilde{s}(t)) - D)x(t)$$

$x(t)$ and $s(t)$ are observed

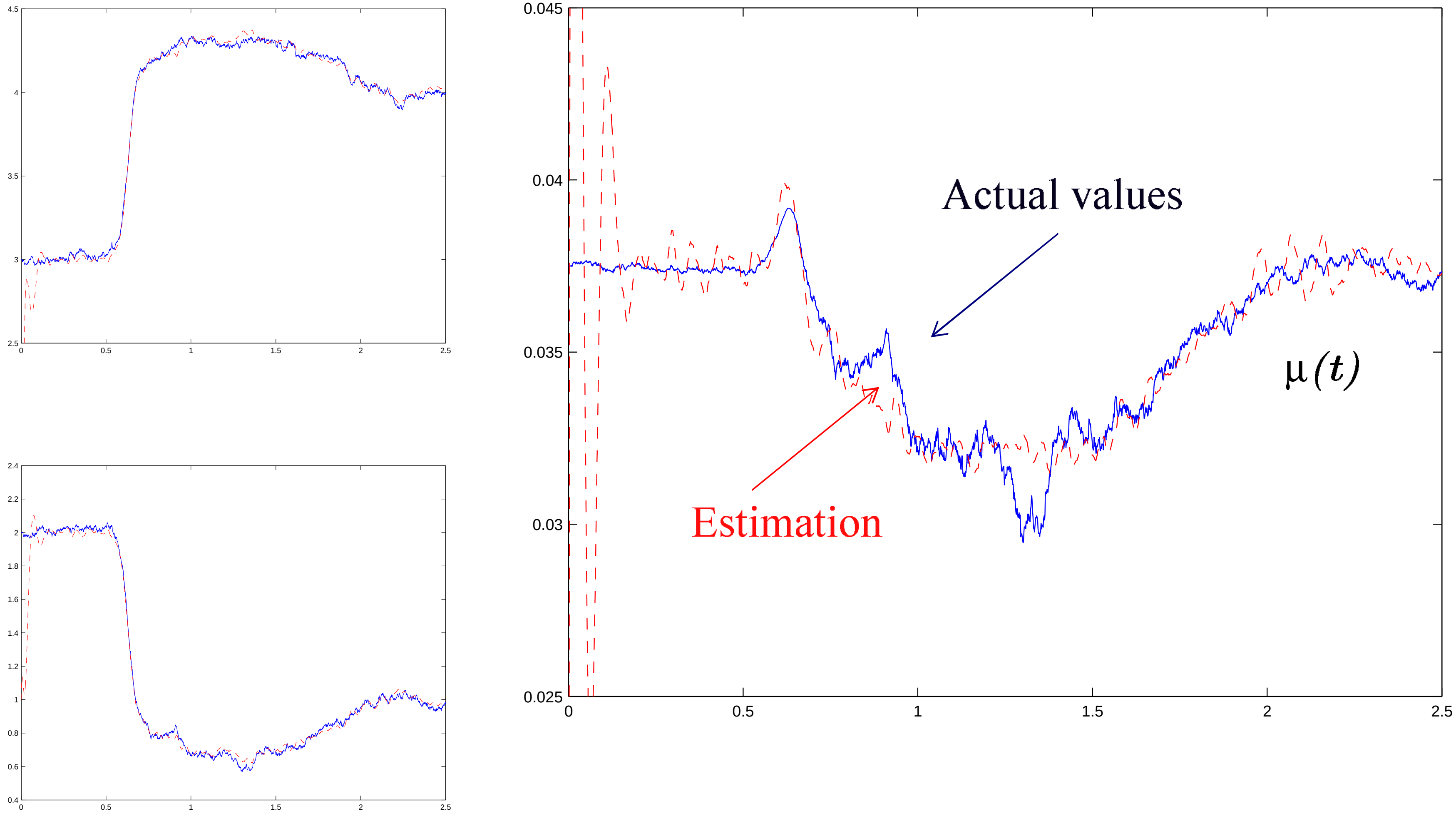
Let us denote $z(t) = \mu(s(t))\, x(t)$, and assume that $\frac{d^k z}{dt^k} = 0$

$$\begin{cases} \dot{s} &= -z + D(t)(S_{in} - s) \\ \dot{x} &= z - D(t)\ x \\ \dot{z} &= z_1 \cdots \\ \dot{z}_{k-2} &= z_{k-1} \\ \dot{z}_{k-1} &= 0 \\ y &= (s, x) \end{cases}$$

where $\frac{ds}{dt} = \dot{s}$

Linear (optimal) Kalman observer

Linear Kalman observer, $y=(x,s)$



$s(t)$ is measured

$$\left\{\begin{array}{rcl} X & = & x+s \\ \widetilde{D}(t) & = & \int_0^t D(\tau)\,d\tau \\ \Lambda(t) & = & e^{\tilde{D}(t)}(s-S_{in})+S_{in} \end{array}\right.$$

$$\begin{array}{rcl} \dot{\Lambda} & = & -e^{\tilde{D}(t)}(X-s)\mu(s) \\ & = & (\Lambda - X_0)\mu(s) \\ \text{with } \Lambda(0) & = & s(0) \end{array}$$

If $s(t_0) = s(t_1)$, $t_0 < t_1$ then

$$\frac{\dot{\Lambda}(t_0)}{\Lambda(t_0)-X_0} = \mu(s(t_0)) = \mu(s(t_1)) = \frac{\dot{\Lambda}(t_1)}{\Lambda(t_1)-X_0}$$

gives X_0 hence $\mu(s(t)) = \frac{\dot{\Lambda}(t)}{\Lambda(t)-X_0}$

Result

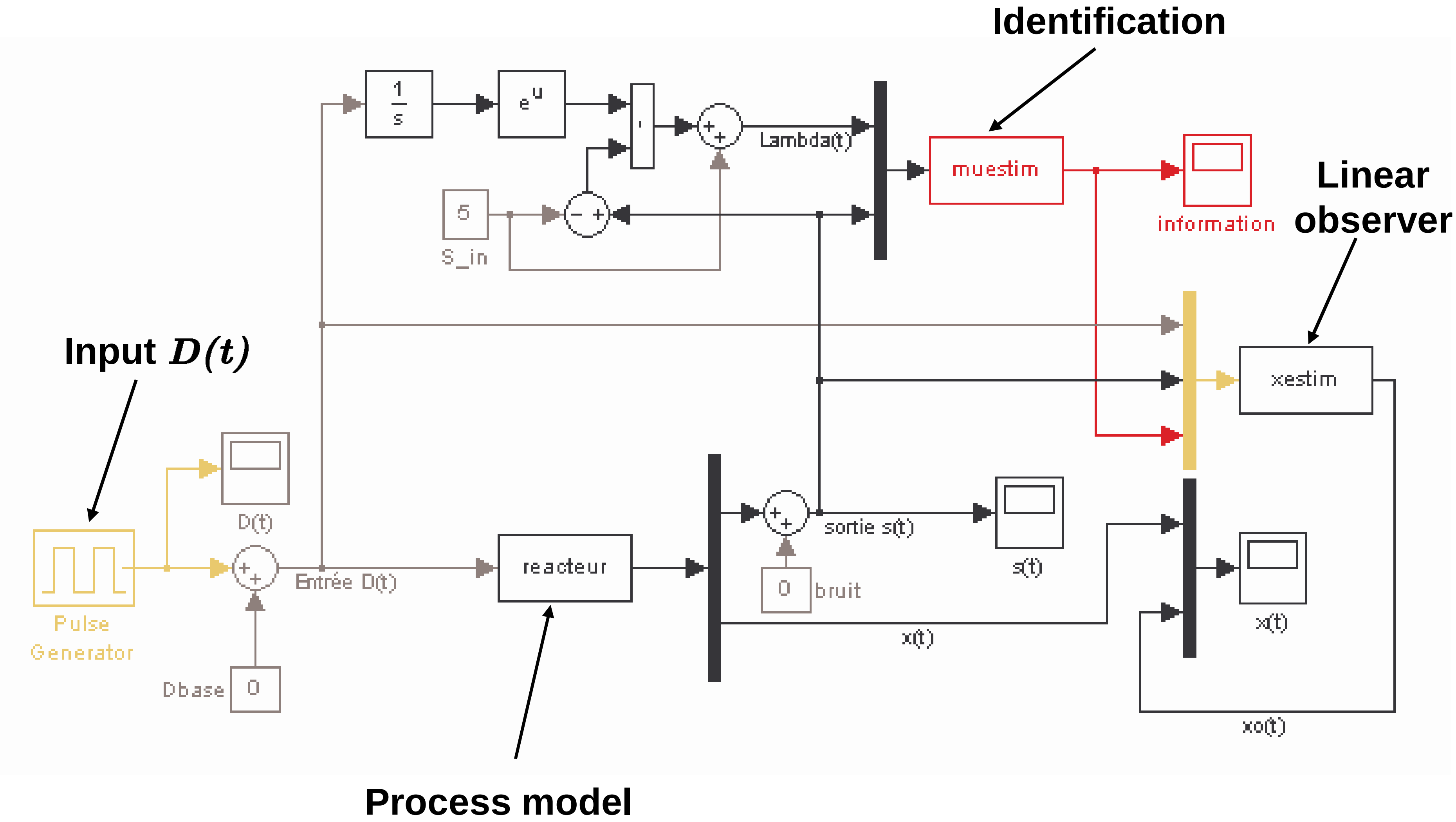
$\mu(s)$ is identifiable $\Longleftrightarrow$ $s(t)$ visits twice the same value

1. μ is identified at sample values $k\Delta s$, at time t, giving $\widehat{\mu}_t(h\Delta s)$;

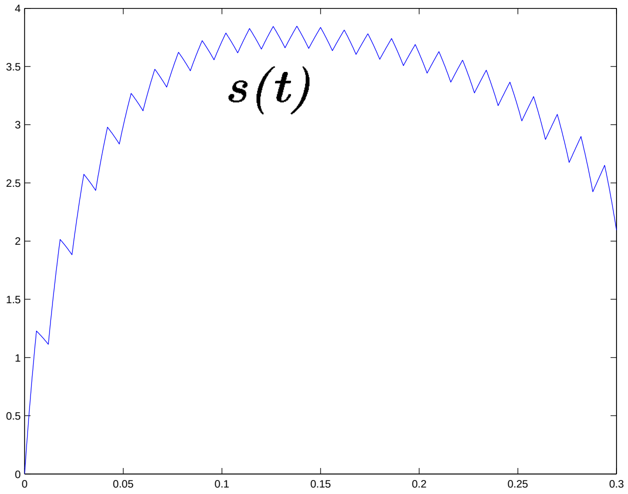
2. $x(t)$ is estimated using a linear Kalman filter and $\widehat{\mu}_t(h\Delta s)$

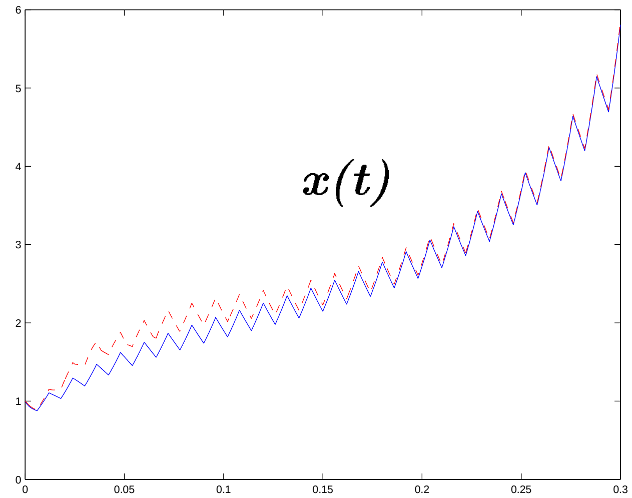
Simulation:
$\mu(s)$ is the Haldane law, $\widehat{\mu}_0(s)$ is the Monod law,

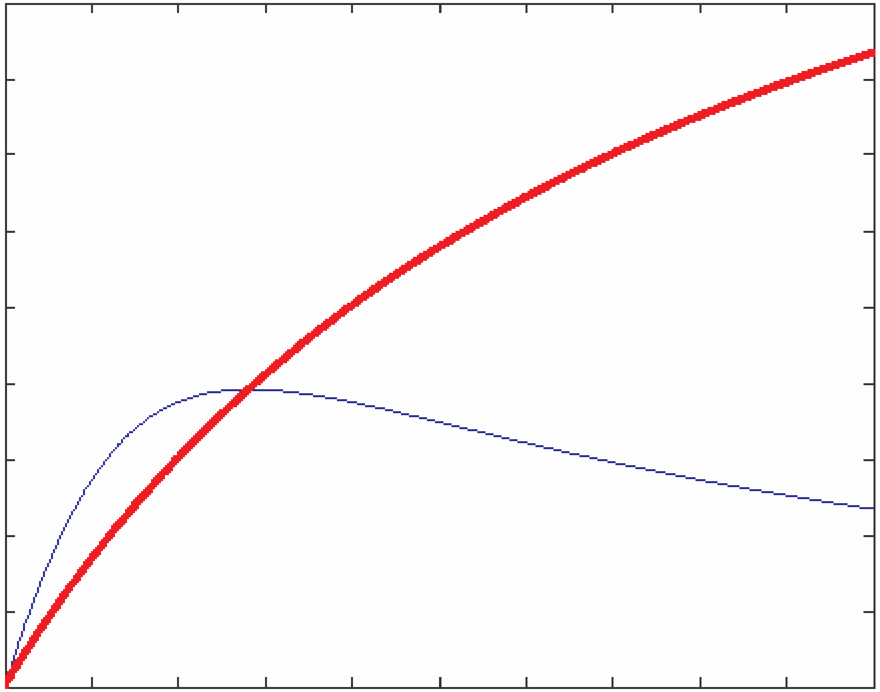
Simulation model using Matlab/Simulink



Identification de μ(t)







Monod's (initial guess) law

Haldane's (actual) law

Identified part of $\mu(s)$