ON DETERMINING UNKNOWN FUNCTIONS IN DIFFERENTIAL SYSTEMS, WITH AN APPLICATION TO BIOLOGICAL REACTORS.

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Abstract. In this paper, we consider general nonlinear systems with observations, containing a (single) unknown function φ. We study the possibility to learn about this unknown function via the observations: if it is possible to determine the [values of the] unknown function from any experiment [on the set of states visited during the experiment], and for any arbitrary input function, on any time interval, we say that the system is "identifiable".

For systems without controls, we give a more or less complete picture of what happens for this identifiability property. This picture is very similar to the picture of the "observation theory" in [7]:

If the number of observations is three or more, then, systems are generically identifiable.

If the number of observations is 1 or 2, then the situation is reversed. Identifiability is not at all generic. In that case, we add a more tractable infinitesimal condition, to define the "infinitesimal identifiability" property.

This property is so rigid, that we can almost characterize it (we can characterize it by geometric properties, on an open-dense subset of the product of the state space X by the set of values of φ). This, surprisingly, leads to a non trivial classification, and to certain corresponding "identifiability normal forms".

Contrarily to the case of the observability property, in order to identify in practice, there is in general no hope to do something better than using "approximate differentiators", as show very elementary examples.

As an illustration of what may happen in controlled cases, we consider the equations of a biological reactor, [2], [4], in which a population is fed by some substrate. The model heavily depends on a "growth function", expressing the way the population grows in presence of the substrate.

The problem is to identify this "growth function". Our result, in the case where the observed variable is the concentration of the substrate, is as follows: the system is identifiable along a trajectory, if and only if this trajectory **visits the same value of the output at least twice**.

We propose also a practical methodology for identification, which shows very reasonable performances.

1. Introduction

1.1. **Notations, definitions.** In this paper, depending on the context, smooth will mean C^ω (real-analytic), or C^∞, or C^k, for an integer k large enough.

Very often, in nonlinear control systems, certain special variables with "physical meaning" appear, (called here the "**internal variables**") and the system depends

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on certain functions of these variables. These functions describe some physical characteristic inside the system, and it may happen that they are not well known, and have to be determined on the basis of experiments.

We consider general smooth nonlinear systems:

$$\Sigma : \left\{ \begin{array}{l} \frac{dx}{dt} = f(x, u(t), \varphi \circ \pi(x(t))); \\ y = h(x, u(t), \varphi \circ \pi(x(t))), \end{array} \right. \tag{1.1}$$

or "**uncontrolled**" such systems:

$$\begin{array}{rcl} \dfrac{dx}{dt} & = & f(x, \varphi \circ \pi(x(t))); \\ y & = & h(x, \varphi \circ \pi(x(t))), \end{array} \tag{1.2}$$

where $x \in X$ denotes the state, y denotes the observation, $u(.)$ is the control function, $z = \pi(x)$ is the "**internal variable**", π is called the **internal mapping**, $\pi : X \to Z$. We assume that X and Z are given analytic connected manifolds, both Hausdorff and paracompact. $\dim(X) = n$.

The (smooth) "**unknown function**" is denoted by $\varphi : Z \to I$. Here, I will denote a compact interval of $\mathbb{R}$. To finish, $y \in \mathbb{R}^{d_y}$, $u \in U \subset \mathbb{R}^{d_u}$, where U is a compact subanalytic subset of $\mathbb{R}^{d_u}$. Also, f is a $(u, \varphi)-$parametrized smooth vector field, and h, the **observation mapping**, is a smooth mapping: $X \times U \times I \to \mathbb{R}^{d_y}$.

In the following, the systems $\Sigma = (f, h)$ will vary, but the manifolds X, Z, the (smooth) internal mapping π, the space $\mathbb{R}^{d_y}$, and the sets I and U are given and fixed.

Associated with such a system Σ, we consider the "**input-output mapping**" P_Σ :

$$P_\Sigma : X \times L^\infty[U] \times L^\infty[I] \rightharpoonup L^\infty[\mathbb{R}^{d_y}];$$

$$(x_0, u(.), \hat{\varphi}(.)) \rightharpoonup y(.),$$

where $L^\infty[U]$, $L^\infty[I]$, $L^\infty[\mathbb{R}^{d_y}]$ denote the set of $U-$valued (resp. $I-$valued, $\mathbb{R}^{d_y}-$valued) measurable, bounded functions, defined on semi-open interval $[0, T_u[$, $[0, T_{\hat{\varphi}}[$, $[0, T_y[$. The mapping P_Σ is defined as follows:

For any input $u(.) \in L^\infty[U]$, any $\hat{\varphi}(.) \in L^\infty[I]$, any $x_0 \in X$, then, the solution $x(t)$ of the Cauchy problem:

$$\frac{dx(t)}{dt} = f(x(t), u(t), \hat{\varphi}(t)), \ x(0) = x_0; \tag{1.3}$$

is defined on a maximum semi-open interval $[0, e(x_0, u, \hat{\varphi})[$, where $0 < e(x_0, u, \hat{\varphi}) \leq \min(T_u, T_{\hat{\varphi}})$. If $e(x_0, u, \hat{\varphi}) < \min(T_u, T_{\hat{\varphi}})$, then $e(x_0, u, \hat{\varphi})$ is the positive escape time of x_0 for the time dependant vector field f in (1.3). For fixed $u, \hat{\varphi}$, the mapping $x_0 \to e(x_0, u, \hat{\varphi}) \in \bar{R}_*^+$ is lower semicontinuous ($\bar{R}_*^+ = \{t | 0 < t \leq \infty\}$).

$P_\Sigma(x_0, u, \hat{\varphi})$ is the function $\hat{y} : [0, e(x_0, u, \hat{\varphi})[\to \mathbb{R}^{d_y}$, defined by $\hat{y}(t) = h(x(t), u(t), \hat{\varphi}(t))$.

We say that $(u(.), y(.)) \in L^\infty[U] \times L^\infty[\mathbb{R}^{d_y}]$, with $T_u = T_y$, is an "**admissible input-output trajectory**", if there exits a couple $(x_0, \hat{\varphi}) \in X \times L^\infty[I]$, such that $y(t) = P_\Sigma(x_0, u, \hat{\varphi})(t)$ for almost all $t \in [0, T_u[$ (which means in particular $e(x_0, u, \hat{\varphi}) = T_u$), and $\hat{\varphi}(t) = \varphi \circ \pi(x(t))$, where φ is some smooth function, $\varphi : Z \to I$. (Of course, this φ depends on the input-output trajectory (or the "experiment") $(u(.), y(.))$).

Note that, as a consequence, if $(u(.), y(.))$ is an admissible i.o. trajectory, then, $\hat{\varphi}(.)$ in the definition is in fact at least continuous. In the uncontrolled case, it is smooth.

We define now the natural notion of identifiability:

Definition 1. *The system Σ is said "identifiable at" $(u(.), y(.)) \in L^\infty[U] \times L^\infty[\mathbb{R}^{d_y}]$, with $T_u = T_y$, if there is* ***at most*** *a single couple $(x_0, \hat{\varphi}), \in X \times L^\infty([0, T_u[, I)$ such that, for almost all $t \in [0,\ T_u[$:*

$$P_\Sigma(x_0, u, \hat{\varphi})(t) = y(t),$$

and $\hat{\varphi}(t) = \varphi \circ \pi(x(t))$ for some smooth function $\varphi : Z \to I$.

Given a system Σ, the "identifiability set" of Σ is the subset of $L^\infty[U] \times L^\infty[\mathbb{R}^{d_y}]$ formed by the admissible i-o trajectories $(u(.), y(.))$ at which Σ is identifiable. If this set is exactly the set of admissible i-o trajectories, then Σ is said "identifiable".

Again, in this definition, the smooth function φ depends on the i.o. trajectory (or the "experiment") $(u(.), y(.))$. Also, again, $\hat{\varphi}$ is in fact at least continuous, and smooth in the uncontrolled case.

Of course, this definition is not very tractable in practice, and, in order to work, we will need a few other definitions.

For arbitrary $k-$jets of smooth functions $\hat{\varphi}, \hat{u}$ at $t = 0$,

$$\begin{aligned} \hat{\varphi} &: [0, \varepsilon[\to I, \hat{u} : [0, \varepsilon[\to U, \\ j^k(\hat{\varphi}) &= (\hat{\varphi}(0), \hat{\varphi}'(0), ..., \hat{\varphi}^{(k-1)}(0)), j^k(\hat{u}) = (\hat{u}(0), \hat{u}'(0), ..., \hat{u}^{(k-1)}(0)), \end{aligned}$$

and for any $x_0 \in X$, the corresponding $k-$jet $j^k\hat{y} = (\hat{y}, \hat{y}', ..., \hat{y}^{(k-1)})$ is well defined, in such a way that the mapping $\Phi_k^\Sigma : (x_0, j^k(\hat{u}), j^k(\hat{\varphi})) \to j^k\hat{y}$ be continuous. $\Phi_k^\Sigma : D_k\Phi = X \times (U \times \mathbb{R}^{(k-1)d_u}) \times (I \times R^{k-1}) \to \mathbb{R}^{kd_y}$.

Remark 1. *This mapping Φ_k^Σ is smooth with respect to x_0, $\hat{\varphi}(0)$, $\hat{u}(0)$, and algebraic with respect to $\hat{\varphi}'(0), ...\hat{\varphi}^{(k-1)}(0)$ and $\hat{u}'(0), ...\hat{u}^{(k-1)}(0)$. Also, Φ_k^Σ depends only on the $k-$jet $j^k\Sigma$ of Σ.*

We define $D_k\Phi_2^*$ as the set of couples $((x_1, j^k(\hat{u}), j^k(\hat{\varphi}_1)), ((x_2, j^k(\hat{u}), j^k(\hat{\varphi}_2)) = (z_1, z_2)$, with $z_1 = (x_1, j^k(\hat{u}), j^k(\hat{\varphi}_1)) \neq ((x_2, j^k(\hat{u}), j^k(\hat{\varphi}_2)) = z_2$.

Δ_k is the diagonal in $\mathbb{R}^{kd_y} \times \mathbb{R}^{kd_y}$. We denote by $\Phi_{k,2}^{\Sigma,*}$ the mapping: $D_k\Phi_2^* \to \mathbb{R}^{kd_y} \times \mathbb{R}^{kd_y}$, $(z_1, z_2) \to (\Phi_k^\Sigma(z_1), \Phi_k^\Sigma(z_2))$

Definition 2. *The system Σ is said "differentially identifiable" of order k if the mapping $\Phi_{k,2}^{\Sigma,*}$ has the following property: If*

$$\Phi_{k,2}^{\Sigma,*}(z_1, z_2) \in \Delta_k,$$

then, $(x_1, \hat{\varphi}_1(0)) = (x_2, \hat{\varphi}_2(0))$.

This means that, for all controls (sufficiently differentiable), all couples (initial state, value of φ) are distinguished between them, by the observations and their $k - 1$ first derivatives, whatever the other time derivatives of φ are.

Equivalently, one can reconstruct the state and the values of φ, (at points x_0 visited by the system) in terms of the controls and their $k - 1$ first derivatives, the outputs and their $k - 1$ first derivatives.

Remark 2. *One could think that a good definition of differential identifiability of order k is just: the map Φ_k^Σ is injective. Unfortunately, this property is not generic (for uncontrolled systems), if there is less outputs than states ($d_y \leq n$).*

Our notion of **differential identifiability** will be useful later on: In the uncontrolled case, we will show that differential identifiability at certain orders k is generic, for $d_y \geq 3$ (Section 2).

It will be also useful, following the ideas developed in the book [7], and in the papers [5], [6], (in the context of **observability**), to define and **infinitesimal** notion of identifiability.

We give such a definition, very natural, just below. For this purpose, (as in the context of observability theory, [7]), we need an adequate concept of the "linearization" of a system.

The mapping $f : X \times U \times I \to TX$ induces a partial tangent mapping $T_{x,\varphi}f$: $TX \times TI \times U \to TTX$. (Here, $TI \sim I \times \mathbb{R}$). If ω denotes the canonical involution of TTX, then, $\omega \circ T_{x,\varphi}f$ defines a parametrized vector field on TX, also denoted by $T_x f$: it is parametrized by the elements $(u, (\varphi, \eta))$ of $U \times TI$.

Similarly, the function $h : X \times U \times I \to R^{d_y}$, has a partial differential $d_{x,\varphi}h$, and the **linearization of** Σ (or the **first variation** of Σ) is the following system on TX :

$$T\Sigma \begin{cases} \frac{d\xi}{dt} = T_{x,\varphi}f(x, u, \varphi; \xi, \eta); \\ \hat{y} = d_{x,\varphi}h(x, u, \varphi; \xi, \eta), \end{cases} \tag{1.4}$$

where $(\xi, \eta) \in T_x X \times T_\varphi I$.

Denote by Π the canonical projection: $TX \to X$. If $\xi : [0, T_\xi[\to TX$ is a trajectory of $T\Sigma$ for the control $u(.)$, function $\varphi(.)$, and the "variation" $\eta(.)$, (all belonging to $L^\infty_{[0,T_\xi[}$), then, $\Pi\xi$ is a trajectory of Σ corresponding to the same control $u(.)$ and function $\varphi(.)$. Conversely, if $\phi_\tau(x_0, u, \hat{\varphi}) : [0, e(x_0, u, \hat{\varphi})[\to X$ is a trajectory of Σ starting from x_0, and corresponding to the control u and the function $\hat{\varphi} : [0, T_{\hat{\varphi}}[\to I$, then, for all $0 < \tau < e(x_0, u, \hat{\varphi})$, the map:

$$(x, \varphi) \to \phi_\tau(x, u, \varphi);$$

is defined on a neighborhood of $(x_0, \hat{\varphi})$ in $X \times L^\infty([0, \tau], I)$, and is differentiable at $(x_0, \hat{\varphi})$. This differential $T_{x,\varphi}\phi_\tau$ is in the usual sense with respect to x, and in the Frechet sense with respect to φ.

For all $(\xi_0, \eta) \in TX \times L^\infty([0, \tau], \mathbb{R})$, for almost all $t \in [0, \tau]$:

$$P_{T\Sigma}(\xi_0, \eta)(t) = d_{x,\varphi}h(u, \hat{\varphi}; T_{x,\varphi}\phi_t(u, \hat{\varphi}; \xi_0, \eta), \eta) = T_{x,\varphi}P^t_\Sigma(\xi_0, \eta). \tag{1.5}$$

The right-hand side of this equality is the differential of the mapping $P^t_\Sigma(x, u, \hat{\varphi})$ with respect to (x, φ), at the point $(x_0, \hat{\varphi})$, with $P^t_\Sigma(x, u, \hat{\varphi}) = P_\Sigma(x, u, \hat{\varphi})(t)$.

Definition 3. *The system Σ is called "infinitesimally identifiable" at $(x_0, u(.), \hat{\varphi}(.)) \in X \times L^\infty[U] \times L^\infty[I]$ if all the linear mappings $P_{T\Sigma} : T_{x_0}X \times L^\infty_{([0,\tau],\mathbb{R})} \to L^\infty_{([0,\tau],\mathbb{R}^{d_y})}$ in (1.5), (for all $0 < \tau < e(x_0, u, \hat{\varphi})$), are injective. It is called "uniformly infinitesimally identifiable", if it is infinitesimally identifiable at all $(x_0, u(.), \hat{\varphi}(.)) \in X \times L^\infty[U] \times L^\infty[I]$.*

Remark 3. *The definitions 2, 3, do not depend on the internal variables and the internal mapping. On the contrary, Definition 1 does.*

A clear comparison between our 3 definitions of identifiability is not obvious at all, at the level of this introduction. Definition 1 is a natural and general definition. Definitions 2, 3, are adapted respectively to the generic and non generic cases (uncontrolled), as we shall see.

It will be shown (Theorem 6, Section 2) that, **for uncontrolled systems**, differential identifiability at some order k implies identifiability. Also, uniform infinitesimal identifiability implies identifiability of the restrictions of the system to certain open subsets. This last point will be proved in Sections 3, 4.

Very clearly, our identifiability properties are related to the notion(s) of "invertibility", introduced by Hirschorn in the papers [12], [13]. In fact, identifiability is a stronger property than invertibility, even for linear single input-output systems.

In particular, invertibility is generic, and moreover the set of noninvertible systems has infinite codimension.

1.2. **An example: biological reactors.** A simple biological reactor is a process where a population grows in presence of some substrate by eating the substrate. The concentration of the biomass (the population) in the mixture is denoted by x, and the concentration of the substrate is s. Here, x and s belong to R^+. The reactor is fed continuously by some flow of substrate, with concentration S_{in}, and flowrate $D(t) > 0$. S_{in} is a constant (in first approximation), and $D(t)$ is usually a control variable.

The way the population grows in presence of substrate is described by a "growth function" $\mu(x, s)$. The total volume is maintained constant by perpetually throwing out the same volume of mixture as the volume of substrate entering the reactor.

Hence, the equations are:

$$\begin{aligned} \frac{ds}{dt} &= -\mu.x + D(S_{in} - s) \\ \frac{dx}{dt} &= (\mu - D)x. \end{aligned} \tag{1.6}$$

The questions of observation and control of such biological reactor have been treated by several authors. See for instance [4], and the book [2].

In the literature, there are many possible choices of the function μ. One can find a cornucopia of them in the book [2].

Here, we will consider the case where this function μ is an unknown function, of either the variable s only, or the variable x only, or of both. We will give several conclusion about its identifiability.

In fact, the general study in this paper was initially motivated by this very interesting simple example.

1.3. **Our main theoretical results.** We will mainly consider the "uncontrolled case", i.e. there is no control variable $u(t)$.

In that case, we will get a lot of general results. To state them, let us endow the set S of uncontrolled systems, of the form (1.2) with the C^∞ Whitney topology.

1.3.1. *The generic case.* In section 2, we will show the following important result:

Theorem 1. *If the number of outputs is larger or equal to* 3, *then, differential identifiability of order* $2n+1$ $(n = \dim X)$, *is a generic property. This is true in the Baire sense only (i.e. the set of differentially identifiable systems is residual). This implies in particular that identifiability is a generic property.*

Remark 4. *This theorem has to be compared with the corresponding result for the observability theory (see* [7], [6]*). In the case of "single input observability", the number of outputs has to be larger or equal to* 2 *only. If one thinks about the function to be identified, as a control variable (which is what we do here in), it is surprising that the number of outputs for generic identifiability be 3. At a first glance, it should be 2.*

In section 2, we will show an open set of systems with 2 outputs, which is not identifiable.

1.3.2. *The single output case.* This case will be the subject of section 3.

In that case, by the previous section, differential identifiability is not generic. Again similarly to the observability theory of [7], [5], we will consider the uniform infinitesimal identifiability property. It is a so rigid property (infinite codimension, in fact) that it can be completely characterized.

We will show the following theorem, in the analytic case (i.e. the system is C^ω).

In the following, $\dim(X) = n$, and L_f is the Lie-derivative operator on X. Also, f_φ denotes the vector field $f(.,\varphi)$, for $\varphi \in I$, and $h_\varphi : X \to R^{d_y}$ is the map $h(.,\varphi)$. The symbol d_x means differential with respect to x only.

Theorem 2. *If* Σ *is uniformly infinitesimally identifiable, then, there is a subanalytic closed subset* Z *of* X, *of codimension 1 at least, such that on the open set* $X\backslash Z$, *the following two equivalent properties 1 and 2 below hold:*

1.a. $\frac{\partial}{\partial\varphi}\{(L_{f_\varphi})^k h_\varphi \equiv 0$, *for* $k = 0, ..., n-1$, *b.* $\frac{\partial}{\partial\varphi}\{(L_{f_\varphi})^n h_\varphi \neq 0$ *(in the sense that it* ***never*** *vanishes), c.* $d_x h_\varphi \wedge ... \wedge d_x L_{f_\varphi}^{n-1} h_\varphi \neq 0$,

2. any $x_0 \in X\backslash Z$ *has a coordinate neighborhood* $(x_1, ..., x_n, V_{x_0})$, $V_{x_0} \subset X\backslash Z$ *in which* Σ *(restricted to* V_{x_0}*) can be written:*

$$\left\{\begin{array}{c} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ . \\ . \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = \psi(x,\varphi); \\ y = x_1; \end{array}\right. \tag{1.7}$$

where $\frac{\partial}{\partial\varphi}\psi(x,\varphi)$ *never vanishes.*

This theorem has the following pseudo-converse:

Theorem 3. *Assume that* Σ *meets the equivalent conditions of the previous theorem.*

Then, any x_0 *has a neighborhood* V_{x_0} *such that the restriction* $\Sigma_{|V_{x_0}}$ *of* Σ *to* V_{x_0} *is uniformly infinitesimally identifiable,* ***identifiable and differentially identifiable of order*** $n+1$.

Notice that Theorem 2 has a global character: it is almost everywhere on X, but it is global with respect to $\varphi \in I$.

1.3.3. *The 2-output case.* This case will be the purpose of section 4.

Again, (differential) "identifiability" being not generic, we will consider the "uniform infinitesimal identifiability" property, that is very rigid (infinite codimension), so that we will be able to characterize it completely in a geometric way.

Since this geometric description is non obvious, and since there is (surprisingly) a small but nontrivial zoology (3 distinct cases), we do not give the intrinsic geometric characterization here. This is done in section 4 (Theorem 8). In this introduction, we state an equivalent theorem (Theorem 4) in terms of normal forms.

Theorem 4. *If Σ is uniformly infinitesimally identifiable, then, there is an open-dense subanalytic subset $\tilde{U}$ of $X \times I$, such that each point (x_0, φ_0) of $X \times I$, has a neighborhood $V_{x_0} \times I_{\varphi_0}$, and coordinates x on V_{x_0} such that the system Σ restricted to $V_{x_0} \times I_{\varphi_0}$, denoted by $\Sigma_{|V_{x_0} \times I_{\varphi_0}}$, has one of the three following normal forms:*

-type 1 normal form: *(in that case, $n > 2k$)*

$$
\begin{aligned}
y_1 &= x_1,\ y_2 = x_2, \\
\dot{x}_1 &= x_3,\ \dot{x}_2 = x_4, \\
&.. \\
\dot{x}_{2k-3} &= x_{2k-1},\ \dot{x}_{2k-2} = x_{2k}, \\
\dot{x}_{2k-1} &= f_{2k-1}(x_1, ..., x_{2k+1}), \\
\dot{x}_{2k} &= x_{2k+1}, \\
&.. \\
\dot{x}_{n-1} &= x_n, \\
\dot{x}_n &= f_n(x, \varphi),\ \textit{with}\ \frac{\partial f_n}{\partial \varphi} \neq 0\ \textit{(never vanishes)},
\end{aligned}
\tag{1.8}
$$

-type 2 normal form:

$$
\begin{aligned}
y_1 &= x_1,\ y_2 = x_2, \\
\dot{x}_1 &= x_3,\ \dot{x}_2 = x_4, \\
&.. \\
\dot{x}_{2r-3} &= x_{2r-1},\ \dot{x}_{2r-2} = x_{2r}, \\
\dot{x}_{2r-1} &= \Phi(x, \varphi),\ \dot{x}_{2r} = F_{2r}(x_1 ..., x_{2r+1}, \Phi(x, \varphi)), \\
\dot{x}_{2r+1} &= F_{2r+1}(x_1 ..., x_{2r+2}, \Phi(x, \varphi)), \\
&.. \\
\dot{x}_{n-1} &= F_{n-1}(x, \Phi(x, \varphi)), \\
\dot{x}_n &= F_n(x, \varphi),
\end{aligned}
\tag{1.9}
$$

with $\frac{\partial \Phi}{\partial \varphi} \neq 0,\ \frac{\partial F_{2r}}{\partial x_{2r+1}} \neq 0, ...,\ \frac{\partial F_{n-1}}{\partial x_n} \neq 0,$

-type 3 normal form:

$$
\begin{array}{rcl}
y_1 & = & x_1,\ y_2 = x_2, \\
\dot{x}_1 & = & x_3,\ \dot{x}_2 = x_4, \\
 & & .. \\
\dot{x}_{n-3} & = & x_{n-1},\ \dot{x}_{n-2} = x_n, \\
\dot{x}_{n-1} & = & f_{n-1}(x,\varphi),\ \dot{x}_n = f_n(x,\varphi),
\end{array} \tag{1.10}
$$

where $(\frac{\partial f_{n-1}}{\partial \varphi}\ ,\ \frac{\partial f_n}{\partial \varphi})$ *never vanishes.*

Also, in Section 4, we will give a lot of complementary results, examples, weak converses of Theorems 4, 8, and a few "global" results.

1.4. **Results for the biological reactor.** One of our results for the biological reactor will be like that: we consider the case of a single observation, the concentration s of the substrate.

Then, the biological reactor is not identifiable in the sense of definition 1.

Also, if we consider only constant input functions, ($D(.) = ct.$), then, the (single output) system is not uniformly infinitesimally identifiable in the sense of Definition 3, because it does not meet the necessary conditions of Theorem 2.

Assuming that the growth function depends on s only ($\pi : (x,s) \to s$ is the internal mapping), then, we prove (easily) the following theorem:

Theorem 5. *The bioreactor is identifiable at an admissible i-o trajectory* $(y(.), u(.))$ $= (s(.), D(.)$ *iff the output trajectory* $y(.) = s(.)$ *visits twice the same value.*

This result, and others, are the subject of Section 6.

1.5. **Identification.** In Section 5, we say a few words about the practical problem of identification, specially in the cases where identifiability is not generic: 1 and 2 outputs. In these cases, a reasonable practical methodology for identification is proposed.

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2. The Generic Case

The purpose of this section is to prove Theorem 1, and to give some complementary results and examples.

2.1. **Comparison between differential identifiability and identifiability.** We state a result in the **uncontrolled case** only. In the controlled case, the situation is much more complicated. It is possible to prove something very strong, using techniques developed in the book [7], for observability. But, this is a great complication, and it will be the purpose of another paper, since it is beyond the scope of this general study.

Theorem 6. ***(uncontrolled systems)*** *In the uncontrolled case, differential identifiability at some order implies identifiability.*

Proof. Let us assume that an uncontrolled system Σ is not identifiable. It means that there is an admissible output trajectory $\hat{y} : [0, \tau[\to \mathbb{R}^{d_y}$, such that Σ is not identifiable at $\hat{y}$, i.e., there are two trajectories $x_1(t), x_2(t)$, and two corresponding functions $\hat{\varphi}_1(t), \hat{\varphi}_2(t)$, $\hat{\varphi}_1(t) = \varphi_1(\pi(x_1(t)))$, $\hat{\varphi}_2(t) = \varphi_2(\pi(x_2(t)))$, such that φ_1 and φ_2 are smooth, hence $\hat{\varphi}_1(t), \hat{\varphi}_2(t)$, and $\hat{y}(t)$ are smooth. Moreover, $\hat{\varphi}_1(t_0) \neq \hat{\varphi}_2(t_0)$ for some $t_0 \in [0, \tau[$, or $x_1(0) \neq x_2(0)$. Restricting the interval $[0, \tau[$, we may assume that $t_0 = 0$. Then, for an arbitrary positive integer k, consider the two k-jets at time zero, $j^k\hat{\varphi}_1$, $j^k\hat{\varphi}_2$. Consider the mapping Φ_k^Σ defined in the introduction. Set $z_1 = (x_1(0), j^k\hat{\varphi}_1)$, $z_2 = (x_2(0), j^k\hat{\varphi}_2)$. Then, $(\Phi_k^\Sigma(z_1), \Phi_k^\Sigma(z_2)) \in \Delta_k$, the diagonal of R^{kd_y}. Therefore, Σ is not differentially identifiable of order k, since $\hat{\varphi}_1(0) \neq \hat{\varphi}_2(0)$ or $x_1(0) \neq x_2(0)$. ∎

2.2. **Preliminaries for the proof of Theorem 1.** Let S denote the set of C^∞ systems $\Sigma = (f, h)$ of the form 1.2, i.e elements of S are couples (f, h) of φ-parametrized vector fields f and functions h. We endow S with the C^∞ Whitney topology. Let J^kS denote the bundle of $k-$jets of systems in S. It is the fiber product $J^kF \times_{X\times I} J^kH$ of the bundles J^kF, J^kH over $X \times I$, that are respectively the bundles of $k-$jets of smooth sections of $TX \times I \to X \times I$ and $X \times I \times \mathbb{R}^{d_y} \to X \times I$.

Let us also denote by $J^kS_2^*$ the restriction of $J^kS^2 = J^kS \times J^kS$ to $((X \times I) \times (X \times I))\backslash\Delta(X \times I)$, where $\Delta(X \times I)$ is the diagonal in $((X \times I) \times (X \times I))$.

Let us recall that the mappings Φ_k^Σ, $\Phi_{k,2}^{\Sigma,*}$, defined in the introduction, depend only on the $k-$jet $j^k\Sigma$ of Σ. When we want to stress this fact, we write $\Phi^{j^k\Sigma}$ in place of Φ_k^Σ.

2.2.1. *The bad sets.*

Definition 4. *$B_1(k)$ is the subset of J^kS^2 of all couples $(j^k\Sigma(p), j^k\Sigma(q))$, such that: (1) $p \neq q$, $p = (x_1, \varphi_1)$, $q = (x_2, \varphi_2)$, (2) $f(p) = f(q) = 0$, (3) $h(p) = h(q)$.*

Definition 5. *a. Let $\hat{B}_2(k)$ be the subset of $J^kS^2 \times \mathbb{R}^{(k-1)} \times \mathbb{R}^{(k-1)}$, of all tuples $(j^k\Sigma(p), j^k\Sigma(q), v_1, v_2)$ such that: (1) $p \neq q$, $p = (x_1, \varphi_1)$, $q = (x_2, \varphi_2)$, (2) $f(p) \neq 0$ or $f(q) \neq 0$, (3) $\Phi_k^\Sigma(x_1, (\varphi_1, v_1)) = \Phi_k^\Sigma(x_2, (\varphi_2, v_2))$,*

b. $B_2(k)$ denotes the canonical projection of $\hat{B}_2(k)$ in J^kS^2.

The two following lemmas are obvious.

Lemma 1. *$B_1(k)$, $\hat{B}_2(k)$, $B_2(k)$, are respectively partially semi-algebraic subbundles of $J^kS_2^*$, $J^kS_2^* \times R^{(k-1)} \times R^{(k-1)}$, $J^kS_2^*$ (this means that heir typical fiber is a semi-algebraic subset of the fibers of the ambient bundles).*

Lemma 2. *If the map*

$$\Theta : ((X \times I) \times (X \times I))\backslash\Delta(X \times I) \to j^kS_2^*$$

$$(p, q) \to (j^k\Sigma(p), j^k\Sigma(q)),\ p = (x_1, \varphi_1),\ q = (x_2, \varphi_2),$$

avoids $B_1(k) \cup B_2(k)$, then Σ is differentially identifiable of order k.

2.3. **Proof of Theorem 1.** a. **Estimation of the codimension of $B_1(k)$ in $J^kS_2^*$** : It is obvious that this codimension is $2\ n + d_y$.

b. **Estimation of the codimension of $B_2(k)$ in $J^kS_2^*$** : We treat only the case $f(p) \neq 0$. The other case $f(q) \neq 0$ is similar.

Let $(x,\varphi,y,\psi)\in((X\times I)\times(X\times I))\backslash\Delta(X\times I)$. The typical fiber $\hat{B}_2(k,x,\varphi,y,\psi)$ of $\hat{B}_2(k)$ in $J^kS(x,\varphi)\times J^kS(y,\psi)\times\mathbb{R}^{k-1}\times\mathbb{R}^{k-1}$ is characterized by the following properties: (i) $f(p)\neq 0$ and (ii) $\Phi_k^{\Sigma}(x,\ (\varphi,v_1))-\Phi_k^{\Sigma}(y,\ (\psi,v_2))=0$.

Let G be the subset of $J^kS(x,\varphi)\times J^kS(y,\psi)\times\mathbb{R}^{k-1}\times\mathbb{R}^{k-1}$ of all tuples $(j^k\Sigma(x,\varphi),v_1,\ j^k\Sigma(y,\psi),v_2)$ such that $f(x,\varphi)\neq 0$ and let $\chi:G\to\mathbb{R}^k$ be the mapping $\chi(j^k\Sigma(x,\varphi),v_1,\ j^k\Sigma(y,\psi),v_2)=\Phi_k^{\Sigma}(x,\ (\varphi,v_1))-\Phi_k^{\Sigma}(y,\ (\psi,v_2))$. Then, $\hat{B}_2(k,x,\varphi,y,\psi)=\chi^{-1}(0)$.

The map χ is an algebraic mapping, affine in $j^kh(x,\varphi)$.

By Lemma 3 in Appendix 2.5 below, for fixed $v_1\in\mathbb{R}^k$ and $j^kf(x,\varphi)$, the linear mapping $j^kh(x,\varphi)\in J^kH(x,\varphi)\to\Phi^{j^k\Sigma}(x,\varphi,v_1)$ is surjective. This shows that the map $\chi:G\to\mathbb{R}^k$ is a submersion. Since $\hat{B}_2(k,x,\varphi,y,\psi)=\chi^{-1}(0)$,

$$\mathit{codim}(\hat{B}_2(k,x,\varphi,y,\psi),J^kS(x,\varphi)\times J^kS(y,\psi)\times\mathbb{R}^{k-1}\times\mathbb{R}^{k-1})=kd_y.$$

Hence:

$$\mathit{codim}(\hat{B}_2(k),J^kS_2^*\times\mathbb{R}^{k-1}\times\mathbb{R}^{k-1})=kd_y.$$

It follows that $\mathit{codim}(B_2(k),J^kS_2^*)\geq k(d_y-2)+2$.

c. **Final estimation of codim**$(B_1(k)\cup B_2(k),J^kS_2^*)$.

It follows from a. and b. above that:

$$\mathit{codim}(B_1(k)\cup B_2(k),J^kS_2^*)\geq\min(2\ n+d_y,k(d_y-2)+2). \tag{2.1}$$

d. **Proof of Theorem** 1. Let $k\geq 2\ n+1$, and $d_y\geq 3$. Then, $\mathit{codim}(B_1(k)\cup B_2(k),J^kS_2^*)\geq 2n+3$. We apply the standard multijet transversality theorems ([1], [11]) to the map

$$\rho:S\times((X\times I)\times(X\times I))\backslash\Delta(X\times I)\to J^kS_2^*,$$

$$(\Sigma,x,\varphi,y,\psi)\to(j^k\Sigma(x,\varphi),\ j^k\Sigma(x,\varphi)).$$

This allows to conclude that the set of $\Sigma\in S$ such that ρ_Σ is transverse to $B_1(k)\cup B_2(k)$ is residual. But, $\dim(((X\times I)\times(X\times I))\backslash\Delta(X\times I))=2n+2<\mathit{codim}(B_1(k)\cup B_2(k),J^kS_2^*)\geq 2n+3$.

Hence, the set of $\Sigma\in S$ such that ρ_Σ avoids $B_1(k)\cup B_2(k)$ is residual. By Lemma 2, this ends the proof of Theorem 1.

2.4. **A counterexample.** We consider, on the circle S^1, for values of φ in the interval $I=[-\pi,\pi]$, systems of the form: $\Sigma=\Sigma_0+\delta\Sigma$, where $\delta\Sigma$ is C^∞ but C^1 small, and Σ_0 is the system:

$$\Sigma_0\left\{\begin{array}{c}\dot{x}=\psi(x)-\varphi=f_0(x,\varphi)\\ y_1=\cos(x)=h_0^1(x,\varphi);\\ y_2=\sin(2x)=h_0^2(x,\varphi).\end{array}\right.$$

The cyclic coordinate on S^1 is x and $\psi : S^1 \to \mathbb{R}$ is such that $\psi(x) = x$ on the interval $[-\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon]$, for $\varepsilon > 0$ small. We want to solve the system of equations:

$$(2.2) \qquad \begin{aligned} (f_0 + \delta f)(x_1, \varphi_1) &= 0; \\ (f_0 + \delta f)(x_2, \varphi_2) &= 0; \\ (h_1 + \delta h_1)(x_1, \varphi_1) - (h_1 + \delta h_1)(x_2, \varphi_2) &= 0; \\ (h_2 + \delta h_2)(x_1, \varphi_1) - (h_2 + \delta h_2)(x_2, \varphi_2) &= 0; \end{aligned}$$

around the trivial solution $\delta f = 0, \delta h_1 = 0, \delta h_2 = 0$, $x_1 = \frac{\pi}{2}, \varphi_1 = \frac{\pi}{2}$, $x_2 = -\frac{\pi}{2}, \varphi_2 = -\frac{\pi}{2}$.

Here, the set S of systems is identified with a subspace of the Banach space BS of C^1 triples $(\delta f, \delta h_1, \delta h_2) \in (C^1(S^1 \times I))^3$ with the C^1 topology .

It is trivial to check that the Jacobian matrix of the system (2.2), with respect to the variables $(x_1, \varphi_1, x_2, \varphi_2)$ at the point $(\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}, -\frac{\pi}{2})$, is invertible. This means, applying the implicit function theorem in the Banach manifold $BS \times S^1 \times I \times S^1 \times I$, that for all C^∞ variations $\delta\Sigma$ small enough (C^1), we can find solutions to Equations (2.2), and these solutions are close to $x_1 = \frac{\pi}{2}, \varphi_1 = \frac{\pi}{2}$, $x_2 = -\frac{\pi}{2}, \varphi_2 = -\frac{\pi}{2}$.

But, such a solution of (2.2) produces a system Σ and two couples $(x_1, \varphi_1) \neq (x_2, \varphi_2)$ such that, for the **constant** control functions φ_1, φ_2, and for the initial conditions x_1, x_2, the corresponding couples of outputs (y_1, y_2) are constant functions of the time, that are equal.

Hence, there is an open neighborhood (C^1 open in C^∞) of systems, that are not differentially identifiable at any order k (and also not identifiable).

Remark 5. *Notice that, in the formula 2.1, the 2 bounds on the codimension are* $2n + 2$ *and* 2, *for* $d_y = 2$. *Hence, there should be* ***two typologies of counterexamples*** *for* $d_y = 2$.

In this example, we have exhibited the simplest one, corresponding to codimension $2n + 2$. *But, from the results of section 4, (the case* $d_y = 2$), *counterexamples corresponding to the other typology are easy to construct.*

2.5. **Appendix.** For the sake of completeness, we give here a (very simple) lemma which is already contained in the paper [6], and in the book [7].

If V is a vector space, $Sym^a(V)$ denotes the space of symmetric tensors of degree a on V, that can be canonically identified with the space of homogeneous polynomials of degree a over V^*, dual space of V. The symbol $\odot$ means symmetric tensor product or power.

Let $(x, \varphi, v) \in X \times I \times \mathbb{R}^{k-1}$ be given, and set $p = (x, \varphi)$. Let $f \in F$ (the space of smooth $I-$parametrized vector fields on X), such that $f(p) \neq 0$.

Lemma 3. *The mapping* $\Theta : J^k H \to \mathbb{R}^{kd_y}$, $j^k h \to \Phi^{j^k\Sigma}(x, \varphi, v)$ *is linear and surjective.*

Proof. Let f be a representative of $j^k f(p)$. Take a coordinate system $(O, x_1, ...x_n)$ for X at x. Then, if $y^{(r)}$ denotes the r^{th} derivative of the output at time 0, we have:

$$\begin{aligned} y^{(r)}(p, v) &= d_x^r h(p; f(p)^{\odot r}) + \sum_{a=0}^{r-1} d_x^a h(p, R_{a,r}(j^k f(p), v) + \\ & \sum_{\substack{l=1,..r, \\ s+l=r}} d_x^s d_\varphi^l h(p; T_{s,l}^r(j^k f(p), v)). \end{aligned}$$

The expressions $R_{a,r}$, $T^r_{s,l}$, are universal polynomial mappings : $J^kF(p)\times\mathbb{R}^{k-1}\to Sym^a(T_xX)$ (resp. $J^kF(p)\times R^{k-1}\to Sym^s(T_xX)\otimes\mathbb{R}^l$). This implies the result because $f(p)\neq 0$. ∎

3. The single-output case.

The purpose of this section is to prove Theorem 2 and Theorem 3

3.1. **Proof of Theorem 2.**

Let us assume (A) that $\frac{\partial}{\partial\varphi}(L_{f_\varphi})^i h_\varphi\equiv 0$, $i=0,...,k-1$, and $\frac{\partial}{\partial\varphi}(L_{f_\varphi})^k h_\varphi\neq 0$, for $k\leq n-1$.

Then, (B): the closed analytic subset of X : $Z_{k-1}=\{x|d_xh\wedge...\wedge d_xL_f^{k-1}h(x)=0\}$, has codimension 1 at least. Then, for all $x_0\in X\backslash Z_{k-1}$, it can be completed by functions $h_k(x),...,h_n(x)$, in order to form a local coordinate system on a neighborhood U_{x_0} of x_0.

Were it otherwise, then, by connectedness, $d_xh\wedge d_xL_fh\wedge...\wedge d_x(L_f)^{k-1}h\equiv 0$. Take $j\leq k-1$, the first integer such that $d_xh\wedge d_xL_fh\wedge...\wedge d_x(L_f)^jh\equiv 0$. Then, set, on some open subset of X : $x_1=h,....,x_j=(L_f)^{j-1}h$. complete this set of functions by $x_{j+1},...,x_n$, in order to form a coordinate system.

In these coordinates, with a straightforward computation, the system Σ can be rewritten:

$$\Sigma\begin{cases} y=x_1,\\ \dot{x}_1=x_2,\\ .\\ .\\ \dot{x}_{j-1}=x_j,\\ \dot{x}_j=\psi_j(x_1,...,x_j),\\ \dot{x}_{j+1}=\psi_{j+1}(x,\varphi)\\ .\\ \dot{x}_n=\psi_n(x,\varphi),\end{cases}$$

which is obviously not uniformly infinitesimally identifiable: take in the first variation, $\xi(0)\neq 0,\xi_1(0)=..=\xi_j(0)=0$, and the "variation" $\eta(.)\equiv 0$. The output of the first variation system is identically zero, whatever $\varphi(.)$. A contradiction with uniform infinitesimal identifiability.

This shows (B).

Now, let us show that (A) is impossible.

In the coordinate system defined as in (B), the system writes, on some on some open subset:

$$\sum \begin{cases} y = x_1, \\ \dot{x}_1 = x_2, \\ \cdot \\ \cdot \\ \dot{x}_{k-1} = x_k, \\ \dot{x}_k = \psi_k(x, \varphi), \\ \cdot \\ \cdot \\ \dot{x}_n = \psi_n(x, \varphi). \end{cases} \tag{3.1}$$

Now, the linearized system writes, in these coordinates (and the associated coordinates ξ_i on TX):

$$\begin{cases} \dot{x} = f(x, \varphi), \\ \dot{\xi}_1 = \xi_2, \\ \dot{\xi}_{k-1} = \xi_k, \\ \dot{\xi}_i = d_x\psi_i(x, \varphi)\xi + d_\varphi\psi_i(x, \varphi)\eta, \\ i = k, ..., n, \end{cases} \tag{3.2}$$

where f is as in 3.1. Let us take $\xi(0) \neq 0$, $\xi_1(0) = 0, ..., \xi_k(0) = 0$. Our assumption (A) implies that on an open dense analytic subset D of $X \times I$, $d_\varphi\psi_k(x, \varphi) \neq 0$. Around a point $p_0 = (x_0, \xi(0), \varphi_0) \in TX \times I$, $(x_0, \varphi_0) \in D$, with $\xi(0)$ just defined, we consider the feedback function for the system 3.2: $\eta(x, \varphi, \xi) = \frac{-d_x\psi_k(x,\varphi)\xi}{d_\varphi\psi_k(x,\varphi)}$. We consider also the function $\varphi(t) \equiv \varphi_0$. Then, the feedback system 3.2 has a unique trajectory $(\xi(t), x(t))$ around p_0, starting from (x_0, ξ_0) at time 0. By construction, this trajectory is such that $\xi_1(t), ..., \xi_k(t) = 0$ for all t small enough. This contradicts the infinitesimal identifiability assumption. Hence, $\frac{\partial}{\partial\varphi}(L_{f_\varphi})^k h_\varphi = 0$, (A) is impossible.

Now, at this point, identically on $X \times I$, $\frac{\partial}{\partial\varphi}\{(L_{f_\varphi})^k h_\varphi = 0$, for $k = 0, ..., n-1$. This is the property 1.a. of Theorem 2. Property 1.c. also holds, by the proof of (B) above, which works also in that case. Moreover, we already know that, on an open dense semianalytic subset D of X, our system can be rewritten (in local coordinates, around any point $x_0 \in D$):

$$\begin{cases} y = x_1, \\ \dot{x}_1 = x_2, \\ \cdot \\ \cdot \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = \psi(x, \varphi). \end{cases} \tag{3.3}$$

Consider the closed analytic subset of $S \subset X \times I$, formed by the (x, φ)'s satisfying $\frac{\partial\psi}{\partial\varphi} = 0$ (or equivalently, $\frac{\partial}{\partial\varphi}\{(L_{f_\varphi})^n h_\varphi = 0)$. If this subset has nonempty interior, by analyticity and connectedness, it is the whole $X \times I$, and ψ does not depend on φ, which is easily seen as a contradiction with infinitesimal identifiability. Then, S has codimension 1 at least. Let ΠS be the projection of S on X, $\Pi : X \times I \to X$. Since I is compact, ΠS is subanalytic. By Hardt's Theorem ([9]), we can stratify the mapping $\Pi : S \to \Pi S$. Let S_1, ΠS_1 be two strata such that ΠS_1 has maximal dimension, and Π maps S_1 submersively onto ΠS_1. Let $\hat{\varphi} : \Pi S_1 \to S_1$ be a smooth section of Π. (See the footnote at the beginning of the appendix, Section 7).

Assume that $\dim(S_1) = \dim X = n$. Then, $\hat{\varphi}$ is a smooth mapping from an open subset ΠS_1 of X to $X \times I$.

Let us apply this (feedback) function $\hat{\varphi}$ to our linearized system restricted to $\Pi S_1 \times I$. In the coordinates defined above, it rewrites:

$$
\begin{aligned}
y &= x_1, \\
\dot{x}_1 &= x_2, ..., \dot{x}_{n-1} = x_n, \dot{x}_n = \psi(x, \hat{\varphi}(x)), \\
\dot{\xi}_1 &= \xi_2, ..., \dot{\xi}_{n-1} = \xi_n, \dot{\xi}_n = d_x\psi(x, \hat{\varphi}(x))\xi + 0. \\
\hat{y} &= \xi_1.
\end{aligned}
\tag{3.4}
$$

This equation 3.4 is independent of η, the input of the linearized system. Then, let us take $\xi(0) = 0$, $\eta(t) \neq 0$. Then $\hat{y}(t)$ is identically zero, which contradicts the infinitesimal identifiability.

Therefore, ΠS_1 is not open in X, and therefore, it has codimension 1. ΠS has codimension 1, and this shows exactly the property 1.b. of Theorem 2, together with the property 2. (the normal form), in the statement of the theorem. This ends the proof.□

3.2. **Proof of Theorem 3.** Assume that Σ is a system in normal form 3.3, on some open neighborhood O of $x = 0$ in $\mathbb{R}^n$, with $\frac{\partial\psi}{\partial\varphi}(x)$ never vanishing.

It is clear that admissible output trajectories (there is no input in our case), are smooth. Given any smooth function $y(.) : [0, T[\to \mathbb{R}$, an immediate computation with the normal form 3.3 shows that $x_1(t) = y(t)$, $x_2(t) = \dot{y}(t), ...$, $x_n(t) = \frac{d^{n-1}y}{dt^{n-1}}(t)$. Also,

$$\frac{d^n y}{dt^n}(t) = \psi(y, \dot{y}, ..., y^{(n-1)}(t), \varphi(t)). \tag{3.5}$$

Assume that $y(.)$ is an admissible output trajectory. For $y, \dot{y}, ..., y^{(n-1)}$ fixed, set:

$$\psi_{y,\dot{y},...,y^{(n-1)}}(\varphi) = \psi(y, \dot{y}, ..., y^{(n-1)}, \varphi).$$

The function $\psi_{y,\dot{y},...,y^{(n-1)}}$ is monotonous: $\frac{\partial\psi}{\partial\varphi}(.)$ never vanishes. Since $y(.)$ is an admissible output trajectory, then, the equation 3.5 has a solution $\varphi(t)$ for all $t \in [0, T]$, (and this solution is smooth w.r.t. t). By the monotonicity, this solution is unique. This means that Σ is identifiable, for $Z = O$ and $\pi : O \to O$ being the trivial "internal mapping". For the same reason, it is also identifiable if $\pi : O \to Z$, is nontrivial.

To finish, by the normal form, it is just a matter of trivial computation to show that $\Sigma_{|O}$ is differentially identifiable of order $n + 1$, and the uniform infinitesimal identifiability of $\Sigma_{|O}$ is also obvious, from the normal form.

4. The two-output case

Here, we want to prove Theorem 4, give a series of intrinsic conditions corresponding to these normal forms, state and prove several weak converses of Theorem 4 for all these normal forms, and give a few examples.

4.1. **Preliminaries, definitions.** Here, as above, L is the Lie derivative operator on X. Hence, if $f(x,\varphi)$ is a φ-dependant vector field, L_f is the Lie derivative operator with respect to the vector fields $f_\varphi(x) = f(x,\varphi)$, for φ fixed in I.

Let $N(l)$ be the rank at generic points of $X \times I$ of the family E_l of one-forms on X:

$$E_l = \{d_x h_i, d_x L_f h_i, ..., d_x L_f^{l-1} h_i, i = 1, 2\}.$$

Set $N(0) = 0$.

This set of generic points U_l, is the intersection of the open sets $\tilde{U}_i$, $i \leq l$, where E_i has maximal rank. U_l is semianalytic, open and dense in $X \times I$. Moreover, $U_{l+1} \subset U_l$.

It is easy to check that $N(l)$ increases strictly by steps of 2, up to $l = k$, and after, (eventually), it increases by steps of 1 up to $l = l_M$, $N(l_M) \leq n$.

It may happen that $k = 0$, *i.e.* $N(1) = 1$.

Lemma 4. *If Σ is uniformly infinitesimally identifiable, then, $N(l_M) = n$.*

The idea in this lemma is that, if it is not true, then, for constant functions $\varphi(.)$, infinitesimal identifiability will be contradicted.

Proof. If $N(l_M) < n$, let $(x_1, ...x_n, \varphi)$ be a coordinate system in an open subset of U_{l_M} formed by φ, and by $N(l_M)$ functions of (x,φ) chosen, in the family $\{h_i, L_f h_i, ..., L_f^{l_M-1} h_i, i = 1, 2\}$, (the $N(l_M)$ first coordinates), and other x-coordinates. In these coordinates, it is easy to see that, for the constant function $\varphi(.) \equiv \varphi_0$, Σ can be rewritten as:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, ..., x_{N(l_M)}, \varphi_0), \\
& . \\
\dot{x}_{N(l_M)} &= f_1(x_1, ..., x_{N(l_M)}, \varphi_0), \\
\dot{x}_{N(l_M)+1} &= f_1(x_1, ..., x_n, \varphi_0), \\
& . \\
\dot{x}_n &= f_1(x_1, ..., x_n, \varphi_0), \\
y_1 &= h_1(x_1, ..., x_{N(l_M)}, \varphi_0), \\
y_2 &= h_2(x_1, ..., x_{N(l_M)}, \varphi_0).
\end{aligned}$$

Then, taking $\eta \equiv 0$, and $\xi_1(0) = ... = \xi_{N(l_M)}(0) = 0$, $\xi_{N(l_M)+1}(0) \neq 0$, in the equation of the first variation, we see that the solution $\xi(t)$ verifies $\xi_1(t) = ... = \xi_{N(l_M)}(t) = 0$, on a small time interval $[0,T]$, $T > 0$. This implies that on this time interval, the output $\hat{y} = (\hat{y}_1, \hat{y}_2)$ of the first variation is identically zero. A contradiction with infinitesimal identifiability for $\varphi(.) \equiv \varphi_0$. ∎

Definition 6. *We define r, the **"order"** of the system, as the first integer such that $d_\varphi L_f^r h$ does not vanish identically.*

Lemma 5. *If Σ is uniformly infinitesimally identifiable, then, $r \leq l_M$.*

Proof. Assume $r \geq l_M + 1$. Let us take again a coordinate system on an open subset of X, formed by functions of the family $\{h_i, L_f h_i, ..., L_f^{l_M-1} h_i, i = 1, 2\}$. These functions are functions of x only, since $r \geq l_M + 1$. By the previous lemma, this is possible. It is obvious that, in these coordinates, the system can be rewritten:

$$\dot{x} = f(x), \quad y = h(x).$$

Let us take $\xi(0)=0$, but a function $\eta(t)$ nonzero, in the first variation. Then, the output $\hat{y}$ of the first variation is identically zero, on some open time interval. This contradicts the infinitesimal identifiability. ∎

Definition 7. *A system Σ is **regular** if $N(l_M)=n$ and $r\leq l_M$.*

If a system is uniformly infinitesimally identifiable, then it is regular, by the 2 previous lemmas. From now on, in this section, we will assume that systems Σ under consideration are regular.

The integer k is the first with the following properties:

$$\begin{aligned} d_xh_1\wedge d_xh_2\wedge d_xL_fh_1\wedge ...\wedge d_xL_f^kh_1\wedge d_xL_f^kh_2 &\equiv 0,\quad \text{but} \\ d_xh_1\wedge d_xh_2\wedge d_xL_fh_1\wedge ...\wedge d_xL_f^{k-1}h_1\wedge d_xL_f^{k-1}h_2 &\neq 0 \text{ (not identically zero).}\end{aligned}$$

If $r=k$, there are three possibilities:

A. $n=2k$;

B.

B.1.

$$d_xh_1\wedge d_xh_2\wedge d_xL_fh_1\wedge ...\wedge d_xL_f^{k-1}h_2\wedge d_xL_f^kh_1\neq 0$$

(hence $n>2k$) and $d_\varphi L_f^kh_2\neq 0$; or,

B.2.

$$d_xh_1\wedge d_xh_2\wedge d_xL_fh_1\wedge ...\wedge d_xL_f^{k-1}h_2\wedge d_xL_f^kh_2\neq 0$$

(hence $n>2k$) and $d_\varphi L_f^kh_1\neq 0$;

C.

C.1

$$d_xh_1\wedge d_xh_2\wedge d_xL_fh_1\wedge ...\wedge d_xL_f^{k-1}h_2\wedge d_xL_f^kh_1\neq 0$$

(hence $n>2k$) and $d_\varphi L_f^kh_2\equiv 0$, $d_xh_1\wedge d_xh_2\wedge d_xL_fh_1\wedge ...\wedge d_xL_f^{k-1}h_2\wedge d_xL_f^kh_2\equiv 0$, or

C.2

$$d_xh_1\wedge d_xh_2\wedge d_xL_fh_1\wedge ...\wedge d_xL_f^{k-1}h_2\wedge d_xL_f^kh_2\neq 0$$

(hence $n>2k$) and $d_\varphi L_f^kh_1\equiv 0$, $d_xh_1\wedge d_xh_2\wedge d_xL_fh_1\wedge ...\wedge d_xL_f^{k-1}h_2\wedge d_xL_f^kh_1\equiv 0$

Definition 8. *Let Σ be a **regular** system. We say that Σ has:*
*-**type 1** if $r>k$, or $r=k$ but **C**. is satisfied,*
*-**type 2** if $r<k$, or $r=k$ but **B**. is satisfied,*
*-**type3** if $r=k$ and **A**. is satisfied.*

Lemma 6. *Types 1, 2 and 3 exhaust the class of regular systems, and form a partition of this class.*

Proof. For a system with $r\neq k$, it is clear that it is either of type 1 or type 2 and not both. There can be only problems for $r=k$. If we show that, for $r=k$:
i) cases B. and C. exhaust all regular systems with $n>2k$,
ii) cases B. and C. do not intersect;
then, the theorem is proved since $n\geq 2k$.

Assume that Σ is simultaneously B. and C., then:

If Σ is B.1., it cannot be C.2. which contradicts $d_xh_1 \wedge d_xh_2 \wedge d_xL_fh_1\wedge \ ... \wedge d_xL_f^{k-1}h_2 \wedge d_xL_f^kh_1 \neq 0$, and it cannot be C.1., which contradicts $d_\varphi L_f^k h_2 \neq 0$.

On the same way, if Σ is B.2., it cannot be C.1. which contradicts $d_xh_1 \wedge d_xh_2 \wedge d_xL_fh_1\wedge \ ... \wedge d_xL_f^{k-1}h_2 \wedge d_xL_f^kh_2 \neq 0$, and it cannot be C.2., which contradicts $d_\varphi L_f^k h_1 \neq 0$.

This shows ii).

Proof of i): By definition of $k:$

$$\begin{aligned} d_xh_1 \wedge d_xh_2 \wedge d_xL_fh_1 \wedge ... \wedge d_xL_f^kh_1 \wedge d_xL_f^kh_2 &\equiv 0, \\ d_xh_1 \wedge d_xh_2 \wedge d_xL_fh_1 \wedge ... \wedge d_xL_f^{k-1}h_1 \wedge d_xL_f^{k-1}h_2 &\neq 0. \end{aligned}$$

Since $n > 2k$,either:

$$(a) \quad d_xh_1 \wedge d_xh_2 \wedge d_xL_fh_1 \wedge ... \wedge d_xL_f^{k-1}h_1 \wedge d_xL_f^{k-1}h_2 \wedge d_xL_f^kh_1 \neq 0,$$

or:

$$(b) \quad d_xh_1 \wedge d_xh_2 \wedge d_xL_fh_1 \wedge ... \wedge d_xL_f^{k-1}h_1 \wedge d_xL_f^{k-1}h_2 \wedge d_xL_f^kh_2 \neq 0,$$

or both: were it otherwise, $l_M = k, N(l_M) = 2k$, and $n = 2k$, by Lemma 4.

Assume that (a) and (b) hold simultaneously.

Then, since $d_\varphi L_f^k h = d_\varphi L_f^r h \neq 0$ by definition of r, either (α), $d_\varphi L_f^k h_1 \neq 0$, or (β), $d_\varphi L_f^k h_2 \neq 0$ Assume (α) (the case (β) is symmetric). Then we are in case B.2.

Assume that (a) holds, but not (b) (the case (b) holds but not (a) is symmetric). Then:

$$\begin{aligned} d_xh_1 \wedge d_xh_2 \wedge d_xL_fh_1 \wedge ... \wedge d_xL_f^{k-1}h_1 \wedge d_xL_f^{k-1}h_2 \wedge d_xL_f^kh_1 &\neq 0, \\ d_xh_1 \wedge d_xh_2 \wedge d_xL_fh_1 \wedge ... \wedge d_xL_f^{k-1}h_1 \wedge d_xL_f^{k-1}h_2 \wedge d_xL_f^kh_2 &\equiv 0. \end{aligned}$$

If $d_\varphi L_f^k h_2 \neq 0$, we are in case B.1., if $d_\varphi L_f^k h_2 \equiv 0$, we are in case C.1. This ends the proof. ∎

Type 2 regular systems:

For a regular system of type 2, eventually interchanging the role of h_1, h_2, we can assume that $d_\varphi L_f^r h_2(x,\varphi) \neq 0$. In a neighborhood of a point $(x_0, \varphi_0) \in U_{l_M}$, such that $L_f^r h_2(x_0, \varphi_0) = u_0$ and $d\varphi L_f^r h_2(x_0, \varphi_0) \neq 0$, there is an analytic function $\Phi^*(x,u)$, such that $L_f^r h_2(x, \Phi^*(x,u)) = u$. Let us consider the "auxiliary system" $\Sigma_A :$

$$\Sigma_A \left\{ \begin{array}{l} \dot{x} = f(x, \Phi^*(x, \tilde{\varphi})) = F(x, \tilde{\varphi}) \\ y = h(x, \Phi^*(x, \tilde{\varphi})) = H(x, \tilde{\varphi}). \end{array} \right.$$

This system is well defined and intrinsic, over an open set $V_{x_0} \times V_{\varphi_0} \subset X \times I$.

By construction, the integer r (the order) associated with this auxiliary system is the same as the one of the given system Σ.

Moreover, the following flags D and D^A of integrable distributions over V_{x_0} :

$$\begin{aligned} D_0(x) &= T_xX, \ D_1(x) = Ker(d_xh(x)), \ ..., D_r(x) = D_{r-1}(x) \cap Ker(d_xL_f^{r-1}h(x)), \\ D &= \{D_0 \supset D_1 \supset ... \supset D_r\}; \end{aligned}$$

and

$$\begin{aligned} D_0^A(x) &= T_xX,\ D_1^A(x) = Ker(d_xh(x)),\ ..., D_r^A(x) = D_{r-1}^A(x) \cap Ker(d_xL_f^{r-1}H(x)), \\ D^A &= \{D_0^A \supset D_1^A \supset ... \supset D_r^A\}, \end{aligned}$$

are equal.

Let us "prolong" the auxiliary flag D^A, in the following way:

$$\begin{aligned} D_{r+1}^A(x,\tilde{\varphi}) &= D_r^A(x) \cap Ker(d_xL_F^rH_1(x,\tilde{\varphi})), \\ D_{i+1}^A(x,\tilde{\varphi}) &= D_i^A(x) \cap Ker(d_xL_F^iH_1(x,\tilde{\varphi})), \\ D^A(\tilde{\varphi}) &= \{D_0^A \supset D_1^A \supset .. \supset D_r^A \supset D_{r+1}^A(\tilde{\varphi}) \supset .. \supset D_l^A(\tilde{\varphi}) = D_{l+1}^A(\tilde{\varphi})\}, \end{aligned}$$

where l is the first integer such that $D_l^A(x,\tilde{\varphi}) = D_{l+1}^A(x,\tilde{\varphi})$ at generic points.

Definition 9. *The auxiliary flag $D^A(\tilde{\varphi})$ is* ***regular*** *on an open subset $U \subset X \times I$, if $D_l^A(\tilde{\varphi}) = \{0\}$, and all the other $D_i^A(\tilde{\varphi})$ have constant rank first $n-2i$ $(i \leq r)$, second $n-r-i$ $(r < i < l)$, third, 0 $(i \geq l = n-r)$; on this open set.*

Definition 10. *The auxiliary flag $D^A(\tilde{\varphi})$ is* ***uniform*** *on an open subset $U \subset X \times I$, if it is regular, and independent of $\tilde{\varphi}$.*

4.2. **Normal form for a uniform auxiliary flag (systems of type 2).** Here, we consider a regular system Σ of type 2, with a uniform auxiliary flag over $X \times I$. The flag being integrable (in the sense that it is a flag of integrable distributions of constant rank), around each point of $X \times I$, we can find coordinates x such that:

$$\begin{aligned} D_{l-1}^A(\tilde{\varphi}) &= Span\{\frac{\partial}{\partial x_n}\}, ..., D_{r+1}^A(\tilde{\varphi}) = Span\{\frac{\partial}{\partial x_n}, ..., \frac{\partial}{\partial x_{2r+2}}\}, \\ D_r^A(\tilde{\varphi}) &= Span\{\frac{\partial}{\partial x_n}, ..., \frac{\partial}{\partial x_{2r+1}}\}, D_1^A(\tilde{\varphi}) = Span\{\frac{\partial}{\partial x_n}, ..., \frac{\partial}{\partial x_3}\}. \end{aligned}$$

Moreover, we can take as x coordinates:

$$x_1 = h_1(x), x_2 = h_2(x), ..., x_{2r-1} = L_F^{r-1}h_1(x), x_{2r} = L_F^{r-1}h_2(x).$$

Then, the auxiliary system Σ_A can be written:

$$\begin{aligned} \dot{x}_1 &= x_3,\ \dot{x}_2 = x_4, \\ &.. \\ \dot{x}_{2r-3} &= x_{2r-1},\ \dot{x}_{2r-2} = x_{2r}, \\ \dot{x}_{2r-1} &= F_{2r-1}(x,\tilde{\varphi}),\ \dot{x}_{2r} = F_{2r}(x,\tilde{\varphi}) = \tilde{\varphi}, \\ &.. \\ \dot{x}_n &= F_n(x,\tilde{\varphi}), \\ y_1 &= x_1,\ y_2 = x_2. \end{aligned} \tag{4.1}$$

Since $D_{r+1}^A(\tilde{\varphi}) = Span\{\frac{\partial}{\partial x_n}, ..., \frac{\partial}{\partial x_{2r+2}}\}$, we must have $\frac{\partial F_{2r-1}}{\partial x_{2r+2}} = .. = \frac{\partial F_{2r-1}}{\partial x_n} = 0$, and $\frac{\partial F_{2r-1}}{\partial x_{2r+1}} \neq 0$, or, locally:

$$F_{2r-1} = F_{2r-1}(x_1, ..., x_{2r+1}, \tilde{\varphi}),\ \frac{\partial F_{2r-1}}{\partial x_{2r+1}} \neq 0.$$

Repeating this reasoning, we get that:

$$\begin{aligned} F_{n-2} &= F_{n-2}(x_1,...,x_{n-1},\tilde{\varphi}),\ \frac{\partial F_{n-2}}{\partial x_{n-1}} \neq 0, \\ F_{n-1} &= F_{n-1}(x_1,...,x_n,\tilde{\varphi}),\ \frac{\partial F_{n-1}}{\partial x_n} \neq 0, \\ F_n &= F_n(x,\tilde{\varphi}). \end{aligned}$$

Conversely, if the system Σ^A is like that, then it is just a trivial computation to check that the auxiliary flag is uniform.

Hence, replacing $\tilde{\varphi}$ by $\Phi(x,\varphi) = L_f^r h_2(x,\varphi)$, and reversing the role of h_1, h_2, we get the following result:

Theorem 7. ***(Normal form for a uniform auxiliary flag)*** *A system Σ has a uniform auxiliary flag around (x_0,φ_0), iff there is a neighborhood $V_{x_0}\times I_{\varphi_0}$ of (x_0,φ_0), and coordinates on V_{x_0} such that Σ can be written:*

$$\begin{aligned} y_1 &= x_1,\ y_2 = x_2, \\ \dot{x}_1 &= x_3,\ \dot{x}_2 = x_4, \\ &.. \\ \dot{x}_{2r-3} &= x_{2r-1},\ \dot{x}_{2r-2} = x_{2r}, \\ \dot{x}_{2r-1} &= \Phi(x,\varphi),\ \dot{x}_{2r} = F_{2r}(x_1...,x_{2r+1},\Phi(x,\varphi)), \\ \dot{x}_{2r+1} &= F_{2r+1}(x_1...,x_{2r+2},\Phi(x,\varphi)), \\ &.. \\ \dot{x}_{n-1} &= F_{n-1}(x,\Phi(x,\varphi)), \\ \dot{x}_n &= F_n(x,\varphi), \end{aligned}$$

with $\frac{\partial \Phi}{\partial \varphi} \neq 0,\ \frac{\partial F_{2r}}{\partial x_{2r+1}} \neq 0, ...,\ \frac{\partial F_{n-1}}{\partial x_n} \neq 0.$

4.3. **Statement of the results for the two-output case.**

Theorem 8. ***(main result in the 2-output case)*** *If Σ is uniformly infinitesimally identifiable, (**hence regular**), then, there is an open-dense subanalytic subset $\tilde{U}$ of $X\times I$, such that at each point (x_0,φ_0) of $\tilde{U}$, Σ has the following properties, on a neighborhood of (x_0,φ_0):*

-If Σ has type 2, the auxiliary flag is uniform,

-If Σ has type 1, then, $N(r) = n$.

Remark 6. *a. In the case where Σ has type 3, then, there is no other requirement.*

*b. In the case of type 1 assume $r = k$. Then, we have **C.,** which implies $n > 2k$. But $N(r) = n = N(k) = 2k$. Then, **for type 1 systems,** $r = k$ **cannot happen for a uniformly infinitesimally identifiable system.***

This theorem is in fact equivalent to the theorem in the introduction, Theorem 4.

These two equivalent theorems (Theorems 8, 4) have a weak converse:

Theorem 9. *Assume that Σ satisfies the equivalent conditions of theorems 8, 4, on some subset $V_{x_0}\times I_{\varphi_0}$ of $X\times I$ (so that, taking V_{x_0}, I_{φ_0} small enough, the restriction $\Sigma_{|V_{x_0}\times I_{\varphi_0}}$ has one of the three normal forms above on $V_{x_0}\times I_{\varphi_0}$). Then,* *in case type 1, type 2, (normal forms 1.8, 1.9)* $\Sigma_{|V_{x_0}\times I_{\varphi_0}}$ *is uniformly infinitesimally identifiable* ***and identifiable****. In case type 3 (normal form 1.10), this is also true, eventually restricting the neighborhoods* V_{x_0}, I_{φ_0}.

Also, in the special case of type 1, there is a stronger result:

Theorem 10. *Assume* Σ *is uniformly infinitesimally identifiable, (****hence regular****). Assume that* Σ *has type 1. Then, there is an open-dense subanalytic subset* $\tilde{X}$ *of* X*, such that each point* x_0 *of* $\tilde{X}$*, has a neighborhood* V_{x_0}*, and coordinates* x *on* V_{x_0} *such that the system* Σ *restricted to* $V_{x_0}\times I$*, denoted by* $\Sigma_{|V_{x_0}}$*, has the normal form 1.8 (globally over* $V_{x_0}\times I$*). Conversely, if it is the case, then, the restriction* $\Sigma_{|V_{x_0}}$ *is uniformly infinitesimally identifiable* ***and identifiable****.*

Several points in this last theorem are not true in the case of types 2 and 3.

Example 1. *(Type 2) consider the type 2 system:*

$$\begin{aligned} y_1 &= x_1, y_2 = x_2, \\ \dot{x}_1 &= x_3\cos\varphi,\ \dot{x}_2 = x_3\sin\varphi, \\ \dot{x}_3 &= f(x). \end{aligned}$$

where $x_3 > 0$ $(x\in X = \mathbb{R}^2\times\mathbb{R}_+)$*, and* $I = [-A, A]$*, for* $A > 0$*, sufficiently large.*

For this system, $k = 1$, $r = 1$, and **B**. is always satisfied.

(a) The normal form 1.9 is met only locally with respect to φ (changing the role of h_1 and h_2 depending on φ);

(b) For any open subset $\tilde{X}$ of X, $\Sigma_{|\tilde{X}}$ is never identifiable: for $\varphi(t)$ and $x(0)$ arbitrary, $(x(0), \varphi(t))$ and $(x(0), \varphi(t) + 2\pi)$ produce the same output.

Example 2. *(Type3) Consider the type 3 system on* $\mathbb{R}^2$:

$$\begin{aligned} y_1 &= x_1, y_2 = x_2, \\ \dot{x}_1 &= \cos\varphi,\ \dot{x}_2 = \sin\varphi, \end{aligned}$$

where, as above, $I = [-A, A]$, for $A > 0$, sufficiently large. Again, $(x(0), \varphi(t))$ and $(x(0), \varphi(t) + 2\pi)$ produce the same output.

Example 3. *(Type 2) Consider the system on a subset* $X\times I$ *of* $\mathbb{R}^2\times\mathbb{R}$*, with* X *open:*

$$\begin{aligned} y_1 &= \frac{1}{2}(\varphi - x_2)^2,\ y_2 = x_1 \\ \dot{x}_1 &= x_2,\ \dot{x}_2 = 0. \end{aligned}$$

For this system, $r = 0$, $k = 1$.

If we take for X and I neighborhoods of zero in $\mathbb{R}^2$ and $\mathbb{R}$, then, the auxilliary flag is not uniform (on any open subset with zero in its closure), and the normal form 1.9 is not met. On the contrary, if X is a neighborhood of $(x_1(0), x_2(0))$, and I is a neighborhood of φ_0, provided that these neighborhoods are small and $\varphi_0 \neq x_2(0)$, the auxilliary flag is uniform, and the normal form is met.

4.4. **Proof of the results for the 2-output case.**

First, let us show that **theorems 8, 4 are equivalent.**

It is just a matter of simple computations to see that normal form 1.9 (resp. 1.8, 1.10) in Theorem 4 imply the conditions "type 2" (resp. "type 1", "type 3" of theorem 8.

Conversely it is easy to check that, in Theorem 8, conditions "type 1", "type 3" imply the corresponding normal forms in Theorem 4. For type 3, see the (trivial) details in the proof of Theorems 8, 4, below. Condition "type 2" is equivalent to the normal form 1.9 by Theorem 7.

Proof. (of Theorem 9).

1. Normal form 1.8. From the normal form, computing the first variation, it follows immediately that $\hat{y}(t) = 0$ on some time interval $[0, \varepsilon]$ ($\hat{y}$ the output of the first variation), implies $\xi(0) = 0$ on the same time interval, where $\xi(0) \in T_{x_0}X$ is the initial condition for the first variation. Deriving once more, we get that $\eta(.)$ (the variation control) vanishes for almost all t. This shows that $\Sigma_{|V_{x_0}\times I_{\varphi_0}}$ is uniformly infinitesimally identifiable.

Now, from the normal form, differentiating the output $y(t)$ a certain number of times, one reconstructs the full state $x(t) = (x_1(t), ..., x_n(t))$ of the system.

Knowing $x(t)$, the equation $\dot{x}_n(t) = f_n(x(t), \varphi(t))$ can be solved with respect to φ, at almost all $t \in [0, \varepsilon]$: if $\frac{\partial f_n}{\partial \varphi}$ never vanishes, then the function of φ: $\dot{x}_n(t) - f_n(x(t), \varphi)$ is a monotonous function of φ. Then, its values determine φ uniquely. This shows that $\Sigma_{|V_{x_0}\times I_{\varphi_0}}$ is identifiable.

2. Normal form 1.10. Again, computing the first variation (with output $\hat{y}$), we see immediately that $\hat{y}(t) = 0$ on some time interval $[0, \varepsilon]$ implies that $\xi(t) = 0$ on this interval. The condition that $(\frac{\partial f_{n-1}}{\partial \varphi}\ ,\ \frac{\partial f_n}{\partial \varphi})$ does not vanish implies that $\eta(t) = 0$ for almost all $t \in [0, \varepsilon]$ if $\hat{y}(.) = 0$. This shows that $\Sigma_{|V_{x_0}\times I_{\varphi_0}}$ is uniformly infinitesimally identifiable.

Now, from the normal form, any output trajectory $y(t)$ determines $x(t)$ by differentiation. The condition that $(\frac{\partial f_{n-1}}{\partial \varphi}\ ,\ \frac{\partial f_n}{\partial \varphi})$ does not vanish implies that (restricting the neighborhood $V_{x_0} \times I_{\varphi_0}$, and eventually exchanging the role of h_1, h_2) that $\frac{\partial f_{n-1}}{\partial \varphi}$ never vanishes. Then the same argument as for the normal form 1.8 shows that $\varphi(t)$ is determined almost everywhere by one more differentiation. $\Sigma_{|V_{x_0}\times I_{\varphi_0}}$ is identifiable.

3. Normal form 1.9. Computing with the first variation, assuming that the output $\hat{y}(t)$ is identically zero on some interval $[0, \varepsilon]$ we get that $\xi_1(t), ...\xi_{2r}(t)$ are identically zero on the same interval. The two equations:

$$\dot{x}_{2r-1} = \Phi(x, \varphi),\ \dot{x}_{2r} = F_{2r}(x_1..., x_{2r+1}, \Phi(x, \varphi)),$$

with $\frac{\partial \Phi}{\partial \varphi} \neq 0$, $\frac{\partial F_{2r}}{\partial x_{2r+1}} \neq 0$, show that $d_x\Phi.\xi(t) + d_\varphi\Phi.\eta(t)$ and $\xi_{2r+1}(t)$ are identically zero for almost all t, and by continuity, $\xi_{2r+1}(t) = 0$ on $[0, \varepsilon]$. By induction, using the fact that $\frac{\partial F_{2r+i}}{\partial x_{2r+i+1}} \neq 0$, we get that $\xi_{2r+i+1}(t)$ is identically zero, and at the end, $\xi(t)$ is identically zero.

The equation of ξ_{2r-1} again, shows that η is zero almost everywhere: $0 = d_x\Phi.\xi(t) + d_\varphi\Phi.\eta(t)$ *a.e.* Hence, Σ is uniformly infinitesimally identifiable.

Now, the output $y(t) = (x_1, x_2)(t)$, by differentiation, determines $(x_3, x_4)(t)$ for all $t \in [0, \varepsilon]$. By differentiation of $(x_3, x_4)(t)$, we get $(x_5, x_6)(t)$, and so on. Once we know (x_{2r-1}, x_{2r}), with the same reasoning, we determine $\Phi(x, \varphi)$ and $F_{2r}(x_1..., x_{2r+1}, \Phi(x, \varphi))$ almost everywhere w.r.t. t. Now, $\frac{\partial F_{2r}}{\partial x_{2r+1}} \neq 0$ (never vanishes), shows that F_{2r} is monotonous w.r.t. x_{2r+1}, the other variables being fixed.

Hence, since we know Φ and $x_1, ..., x_{2r}$, we can determine uniquely $x_{2r+1}(t)$, $t \in [0, \varepsilon]$ (by continuity) from the knowledge of the values of F_{2r} a.e. By induction, we determine $x(t)$ for all $t \in [0, \varepsilon]$.

Solving the equation $\dot{x}_{2r-1} = \Phi(x(t), \varphi(t))$ (using again the monotonicity of Φ w.r.t φ, since $\frac{\partial \Phi}{\partial \varphi} \neq 0$), determines $\varphi(t)$ for almost all t, hence determines φ as an $L_\infty([0, \varepsilon], I_{\varphi_0})$ function. Hence, $\Sigma_{|V_{x_0} \times I_{\varphi_0}}$ is identifiable. ∎

Proof. (of Theorem 10.)

Assume that Σ is regular, type 1. Then, consider the subset $\tilde{X}$ of X where

$$d_x h_1 \wedge d_x h_2 \wedge d_x L_f h_1 \wedge ... \wedge d_x L_f^{k-1} h_1 \wedge d_x L_f^{k-1} h_2 \neq 0.$$

$\tilde{X}$ is semi-analytic, open dense. On a neighborhood of each point of $\tilde{X}$, we can chose as coordinates $(x_1, x_2) = h(x)$, ..., $(x_{2k-1}, x_{2k}) = L_f^{k-1} h(x)$. Let us assume that $r > k$ or **C.2** is satisfied (the case **C.1** is obtained by exchanging the role of h_1 and h_2). Then, Σ can be written locally in x :

$$\begin{aligned} y_1 &= x_1, \ y_2 = x_2, \\ \dot{x}_1 &= x_3, \ \dot{x}_2 = x_4, \\ & .. \\ \dot{x}_{2k-3} &= x_{2k-1}, \ \dot{x}_{2k-2} = x_{2k}, \\ \dot{x}_{2k-1} &= f_{2k-1}(x_1, ..., x_{2k+1}), \\ \dot{x}_{2k} &= x_{2k+1}, \\ & .. \\ \dot{x}_{N(r)-1} &= x_{N(r)}, \\ \dot{x}_{N(r)} &= f_{N(r)}(x, \varphi), \\ & .. \\ \dot{x}_n &= f_n(x, \varphi). \end{aligned}$$

with $\frac{\partial f_{N(r)}}{\partial \varphi}$ nonidentically zero.

Moreover, if $r = k$, then $f_{2k-1} = f_{2k-1}(x_1, ..., x_k)$, and $\dot{x}_{2k} = f_{2k}(x, \varphi) = f_{N(r)}(x, \varphi)$.

Assume that $N(r) < n$. Then, let us consider the initial condition $\xi(0)$ for the first variation: $\xi_1(0) = ... = \xi_{N(r)}(0) = 0$, $\xi_{N(r)+1}(0) \neq 0$. Chose the feedback function $\eta(x, \varphi, \xi) = -\frac{d_x f_{N(r)}(x,\varphi)\xi}{d_\varphi f_{N(r)}(x,\varphi)}$. For this, chose any function φ (φ constant for instance). We have, for the first variation:

$$\begin{aligned} \dot{\xi}_1 &= \xi_3, \ \dot{\xi}_2 = \xi_4 \\ & .. \\ \dot{\xi}_{2k-1} &= \frac{\partial f_{2k-1}}{\partial x_1} \xi_1 + ... + \frac{\partial f_{2k-1}}{\partial x_{2k+1}} \xi_{2k+1}, \\ \dot{\xi}_{2k} &= \xi_{2k+1}, \\ & .. \\ \dot{\xi}_{N(r)-1} &= \xi_{N(r)}, \\ \dot{\xi}_{N(r)} &= 0 \ \text{ by construction.} \end{aligned}$$

Moreover, if $r = k$, $\frac{\partial f_{2k-1}}{\partial x_{2k+1}} \equiv 0$, and $\dot{\xi}_{N(r)} = \dot{\xi}_{2k} = 0$.

Hence, for $t > 0$ small, $(\xi_1(t), \xi_2(t)) = \hat{y}(t) = 0$ (remind that $\hat{y}$ is the output of the first variation). This contradicts the uniform infinitesimal identifiability.

Hence, $N(r) = n$. Let $E = \{(x, \varphi)|d_\varphi L_f^r h(x, \varphi) = 0\}$. Let πE be the projection of E on X. $X \backslash \pi E$ is subanalytic. Assume that πE contains an open set $\tilde{X}$. By Hardt's Theorem on the stratification of mappings, there is a smooth (analytic) mapping $\hat{\varphi} : \tilde{X} \to E$ (restricting $\tilde{X}$ eventually). Then, choosing for the first variation the initial condition $\xi(0) = 0$, and any nonzero variation $\eta(t)$, but the feedback "control" $\varphi(t) = \hat{\varphi}(x(t))$, (for small times), we get by construction that the trajectory $\xi(t)$ of the first variation is in the zero section of TX. A contradiction with the uniform infinitesimal identifiability.

We conclude that πE has codimension 1, and $\tilde{X} = X \backslash \pi E$ contains an open dense set. This shows the first part of the theorem.

Since on $\tilde{X} \times I$, $d_\varphi L_f^r h(x, \varphi)$ never vanishes by construction, the proof of the last part of the theorem is exactly the same as the proof of Theorem 9, part "type 1". ∎

Proof. (of Theorems 8, 4.)

We already know, by the beginning of this section, that these theorems are equivalent.

The proof of Theorem 4, **"type 1"**, is already contained in the proof of Theorem 10 (which is stronger, for type 1 systems).

Type 3: (the most simple case). It is clear that if $r = k, n = 2k$, the system can be locally written under normal form 1.10 around any point of an open dense semi-analytic subset of $X \times I$: $x_1 = h_1(x)$, $x_2 = h_2(x), ..., x_{2k-1} = L_f^{k-1} h_1(x)$, $x_{2k} = L_f^{k-1} h_2(x)$ are adequate coordinates around any point where they are independent. The set of these points is semianalytic open, dense in X. The fact that $r = k$ implies that $d_\varphi L_f^k h(x, \varphi)$ is not identically zero. Hence, it is never zero on a semi-analytic open dense subset of $X \times I$. On the intersection of these two semianalytic subsets of $X \times I$, (the first one can be considered as such), in the coordinates just defined, the system is under the normal form 1.10.

Type 2: This is the most difficult case. In that case, by the definition of type 2, eventually interchanging h_1, h_2, we may assume that, around any point (x_0, φ_0) of the complement $(X \times I) \backslash \tilde{U}$ of the codimension 1 analytic set $\tilde{U} \subset X \times I$:

$$\tilde{U} =$$

$$\begin{aligned} \{(x, \varphi)|d_x h_1 \wedge d_x h_2 \wedge d_x L_f h_1 \wedge ... \wedge d_x L_f^{r-1} h_2 \wedge d_x L_f^r h_1 &= 0\} \\ \cup \{(x, \varphi)|d_\varphi L_f^r h_2 &= 0\}, \end{aligned}$$

we can find coordinates x such that the system can be written:

$$\begin{aligned} y_1 &= x_1,\ y_2 = x_2, \\ \dot{x}_1 &= x_3,\ \dot{x}_2 = x_4, \\ & .. \\ \dot{x}_{2r-3} &= x_{2r-1},\ \dot{x}_{2r-2} = x_{2r}, \\ \dot{x}_{2r-1} &= f_{2r-1}(x,\varphi), \dot{x}_{2r} = f_{2r}(x,\varphi), \\ & .. \\ \dot{x}_n &= f_n(x,\varphi), \end{aligned} \tag{4.2}$$

with $\frac{\partial f_{2r}}{\partial \varphi} \neq 0$, and $dx_1 \wedge ... \wedge dx_{2r} \wedge d_x f_{2r-1} \neq 0$.

Here, in case $r = k$, we treat **B.1**. only. In case $r < k$, we chose h_2 for $d_\varphi L_f^r h_2 \neq 0$, and h_1 is the other. This ensures that $\tilde{U}$ has codimension 1.

Let $\hat{y}(t)$ be an output function of the first variation of the system, which is identically zero on some time interval $[0, \varepsilon]$. Let $x(t)$, $\xi(t)$, $\varphi(t)$, $\eta(t)$ be the corresponding trajectories. We know that the couple $(\xi(t), \eta(t))$ has to be identically zero, by the uniform infinitesimal identifiability. Differentiating $\hat{y}(t) = 0$ r times, we get:

$\xi_1(t) = ... = \xi_{2r}(t) = 0$ for all $t \in [0, \varepsilon]$, and:

$$\begin{aligned} d_x f_{2r-1}(x(t),\varphi(t))\xi(t) + d_\varphi f_{2r-1}(x(t),\varphi(t))\eta(t) &= 0, \\ d_x f_{2r}(x(t),\varphi(t))\xi(t) + d_\varphi f_{2r}(x(t),\varphi(t))\eta(t) &= 0, \end{aligned} \tag{4.3}$$

for almost all $t \in [0, \varepsilon]$.

Hence, the system of equations (4.3), must have no smooth solution $(\eta, \varphi)(x, \xi)$, in a neighborhood of $(x_0, \xi(0))$, $\xi(0) \neq 0$, in $X \times \mathbb{R}^{n-2r} = \{(x, \xi(0)) | \xi_1(0) = ... = \xi_{2r}(0) = 0\}$, with φ close to φ_0 and η arbitrary:

Indeed, assume that there is a solution $(\eta, \varphi)(x, \xi)$. Then, consider the feedback system:

$$\begin{aligned} \dot{x} &= f(x, \varphi(x,\xi)), \\ \dot{\xi}_{2r+1} &= d_x f_{2r+1}(x, \varphi(x,\xi))\xi + d_\varphi f_{2r+1}(x, \varphi(x,\xi))\eta(x,\xi), \\ & ... \\ \dot{\xi}_n &= d_x f_n(x, \varphi(x,\xi))\xi + d_\varphi f_n(x, \varphi(x,\xi))\eta(x,\xi), \end{aligned}$$

in which $\xi_1 = ... = \xi_{2r} = 0$.

This is a smooth differential equation on an open subset of $X \times \mathbb{R}^{n-2r}$. It has a smooth solution $x(t), \xi(t)$. We set $\hat{\varphi}(t) = \varphi(x(t), \xi(t))$, $\hat{\eta}(t) = \eta(x(t), \xi(t))$, with $\xi(t) \neq 0$.

By construction, we have:

$$\begin{aligned} \dot{x} &= f(x, \hat{\varphi}(t)), \\ \dot{\xi} &= T_x f(x, \hat{\varphi}(t))\xi + d_\varphi f(x, \hat{\varphi}(t))\hat{\eta}(t), \end{aligned}$$

because, for $\xi_1, ..., \xi_{2r}$, these equations read $\xi_1(t) = ... = \xi_{2r}(t) = 0$ for all t (small).

Hence, in particular $\hat{y}_1(t) = 0$, $\hat{y}_2(t) = 0$. This is impossible, by uniform infinitesimal identifiability.

Therefore, by the crucial Lemma 7, Section 7, we get that:

there are neighborhoods V_{x_0}, V_{φ_0}, and coordinates $\tilde{x}$ on V_{x_0}, with $\tilde{x}_1 = x_1, ..., \tilde{x}_{2r} = x_{2r}$, such that:

$$\tilde{x}_{2r+1} = \Phi_{\varphi_0}(\tilde{x}_1, ..., \tilde{x}_{2r}, f_{2r-1}(\tilde{x}, \varphi), f_{2r}(\tilde{x}, \varphi)), \tag{4.4}$$

for all $(\tilde{x},\varphi)\in V_{x_0}\times V_{\varphi_0}$.

Moreover,

$$(4.5)\qquad (\frac{\partial\Phi_{\varphi_0}}{\partial f_{2r-1}},\frac{\partial\Phi_{\varphi_0}}{\partial f_{2r}})\ \text{ never vanishes,}$$

$$(4.6)\qquad \frac{\partial f}{\partial\varphi}\,\bar{\wedge}\,\frac{\partial f}{\partial\tilde{x}_{p+1}}\ \text{ never vanishes,}$$

on $V_{x_0}\times V_{\varphi_0}$, where $\frac{\partial f}{\partial\varphi}\,\bar{\wedge}\,\frac{\partial f}{\partial\tilde{x}_{p+1}}$ denotes the determinant of the 2×2 matrix formed by $\frac{\partial(f_{2r-1},f_{2r})}{\partial\varphi}$ and $\frac{\partial(f_{2r-1},f_{2r})}{\partial\tilde{x}_{p+1}}$.

In fact, $\frac{\partial\Phi_{\varphi_0}}{\partial f_{2r-1}}\neq 0$, because, if it is zero, then $\frac{\partial\Phi_{\varphi_0}}{\partial f_{2r}}\neq 0$ by (4.5), and, differentiating (4.4) with respect to φ, we get a contradiction (since $\frac{\partial f_{2r}}{\partial\varphi}\neq 0$).

Therefore, we can apply the implicit function theorem to (4.4): restricting may be our neighborhood $V_{x_0}\times V_{\varphi_0}$, we find a smooth function $\bar{\Phi}$ such that

$$(4.7)\qquad f_{2r-1}(\tilde{x},\varphi)=\bar{\Phi}(\tilde{x}_1,...,\tilde{x}_{2r},\tilde{x}_{2r+1},f_{2r}(\tilde{x},\varphi)),$$

with $\frac{\partial\bar{\Phi}}{\partial\tilde{x}_{2r+1}}\neq 0$.

Now, set $\Delta=\frac{\partial(f_{2r-1},f_{2r})}{\partial\varphi}\,\bar{\wedge}\,\frac{\partial(f_{2r-1},f_{2r})}{\partial\tilde{x}_{p+1}}$. An easy computation shows that $\Delta=-\frac{\partial f_{2r}}{\partial\varphi}\frac{\partial\bar{\Phi}}{\partial\tilde{x}_{2r+1}}$. This says no more than $\frac{\partial f_{2r}}{\partial\varphi}\neq 0$, which we already know. Equation (4.3) becomes:

$$(4.8)\qquad \begin{array}{lrcl} 1. & \frac{\partial\bar{\Phi}}{\partial\tilde{x}_{2r+1}}\xi_{2r+1}+\frac{\partial\bar{\Phi}}{\partial f_{2r}}d_xf_{2r}(\tilde{x},\varphi)\xi+\frac{\partial\bar{\Phi}}{\partial f_{2r}}d_\varphi f_{2r}\eta & = & 0,\\ 2. & d_xf_{2r}(\tilde{x},\varphi)\xi+d_\varphi f_{2r}\eta & = & 0.\end{array}$$

Equation (4.8, 2) implies $\eta=-\frac{d_xf_{2r}(\tilde{x},\varphi)\xi}{d_\varphi f_{2r}}$, which replaced in (4.8,1) gives $\frac{\partial\bar{\Phi}}{\partial\tilde{x}_{2r+1}}\xi_{2r+1}=0$, $\xi_{2r+1}\equiv 0$. We have shown that $\hat{y}\equiv 0$ implies $(\xi_1,...,\xi_{2r+1})\equiv 0$, and, making the change of notations $x:=\tilde{x}$, (4.2) rewrites:

$$(4.9)\qquad \begin{array}{rcl} y_1 & = & x_1,\ y_2=x_2,\\ \dot{x}_1 & = & x_3,\ \dot{x}_2=x_4,\\ & & ...\\ \dot{x}_{2r-3} & = & x_{2r-1},\ \dot{x}_{2r-2}=x_{2r},\\ \dot{x}_{2r-1} & = & f_{2r-1}(x_1,...,x_{2r+1},f_{2r}(x,\varphi)),\dot{x}_{2r}=f_{2r}(x,\varphi),\\ \dot{x}_{2r+1} & = & f_{2r+1}(x,\varphi),\\ & & ...\\ \dot{x}_n & = & f_n(x,\varphi),\end{array}$$

$\frac{\partial f_{2r}}{\partial \varphi} \neq 0$, and $\frac{\partial f_{2r-1}}{\partial x_{2r+1}} \neq 0$. With $\xi_1(0) = ... = \xi_{2r+1}(0) = 0$, we obtain, for the first variation:

$$\begin{aligned}
\hat{y}_1 &= \xi_1, \ \hat{y}_2 = \xi_2, \\
\dot{\xi}_1 &= \xi_3, \ \dot{\xi}_2 = \xi_4 \\
& ... \\
\dot{\xi}_{2r-3} &= \xi_{2r-1}, \ \dot{\xi}_{2r-2} = \xi_{2r}, \\
\dot{\xi}_{2r-1} &= d_x f_{2r-1}(\xi_1, .., \xi_{2r+1}) + \frac{\partial f_{2r-1}}{\partial f_{2r}}(d_x f_{2r}\xi + d_\varphi f_{2r}\eta), \\
\dot{\xi}_{2r} &= d_x f_{2r}\xi + d_\varphi f_{2r}\eta, \\
\dot{\xi}_{2r+1} &= d_x f_{2r+1}\xi + d_\varphi f_{2r+1}\eta, \\
& ... \\
\dot{\xi}_n &= d_x f_n \xi + d_\varphi f_n \eta.
\end{aligned} \tag{4.10}$$

If $n = 2r + 1$, our theorem is already proved (exchanging h_1, h_2). Assume $n > 2r + 1$.

If we can find a feedback solution $(\hat{\eta}, \hat{\varphi})(x, \xi)$ of :

$$\begin{aligned}
d_x f_{2r}(x, \varphi)\xi + d_\varphi f_{2r}(x, \varphi)\eta &= 0, \\
d_x f_{2r+1}(x, \varphi)\xi + d_\varphi f_{2r+1}(x, \varphi)\eta &= 0,
\end{aligned}$$

for $(\xi_1, ..., \xi_{2r+1}) = 0$, $(\xi_{2r+2}, ..., \xi_n) \neq 0$, on some open set in the space of the other variables $\xi_{2r+2}, .., \xi_n, x$, then, this will contradict the uniform infinitesimal identifiability, with the same reasoning as above. Therefore, another application of Lemma 7 shows that, we can change the coordinates x, keeping $x_1, ..., x_{2r+1}$ unchanged, for:

$$f_{2r+1} = f_{2r+1}(x_1, ..., x_{2r+2}, f_{2r}(x, \varphi)),$$

where $\frac{\partial f_{2r+1}}{\partial x_{2r+2}} \neq 0$ (never vanishes on some neighborhood $V_{x_0} \times V_{\varphi_0}$). Iterating the process, and at the end, exchanging the roles of $y_1 = x_1$ and $y_2 = x_2$, ends the proof of the theorem. ∎

5. Identification

We will not consider the case where identifiability is a generic property ($d_y \geq 3$). In this case, there is a new difficulty, and it will be treated in another paper. For the cases $d_y = 1$, $d_y = 2$, we will be very short, and give only the ideas, leaving all details to the reader. Also, we will focus on the problem of "on line" identification (that is, the process of learning about the unknown function runs simultaneously to the process of observing the data $y(t)$).

In the case $d_y = 1$, (normal form 1.7), the most simple example is:

$$y^{(n)} = \varphi.$$

This trivial example shows that there is in general no hope to do something better than approximate differentiation: the problem of learning about the graph of φ is just the problem of estimating the n first derivatives of the output.

5.1. Identification using approximate derivators.

Theorem 11. *We consider, on $X \times I$, (X an open subset of $\mathbb{R}^n$), a system which is globally in normal form 1.7, ($d_y = 1$) or in normal forms 1.8, 1.9, ($d_y = 2$), or 1.10 with the additional requirement that $\frac{\partial f_n}{\partial \varphi}$ never vanishes ($d_y = 2$). The points of the graph of the functions $\varphi(x)$, or $\varphi \circ \pi(x)$ visited during some experiment can be reconstructed by (a finite number of) differentiations of the output(s).*

Proof. Several facts in this theorem have been proved already. Nevertheless, we shortly prove everything.

1. **Normal forms 1.7, 1.8, 1.10.**

Differentiating the outputs, we can reconstruct the state $x(t)$ of the system. Differentiating once more, we reconstruct respectively $\psi(x, \varphi)(t)$ in (1.7), and $f_n(x, \varphi)(t)$ in (1.8), (1.10). These functions, as functions of φ, are monotonous. Hence, $x(t)$ being known, we can reconstruct $\varphi(t)$ from the knowledge of their values.

2. **Normal form** 1.9.

Differentiating a certain number of times, we reconstruct $x_1, ..., x_{2r}$, and $\Phi(x, \varphi)$. Differentiating once more, we reconstruct $F_{2r}(x_1, .., x_{2r+1}, \Phi)$. It is a monotonous function of x_{2r+1}. Hence, x_{2r+1} can be obtained from these values. Iterating the result, we reconstruct $x(t)$. Once $x(t)$ is known, the function Φ being monotonous with respect to φ, we get $\varphi(t)$. ∎

In fact, this procedure is not so far from what is done for the identification of linear systems.

5.2. Identification using nonlinear observers.

We may assume, along the trajectories visited, a "local model" for φ as a function of time. For instance, the most simple model is $\varphi^{(k)} \equiv 0$, i.e. φ is a polynomial function of the time. Of course, the coefficients of this polynomial will be perpetually reestimated. And hence, the question is not that they model the function φ globally as a function of time, but only locally, on reasonable time intervals (reasonable with respect to the performances of the observer).

Let us consider again the 4 normal forms 1.7, 1.8, 1.9, 1.10, (with the same additional requirement, in case 1.10, that $\frac{\partial f_n}{\partial \varphi}$ never vanishes). Adding φ and its k first derivatives as extra state variables, the problem is now reduced to the problem of estimation of the state.

It turns out that, in all these cases, the extended systems we get have **very strong observability properties**, and that the "High Gain Construction", presented in the book [7], generalizes to these cases, allowing to reconstruct (approximately) the state of the extended system, and hence to estimate the corresponding points of the graph of φ.

5.3. A more robust solution.

High gain observers may be rather sensitive to noise. A more robust solution is proposed in [3]. This construction also works for all the cases under consideration here in, if $d_y = 1$ or 2.

As an example, let us consider just the case of the normal form (1.7), which gives, adding derivatives of φ as state variables:

$$
\begin{aligned}
\frac{d^n y}{dt^n} &= \psi(y, ..., y^{(n-1)}, \varphi), \\
\dot{\varphi} &= \varphi_1, \\
&.. \\
\dot{\varphi}_{k-2} &= \varphi_{k-1}, \\
\dot{\varphi}_{k-1} &= 0,
\end{aligned}
\tag{5.1}
$$

where, as usual, $\frac{\partial\psi}{\partial_\varphi}$ never vanishes.

The map $\Xi : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$, $(y, \dot{y}, ..., y^{n-1}, \varphi, ..., \varphi^{k-1}) \to (y, \dot{y}, ..., y^{(n+k-1)})$, is a diffeomorphism, as it is easily checked. Hence, (see again [7] for details), this system is diffeomorphic to a system on $\mathbb{R}^{n+k}$, of the form:

$$\dot{\xi}_1 = \xi_2, ..., \dot{\xi}_{n+k-1} = \xi_{n+k}, \dot{\xi}_{n+k} = \hat{\psi}(\xi), \quad y = \xi_1,$$

for some smooth function $\hat{\psi}$. This is exactly what is needed for applying the technique developed in [3].

We will exploit a variation of this idea for the biological reactor in the next section.

6. The biological reactor

6.1. **The model and its basic properties.** Let us recall the equations of the model of bioreactor presented in the introduction:

$$
\begin{aligned}
\frac{ds}{dt} &= -\mu.x + D(S_{in} - s) \\
\frac{dx}{dt} &= (\mu - D)x.
\end{aligned}
\tag{6.1}
$$

The growth function, $\mu(s, x)$, is smooth, positive or zero, and, for obvious physical reasons, $\mu(0, x) = 0$: if there is no substrate to eat, the population cannot grow. On the contrary, if there is something to eat, the population grows: $\mu(s, x) > 0$ for $s > 0$. The control function $D(t)$ verifies $D(t) > \varepsilon > 0$: this means that the reactor is always fed. The constant S_{in} is assumed to be strictly positive.

Now, the subset $X = \mathbb{R}^+ \times \mathbb{R}^+$ is invariant by the dynamics (6.1) of the bioreactor: if $x = 0$, then $\dot{x} = 0$, and if $s = 0$, then $\dot{s} = D.\ S_{in} > 0$. Therefore, we may consider that $x(t)$, $s(t)$ are always strictly positive.

The function $\mu(s, x)$ is often considered in the literature as a function of s only, $\mu(0) = 0$, $\mu \geq 0$. This means that the internal variable is $s \in \mathbb{R}^+$, and the internal mapping $\pi : X \to \mathbb{R}^+$ is the mapping $(x, s) \to s$.

Typical expressions of the growth function in that case are the Monod model:

$$\mu(s) = \frac{\mu_0 s}{k_m + s}, \tag{6.2}$$

or the Haldane model:

$$\mu(s) = \frac{\mu_0 s}{k_m + s + \frac{s^2}{k_i}}. \tag{6.3}$$

6.2. **Observation of s only.** Setting $X = x + s$, we get:

$$\begin{aligned} \frac{ds}{dt} &= -\mu.x + D(S_{in} - s), \\ \frac{dX}{dt} &= D(S_{in} - X), \end{aligned} \tag{6.4}$$

hence, setting $\tilde{D}(t) = \int_0^t D(\tau)d\tau$, we get:

$$X = e^{-\tilde{D}(t)}X_0 + (1 - e^{-\tilde{D}(t)})S_{in}. \tag{6.5}$$

Let us set:

$$\begin{aligned} (1)\ \ \Lambda(t) &= e^{\tilde{D}(t)}(s - S_{in}) + S_{in}, \ \ \text{or:} \\ (2)\qquad s &= e^{-\tilde{D}(t)}\Lambda(t) + S_{in}(1 - e^{-\tilde{D}(t)}), \ \ s(0) = \Lambda(0). \end{aligned} \tag{6.6}$$

By (6.4), we get:

$$\dot{\Lambda} = -e^{\tilde{D}(t)}(X - s)\mu, \tag{6.7}$$

and with (6.5):

$$\dot{\Lambda} = (\Lambda - X_0)\mu. \tag{6.8}$$

Let us assume, as we said in the previous section, that μ is a function of s only. Let us also assume that $s(.)$ visits twice the same value, i.e, $T_0 < T_1$, $s(T_0) = s(T_1)$. Then, with $X_0 = X(0) = x(0) + s(0)$:

$$\mu(s(T_0)) = \frac{\dot{\Lambda}(T_0)}{\Lambda(T_0) - X_0} = \frac{\dot{\Lambda}(T_1)}{\Lambda(T_1) - X_0} = \mu(s(T_1)).$$

Observe, with Equation (6.8), that Λ is **everywhere continuously differentiable**, even if $D(.)$ is only measurable, bounded. From (6.7), and the fact that $\mu(s) > 0$ for $s > 0$, we get that $\Lambda(t)$ is a strictly decreasing function, and $\Lambda(0) = s(0)$, $X_0 = x(0) + s(0)$, $x(0) > 0$, implies that $\Lambda(t) - X_0$ is never zero for $t \geq 0$.

Also, $s(t)$ being observed, and $D(t)$, the control, being known, we may consider, by the definition (6.6, 1) of Λ, that Λ is an observed function. Then, X_0 can be computed:

$$X_0 = \frac{\dot{\Lambda}(T_0)\Lambda(T_1) - \dot{\Lambda}(T_1)\Lambda(T_0)}{\dot{\Lambda}(T_0) - \dot{\Lambda}(T_1)}, \tag{6.9}$$

indeed, $\dot{\Lambda}(T_0) - \dot{\Lambda}(T_1) = (\Lambda(T_0) - \Lambda(T_1))\mu(s(T_0)) \neq 0$.

Now, X_0 being known,

$$\mu(s(t)) = \frac{\dot{\Lambda}(t)}{\Lambda(t) - X_0} \tag{6.10}$$

since $\Lambda(t) - X_0$ never vanishes.

Conversely, let us consider a trajectory, defined on $[0, T]$, such that:

(H.a.) s never visits twice the same value on $[0, T]$, (i.e. s is strictly monotonous),

or, stronger,

(H.b.) $D(.)$ is smooth, (from what it follows that $s(.), x(.), \Lambda(.), X(.)$, are all smooth functions), and $\frac{ds}{dt}(t) \neq 0$ for all $t \in [0, T]$.

Initial conditions corresponding to this trajectory are s_0, x_0, X_0 (all strictly > 0).

Let us consider another arbitrary value, $\tilde{X}_0 = \tilde{x}_0 + s_0 > 0$, close to X_0. Then, since $\Lambda(t) = e^{\tilde{D}(t)}(s - S_{in}) + S_{in}$ is C^1 (even in case, (H.a.)), as we know by (6.8), we may compute $\tilde{\mu}(t) = \frac{\dot{\Lambda}(t)}{\Lambda(t) - \tilde{X}_0}$, on the interval $[0, T]$: indeed, since $\tilde{X}_0$ is close to X_0, then $\Lambda(t) - \tilde{X}_0$ is close to $\Lambda(t) - X_0$, and then never vanishes. Then, under Assumption (H.a.), there is a continuous function $t(s)$, which is the inverse of $s(t)$, and which is smooth under assumption (H.b.). Set $\bar{\mu}(s) = \tilde{\mu}(t(s))$. In case (H.b.), $\bar{\mu}$ is smooth, in case (H.a.), it is continuous only. Set also $\tilde{x}(t) = \tilde{X}(t) - s(t) = e^{-\tilde{D}(t)}\tilde{X}_0 + (1 - e^{-\tilde{D}(t)})S_{in} - s(t)$. In these circumstances, we claim that:

Claim: (a) $s(t)$, $\tilde{x}(t)$ are solutions of the system:

$$\begin{aligned} (1) \quad \frac{ds}{dt} &= -\bar{\mu}(s(t))\tilde{x} + D(t)(S_{in} - s(t)), \\ (2) \quad \frac{d\tilde{x}}{dt} &= (\bar{\mu}(s(t)) - D(t))\tilde{x}(t), \end{aligned} \tag{6.11}$$

(b) if $\tilde{X}_0$ is sufficiently close to X_0, $\tilde{x}(t) > 0$ for all $t \in [0, T]$.

Proof. (of the claim) (a): Let us show first that $\mu(t)x(t) = \tilde{\mu}(t)\tilde{x}(t)$ for all $t \in [0, T]$. By construction:

$$\tilde{\mu}(t) = \frac{\Lambda(t) - X_0}{\Lambda(t) - \tilde{X}_0}\mu(t),$$

then it is sufficient to check that $\frac{\Lambda(t) - X_0}{\Lambda(t) - \tilde{X}_0}\mu(t)\tilde{x}(t) = \mu(t)x(t)$, or, since $\mu(t) > 0$:

$$(\Lambda(t) - X_0)\tilde{x}(t) = (\Lambda(t) - \tilde{X}_0)x(t). \tag{6.12}$$

But $x(t) = X(t) - s(t) = e^{-\tilde{D}(t)}X_0 + (1 - e^{-\tilde{D}(t)})S_{in} - s(t)$, and $\tilde{x}(t) = \tilde{X}(t) - s(t) = e^{-\tilde{D}(t)}\tilde{X}_0 + (1 - e^{-\tilde{D}(t)})S_{in} - s(t)$. Replacing by these expressions and by the expression (6.6, 1) of $\Lambda(t)$, just shows that (6.12) is true. Therefore, since $\frac{ds}{dt} = -\mu.x + D(S_{in} - s)$, we get that also, $\frac{ds}{dt} = -\tilde{\mu}.\tilde{x} + D(S_{in} - s)$. This is (a, 1).

Now,

$$\begin{aligned} \tilde{x}(t) &= \tilde{X}(t) - s(t) = e^{-\tilde{D}(t)}\tilde{X}_0 + (1 - e^{-\tilde{D}(t)})S_{in} - s(t), \\ \frac{d\tilde{x}}{dt}(t) &= -D(t)(\tilde{x}(t) - S_{in} + s(t)) + \mu(t)x(t) - D(t)(S_{in} - s(t)), \\ &= -D(t)\tilde{x}(t) + \mu(t)x(t) = -D(t)\tilde{x}(t) + \tilde{\mu}(t)\tilde{x}(t), \end{aligned}$$

by the proof of (a, 1). Hence, $\frac{d\tilde{x}}{dt}(t) = (\tilde{\mu}(t) - D(t))\tilde{x}(t)$. This is (a, 2).

To prove (b), let us just observe that: $\tilde{x}(t) = e^{-\tilde{D}(t)}\tilde{X}_0 + (1 - e^{-\tilde{D}(t)})S_{in} - s(t)$, and $x(t) = e^{-\tilde{D}(t)}X_0 + (1 - e^{-\tilde{D}(t)})S_{in} - s(t)$, then, $\tilde{x}(t) - x(t) = e^{-\tilde{D}(t)}(\tilde{X}_0 - X_0)$. Hence, for $\tilde{X}_0$ sufficiently close to X_0, or equivalently $\tilde{x}_0$ sufficiently close to x_0, $\tilde{x}(t)$ is strictly positive. ∎

We have shown the following theorem (precise version of Theorem 5 in the introduction):

Theorem 12. *a. The bioreactor is identifiable at any admissible i.o. trajectory* $(s(t), D(t))$*,* $t \in [0, T]$ *such that* $s(t)$ *visits twice the same value,*

b. Conversely, if $(s(t), D(t))$*,* $t \in [0, T]$ *is an admissible i.o. trajectory along which* $s(t)$ *is strictly monotonous, then, there is an infinity of corresponding couples* $(x(.), \mu)$*, with* μ *continuous. If moreover* $D(.)$ *is smooth, and* $\dot{s}(t) \neq 0$ *for all* $t \in [0, T]$*, then, there is an infinity of corresponding couples* $(x(.), \mu)$*, with* μ *smooth.*

In Section 6.5, some numerical investigation will be made on the basis of this theorem.

6.3. **Observation of both** s **and** x**.** In that case, assuming $D(.)$ constant, we are in the situation $d_y = 2$, of Section 4, and our system is regular, and Type 3 (normal form 1.10), with the additional requirement of Theorem 11.

Therefore, we may use the ideas explained briefly in Section 5, Subsections 5.2, 5.3. In fact, in that case, we can do better:

1. we can adapt these ideas to the case where $D(.)$ is not constant,
2. making a small change of variables, we can use a real linear Kalman observer, in place of a high-gain one:

Let us set $z(t) = \mu(s(t)).x(t)$, and let us, (as explained in Section 5), assume a local model for $z(.)$, of the form $\frac{d^k z}{dt^k} = 0$. Then, our system becomes:

$$\begin{aligned} \dot{s} &= -z + D(t)(S_{in} - s); \\ \dot{x} &= z - D(t).x, \\ \dot{z} &= z_1, ..., \dot{z}_{k-2} = z_{k-1}, \dot{z}_{k-1} = 0; \\ y_1 &= s, \quad y_2 = x. \end{aligned} \tag{6.13}$$

Assuming only that $0 \leq D(t) \leq D_{\max}$, this is a linear time-dependant system, which is uniformly observable in the sense of the linear theory. Hence, classical versions of the (time dependant) Linear Kalman Filter work, even in a stochastic context.

This method will be also investigated numerically in Section 6.5.

6.4. **Observation of** x **only.** This case is often considered in practice.

If μ would be a function $\mu(x)$ of x, then, we could consider the system:

$$\begin{aligned} \dot{x} &= (\mu(x) - D)x, \\ y &= x, \end{aligned}$$

for $x \in \mathbb{R}^+$. Then, assuming $D(.)$ constant, we are exactly in the case of uniformly infinitesimally identifiable systems, in the case $d_y = 1$, normal form 1.7. Our ideas of Section 5 can then be used, and work as perfectly as in the case of the previous section 6.3, (even for $D(.)$ non constant).

But, in the case where μ depends only on s, as we assume here, (and as is often assumed in the literature), or if $\mu = \mu(s, x)$ depends on both s and x, then, the system:

$$\begin{aligned} \dot{x} &= (\mu - D)x, \\ \dot{s} &= -\mu.x + D(S_{in} - s), \\ y &= x; \end{aligned}$$

is not uniformly infinitesimally identifiable (it does not verify the necessary conditions of Theorem 2).

Moreover, clearly, it is not identifiable: assume D constant (for simplicity only), and $\mu(x,s)$, or $\mu(s)$ given, smooth. Given a smooth trajectory $(x(t),s(t)), t \in [0,T]$, $x(0)>0, s(0)>0$, of this system. Then,

$$s(t) = e^{-Dt}s(0) + \int_0^t e^{-D(t-\tau)}(-(\mu x)(\tau) + DS_{in})d\tau.$$

Let us chose another $\tilde{s}_0 \neq s(0)$, ($\tilde{s}(0)$ close to $s(0)$), and consider:

$$\tilde{s}(t) = e^{-Dt}\tilde{s}_0 + \int_0^t e^{-D(t-\tau)}(-(\mu x)(\tau) + DS_{in})d\tau.$$

Consider now the trajectory $(x(t),\tilde{s}(t)), t \in [0,T]$, and a smooth function $\tilde{\mu}(x,s)$, or $\tilde{\mu}(s)$, such that $\tilde{\mu}(x(t),\tilde{s}(t)) = \mu(x(t),s(t))$ for $t \in [0,T]$. This is possible, eventually restricting the interval $[0,T]$ to some $[T_0,T_1] \subset [0,T]$. In fact, $(x(t),\tilde{s}(t))$ is a solution of:

$$\begin{aligned} \frac{dx}{dt} &= (\tilde{\mu} - D)x, \\ \frac{d\tilde{s}}{dt} &= -\tilde{\mu}.x + D(S_{in} - \tilde{s}), \end{aligned}$$

$x(0) = x_0$, $\tilde{s}(0) = \tilde{s}_0$.

Hence, some people identify systems that are not identifiable: It is possible, by differentiation of the output x, to obtain complete information on the function $\mu(x(t),s(t))$, as a function of time. But it is not possible to deduce from this, any information about the function $\mu(s,x)$, or $\mu(s)$.

In fact, the reason why they obtain "some results" in practice (even for $D(.)$ nonconstant) is the following: because of Equation 6.5, we see that, whatever the control $D(.) > \varepsilon > 0$, $X(t)$ tends to S_{in} when $t \to +\infty$. Hence, after some time, $X = x+s$ is close to S_{in}, and s is close to $S_{in}-x$. Then, the estimate $\hat{\mu}(t)$, obtained after differentiation of the output $x(t)$, is such that $(x(t), S_{in} - x(t), \hat{\mu}(t))$, is close to a point of the graph of μ, for t large enough.

6.5. **Numerical simulations.**

6.5.1. *Simulation with complete measurements.* In this section, we simulate the bioreactor model using the Monod growth function (6.2)

$$\mu(s) = \frac{0.15\,s}{2+s}$$

We assume S_{in} constant, equal to 5. $D(t)$ is a periodic function with period shown on Figure 1. Initial conditions are $x(0) = 3$ and $s(0) = 2$.

In this first case, we assume that both the substrate and the biomass concentration are measured. Hence the output is $y(t) = (s(t), x(t))$. A colored noise is added to both variables to simulate noisy measurement. More precisely, this colored noise has been simulated using

$$dU_t = -a\,U_t\,dt + \sigma\sqrt{2a}\,dW_t$$

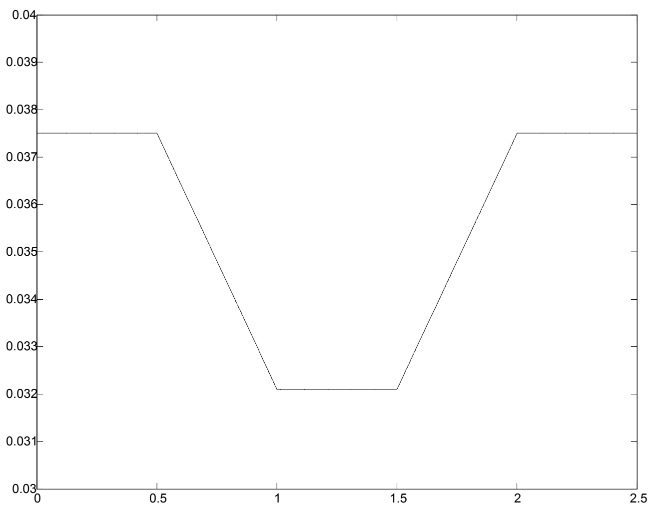

FIGURE 1. $D(t)$

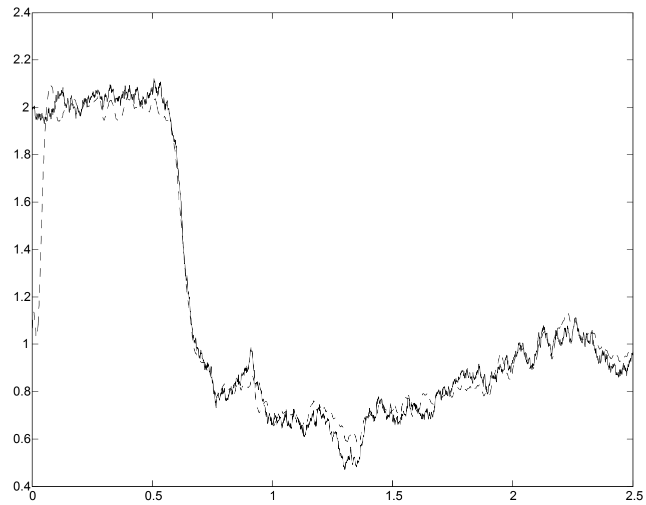

FIGURE 2. $s(t)$

where W_t is a normalized Brownian motion. So U_t is an Ornstein–Uhlenbeck process, that is a stationary process with mean 0 and with covariance function

$$\Gamma_U(t,s) = E[U_t U_s] = \sigma^2 \mathrm{e}^{-a|t-s|}$$

and hence U_t is a reasonable approximation of a realistic noise.

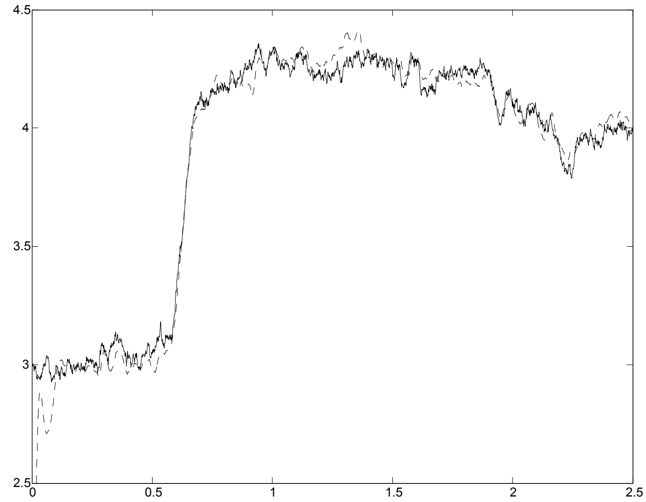

FIGURE 3. $x(t)$

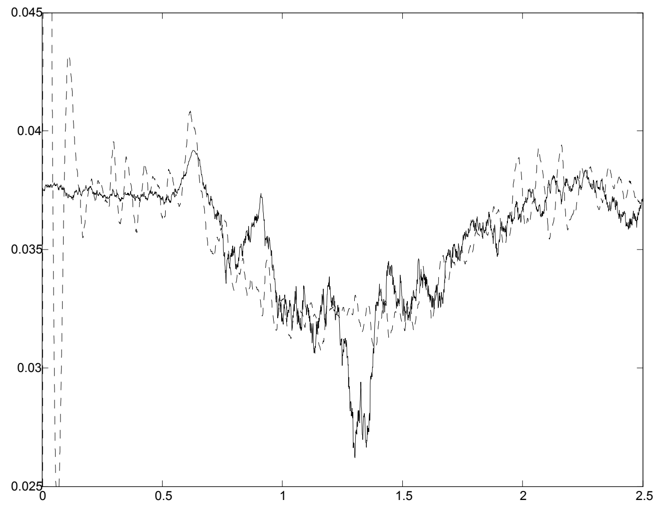

FIGURE 4. $\mu(t)$

As explained in section 6.3, we set $z(t) = \mu(s(t))\, x(t)$, and we assume a local model for $z(t)$ of the form

$$\begin{cases} \dfrac{dz(t)}{dt} & = & z_1(t) \\ \dfrac{dz_1(t)}{dt} & = & z_2(t) \\ \dfrac{dz_2(t)}{dt} & = & 0 \end{cases}$$

We add to these equations the two equations of the bioreactor

$$\left\{\begin{array}{lcl}\dfrac{ds(t)}{dt} & = & -z(t)+D(t)\,(S_{\text{in}}-s(t))\\ \dfrac{dx(t)}{dt} & = & z(t)-D(t)\,x(t)\end{array}\right.$$

and we apply the classical Kalman filter to these five equations, *i.e.* setting $X(t) = (s(t), x(t), z(t), z_1(t), z_2(t))$

$$\begin{array}{lcl}\dfrac{d\widehat{X}(t)}{dt} & = & A(t)\,\widehat{X}(t)+b(t)+PC^TR^{-1}\left(y(t)-C\widehat{X}(t)\right)\\ \dfrac{dP(t)}{dt} & = & A(t)\,P(t)+P(t)\,A(t)^T+Q-P(t)\,C^TR^{-1}CP(t)\end{array}$$

with

$$A(t) = \begin{pmatrix} -D(t) & 0 & -1 & 0 & 0\\ 0 & -D(t) & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{pmatrix},\ b(t) = \begin{pmatrix} D(t)\,s_{\text{in}}\\ 0\\ 0\\ 0\\ 0\end{pmatrix}$$

$$\text{and } C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\end{pmatrix}$$

Measured outputs are shown on figure 2 and 3 where the continuous line represents noisy outputs and the dashed line represents observer estimation.

Figure 4 represents $\mu(t)$ and $\widehat{\mu}(t) = \dfrac{\widehat{z}(t)}{\widehat{x}(t)}$ with the same convention as in the previous figures. Clearly, after the initial transient, the behavior of the Kalman filter looks good.

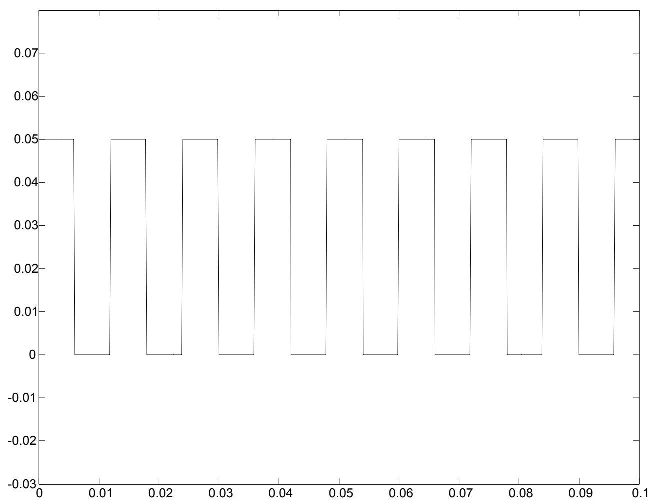

FIGURE 5. $D(t)$

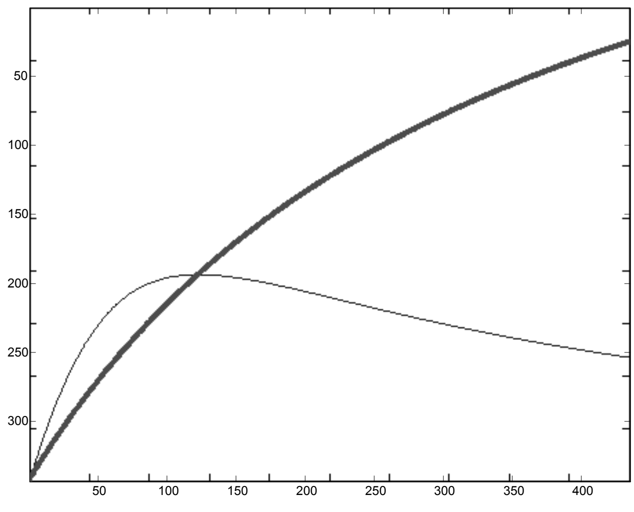

FIGURE 6. $\mu(s)$ (thin line) and its initial guess $\widehat{\mu_0}(s)$ (thick line)

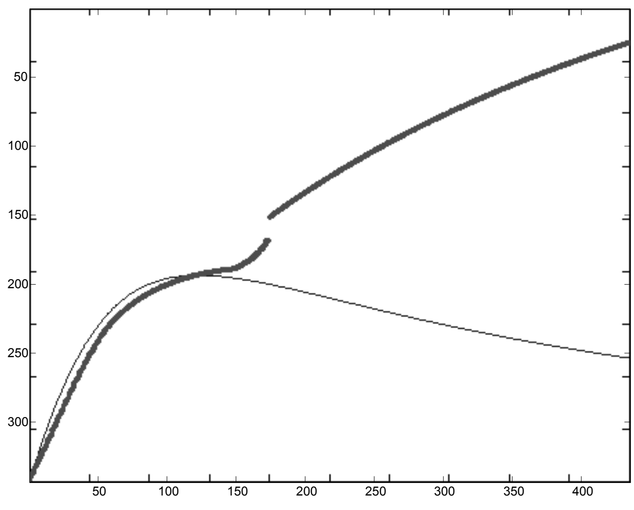

FIGURE 7. $\mu(s)$ and $\widehat{\mu_t}(s)$ at time $t = 0.019$

6.5.2. *Simulation with a single measurement* $s(t)$. If s is the only measurement, we can still expect to reconstruct $\mu(s)$ for all visited value of s provided that at least one value of s has been visited twice, as already explained in section 6.2.

Since this identification process does not depend on the estimation of x, we use the estimation of μ to estimate x. Moreover, once μ is known, the system is a

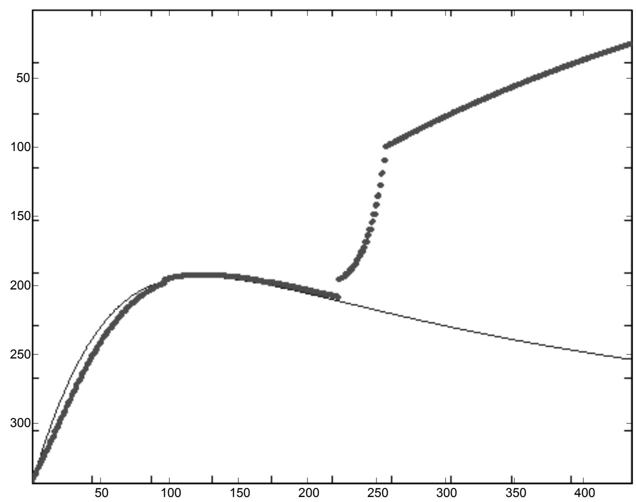

FIGURE 8. $\mu(s)$ and $\widehat{\mu_t}(s)$ at time $t = 0.042$

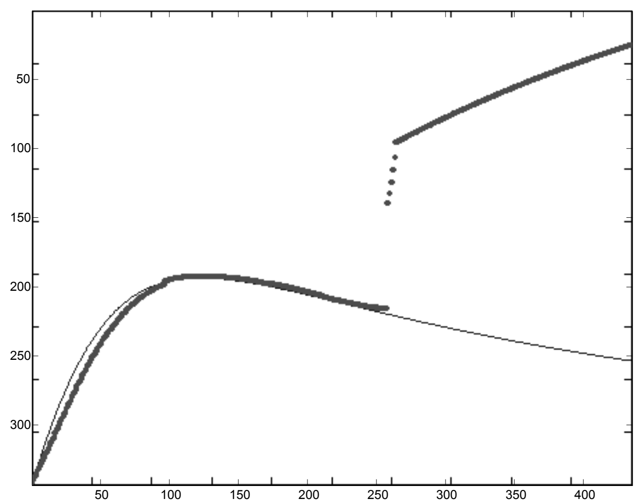

FIGURE 9. $\mu(s)$ and $\widehat{\mu_t}(s)$ at time $t = 0.051$

linear system with a matrix depending on the output s, therefore we can apply linear Kalman filter to estimate x.

Our algorithm is divided in two steps:

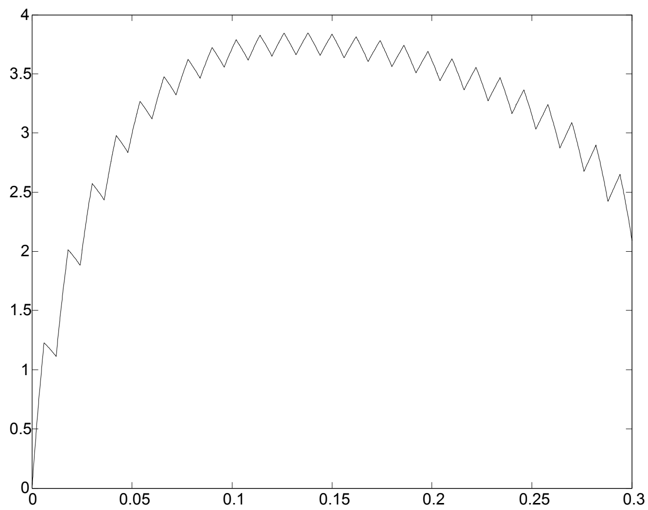

FIGURE 10. $s(t)$

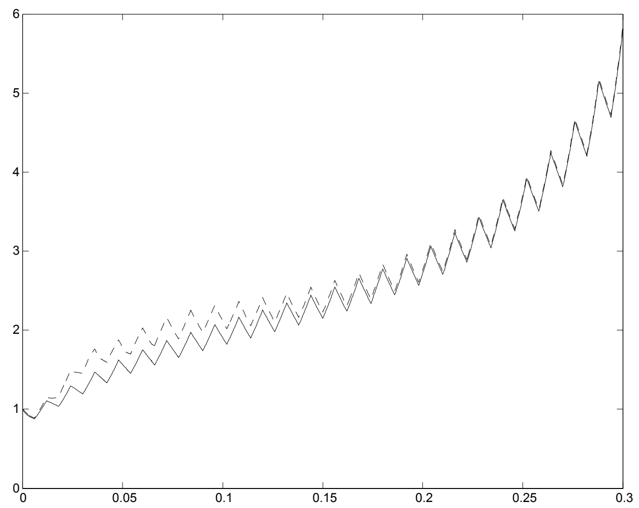

FIGURE 11. $x(t)$

- estimation of μ using the redundant visited values of the measurement s. The estimation of the function $s \to \mu(s)$ at time t is denoted by $s \to \widehat{\mu_t}(s)$;
- estimation of $x(t)$ on the basis of $s(t)$ and $\widehat{\mu_t}$.

The second part of our algorithm is simply standard Kalman filtering. Let us explain the first part.

We first choose a sample time Δt, arbitrarily fixed to 0.2. At each sample time $t = k\Delta t$, we consider the $N+1$ values of s at time $k\Delta t$, $(k-1)\Delta t, \dots, (k-N)\Delta t$, *i.e.* we consider only the history up to time $T = N\Delta t$ in the past.

Since we want to estimate a function, we have to discretize the coordinate space of s in a range corresponding to physical values, with a small enough discretization step, to ensure good accuracy. In this way, we chose $s_{\min}$ and $s_{\max}$ and a step size Δs.

At each sample time $k\Delta t$, we replace the measured trajectory

$$s\left((k-N)\Delta t\right), \dots, s\left(k\Delta t\right) \tag{6.14}$$

by a linearly interpolated trajectory

$$s\left(t_1\right), s\left(t_2\right), \dots, s\left(t_n\right) \tag{6.15}$$

such that $(k-N)\Delta t \leq t_1 < t_2 < \cdots < t_n \leq k\Delta t$ and each $s(t_j)$ is of the form $s_{\min} + p\Delta s$. Let us explain more precisely how we build the list 6.15. Let us assume that we have already build the list from t_1 to t_j considering measurements from time $(k-N)\Delta t$ to time $(k-i)\Delta t$. Then between time $(k-i)\Delta t$ and $(k-i+1)\Delta t$, l and r are such that

$$\begin{aligned} s_{\min} + l\Delta s < s\left((k-i)\Delta t\right) \leq s_{\min} + (l+1)\Delta s \leq s_{\min} + (l+2)\Delta s \leq \cdots \\ \leq s_{\min} + (l+r)\Delta s \leq s\left((k-i+1)\Delta t\right) \leq s_{\min} + (l+r+1)\Delta s \end{aligned}$$

Now we interpolate $t_{j+1}, \dots, t_{j+r}$ so that $s(t_{j+p}) \approx s_{\min} + (l+p)\Delta s$, $p = 1, \dots, r$ and we add $s(t_{j+1}), \dots, s(t_{j+r})$ to the list.

The list 6.15 is then used to look for values appearing at least twice and to estimate X_0 and μ at sample values $s_{\min} + (l+p)\Delta s$.

Nevertheless, in order to give some weight to *a priori* knowledge of μ and to increase robustness with respect to measurement noise, we do not modify completely $\mu(s)$ when a new value is provided by equations (6.9) and (6.10) and the previous algorithm. In fact, we use a first order filter to actualize $\widehat{\mu_t}$, that is, if a new $\mu_t(s)$ is obtained from (6.10) at time t for a discretized s, we modify $\widehat{\mu_t}(s)$ using

$$\widehat{\mu_t}(s) = (1-\beta)\,\widehat{\mu_t}(s) + \beta\,\mu_t(s)$$

Despite this filtering, both on measurement and actualization of μ, this algorithm looks rather sensitive to noise. Here, we illustrate our approach without adding any noise on the output s.

Figure 5 shows $D(t)$ which is the control. Figures 6 to 9 show the estimation $\widehat{\mu_t}(s)$ of $\mu(s)$ at different times. At the beginning of the simulation, we set $\widehat{\mu_0}$ to be the Monod law, although the actual unknown law $\mu(s)$ is the Haldane law. Thin lines represent the actual Haldane growth function. Thick lines and dots represent the estimation of the function μ.

At each sample time, as already mentioned, we use the bioreactor model with $\widehat{\mu_t}(s)$ as growth function to estimate $x(t)$ using a linear Kalman filter. At the beginning of our simulation, since $\widehat{\mu_0}$ is wrong (Monod law instead Haldane law), our observer does not estimate x accurately and there is a bias between the actual $x(t)$ and its estimate $\widehat{x}(t)$. It is expected that when μ is correctly identified, the observer gives an unbiased estimation of x. Indeed, at time 0.15 approximately, $s(t)$ begins to decrease (Figure 10) and then visits again a domain where μ has already been identified. Therefore $\mu(s) \approx \widehat{\mu_t}(s)$ for $t > 0.15$. It is then expected that after time 0.15, estimation of $x(t)$ will be unbiased. This actually happens, see Figure 11.

7. Appendix

7.1. A crucial lemma .

Notations:

1. In this section, we keep the notation $\wedge$ for the exterior product of differential forms on X or on $X \times I$, and, for $V_1, V_2 \in \mathbb{R}^2$, we denote by $V_1 \bar{\wedge} V_2$ the determinant of the 2×2 matrix formed by the vectors V_1, V_2.

2. Again, in this section, for a smooth function f of two variables (x, φ), $x \in \mathbb{R}^n$, $\varphi \in \mathbb{R}^p$, we denote by $d_x f$ (resp $d_\varphi f$), the differentials with respect to the x variable (resp. φ variable) only.

3. For $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we denote by $\underline{x_p} \in \mathbb{R}^p$ the vector $\underline{x_p} = (x_1, ..., x_p)$. For a function $h : \mathbb{R}^n \to \mathbb{R}$, of the variables $x_1, ..., x_n$, the notation $h^{\widehat{i_1, ..., i_p}}$ means that h does not depend on $x_{i_1}, ..., x_{i_p}$.

Lemma 7. *Consider $h : U \times I \to \mathbb{R}^2$, U an open connected subset of $\mathbb{R}^n$, with a given C^ω coordinate system $x = (x_1, ..., x_n)$, and $h(x, \varphi) = (h_1(x, \varphi), h_2(x, \varphi))$, such that $d_\varphi h$ never vanishes on $U \times I$, and the equation*

$$d_x h(x, \varphi)\xi + d_\varphi h(x, \varphi)\eta = 0, \quad \text{with } (\xi_1, ...\xi_p) = 0, \quad p < n, \tag{7.1}$$

has no smooth solution $(\eta, \varphi)(x, \xi)$, on any open subset of $U \times \mathbb{R}^{n-p} \subset \{(x, \xi) | (\xi_{p+1}, ..., \xi_n) \neq 0\}$.

Then, there is $Z \subset U$, a closed subanalytic subset of codimension 1, such that, for all $x_0 \in U \backslash Z$, it does exist a neighborhood V_{x_0} of x_0 and coordinates $\tilde{x} = (\tilde{x}_1, ..., \tilde{x}_n)$ on V_{x_0}, with $\tilde{x}_1 = x_1, ..., \tilde{x}_p = x_p$, and:

for all $\varphi_0 \in I$, there is a neighborhood V_{φ_0} of φ_0, a C^ω real function Φ_{φ_0}, with open domain D in $V_{x_0} \times \Theta_1 \times \Theta_2$, Θ_1, Θ_2 open subsets of $\mathbb{R}$,

with:

$$(F) \ \tilde{x}_{p+1} = \Phi_{\varphi_0}(\underline{\tilde{x}_p}, h_1(\tilde{x}, \varphi), h_2(\tilde{x}, \varphi)),$$

for all $(\tilde{x}, \varphi) \in V_{x_0} \times V_{\varphi_0}$.

Moreover,

$$(G) \ \left(\frac{\partial \Phi_{\varphi_0}}{\partial h_1}, \frac{\partial \Phi_{\varphi_0}}{\partial h_2}\right) \text{ never vanishes on } D,$$

$$(H) \ \frac{\partial h}{\partial \varphi} \bar{\wedge} \frac{\partial h}{\partial \tilde{x}_{p+1}} \text{ never vanishes on } (U \backslash Z) \times I. \tag{7.2}$$

Remark 7. *(G) is a consequence of (F), since $\tilde{x}$ is a C^ω coordinate system on X.*

Proof. (of Lemma 7).

Proof for $p + 1 = n$: Let $E = \{(x, \varphi) | \frac{\partial h}{\partial \varphi} \bar{\wedge} \frac{\partial h}{\partial x_n}(x, \varphi) = 0\}$, and let $\pi_E : E \to U$. By Hardt's theorem on the stratification of proper subanalytic mappings between subanalytic sets, if π_E contains an open set, then, there is , on a (may be smaller) open set Θ, a smooth (C^ω) function $\hat{\varphi}:\Theta\to E$.[1]

Hence, $\frac{\partial h}{\partial \varphi}\,\bar{\wedge}\,\frac{\partial h}{\partial x_n}(x,\hat{\varphi}(x))=0$, for $x\in\Theta$. We chose $x_0\in\Theta$, $\varphi_0=\hat{\varphi}(x_0)$.

If $d_\varphi h_1(x_0,\varphi_0)\neq 0$ (we can assume this by the statement of the lemma, eventually changing h_1 for h_2), then, set:

$$\hat{\eta}(x,\xi)=-\frac{(d_x h_1)(x,\hat{\varphi}(x))\xi}{(d_\varphi h_1)(x,\hat{\varphi}(x))},\ \text{for } \xi=(0,...,0,\xi_n)\neq 0.$$

Then the couple $(\hat{\eta},\hat{\varphi})(x,\xi_n)$ solves Equation (7.1). This is a contradiction. Therefore, $Z=\pi_E$ has codimension 1, and on $(U\backslash Z\times I)$, $\frac{\partial h}{\partial \varphi}\,\bar{\wedge}\,\frac{\partial h}{\partial x_n}$ never vanishes by construction. This proves (H).

Now, $(x_1,...,x_p,h_1,h_2)$ is a coordinate system over small open subsets of $(U\backslash Z)\times I$:

$$\begin{aligned} dx_1\wedge....\wedge dx_p\wedge dh_1\wedge dh_2 &= \\ dx_1\wedge....\wedge dx_p\wedge d\varphi\wedge dx_n(\frac{\partial h}{\partial \varphi} &\ \bar{\wedge}\ \frac{\partial h}{\partial x_n})\neq 0. \end{aligned}$$

Hence, on such a small open set, $x_{p+1}=\Phi(x_1,...,x_p,h_1,h_2)$.

Proof for $p+1<n$: There exists $i>p$ such that $\frac{\partial h}{\partial \varphi}\,\bar{\wedge}\,\frac{\partial h}{\partial x_i}$ does not vanish identically: were it otherwise, for $\xi\neq 0$, $\xi_1=0,...,\xi_p=0$, and in a neighborhood of (x,φ) such that $d_\varphi h_1\neq 0$, $\hat{\eta}=-\frac{d_x h_1.\xi}{d\varphi h_1}$ solves Equation (7.1), which is impossible.

Now, for $\xi\neq 0$, for $\xi_1=0,...,\xi_p=0$,

$$\begin{aligned} &(1)\quad d_x h.\xi\ \ \bar{\wedge}\ \ d_\varphi h(x,\varphi)=0\ \text{ implies:} \\ &(2)\ d\varphi(d_x h.\xi\ \ \bar{\wedge}\ \ d_\varphi h(x,\varphi))=0. \end{aligned}\tag{7.3}$$

Indeed, if it is not true, by the implicit function Theorem, one can solve (7.3,1) with respect to φ, and obtain a smooth solution $\hat{\varphi}(x,\xi)$, on some open set. Setting

$$\hat{\eta}(x,\xi)=-\frac{(d_x h_1)(x,\hat{\varphi}(x,\xi))\xi}{(d_\varphi h_1)(x,\hat{\varphi}(x,\xi))},$$

we solve again Equation (7.1) on an open set, a contradiction.

Statement (7.3) can be rewritten:

$$\sum_{i=p+1}^{n}(d_{x_i}h\,\bar{\wedge}\,d_\varphi h(x,\varphi))\xi_i=0\Rightarrow\sum_{i=p+1}^{n}d_\varphi(d_{x_i}h\,\bar{\wedge}\,d_\varphi h(x,\varphi))\xi_i=0.$$

But, one of the $d_{x_i}h\,\bar{\wedge}\,d_\varphi h(x,\varphi)$does not vanish identically. Hence, in a neighborhood V of some (x_0,φ_0), we have, for all $i=p+1,...,n$, and for a certain analytic function $\lambda(x,\varphi)$:

$$d_\varphi(d_{x_i}h\,\bar{\wedge}\,d_\varphi h(x,\varphi))=\lambda(x,\varphi)(d_{x_i}h\,\bar{\wedge}\,d_\varphi h(x,\varphi)),$$

and then, integrating this (linear) differential equation in φ :

$$d_{x_i}h\,\bar{\wedge}\,d_\varphi h(x,\varphi)=\Lambda(x,\varphi)\omega_i(x),\ \Lambda(x,\varphi)\neq 0,\ i=p+1,...,n.$$

This implies, on V :

$$\omega_j(x)\ d_{x_i}h\,\bar{\wedge}\,d_\varphi h(x,\varphi)-\omega_i(x)d_{x_j}h\,\bar{\wedge}\,d_\varphi h(x,\varphi)=0,$$

[1] Here, in fact, Sard's Theorem plus the implicit function Theorem is enough to obtain this function $\hat{\varphi}$. But Hardt's Theorem is more explicit, and we crucially use subanalyticity elsewhere.

for all $i, j > p$. Hence, since $d\varphi h$ never vanishes,

$$\omega_j(x)\ d_{x_i}h - \omega_i(x)d_{x_j}h = \lambda_{i,j}(x,\varphi)d_\varphi h,\ i,j > p. \tag{7.4}$$

This is true again on V, and for analytic functions $\omega_i(x)$, $\lambda_{i,j}(x,\varphi)$. Moreover, one of the functions $\omega_i(x)$ does not vanish ($\omega_{p+1} \neq 0$, say): remember that $\omega_i(x_0) = d_{x_i}h \barwedge d_\varphi h(x_0,\varphi_0)$, which is nonzero for some i.

Taking $i = p+1, j = p+2$, this implies in particular that h is a ($\mathbb{R}^2-$valued) first integral of the vector field:

$$(\overrightarrow{C})\ \ \varpi_{p+2}(x)\frac{\partial}{\partial x_{p+1}} + \frac{\partial}{\partial x_{p+2}} + \bar{\lambda}(x,\varphi)\frac{\partial}{\partial\varphi}.$$

This is true on V, and for certain analytic functions $\varpi_{p+2}(x)$, $\bar{\lambda}(x,\varphi)$.

The flow of the "characteristic vector field" $\overrightarrow{C}$ is:

$$\exp(t\ \overrightarrow{C})(x) = (x_1,...,x_p,R_{p+2}(t,x),t+x_{p+2},x_{p+3},...,x_n,S_{p+2}(t,x,\varphi)),$$

with $\frac{\partial R_{p+2}}{\partial x_{p+1}} \neq 0$, $\frac{\partial S_{p+2}}{\partial\varphi} \neq 0$. Therefore, $h(\exp(t\ \overrightarrow{C})(x) = h(x)$, and, setting ($x_{p+2} := 0$; $t := x_{p+2}$), we get:

$$h(x_1,...,x_p,\bar{R}_{p+2}(x),x_{p+2},...,x_n,\bar{S}_{p+2}(x,\varphi)) = h(x^{\widehat{p+2}},\varphi),$$

for some analytic functions $\bar{R}_{p+2}(x)$, $\bar{S}_{p+2}(x,\varphi)$, with $\frac{\partial\bar{R}_{p+2}}{\partial x_{p+1}} \neq 0$, $\frac{\partial\bar{S}_{p+2}}{\partial\varphi} \neq 0$. Hence, using the implicit function Theorem:

$$h(x,\varphi) = h(x_1,...,x_p,R^*_{p+2}(x),0,x_{p+3},...,x_n,S^*_{p+2}(x,\varphi)),$$

or, setting $\tilde{x}_{p+1} = R^*_{p+2}(x)$, $\tilde{x}_i = x_i$ for $i \neq p+1$,

$$h(\tilde{x},\varphi) = H(\tilde{x}^{\widehat{p+2}},S^*_{p+2}(\tilde{x},\varphi)),\ \ \frac{\partial S^*_{p+2}}{\partial\varphi} \neq 0.$$

Let us denote these new coordinates $\tilde{x}$ by x. We get:

$$h(x,\varphi) = H(x^{\widehat{p+2}},S_{p+2}(x,\varphi)),\ \ \frac{\partial S_{p+2}}{\partial\varphi} \neq 0.$$

Again (for the same reason as above), there is $i > p$ such that $d_{x_i}h \barwedge d_\varphi h(x,\varphi)$ does not vanish identically. Let us assume $d_{x_{p+1}}h \barwedge d_\varphi h(x,\varphi) \neq 0$. Hence, $(d_{x_{p+1}}H + d_S H\frac{\partial S_{p+2}}{\partial x_{p+1}}) \barwedge d_S H\frac{\partial S_{p+2}}{\partial\varphi} \neq 0$, which implies $d_{x_{p+1}}H \barwedge d_S H \neq 0$.

Also for the same reason as above, we obtain (7.4), for $i = p+1$, $j > p+2$, and dividing by $\omega_i(x) \neq 0$:

$$\omega_j(x)\ d_{x_{p+1}}h - d_{x_j}h = \lambda_{p+1,j}(x,\varphi)d_\varphi h,$$

which gives:

$$\begin{aligned}\omega_j(x)\ d_{x_{p+1}}(H(x,S(x,\varphi))) - d_{x_j}(H(x,S(x,\varphi))) &= \lambda_{p+1,j}(x,\varphi)d_\varphi(H(x,S(x,\varphi))),\\ d_\varphi S &\neq 0.\end{aligned}$$

or:

$$\omega_j(x)\ (d_{x_{p+1}}H)(x,\theta) - (d_{x_j}H)(x,\theta) = \bar{\lambda}_{p+1,j}(x,\theta)(d_\theta H)(x,\theta).$$

For $j = p+3$, with the same reasoning as above, we get:

$$H(x_1,...x_p,\bar{R}_{p+3}(x),x_{p+3},...,x_n,\bar{S}_{p+3}(x,\varphi)) = H(x^{\widehat{p+2,p+3}},\varphi),$$

with $\frac{\partial \bar{R}_{p+3}(x)}{\partial x_{p+1}} \neq 0$, $\frac{\partial S_{p+3}(x)}{\partial \varphi} \neq 0$, or:

$$\begin{aligned} & H(x_1, ...x_p, x_{p+1}, x_{p+3}, ..., x_n, \varphi) \\ = \;& H(x_1, ...x_p, R^*_{p+3}(x), x_{p+4}, ..., x_n, S^*_{p+3}(x, \varphi)). \end{aligned}$$

Making the change of coordinates $x_{p+1} := R^*_{p+3}(x)$, we get:

$$H(x, \varphi) = H(x_1, ...x_p, x_{p+1}, x_{p+4}, ..., x_n, S^*_{p+3}(x, \varphi)).$$

At the end, iterating the process, we get that:

$$h(x, \varphi) = H(x_1, ..., x_{p+1}, S(x, \varphi)), \quad \frac{\partial S}{\partial \varphi} \neq 0, \tag{7.5}$$

on some open subset of $U \times I$. The coordinates x_i, $i = 1, ..., p$, are unchanged.

As a consequence, we finally get that there is an open dense subanalytic subset $U \backslash Z$ of U, and for each $x_0 \in U \backslash Z$, a coordinate neighborhood of x_0, (U_{x_0}, x), with coordinates x_i, $i = 1, ..., p$, unchanged, and with:

$$dh_1 \wedge dh_2 \wedge dx_1 \wedge ... \wedge dx_{p+1} \equiv 0,$$

identically on $U_{x_0} \times I$, by analyticity.

Now, let $E_{x_0} = \{(x, \varphi) | dh_1 \wedge dh_2 \wedge dx_1 \wedge ... \wedge dx_p = 0\}$, and let πE_{x_0} be the canonical projection of E_{x_0} on U_{x_0}. If πE_{x_0} contains an open set, again by Hardt's Theorem on stratification of proper subanalytic maps, we can find another open subset Θ of U, and a smooth mapping $\hat{\varphi} : \Theta \to E_{x_0}$. Then:

$$(dh_1 \wedge dh_2 \wedge dx_1 \wedge ... \wedge dx_p)_{|(x, \hat{\varphi}(x))} = 0,$$

and, in particular, for $j > p$, $(\frac{\partial h_1}{\partial x_j} \frac{\partial h_2}{\partial \varphi} - \frac{\partial h_1}{\partial \varphi} \frac{\partial h_2}{\partial x_j})_{|(x, \hat{\varphi}(x))} = 0$. Therefore, since $d_\varphi h$ is nonzero, Equation (7.1) can still be solved, in the following way:

$$\eta = -\frac{d_x h_1(x, \hat{\varphi}(x)) \xi}{d_\varphi h_1(x, \hat{\varphi}(x)),}$$

for $\xi \neq 0$ (if $d_\varphi h_1(x_0, \hat{\varphi}(x_0)) \neq 0$, and using h_2 if not).

This contradicts the assumptions of the lemma. Hence, there is a codimension 1 subanalytic closed subset of U, called again Z, such that, over $U \backslash Z$, each x_0 has a coordinate neighborhood (U_{x_0}, x), x_i, $i = 1, ..., p$, unchanged, where $dh_1 \wedge dh_2 \wedge dx_1 \wedge ... \wedge dx_p$ never vanishes, and $dh_1 \wedge dh_2 \wedge dx_1 \wedge ... \wedge dx_{p+1}$ is everywhere zero, and this is true over $U_{x_0} \times I$.

Since (if $d_\varphi h_1 \neq 0$), $dx_1 \wedge ... \wedge dx_n \wedge dh_1 \neq 0$, $dh_1 \wedge dh_2 \wedge dx_1 \wedge ... \wedge dx_{p+1} \equiv 0$ implies:

$$h_2 = H_2(h_1, x_1, ..., x_{p+1}). \tag{7.6}$$

The condition $dh_1 \wedge dh_2 \wedge dx_1 \wedge ... \wedge dx_p \neq 0$ can be rewritten $\frac{\partial H_2}{\partial x_{p+1}} \neq 0$. Hence:

$$d\varphi h \,\bar{\wedge}\, dx_{p+1} h = d_\varphi h_1 . d_{x_{p+1}} H_2 \neq 0.$$

This is (H), in (7.2).

Now, since $\frac{\partial H_2}{\partial x_{p+1}} \neq 0$,

$$\begin{aligned} & dh_1 \wedge dh_2 \wedge dx_1 \wedge ... \wedge dx_p \wedge dx_{p+2} \wedge ... \wedge dx_n \\ = \;& (d\varphi h \,\bar{\wedge}\, dx_{p+1} h) d\varphi \wedge dx_{p+1} \wedge dx_1 \wedge ... \wedge dx_p \wedge dx_{p+2} \wedge ... \wedge dx_n \neq 0. \end{aligned}$$

This shows that $h_1, h_2, x_1, ..., x_p, x_{p+2}, ..., x_n$ is a coordinate system on some neighborhood $U_{x_0,\varphi_0} \times V_{\varphi_0}$, for all $\varphi_0 \in I$, and then, since $dh_1 \wedge dh_2 \wedge dx_1 \wedge ... \wedge dx_{p+1}$ is identically zero,

$$x_{p+1} = \Phi_{\varphi_0}(h_1, h_2, \underline{x_p}).$$

Now, I being compact, we may cover $\{x_0\} \times I \subset X \times I$ by a finite number of such open neighborhoods $U_{x_0,\varphi_0} \times V_{\varphi_0}$, on which (F), (G), (H) are satisfied. Hence the neighborhood U_{x_0} can be taken fixed, independantly of φ_0. This ends the proof. ∎

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