The λ-calculus as a 2-dimensional clone

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Based on joint (ongoing) work with Federico Olimpieri

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Outline

The zoo of multicategorical structures

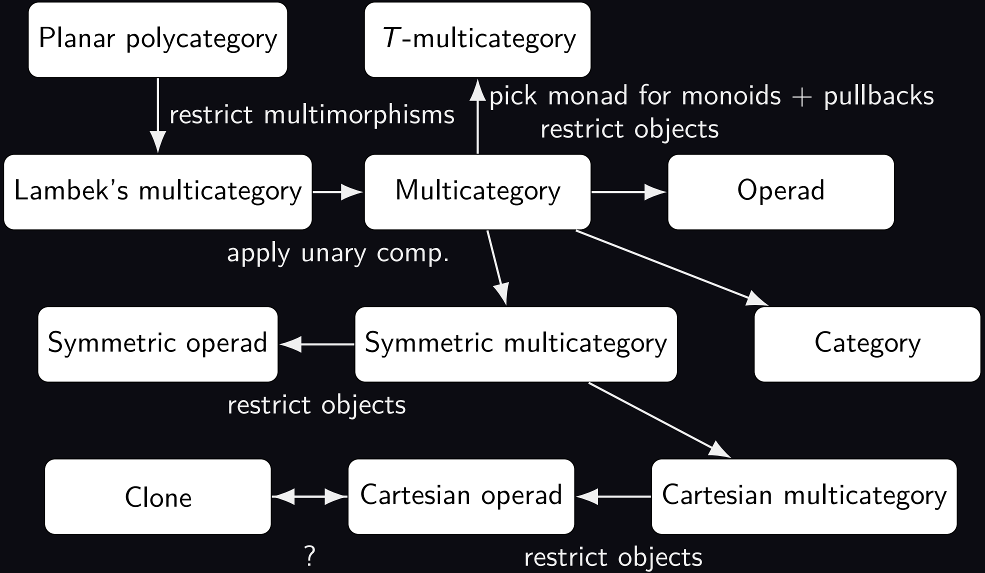
The untyped λ-calculus as a closed clone [Hyland2015]

The untyped λ-calculus as a 2-dimensional structure (PhD project)

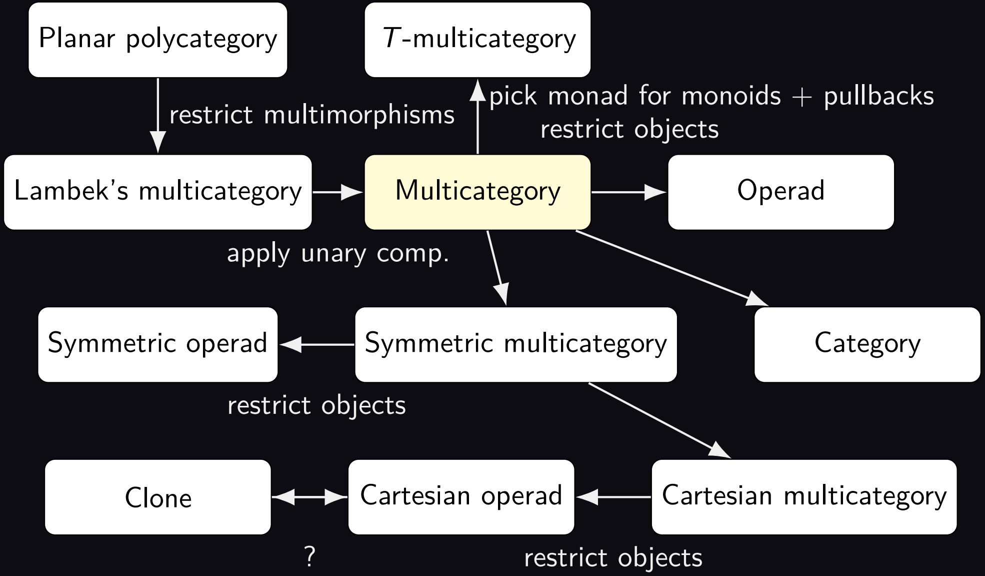
Take-away messages

The zoo of multicategorical structures

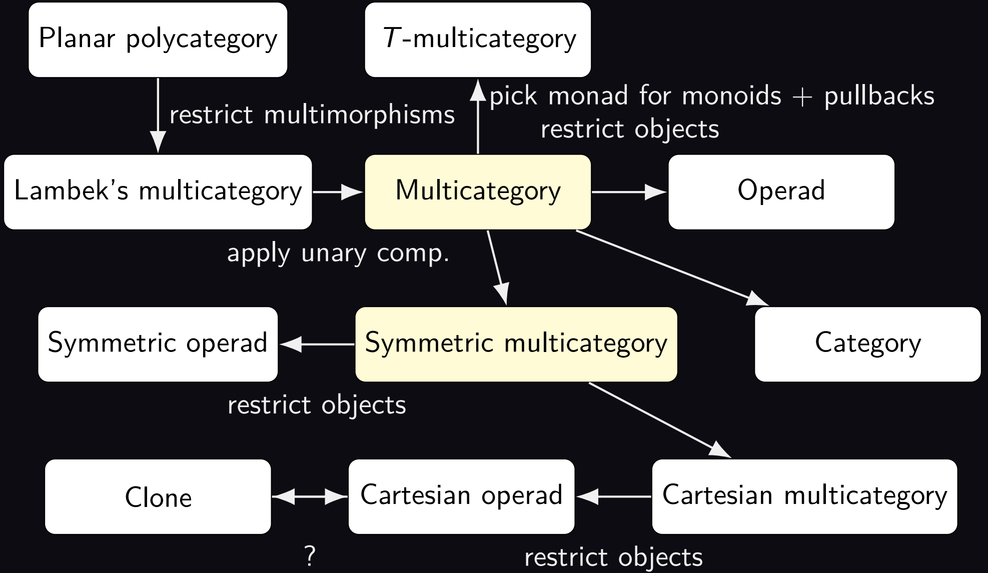
Relationships between multicategorical structures



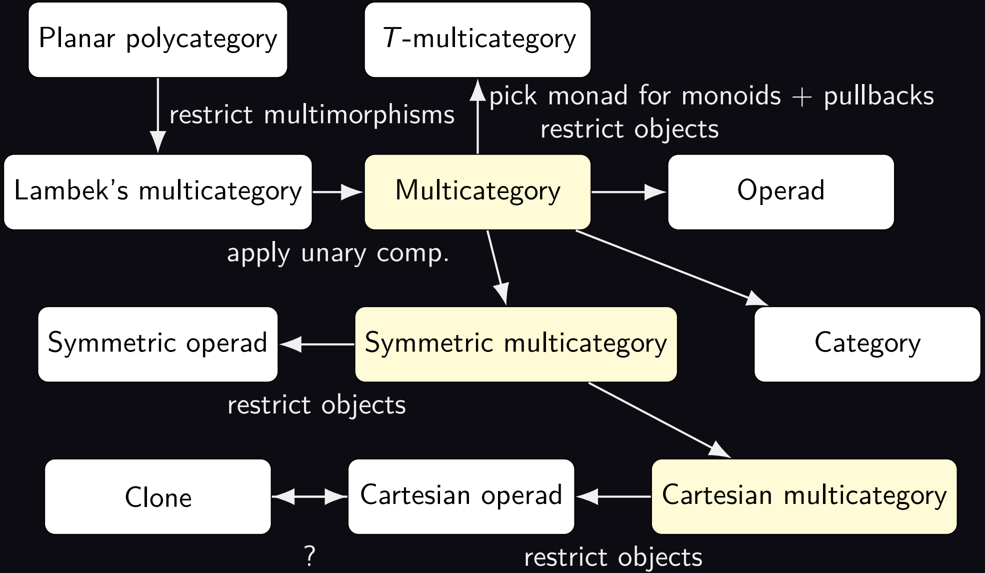
Relationships between multicategorical structures



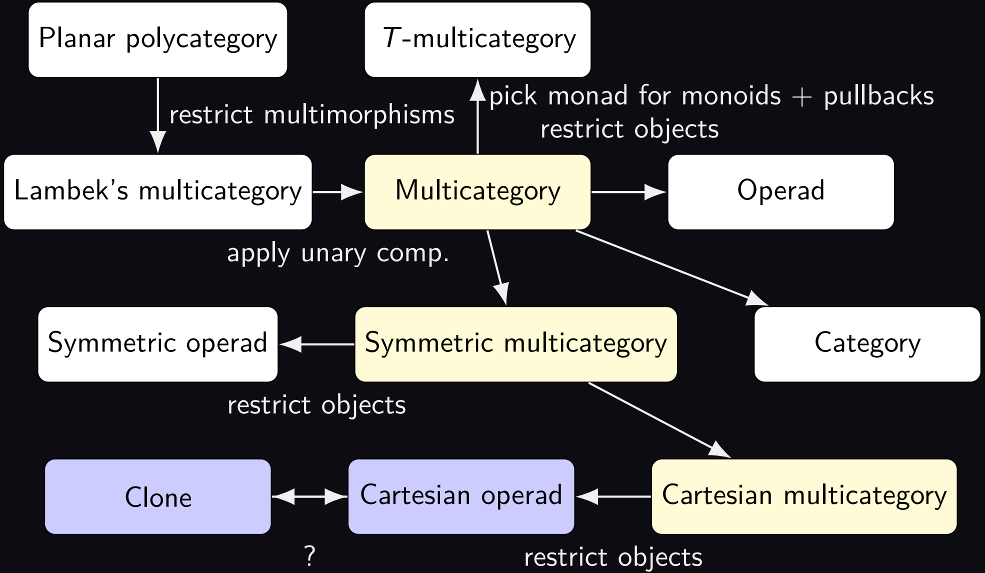
Relationships between multicategorical structures



Relationships between multicategorical structures



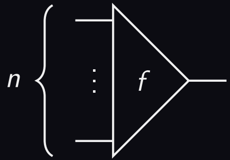
Relationships between multicategorical structures



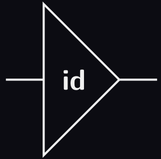
What is an operad?

An operad $\mathcal{O}$ is given by:

- For each $n \in \mathbb{N}$, a set of multimorphisms $\mathcal{O}(n)$

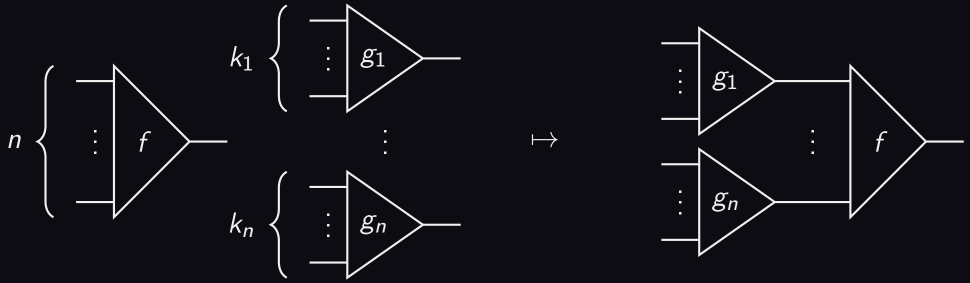


- An identity $\mathbf{id} \in \mathcal{O}(1)$



- For each $n, k_1, \ldots, k_n \in \mathbb{N}$, a composition

$$\mathcal{O}(n) \times \prod_{i=1}^{n} \mathcal{O}(k_i) \to \mathcal{O}(\sum_{i=1}^{n} k_i)$$



Axioms: associativity, unitality

What is a cartesian operad?

A cartesian operad is given by:

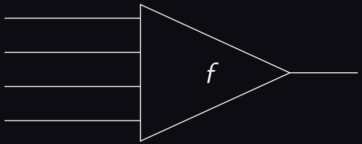
- An operad $\mathcal{O}$
- For each $n, m \in \mathbb{N}$ and each function $f : \{1, \ldots, n\} \to \{1, \ldots, m\}$, a structural map

$$- \cdot f : \mathcal{O}(n) \to \mathcal{O}(m)$$

Axioms describing the interaction of the structural maps with the composition of $\mathcal{O}$

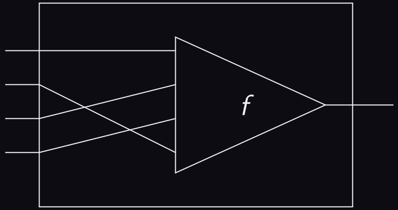
$$\sigma : \{1, \dots, 4\} \to \{1, \dots, 4\}$$
$$1 \mapsto 1$$
$$2 \mapsto 4$$
$$3 \mapsto 2$$
$$4 \mapsto 3$$

$- \cdot \sigma :$



$\mapsto$

$f \cdot \sigma$



What is a clone?

A clone $\mathcal{T}$ is given by:

- For each $n \in \mathbb{N}$, a set of multimorphisms $\mathcal{T}(n)$
- For each $n \in \mathbb{N}^*$ and each $1 \leq k \leq n$, a projection $\mathbf{pr}_k^n \in \mathcal{T}(n)$
- For each $n, k \in \mathbb{N}$, a composition

$$\mathcal{T}(n) \times \mathcal{T}(k)^n \to \mathcal{T}(k)$$

Axioms: associativity, projection laws

Cartesian operads vs clones

COperad	Clone
Objects: cartesian operads	Objects: Clones
Morphisms: Small maps of operads which preserve the cartesian structure	Morphisms: Maps of clones

$$\boxed{\mathsf{COperad} \simeq \mathsf{Clone}}$$

Traditionally, the untyped λ-calculus is...

Let $\mathcal{V}$ be a countably infinite set of variables.

$$\text{The set of terms} \quad \Lambda^- ::= \mathcal{V} \mid \lambda x.\Lambda^- \mid \Lambda^-\Lambda^-$$

$$\text{Substitution} \quad M\{N/x\}$$

Terms are considered equal up to α-equivalence.

$$\beta\text{-reduction} \quad (\lambda x.M)\,N \to_\beta M\{N/x\}$$

$$\eta\text{-expansion} \quad M \to_\eta \lambda x.\,(Mx)$$

Two ways to assign...

...an arity context to a term

Multiplicative rules

$$\frac{}{x \vdash x}\ \text{var}$$

$$\frac{\Gamma, x \vdash M}{\Gamma \vdash \lambda x.M}\ \text{abs}$$

$$\frac{\Gamma \vdash M \qquad \Delta \vdash N \qquad \Gamma \cap \Delta = \emptyset}{\Gamma, \Delta \vdash MN}\ \text{app}$$

$$\frac{x_1, \ldots, x_n \vdash M \qquad f : [n] \to [m]}{z_1, \ldots, z_m \vdash M\{z_{f(1)}, \ldots, z_{f(n)} / x_1, \ldots, x_n\}}\ \text{struct}$$

Additive rules

$$\frac{1 \leq i \leq n+1}{x_1, \ldots, x_{n+1} \vdash x_i}\ \text{var}$$

$$\frac{\Gamma, x \vdash M}{\Gamma \vdash \lambda x.M}\ \text{abs}$$

$$\frac{\Gamma \vdash M \qquad \Gamma \vdash N}{\Gamma \vdash MN}\ \text{app}$$

which allow to prove the same judgments

Deriving $\vdash \lambda x.xx$ using the additive rules

$$\dfrac{\dfrac{}{x \vdash x}\,\text{var} \qquad \dfrac{}{x \vdash x}\,\text{var}}{\dfrac{x \vdash xx}{\vdash \lambda x.xx}\,\text{abs}}\,\text{app}$$

Deriving $\vdash \lambda x.xx$ using the multiplicative rules

$$
\dfrac{\dfrac{\dfrac{}{y \vdash y}\,\text{var} \qquad \dfrac{}{z \vdash z}\,\text{var}}{y, z \vdash yz}\,\text{app} \qquad f : [2] \to [1]}{\dfrac{x \vdash xx}{\vdash \lambda x.xx}\,\text{abs}}\,\text{struct}
$$

The untyped λ-calculus as a closed clone [Hyland2015]

The additive λ-calculus as a **clone**

- For each $n \in \mathbb{N}$,

$$\Lambda(n) := \{M \in \Lambda^- \mid x_1, \ldots, x_n \vdash M\}$$

- For each $n, k \in \mathbb{N}$, the composition $\Lambda(n) \times \Lambda(k)^n \to \Lambda(k)$ written

$$(M, N_1, \ldots, N_n) \mapsto M(N_1, \ldots, N_n)$$

is defined by

$$M(N_1, \ldots, N_n) := {\color{red} M\{N_1, \ldots, N_n/x_1, \ldots, x_n\}}$$

$$\dfrac{\dfrac{\vdots}{x_1, \ldots, x_n \vdash M} \qquad \dfrac{\vdots}{x_1, \ldots, x_k \vdash N_1} \qquad \cdots \qquad \dfrac{\vdots}{x_1, \ldots, x_k \vdash N_n}}{x_1, \ldots, x_k \vdash {\color{red} M\{N_1, \ldots, N_n/x_1, \ldots, x_n\}}}\ \text{multiSubst}$$

- For each $n \in \mathbb{N}^*$ and each $1 \leq k \leq n$,

$$\mathbf{pr}_k^n := ?$$

$$\dfrac{\vdots}{x_1, \ldots, x_n \vdash ?}$$

- For each $n \in \mathbb{N}^*$ and each $1 \leq k \leq n$,

$$\mathbf{pr}_k^n := x_k$$

$$\frac{}{x_1, \ldots, x_n \vdash x_k}\ \mathsf{var}$$

The additive λ-calculus as a **closed clone**

Each clone $\mathcal{T}$ induces the operation

$$\begin{aligned} W : \mathcal{T}(n) &\to \mathcal{T}(n+1) \\ M &\mapsto M(\mathbf{pr}_1^{n+1}, \dots, \mathbf{pr}_n^{n+1}) \end{aligned}$$

called the weakening operation.

In the case of Λ,

$$\frac{\pi \ \triangleright \ x_1, \dots, x_n \vdash M \qquad \overline{x_1, \dots, x_{n+1} \vdash x_1} \qquad \cdots \qquad \overline{x_1, \dots, x_{n+1} \vdash x_n}}{x_1, \dots, x_{n+1} \vdash M\{x_1, \dots, x_n / x_1, \dots, x_n\}}$$

The additive λ-calculus as a **closed clone**

Each clone $\mathcal{T}$ induces the operation

$$\begin{aligned} W : \mathcal{T}(n) &\to \mathcal{T}(n+1) \\ M &\mapsto M(\mathbf{pr}_1^{n+1}, \ldots, \mathbf{pr}_n^{n+1}) \end{aligned}$$

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In the case of Λ,

$$\dfrac{\pi \;\triangleright\; x_1, \ldots, x_n \vdash {\color{red}M} \qquad \dfrac{}{x_1, \ldots, x_{n+1} \vdash x_1} \qquad \cdots \qquad \dfrac{}{x_1, \ldots, x_{n+1} \vdash x_n}}{x_1, \ldots, x_{n+1} \vdash {\color{red}M}}$$

The additive λ-calculus as a **closed clone**

A closed clone is given by:

- a clone $\mathcal{T}$
- a multimorphism $\mathrm{ev} \in \mathcal{T}(2)$

s.t.

$$\begin{aligned} \mathcal{T}(n) &\to \mathcal{T}(n+1) \\ M &\mapsto \mathrm{ev}(W(M), \mathbf{pr}_{n+1}^{n+1}) \end{aligned}$$

is invertible.

$$\dfrac{\dfrac{}{x_1, x_2 \vdash x_1}\,\text{var} \qquad \dfrac{}{x_1, x_2 \vdash x_2}\,\text{var}}{x_1, x_2 \vdash x_1 x_2}\,\text{app}$$

The additive λ-calculus as a **closed clone**

$$\begin{aligned} \Lambda(n) &\to \Lambda(n+1) \\ M &\mapsto x_1 x_2(W(M), \mathbf{pr}^{n+1}_{n+1}) \end{aligned}$$

is invertible?

$$\dfrac{\dfrac{}{x_1, x_2 \vdash x_1}\,\text{var} \qquad \dfrac{}{x_1, x_2 \vdash x_2}\,\text{var}}{x_1, x_2 \vdash x_1 x_2}\,\text{app}$$

The additive λ-calculus as a **closed clone**

$$\begin{aligned}\Lambda(n) &\to \Lambda(n+1)\\ M &\mapsto x_1 x_2\{W(M), \mathbf{pr}^{n+1}_{n+1}/x_1, x_2\}\end{aligned}$$

is invertible?

$$\dfrac{\dfrac{}{x_1, x_2 \vdash x_1}\,\text{var} \qquad \dfrac{}{x_1, x_2 \vdash x_2}\,\text{var}}{x_1, x_2 \vdash {\color{red} x_1 x_2}}\,\text{app}$$

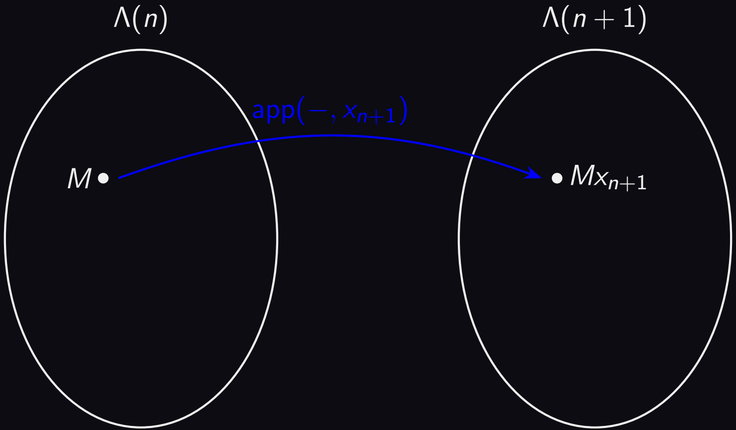
The additive λ-calculus as a **closed clone**

$$\Lambda(n) \to \Lambda(n+1)$$
$$M \mapsto Mx_{n+1}$$

is invertible?

$$\dfrac{\dfrac{}{x_1, x_2 \vdash x_1}\,\text{var} \qquad \dfrac{}{x_1, x_2 \vdash x_2}\,\text{var}}{x_1, x_2 \vdash x_1 x_2}\,\text{app}$$

The additive λ-calculus as a **closed clone**



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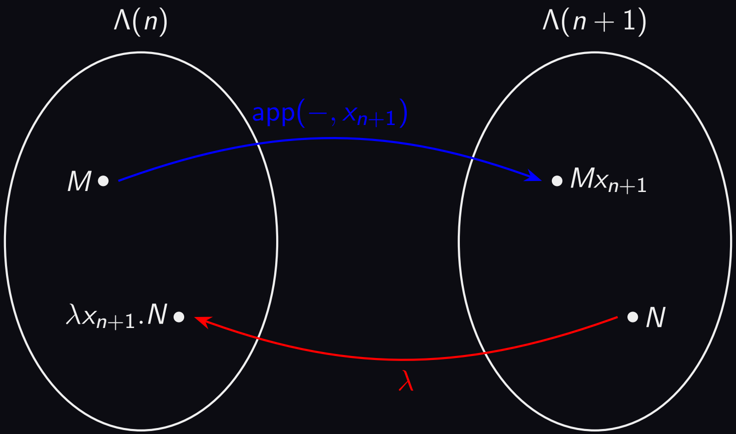

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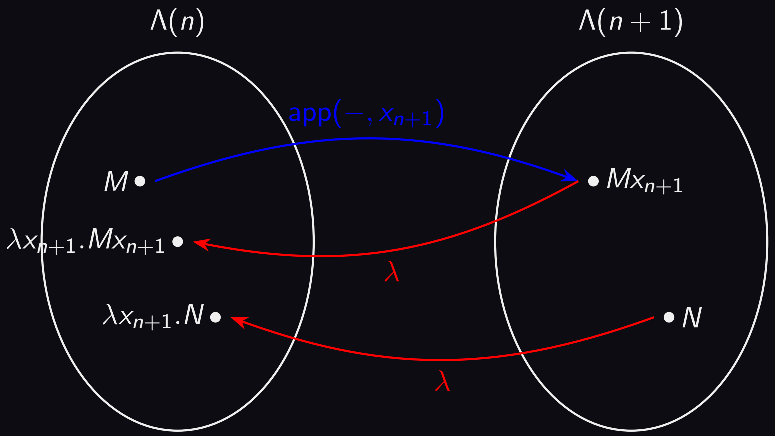

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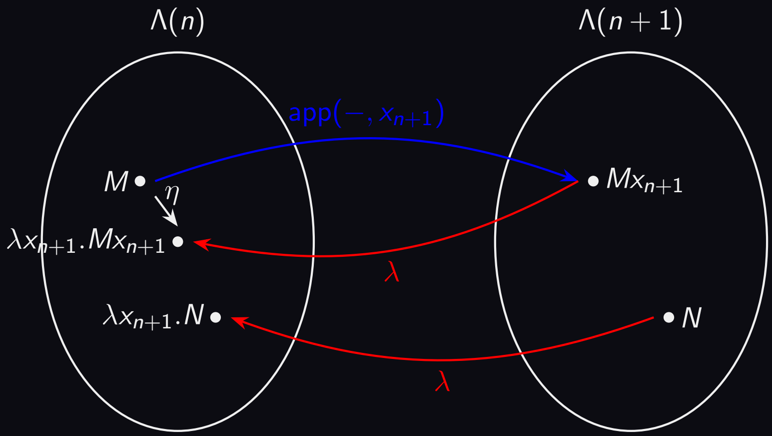

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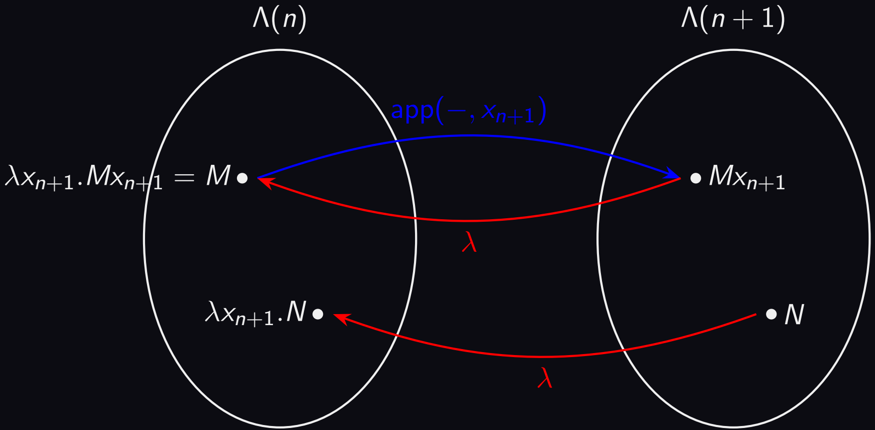


The untyped λ-calculus as a 2-dimensional structure (PhD project)

Recap

The untyped λ-calculus as a closed clone:

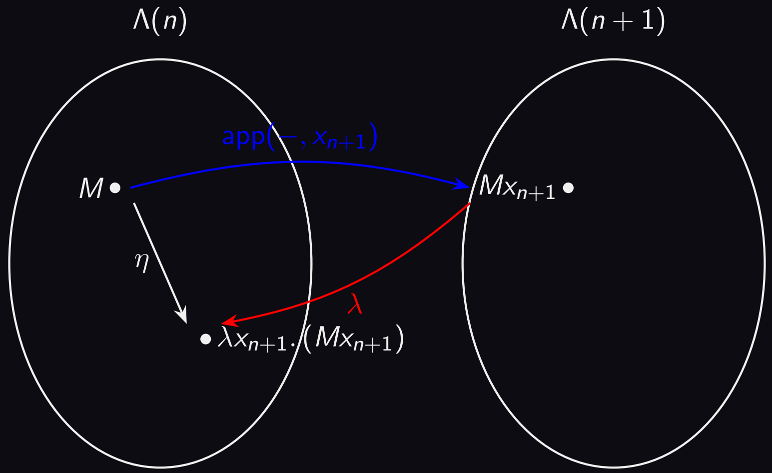
$$\Lambda(n) \xrightarrow{\sim} \Lambda(n+1)$$

The untyped λ-calculus as a 2-dimensional structure (i.e. closed Cat-clone):

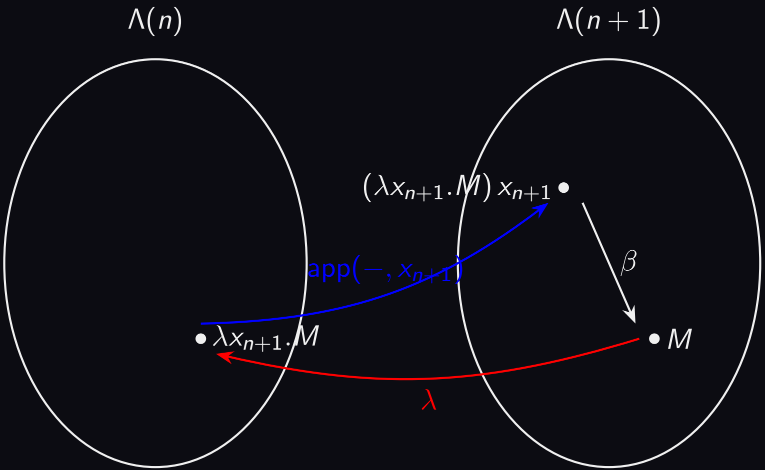
$$\Lambda(n) \underset{\lambda}{\overset{\mathsf{app}(-,x_{n+1})}{\underset{\longleftarrow}{\overset{\longrightarrow}{\dashv}}}} \Lambda(n+1)$$

For the untyped λ-calculus as a 2-operad, see [MazzaPellissierVial2018].

What if...


$\Lambda(n)$
$\Lambda(n+1)$
$\mathsf{app}(-, x_{n+1})$
M
Mx_{n+1}
η
λ
$\lambda x_{n+1}.(Mx_{n+1})$

What if...


$\Lambda(n)$
$\Lambda(n+1)$
$(\lambda x_{n+1}.M)\, x_{n+1}$
$\mathsf{app}(-, x_{n+1})$
β
$\lambda x_{n+1}.M$
M
λ

What does it take to have this adjunction?

1. Each $\Lambda(n)$ has to be category.

Objects $\{M \in \Lambda^- \mid x_1, \ldots, x_n \vdash M\}$

Morphisms A set-theoretic relation over the objects?

What does it take to have this adjunction?

1. Each $\Lambda(n)$ has to be category.

Objects $\{M \in \Lambda^- \mid x_1, \ldots, x_n \vdash M\}$

Morphisms

Chains of reduction terms $\rho ::= \epsilon_M \mid \alpha_1 \cdots \alpha_n$

Reduction terms $\alpha ::= C[r_b]$

Base reduction terms $r_b ::= \beta_{M,M} \mid \eta_M$

One-hole contexts $C ::= - \mid \lambda x.C \mid CM \mid MC$

What are the sources and targets of these chains of reduction terms?

$$\frac{\Gamma, x \vdash M \qquad \Gamma \vdash N}{\Gamma \vdash \beta_{M,N} : (\lambda x.M)N \to M\{N/x\}}\ \beta$$

$$\frac{\Gamma \vdash M}{\Gamma \vdash \eta_M : M \to \lambda x.\,(Mx)}\ \eta$$

$$\frac{\Gamma, x \vdash \alpha : M \to N}{\Gamma \vdash \lambda x.\alpha : \lambda x.M \to \lambda x.N}\ \text{abs}$$

$$\frac{\Gamma \vdash \alpha : M \to M' \qquad \Gamma \vdash N}{\Gamma \vdash \alpha N : MN \to M'N}\ \text{appRight}$$

$$\frac{\Gamma \vdash \alpha : M \to M' \qquad \Gamma \vdash N}{\Gamma \vdash N\alpha : NM \to NM'}\ \text{appLeft}$$

$$\frac{(\Gamma \vdash \alpha_i : M_i \to M_{i+1})_{1 \le i \le n+1}}{\Gamma \vdash_c \alpha_1 \cdots \alpha_n : M_1 \to M_{n+1}}\ \text{comp}$$

$$\frac{\Gamma \vdash M}{\Gamma \vdash_c \epsilon_M : M \to M}\ \text{id}$$

More precisely,

$$\Lambda(n)(M, N) := \{\rho \mid x_1, \ldots, x_n \vdash \rho : M \to N\}$$

Composition

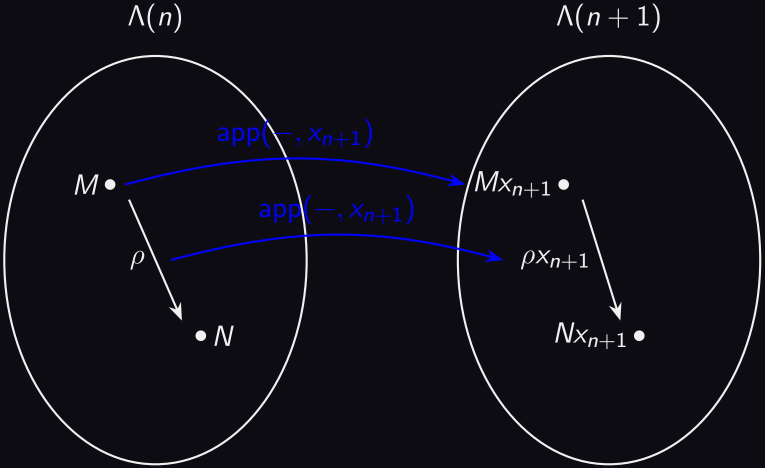
$$\begin{aligned} \Lambda(n)(M, N) \times \Lambda(n)(N, P) &\to \Lambda(n)(M, P) \\ (\rho, \rho') &\mapsto \rho \cdot \rho' \end{aligned}$$

Identities

$$\mathrm{id}_M := \epsilon_M$$

What does it take to have this adjunction?

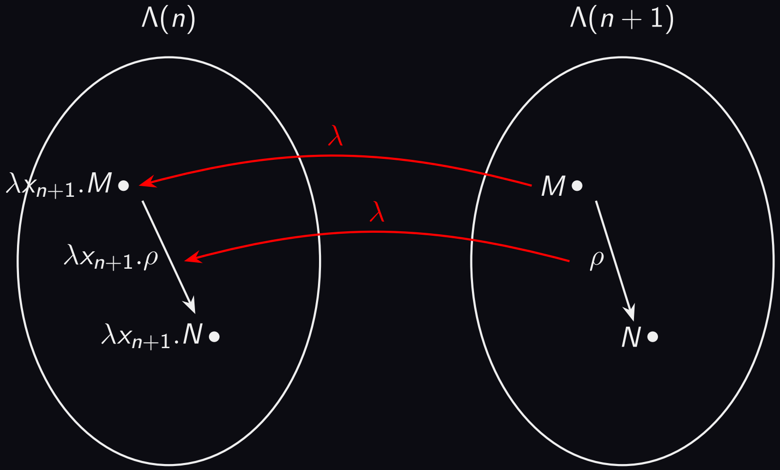
2. The abstraction and application must be functors.



Let ρ be a chain of reduction terms and let $N \in \Lambda^-$. The *right chain application* ρN is defined by induction on ρ.

$$\epsilon_M N := \epsilon_{MN}$$

$$(\alpha_1 \cdots \alpha_n)N := (\alpha_1 N) \cdots (\alpha_n N)$$


$\Lambda(n)$
$\Lambda(n+1)$
λ
λ
$\lambda x_{n+1}.M$
$\lambda x_{n+1}.\rho$
$\lambda x_{n+1}.N$
M
ρ
N

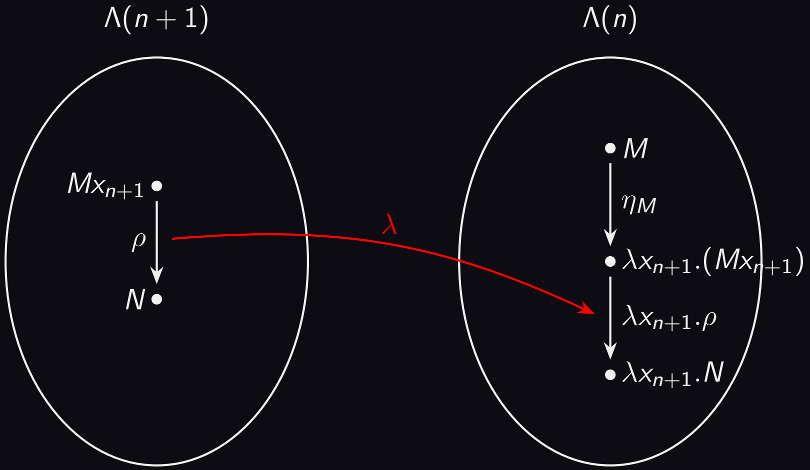
Let ρ be a chain of reduction terms and let $x \in \mathcal{V}$. The *chain abstraction* $\lambda x.\rho$ is defined by induction on ρ.

$$\lambda x.\epsilon_M := \epsilon_{\lambda x.M}$$

$$\lambda x.(\alpha_1 \cdots \alpha_n) := (\lambda x.\alpha_1) \cdots (\lambda x.\alpha_n)$$

What does it take to have this adjunction?

3. $\Lambda(n+1)(Mx_{n+1}, N) \cong \Lambda(n)(M, \lambda x_{n+1}.N)$



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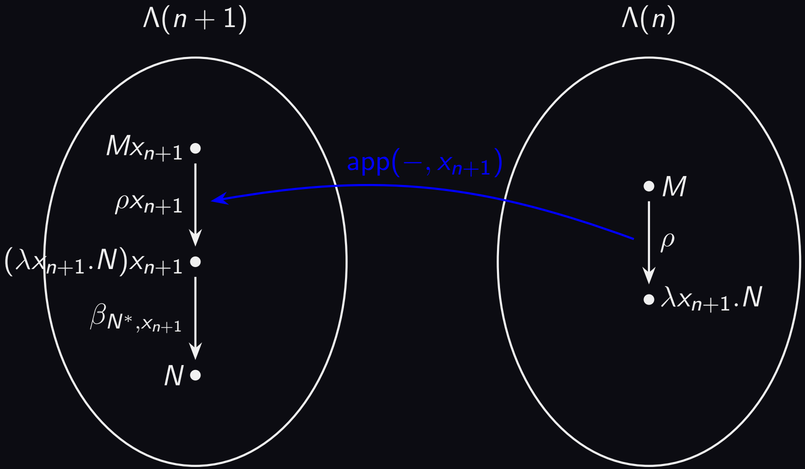

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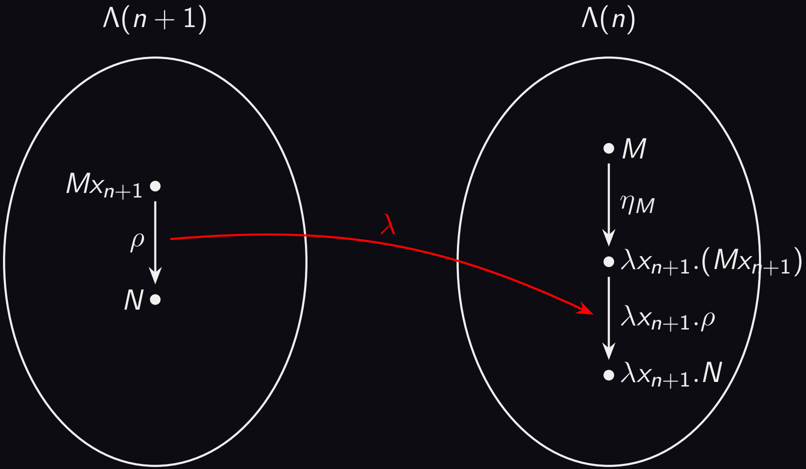

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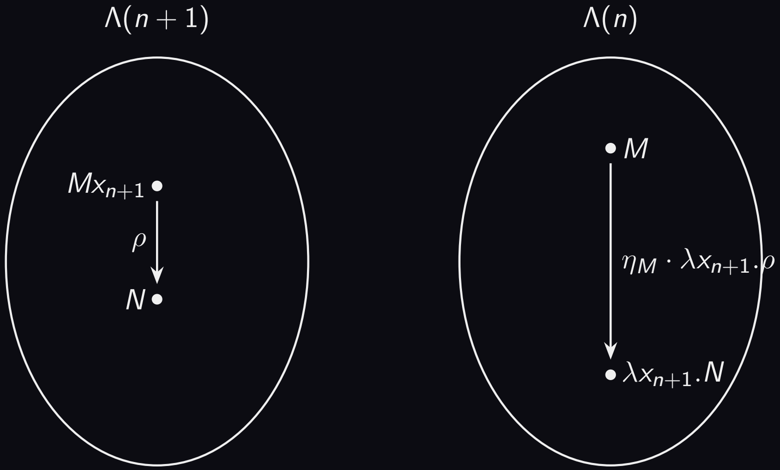

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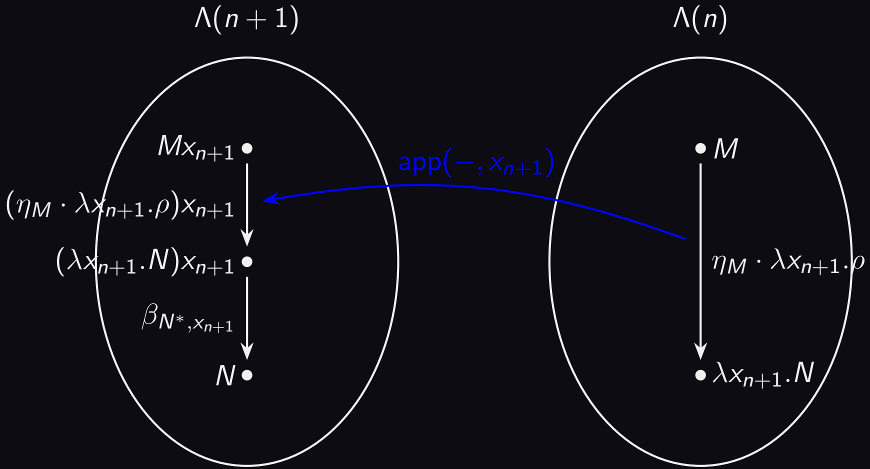

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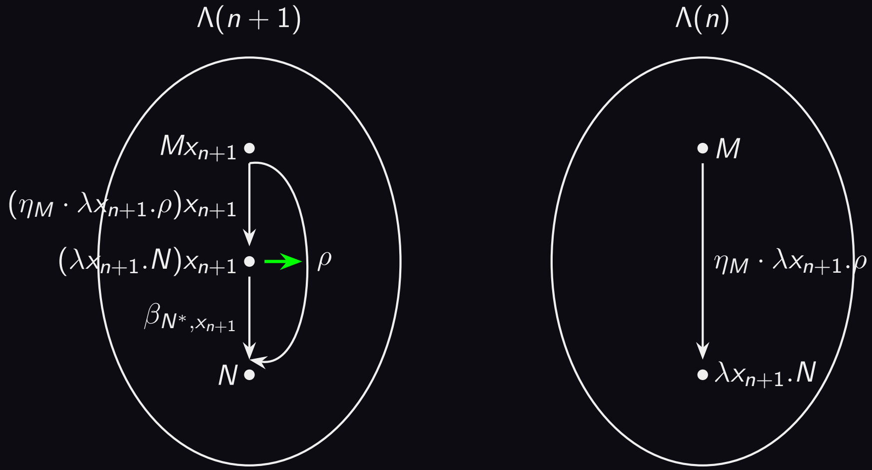

What does it take to have this adjunction?

3. $\Lambda(n+1)(Mx_{n+1}, N) \cong \Lambda(n)(M, \lambda x_{n+1}.N)$

$$(\eta_M \cdot \lambda x_{n+1}.\rho)x_{n+1} \cdot \beta_{M^* x_{n+2}, x_{n+1}} \to \rho$$

What does it take to have this adjunction?

3. $\Lambda(n+1)(Mx_{n+1}, N) \cong \Lambda(n)(M, \lambda x_{n+1}.N)$

$$(\eta_M \cdot \lambda x_{n+1}.\epsilon_{Mx_{n+1}})x_{n+1} \cdot \beta_{M^* x_{n+2}, x_{n+1}} \to \epsilon_{Mx_{n+1}}$$

What does it take to have this adjunction?

3. $\Lambda(n+1)(Mx_{n+1}, N) \cong \Lambda(n)(M, \lambda x_{n+1}.N)$

$$(\eta_M \cdot \epsilon_{\lambda x_{n+1}.(Mx_{n+1})})x_{n+1} \cdot \beta_{M^* x_{n+2}, x_{n+1}} \to \epsilon_{Mx_{n+1}}$$

What does it take to have this adjunction?

3. $\Lambda(n+1)(Mx_{n+1}, N) \cong \Lambda(n)(M, \lambda x_{n+1}.N)$

$$(\eta_M)x_{n+1} \cdot \beta_{M^* x_{n+2}, x_{n+1}} \to \epsilon_{Mx_{n+1}}$$

$$\begin{tikzcd}
Mx_{n+1} \arrow[r, "(\eta_M)x_{n+1}"] \arrow[dr, "\epsilon_{Mx_{n+1}}"'] & (\lambda x_{n+1}.(Mx_{n+1}))x_{n+1} \arrow[d, "\beta_{M^* x_{n+2}, x_{n+1}}"] \arrow[dl, phantom, "\swarrow"] \\
 & Mx_{n+1}
\end{tikzcd}$$

Recap

We dispose of:

- For each $n \in \mathbb{N}$, a category $\Lambda(n)$ of terms and chains of reduction terms
- $\mathsf{app}(-, x_{n+1})$ and λ are adjoint functors.

What's missing to have a closed Cat-clone? $\rightsquigarrow$ a composition and projection maps

The composition

$$\Lambda(n) \times \prod_{i=1}^{n} \Lambda(m) \to \Lambda(m)$$

$$M, N_1, \ldots, N_n \mapsto M\{\vec{N}/\vec{x}\}$$

The composition

$$\Lambda(n) \times \prod_{i=1}^{n} \Lambda(m) \to \Lambda(m)$$

$$M \xrightarrow{\rho} N \quad , \quad M_1 \xrightarrow{\rho_1} N_1 \quad , \quad \dots \quad , \quad M_n \xrightarrow{\rho_n} N_n \mapsto \rho\{\vec{\rho}/\vec{x}\}$$

where $\rho\{\vec{\rho}/\vec{x}\}$ is defined as

The composition

$$\Lambda(n) \times \prod_{i=1}^{n} \Lambda(m) \to \Lambda(m)$$

$$M \xrightarrow{\rho} N \quad , \quad M_1 \xrightarrow{\rho_1} N_1 \quad , \quad \dots \quad , \quad M_n \xrightarrow{\rho_n} N_n \mapsto \rho\{\vec{\rho}/\vec{x}\}$$

where $\rho\{\vec{\rho}/\vec{x}\}$ is defined as

$$\begin{tikzcd}
M\{\vec{M}/\vec{x}\} \arrow[d, "\rho\{\vec{\rho}/\vec{x}\}_{O}"] \\
N\{\vec{M}/\vec{x}\} \arrow[d, "\rho\{\vec{\rho}/\vec{x}\}_{I}"] \\
N\{\vec{N}/\vec{x}\}
\end{tikzcd}$$

The untyped λ-calculus as the initial closed clone

Let $\mathcal{T}$ be a closed clone.

$$\Lambda \xrightarrow{\exists ! f} \mathcal{T}$$

What's f? In particular,

- For each $n \in \mathbb{N}$, a functor $\Lambda(n) \to \mathcal{T}(n)$ defined by induction

$\boxed{\text{It's the interpretation of the untyped } \lambda\text{-calculus in a closed Cat-clone.}}$

Take-away messages

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Cartesian operads	Clones
Multiplicative, untyped λ-calculus	Additive, untyped λ-calculus
- The clone of the additive, untyped λ-calculus:
  - Homsets of terms
  - Multivariate substitution as the composition
  - Terms up to β-reduction and η-expansion $\rightsquigarrow$ **closed** clone
- The closed Cat-clone of the additive, untyped λ-calculus:
  - Homcategories of terms and chains of reduction terms
  - Substitution as the composition

Thank you! Any questions?