NEW RESULTS ON IDENTIFIABILITY OF NONLINEAR SYSTEMS

Eric Busvelle, Jean-Paul Gauthier *

* *Le2i UMR 5158, Université de Bourgogne, BP 47870, 21078, Dijon Cedex France*

Abstract: In this paper, we recall definition of identifiability of nonlinear systems. We prove equivalence between **identifiability** and **smooth identifiability**. This new result justifies our definition of identifiability.
In a previous paper (Busvelle and Gauthier, 2003), we have established that

- If the number of observations is three or more, then, systems are generically identifiable.
- If the number of observations is 1 or 2, then the situation is reversed. Identifiability is not at all generic.

Also, we have completely classified infinitesimally identifiable systems in the second case, and in particular, we gave normal forms for identifiable systems. Here, we will give similar results in the first case.

Keywords: nonlinear systems, observer, observability, identification, identifiability.

We consider smooth (C^ω or C^∞, depending on the context) systems of the form

$$\Sigma_{x_0,\varphi}\begin{cases} \dfrac{dx}{dt} = f(x,\varphi(x)) \\ y = h(x,\varphi(x)) \end{cases} \quad (1)$$

where the state $x = x(t)$ lies in a n–dimensional analytic manifold [1] X, $x(0) = x_0$, the observation y is $\mathbf{R}^{d_y}$–valued, and f, h are respectively a smooth (parametrized) vector field and a smooth function. The function φ is an unknown function of the state. In this paper, each trajectory is supposed to be defined on some interval $[0, T_{x_0,\varphi}[$.

Our goal is to estimate both state variable x and unknown function $\varphi : X \longrightarrow I$, I being a compact interval of $\mathbf{R}$. More precisely, we want to reconstruct the piece of the graph of φ visited during any experiment. Without φ, the problem is an observation problem and we refer to the book from Gauthier–Kupka (Gauthier and Kupka, 2001). In presence of the unknown function φ, the problem is an identification problem, which has been introduced in (Busvelle and Gauthier, 2003). Let us recall some definitions and results from this last paper. For this introduction, we will only consider uncontrolled systems such as Σ. Some results can be extended to controlled systems.

Let $\Omega = X \times L^\infty[I]$, where

$$L^\infty[I] = \{\hat\varphi : [0, T_{\hat\varphi}] \mapsto I,\ \hat\varphi \text{ measurable}\}$$

Then we can define the input/output mapping

$$P_\Sigma : \begin{array}{rcl} \Omega & \longrightarrow & L^\infty\left[\mathbf{R}^{d_y}\right] \\ (x_0, \hat\varphi(\cdot)) & \longrightarrow & y(\cdot) \end{array}$$

Definition 1. Σ is said to be identifiable if P_Σ is injective.

As for observability, we define an infinitesimal version of identifiability. Let us consider the first

[1] analytic manifold stands for analytic connected paracompact Hausdorf manifold

variation of $\Sigma_{x_0,\varphi}$ (where $\hat{\varphi}(t) = \varphi \circ x(t)$):

$$T\Sigma_{x_0,\hat{\varphi},\xi_0,\eta} \begin{cases} \dfrac{dx}{dt} = f(x,\hat{\varphi}) \\ \dfrac{d\xi}{dt} = T_x f(x,\hat{\varphi})\xi + d_\varphi f(x,\hat{\varphi})\eta \\ \hat{y} = d_x h(x,\hat{\varphi})\xi + d_\varphi h(x,\hat{\varphi})\eta \end{cases}$$

and the input/output mapping of $T\Sigma$

$$P_{T\Sigma,x_0,\hat{\varphi}} : T_{x_0}X \times L^\infty[\mathbf{R}] \longrightarrow L^\infty\left[\mathbf{R}^{d_y}\right]$$
$$(\xi_0, \eta(\cdot)) \longrightarrow \hat{y}(\cdot)$$

Definition 2. Σ is said to be infinitesimally identifiable if $P_{T\Sigma,,x_0,\hat{\varphi}}$ is injective for any $(x_0, \hat{\varphi}(\cdot)) \in \Omega$ i.e. $\ker(P_{T\Sigma,x_0,\hat{\varphi}}) = \{0\}$ for any $(x_0, \hat{\varphi}(\cdot))$.

1. Equivalence between "identifiability" and "identifiability for smooth functions"

Both identifiability and infinitesimal identifiability mean injectivity of some mapping. Clearly injectivity depends on the domain. Therefore, it seems that these notions are not well defined. In fact we show that these notions do not depend on the domain, at least for analytic systems.

Theorem 1. (C^ω–case) If Σ is infinitesimally identifiable in the class of analytic functions then it is infinitesimally identifiable in the class of L^∞–functions.

To be more explicit, if Σ is not infinitesimally identifiable because there exists $(x_0, \hat{\varphi}(\cdot)) \in \Omega$ such that $(\xi_0, \eta) \in \ker(P_{T\Sigma,x_0,\hat{\varphi}})$, $(\xi_0, \eta) \neq 0$, then there exists also $\left(\tilde{x}_0, \widetilde{\hat{\varphi}}(\cdot)\right) \in \Omega$ where $\widetilde{\hat{\varphi}}$ is analytic, and $\left(\tilde{\xi}_0, \tilde{\eta}\right) \in \ker(P_{T\Sigma,x_0,\hat{\varphi}})$, $\left(\tilde{\xi}_0, \tilde{\eta}\right) \neq 0$, with $\tilde{\eta}$ analytic.

We have also the following result

Theorem 2. (C^ω–case) If Σ is identifiable in the class of analytic functions then it is identifiable in the class of L^∞–functions.

The proofs of both results are based upon the following lemma which is an immediate adaptation of a lemma in (Gauthier and Kupka, 2001):

Lemma 1. Let $\dfrac{dx}{dt} = f(x,u)$ an analytic system $\bar{\Sigma}$ defined on $X \times U$ where X is an analytic manifold and U a compact subanalytic subset of $\mathbf{R}^d$. Let S be a closed[2] subanalytic subset of $X \times U$. Let $(x(t), u(t))_{t\in[0,T[}$ a L^∞–trajectory of $\bar{\Sigma}$ such that $E = \{t \in [0,T[; \quad (x(t), u(t)) \in S\}$ has strictly positive Lebesgue measure (we say that "the trajectory visits S"). Then there exists an analytic trajectory $(\tilde{x}(t), \tilde{u}(t))_{t\in[0,\tilde{T}[}$ of $\bar{\Sigma}$ such that $(\tilde{x}(t), \tilde{u}(t))$ lies in S for each $t \in \left[0, \tilde{T}\right[$.

[2] This lemma is also true if S is not closed but this is much harder to prove.

Proof of the theorem 1.

Let $Z(\cdot) = (x(\cdot), \xi(\cdot), \varphi(\cdot), \eta(\cdot))$ be a trajectory of $T\Sigma$ such that $y_{T\Sigma}(t) \equiv 0$, $t \in [0,T]$ but $(\xi(\cdot), \eta(\cdot)) \neq 0$.

If $\xi(t) \neq 0$ for some $t \in [0,T]$, we consider on $TX \setminus \{0 \text{ section}\}$

$$T\Sigma \begin{cases} \dot{x} = f(x,\varphi) \\ \dot{\xi} = T_x f(x,\varphi)\xi + T_\varphi f(x,\varphi)\eta \end{cases}$$

and the set

$$Z_A = \{d_x h(x,\varphi)\xi + d_\varphi h(x,\varphi)\eta = 0, \quad |\eta| \leq A\}$$

By the lemma, since we have a trajectory of $T\Sigma$ that remains in Z_A for some $A > 0$, we can find a C^ω one that remains also in Z_A.

If $\xi(\cdot) \equiv 0$, then $(x(\cdot), \varphi(\cdot))$ is a trajectory of Σ such that $(d_\varphi h, T_\varphi f)(x(t), \varphi(t))$ vanishes on a set of strictly positive measure because since $\xi(t) \equiv 0$, $y_{T\Sigma}(t) = 0$ then $d_\varphi h(x,u)\eta(t) = 0$ and $T_\varphi f(x,\varphi)\eta(t) = 0$ for almost all $t \in [0,T]$, and $\eta(t)$ is non zero on a set of strictly positive measure set

$$Z = \{(x,\varphi), \quad (d_\varphi h, T_\varphi f)(x(t), \varphi(t)) = 0\}$$

By lemma 1, we find a C^ω–trajectory of Σ in Z (taking ξ_0, and arbitrary $\eta(\cdot)$ non zero). ■

Proof of the theorem 2.

Assume that Σ is not identifiable for L^∞–function $\hat{\varphi}(\cdot)$. It means that we can find $(x_1, \hat{\varphi}_1(\cdot)) \neq (x_2, \hat{\varphi}_2(\cdot))$ such that the corresponding outputs $y_1(\cdot)$ and $y_2(\cdot)$ are equal on $[0,T]$. Let $x_1(\cdot)$ and $x_2(\cdot)$ be the corresponding trajectories defined for $t \in [0,T]$. Then if $x_1(t) \neq x_2(t)$ for some $t \in [0,T]$, we consider on $(X \times X \setminus \Delta X) \times (I \times I)$ the system

$$\Sigma_2 \begin{cases} \dot{x}_1 = f(x_1, \varphi_1) \\ \dot{x}_2 = f(x_2, \varphi_2) \end{cases}$$

and the set

$$Z = \{(x_1, x_2, \varphi_1, \varphi_2) \in (X \times X \setminus \Delta X) \times I \times I, \quad h(x_1,\varphi_1) - h(x_2,\varphi_2) = 0\}$$

Z is analytic closed.We apply the lemma above to Σ_2 and we obtain an analytic trajectory $(\hat{x}_1, \hat{x}_2)$ of Σ_2 with $\hat{x}_1(0) \neq \hat{x}_2(0)$, but $y_1(\cdot) = y_2(\cdot)$. Then Σ is not identifiable in the C^ω– class.

Now assume that $x_1(t) = x_2(t)$ for each $t \in [0, T]$. For $A > 0$, we define the closed semianalytic subset of $(X \times X) \times (I \times I)$

$$Z'_A = \{(x_1, x_2, \varphi_1, \varphi_2), \quad \varphi_1 - \varphi_2 \geq A, \\ x_1 = x_2, \quad h(x_1, \varphi_1) - h(x_2, \varphi_2) = 0\}$$

For a certain $A > 0$, our given trajectory visits Z'_A for a set of times of positive Lebesgue measure in $[0, T]$ (eventually reversing the role of φ_1 and φ_2). By the lemma again, we find a C^ω–trajectory in Z'_A. As a consequence, theorem 2 is proved.■

2. Normal forms

We consider again a system Σ of the form (1). In (Busvelle and Gauthier, 2003), we have shown that identifiability is a generic property if and only if the number of observation d_y is greater or equal to 3 (Theorem 5 below). On the contrary, if d_y is equal to 1 or 2, identifiability is a very restrictive hypothesis (infinite codimension) and we have completely classified infinitesimally identifiable systems by giving certain geometric properties that are equivalent to the normal forms presented in Theorems 3 and 4 (Busvelle and Gauthier, 2003) below.

Theorem 3. ($d_y = 1$) If Σ is uniformly infinitesimally identifiable, then, there is a subanalytic closed subset Z of X, of codimension 1 at least, such that for any $x_0 \in X \backslash Z$, there is a coordinate neighborhood $(x_1, \ldots, x_n,\ V_{x_0})$, $V_{x_0} \subset X \backslash Z$ in which Σ (restricted to V_{x_0}) can be written:

$$\Sigma_1 \begin{cases} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= \psi(x, \varphi) \\ y &= x_1 \end{cases} \quad \text{and} \quad \frac{\partial}{\partial \varphi}\psi(x, \varphi) \neq 0$$

Theorem 4. ($d_y = 2$) If Σ is uniformly infinitesimally identifiable, then, there is an open-dense semi-analytic subset $\tilde{U}$ of $X \times I$, such that each point (x_0, φ_0) of $\tilde{U}$, has a neighborhood $V_{x_0} \times I_{\varphi_0}$, and coordinates x on V_{x_0} such that the system Σ restricted to $V_{x_0} \times I_{\varphi_0}$, denoted by $\Sigma_{|V_{x_0} \times I_{\varphi_0}}$, has one of the three following normal forms:

-type 1 normal form:

$$\Sigma_{2,1} \begin{cases} y_1 = x_1 & y_2 = x_2 \\ \dot{x}_1 = x_3 & \dot{x}_2 = x_4 \\ \quad\vdots & \quad\vdots \\ \dot{x}_{2k-3} = x_{2k-1} & \dot{x}_{2k-2} = x_{2k} \\ \dot{x}_{2k-1} = f_{2k-1}(x_1, ..., x_{2k+1}) & \\ \dot{x}_{2k} = x_{2k+1} & \\ \quad\vdots & \\ \dot{x}_{n-1} = x_n & \\ \dot{x}_n = f_n(x, \varphi) & \end{cases}$$

with $\frac{\partial f_n}{\partial \varphi} \neq 0$.

-type 2 normal form:

$$\Sigma_{2,2} \begin{cases} y_1 = x_1 & y_2 = x_2 \\ \dot{x}_1 = x_3 & \dot{x}_2 = x_4 \\ \quad\vdots & \quad\vdots \\ \dot{x}_{2r-3} = x_{2r-1} & \dot{x}_{2r-2} = x_{2r} \\ \dot{x}_{2r-1} = \psi(x, \varphi) & \dot{x}_{2r} = F_{2r}(x_1, \ldots, x_{2r+1}, \psi(x, \varphi)) \\ & \dot{x}_{2r+1} = F_{2r+1}(x_1, \ldots, x_{2r+2}, \psi(x, \varphi)) \\ & \quad\vdots \\ & \dot{x}_{n-1} = F_{n-1}(x, \psi(x, \varphi)) \\ & \dot{x}_n = F_n(x, \varphi) \end{cases}$$

with $\frac{\partial \psi}{\partial \varphi} \neq 0$, $\frac{\partial F_{2r}}{\partial x_{2r+1}} \neq 0, ...,\ \frac{\partial F_{n-1}}{\partial x_n} \neq 0$

-type 3 normal form:

$$\Sigma_{2,3} \begin{cases} y_1 = x_1 & y_2 = x_2 \\ \dot{x}_1 = x_3 & \dot{x}_2 = x_4 \\ \quad\vdots & \quad\vdots \\ \dot{x}_{n-3} = x_{n-1} & \dot{x}_{n-2} = x_n \\ \dot{x}_{n-1} = f_{n-1}(x, \varphi) & \dot{x}_n = f_n(x, \varphi) \end{cases}$$

with $\frac{\partial}{\partial \varphi}(f_{n-1}, f_n) \neq 0$

In fact, see (Busvelle and Gauthier, 2003), these sufficient conditions are also almost (in a local sense) necessary.

In (Busvelle and Gauthier, 2003), we did not give an identifiability normal form for identifiable systems with $d_y \geq 3$, but we proved the following result:

Theorem 5. If $d_y \geq 3$, identifiability is a generic property.

We will now gives normal forms for generic identifiable systems. Let us consider a smooth system Σ with 3 observations (we present the case $d_y = 3$ but all results below are also true for systems with more than 3 measurements, i.e. $d_y \geq 3$):

$$\Sigma \begin{cases} \frac{dx}{dt} = f(x, \varphi) \\ y_1 = h_1(x, \varphi) \\ y_2 = h_2(x, \varphi) \\ y_3 = h_3(x, \varphi) \end{cases}$$

2.1. Injectivity

First, let S^* be the open dense set of systems $\Sigma = (f, h)$ such that the set

$$\bar{Z} = \left\{ x \in X, \quad \frac{\partial h}{\partial \varphi}(x, \varphi) \neq 0 \quad \forall \varphi \in I \right\}$$

has the following Property $(\mathcal{P})$: it is a connected open dense subset of X wich is also locally connected in X (in the strong sense that its intersection with any open connected subset of X is connected). For a system $\Sigma \in S^*$, we consider $\bar{\Sigma} = \Sigma|_{\bar{Z}}$ the restriction of Σ to $\bar{Z}$. If $\bar{\Sigma}$ is differentially identifiable of order k (in the sense of (Busvelle and Gauthier, 2003)) then the mapping

$$\Phi_k^{\bar{\Sigma}} : \left(x, \varphi, \ldots, \varphi^{(k-1)}\right) \mapsto \left(y, \ldots, y^{(k-1)}\right)$$

is injective: this follows from the fact that if $j_{y_1}^k = j_{y_2}^k$ then $(x_1, \varphi_1) = (x_2, \varphi_2) \overset{\text{def.}}{=} (\bar{x}, \bar{\varphi})$ by identifiability and then

$$\begin{aligned} \dot{y}_1 &= L_f h\left(\bar{x}, \bar{\varphi}\right) + \frac{\partial h}{\partial \varphi}\left(\bar{x}, \bar{\varphi}\right) \dot{\varphi}_1 \\ &= L_f h\left(\bar{x}, \bar{\varphi}\right) + \frac{\partial h}{\partial \varphi}\left(\bar{x}, \bar{\varphi}\right) \dot{\varphi}_2 \end{aligned}$$

and since $\frac{\partial h}{\partial \varphi}$ does not vanishes on $\bar{Z}$, we conclude that $\dot{\varphi}_1 = \dot{\varphi}_2$. By induction $j_{\varphi_1}^k = j_{\varphi_2}^k$. The following theorem is now a consequence of the genericity of differential identifiability, proved in (Busvelle and Gauthier, 2003):

Theorem 6. There is a residual set S^{**} of systems such that $\bar{\Sigma} = \Sigma|_{\bar{Z}}$ has the following property:

$$\Phi_{2n+1}^{\bar{\Sigma}} \text{ is injective}$$

2.2. Immersivity (We give only the sketch of the proof)

Let us fix $h_3 : X \times I \mapsto \mathbf{R}$, $h_3 \in H$ where H is the open–dense subset of $C^\infty(X \times I)$ of mappings such that $\frac{\partial h_3}{\partial \varphi}$ does not vanishes out of a closed submanifold of codimension 1 of $X \times I$. Let $Z_{h_3} = \left\{(x, \varphi),\quad \frac{\partial h_3}{\partial \varphi}(x, \varphi) = 0\right\}$ and let $\bar{Z}_{h_3}$ be the complement of Z_{h_3} in $X \times I$.

Consider the immersion $\psi : \bar{Z}_{h_3} \to X \times \mathbf{R}, (x, \varphi) \mapsto (x, y_3 = h_3(x, \varphi))$. Let $\bar{W}_{h_3}$ denote the (open) image of ψ.

Consider now locally defined systems $\tilde{\Sigma}$ on $\bar{W}_{h_3}$:

$$\tilde{\Sigma} \begin{cases} \dot{x} &= \tilde{f}(x, y_3) \\ \dot{y}_1 &= \tilde{h}_1(x, y_3) \\ \dot{y}_2 &= \tilde{h}_2(x, y_3) \end{cases}$$

By the book (Gauthier and Kupka, 2001), there are bad sets $\tilde{B}$ in $j_{\tilde{\Sigma}}^k$ (the set of k–jets of these systems), relative to immersivity. These bad sets are pulled back by ψ in the set of k–jets of systems Σ on Z_{h_3}, $B = \psi_*^{-1}\left(\tilde{B}\right)$

$$\Sigma \begin{cases} \dot{x} &= f(x, \varphi) \\ \dot{y}_1 &= h_1(x, \varphi) \\ \dot{y}_2 &= h_2(x, \varphi) \end{cases}$$

If $k > 2n$, $\operatorname{codim}(B) > n + 1$. By the transversality theorems, the set of Σ that avoid B is residual. Notice that a system avoiding B has the following property, in restriction to $\bar{Z}_{h_3}$: the map $\left(x, \varphi, \ldots, \varphi^{(k-1)}\right) \mapsto \left(y, \dot{y}, \ldots, y^{(k-1)}\right)$ is immersive (this point is not completely obvious).

Now, taking just the intersection of 3 residual sets of this type (constructed with h_1, h_2 and h_3), we get the following theorem:

Theorem 7. ($d_y \geq 3$) The set of systems $\Sigma \in S^*$ such that, in restriction to $\bar{Z} \times I$, $\Phi_{\Sigma}^k : \left(x, \varphi, \ldots, \varphi^{(k-1)}\right) \mapsto \left(y, \dot{y}, \ldots, y^{(k-1)}\right)$ is immersive, is residual.

Corollary 1. The set S^{***} of systems such that, in restriction to $\bar{Z} \times I$, $\Phi_{\Sigma}^k : \left(x, \varphi, \ldots, \varphi^{(k-1)}\right) \mapsto \left(y, \dot{y}, \ldots, y^{(k-1)}\right)$ is an injective immersion, is residual.

It follows from this result that systems in S^{***} can be embedded into systems of the form

$$\begin{aligned} y &= z^1 = \left(z_1^1, z_2^1, z_3^1\right) \\ \dot{z}^1 &= z^2 \\ &\vdots \\ \dot{z}^{k-1} &= z^k \\ \dot{z}^k &= F\left(z^1, \ldots, z^{k-1}, \varphi^{(k)}\right) \end{aligned}$$

with

$$\left(x, j^k \varphi\right) = H\left(z^1, \ldots, z^{k-1}\right)$$

and

$$y^{(k)} = G\left(x, \varphi, \ldots, \varphi^{(k)}\right)$$

in restriction to arbitrary compact subsets of $\bar{Z}$.

There is also another normal form, local only, useful for the practical identification purpose: systems are immersed by Φ_{Σ}^k into systems of the form (generic canonical form, $d_y = 3$):

$$\begin{aligned} y &= z^1 = \left(z_1^1, z_2^1\right) \\ \dot{z}^1 &= z^2 \\ &\vdots \\ \dot{z}^{k-1} &= z^k \\ \dot{z}^k &= F\left(z^1, \ldots, z^{k-1}, y_3, \ldots, y_3^{(k)}\right) \end{aligned}$$

and

$$x = \Phi\left(z^1, \ldots, z^{k-1}, y_3, \ldots, y_3^{(k-1)}\right)$$

This normal form holds for the systems in the residual set S^{***} locally around each point $(x, \varphi) \in \bar{Z} \times I$.

Finally, we proved:

Theorem 8. ($d_y \geq 3$) If Σ is an infinitesimally identifiable generic system ($\Sigma \in S^{***}$), then there is a very small subset Z of X (in the sense of Property ($\mathcal{P}$), Section 2.1), such that for any $x_0 \in X \backslash Z$, there exist a smooth C^∞–function F and a $\left(\check{y}, \check{y}', \ldots, \check{y}^{(2n)}\right)$–dependant embedding $\Phi_{\check{y},\ldots,\check{y}^{(2n)}}(x)$ such that outside Z, trajectories of $\Sigma_{x_0,\varphi}$ are mapped via $\Phi_{\check{y},\ldots,\check{y}^{(2n)}}$ into trajectories of the following system

$$\Sigma_{3+}\begin{cases} \dfrac{dz_1}{dt} = z_2, \quad \dfrac{dz_2}{dt} = z_3, \quad \ldots \quad , \\ \dfrac{dz_{2n}}{dt} = z_{2n+1} \\ \dfrac{dz_{2n+1}}{dt} = F\left(z_1, \ldots, z_{2n+1}, \check{y}, \ldots, \check{y}^{(2n+1)}\right) \\ \bar{y} = z_1 \end{cases}$$

where z_i, $i = 1, \ldots, 2n+1$ has dimension $d_y - 1$, and with

$$\begin{cases} x = \Phi^{-1}_{\check{y},\ldots,\check{y}^{(2n)}}(z) \\ \varphi = \Psi(x, \check{y}) \end{cases} \tag{2}$$

($\check{y}$ is a selected output, such as y_3 in previous proofs)

Practical considerations. Several explicit constructions of high–gain observers can be applied to Σ_{3+} in order to recover an estimation $\hat{z}(t)$ of $z(t)$ knowing y, $\check{y}$ and its derivatives ((Busvelle and Gauthier, 2003; Gauthier and Kupka, 2001)). Then x and φ are recovered using (2). In (Busvelle and Gauthier, 2003), we explained how to apply the same type of observers to non generic systems Σ_1 and $\Sigma_{2,i}$.

REFERENCES

Busvelle, E. and J.-P. Gauthier (2003). On determining unknown functions in differential systems, with an application to biological reactors. *ESAIM: COCV* **9**, 509–552.

Gauthier, J.-P. and I. Kupka (2001). *Deterministic Observation Theory and Applications.* Cambridge University Press.

Hironaka, H. (1973). Subanalytic sets. In: *Number Theory, Algebraic Geometry and Commutative Algebra.* pp. 453–493. Kinokuniya. Tokyo. Volume in honor of Y. Akizuki.