But what is β-reduction?

The categorical status of β-reduction

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Based on joint (ongoing) work with Federico Olimpieri

PACM$\wedge$N 2026

Outline

The zoo of multicategorical structures

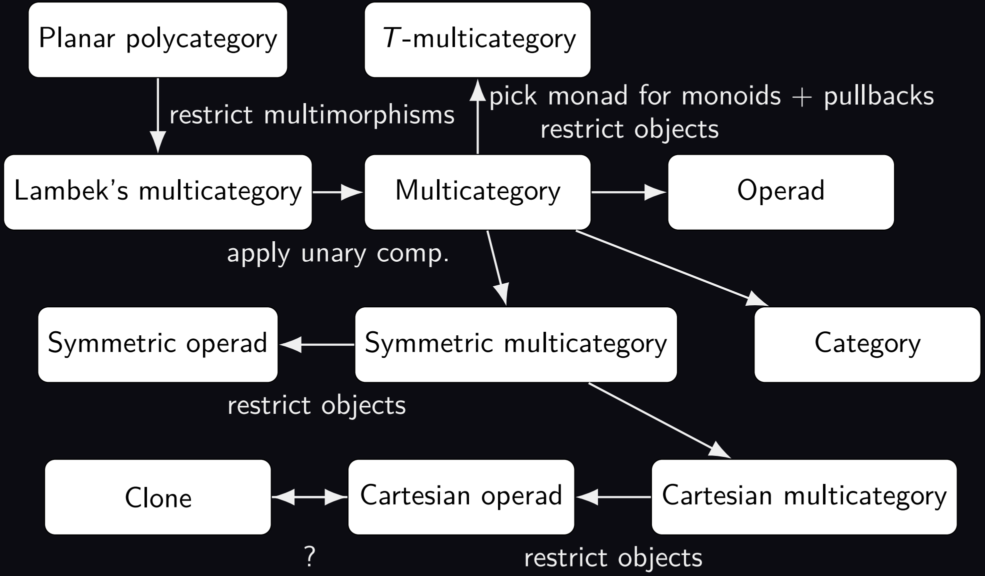
The untyped λ-calculus as a closed clone (state-of-the-art)

The untyped λ-calculus as a 2-dimensional structure (PhD project)

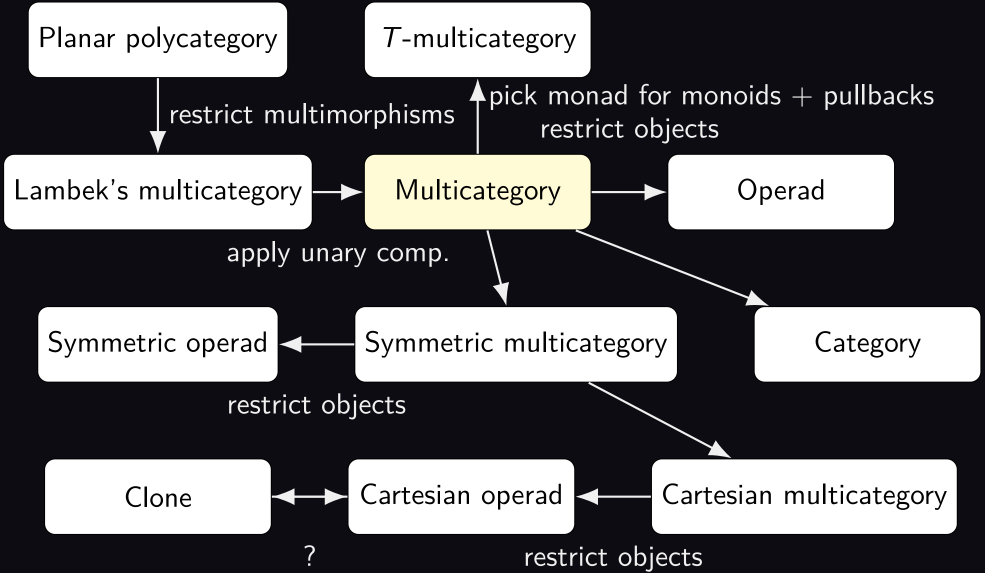
Take-away messages

The zoo of multicategorical structures

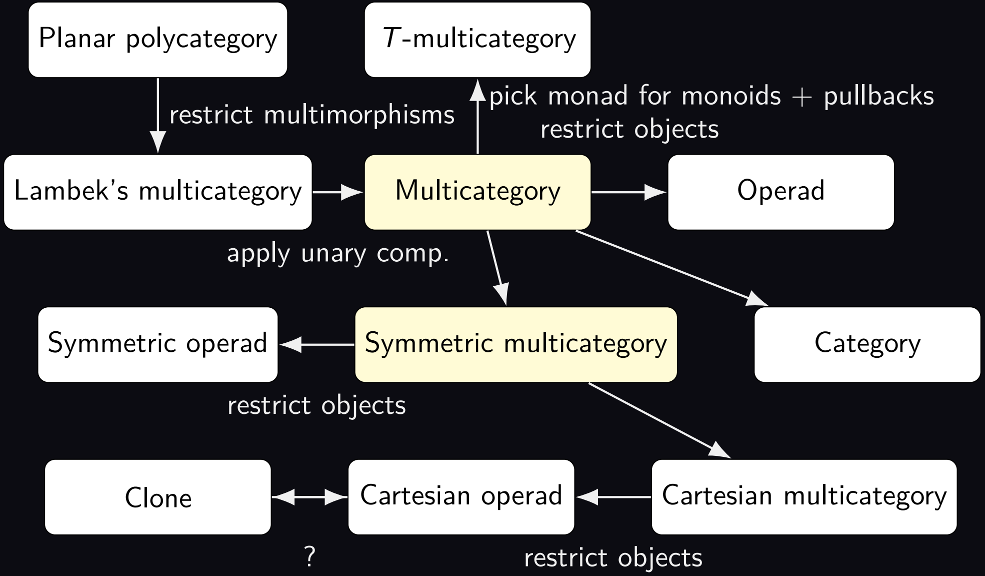
Relationships between multicategorical structures



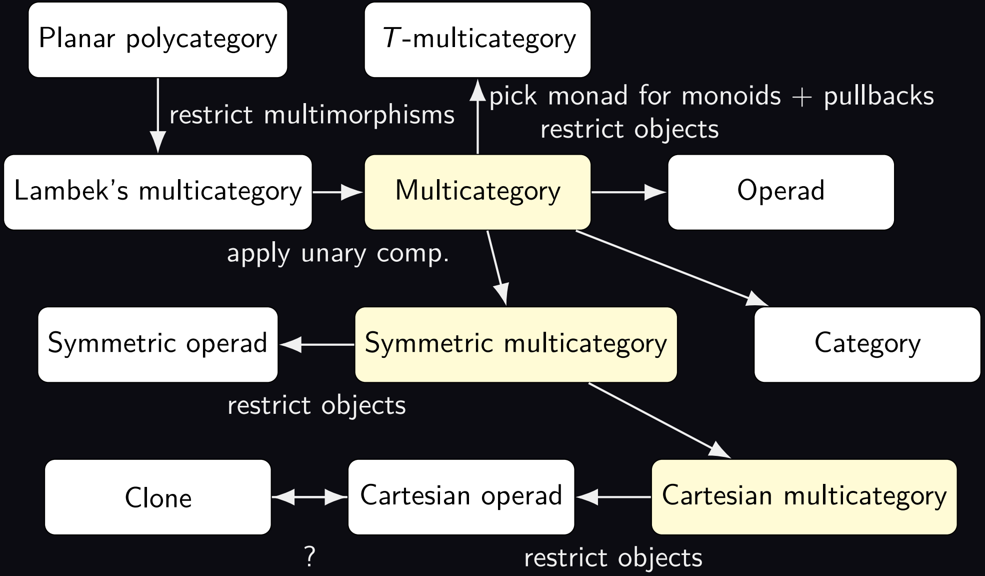
Relationships between multicategorical structures



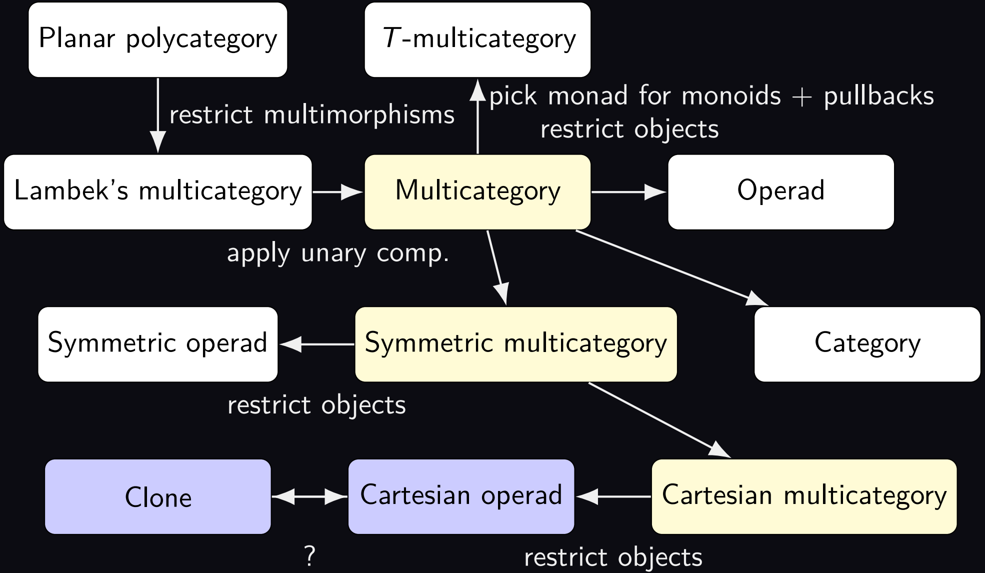
Relationships between multicategorical structures



Relationships between multicategorical structures



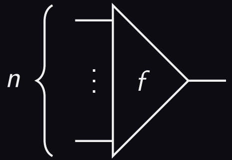
Relationships between multicategorical structures



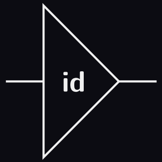
What is an operad?

An operad $\mathcal{O}$ is given by:

- For each $n \in \mathbb{N}$, a set of multimorphisms $\mathcal{O}(n)$

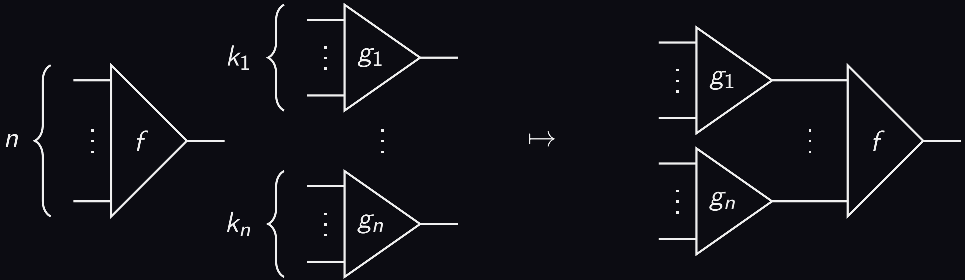


- An identity $\mathbf{id} \in \mathcal{O}(1)$



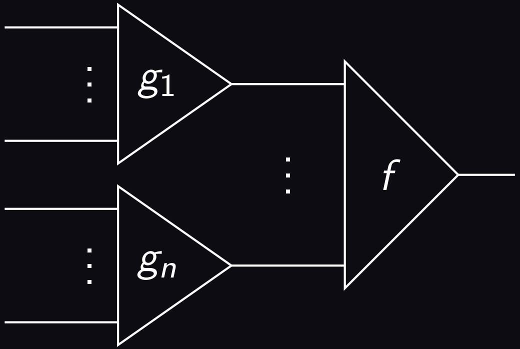
- For each $n, k_1, \ldots, k_n \in \mathbb{N}$, a composition

$$\mathcal{O}(n) \times \prod_{i=1}^{n} \mathcal{O}(k_i) \to \mathcal{O}(\sum_{i=1}^{n} k_i)$$

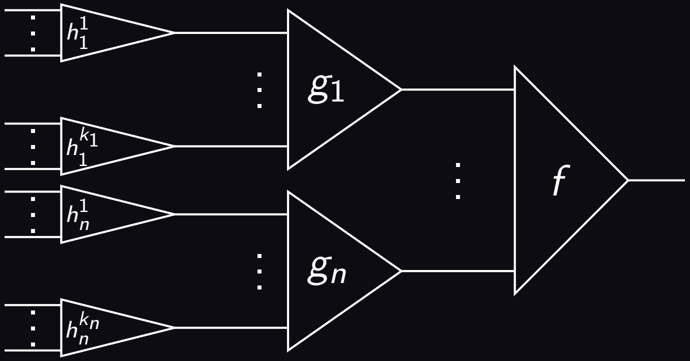


Axioms: associativity, unitality

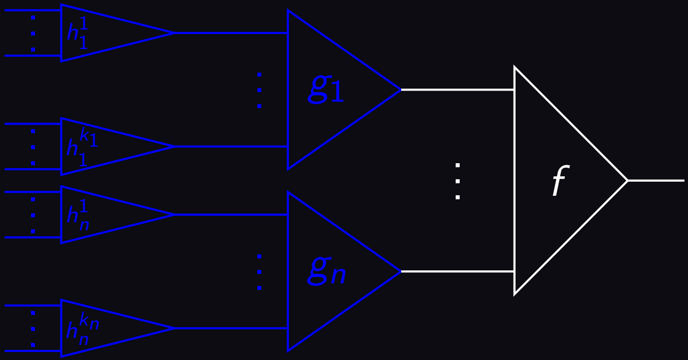
Associativity



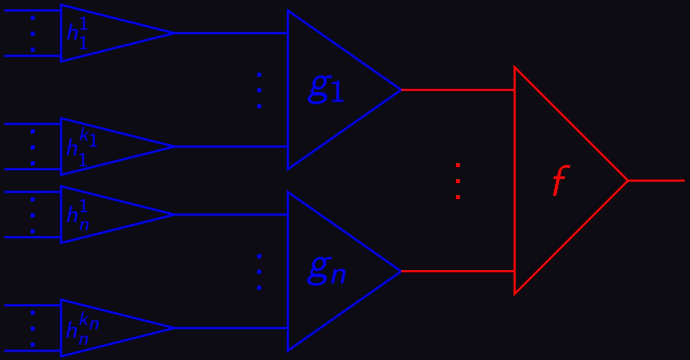
Associativity



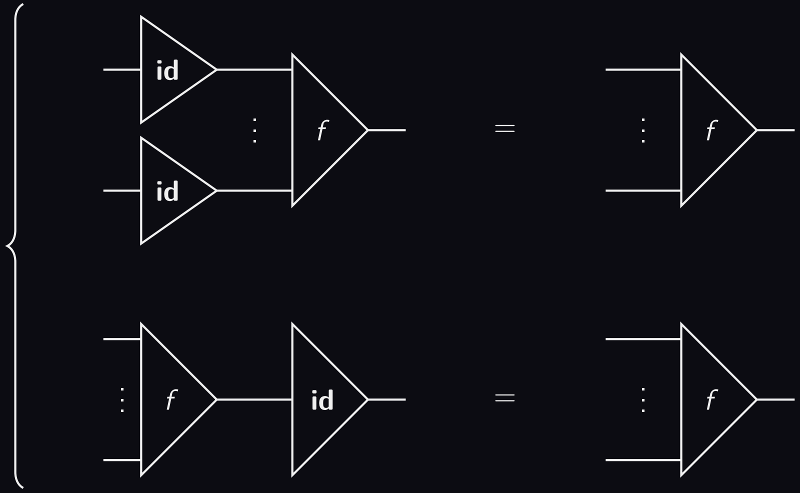
Associativity



Associativity



Unitality



What is a cartesian operad?

A cartesian operad is given by:

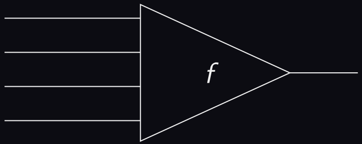
- An operad $\mathcal{O}$
- For each $n, m \in \mathbb{N}$ and each function $f : \{1, \dots, n\} \to \{1, \dots, m\}$, a structural map

$$- \cdot f : \mathcal{O}(n) \to \mathcal{O}(m)$$

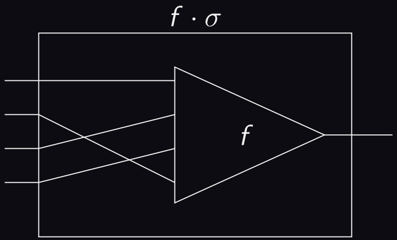
Axioms describing the interaction of the structural maps with the composition of $\mathcal{O}$

$$
\begin{aligned}
\sigma : \mathcal{O}(4) &\to \mathcal{O}(4) \\
1 &\mapsto 1 \\
2 &\mapsto 4 \\
3 &\mapsto 2 \\
4 &\mapsto 3
\end{aligned}
$$

$- \cdot \sigma :$



$\mapsto$



What is a clone?

A clone $\mathcal{T}$ is given by:

- For each $n \in \mathbb{N}$, a set of multimorphisms $\mathcal{T}(n)$
- For each $n \in \mathbb{N}^*$ and each $1 \leq k \leq n$, a projection $\mathbf{pr}_k^n \in \mathcal{T}(n)$
- For each $n, k \in \mathbb{N}$, a composition

$$\mathcal{T}(n) \times \mathcal{T}(k)^n \to \mathcal{T}(k)$$

Axioms: associativity, projection laws

Cartesian operads vs clones

COperad	Clone
Objects: cartesian operads	Objects: Clones
Morphisms: Small maps of operads which preserve the cartesian structure	Morphisms: Maps of clones

$$\boxed{\mathsf{COperad} \simeq \mathsf{Clone}}$$

Traditionally, the untyped λ-calculus is...

Let $\mathcal{V}$ be a countably infinite set of variables.

$$\text{The set of terms} \quad \Lambda^- ::= \mathcal{V} \mid \lambda x.\Lambda^- \mid \Lambda^-\Lambda^-$$

$$\text{Substitution} \quad M\{N/x\}$$

Terms are considered equal up to α-equivalence.

$$\beta\text{-reduction} \quad (\lambda x.M)\, N \to_\beta M\{N/x\}$$

$$\eta\text{-expansion} \quad M \to_\eta \lambda x.\,(Mx)$$

Two ways to assign...

...an arity context to a term

Multiplicative rules	Additive rules
$\dfrac{}{x \vdash x}\,\text{var}$	$\dfrac{1 \leq i \leq n+1}{x_1, \ldots, x_{n+1} \vdash x_i}\,\text{var}$
$\dfrac{\Gamma, x \vdash M}{\Gamma \vdash \lambda x.M}\,\text{abs}$	$\dfrac{\Gamma, x \vdash M}{\Gamma \vdash \lambda x.M}\,\text{abs}$
$\dfrac{\Gamma \vdash M \qquad \Delta \vdash N \qquad \Gamma \cap \Delta = \emptyset}{\Gamma, \Delta \vdash MN}\,\text{app}$	$\dfrac{\Gamma \vdash M \qquad \Gamma \vdash N}{\Gamma \vdash MN}\,\text{app}$
$\dfrac{x_1, \ldots, x_n \vdash M \qquad f : [n] \to [m]}{z_1, \ldots, z_m \vdash M\{z_{f(1)}, \ldots, z_{f(n)}/x_1, \ldots, x_n\}}\,\text{struct}$	

which allow to prove the same judgments

Deriving $\vdash \lambda x.xx$ using the additive rules

$$\dfrac{\dfrac{}{x \vdash x}\,\text{var} \qquad \dfrac{}{x \vdash x}\,\text{var}}{\dfrac{x \vdash xx}{\vdash \lambda x.xx}\,\text{abs}}\,\text{app}$$

Deriving $\vdash \lambda x.xx$ using the multiplicative rules

$$\dfrac{\dfrac{\dfrac{}{y \vdash y}\,\text{var} \qquad \dfrac{}{z \vdash z}\,\text{var}}{y, z \vdash yz}\,\text{app} \qquad f : [2] \to [1]}{\dfrac{x \vdash xx}{\vdash \lambda x.xx}\,\text{abs}}\,\text{struct}$$

The untyped λ-calculus as a closed clone (state-of-the-art)

The additive λ-calculus as a **clone**

- For each $n \in \mathbb{N}$,
$$\Lambda(n) := \{M \in \Lambda^- \mid x_1, \dots, x_n \vdash M\}$$
- For each $n, k \in \mathbb{N}$, the composition $\Lambda(n) \times \Lambda(k)^n \to \Lambda(k)$ written
$$(M, N_1, \dots, N_n) \mapsto M(N_1, \dots, N_n)$$
is defined by
$$M(N_1, \dots, N_n) := M\{N_1, \dots, N_n / x_1, \dots, x_n\}$$

$$\dfrac{\dfrac{\vdots}{x_1, \dots, x_n \vdash M} \qquad \dfrac{\vdots}{x_1, \dots, x_k \vdash N_1} \qquad \cdots \qquad \dfrac{\vdots}{x_1, \dots, x_k \vdash N_n}}{x_1, \dots, x_k \vdash M\{N_1, \dots, N_n / x_1, \dots, x_n\}}\ \text{multiSubst}$$

- For each $n \in \mathbb{N}^*$ and each $1 \leq k \leq n$,

$$\mathbf{pr}_k^n := ?$$

$$\dfrac{\vdots}{x_1, \ldots, x_k \vdash ?}$$

- For each $n \in \mathbb{N}^*$ and each $1 \leq k \leq n$,

$$\mathbf{pr}^n_k := {\color{red} x_k}$$

$$\frac{}{x_1, \ldots, x_k \vdash {\color{red} x_k}}\ \mathsf{var}$$

The additive λ-calculus as a **closed clone**

Each clone $\mathcal{T}$ induces the operation

$$\begin{aligned} W : \mathcal{T}(n) &\to \mathcal{T}(n+1) \\ M &\mapsto M(\mathbf{pr}_1^{n+1}, \dots, \mathbf{pr}_n^{n+1}) \end{aligned}$$

called the weakening operation.

In the case of Λ,

$$\dfrac{\pi \;\triangleright\; x_1, \dots, x_n \vdash M \qquad \dfrac{}{x_1, \dots, x_{n+1} \vdash x_1} \qquad \cdots \qquad \dfrac{}{x_1, \dots, x_{n+1} \vdash x_n}}{x_1, \dots, x_{n+1} \vdash M\{x_1, \dots, x_n / x_1, \dots, x_n\}}$$

The additive λ-calculus as a **closed clone**

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In the case of Λ,

$$\dfrac{\pi \ \triangleright\ x_1, \ldots, x_n \vdash M \qquad \overline{x_1, \ldots, x_{n+1} \vdash x_1} \qquad \cdots \qquad \overline{x_1, \ldots, x_{n+1} \vdash x_n}}{x_1, \ldots, x_{n+1} \vdash M}$$

The additive λ-calculus as a **closed clone**

A clone $\mathcal{O}$ is closed iff $\exists\, \mathsf{ev} \in \mathcal{O}(2)$ such that

$$\begin{aligned} \Lambda(n) &\to \Lambda(n+1) \\ M &\mapsto \mathsf{ev}(W(M), \mathbf{pr}^{n+1}_{n+1}) \end{aligned}$$

is invertible.

$$\dfrac{\dfrac{}{x_1, x_2 \vdash x_1}\,\mathsf{var} \qquad \dfrac{}{x_1, x_2 \vdash x_2}\,\mathsf{var}}{x_1, x_2 \vdash {\color{red} x_1 x_2}}\,\mathsf{app}$$

The additive λ-calculus as a **closed clone**

$$
\begin{aligned}
\Lambda(n) &\to \Lambda(n+1) \\
M &\mapsto x_1 x_2(W(M), \mathbf{pr}^{n+1}_{n+1})
\end{aligned}
$$

is invertible?

$$
\dfrac{\dfrac{}{x_1, x_2 \vdash x_1}\,\text{var} \qquad \dfrac{}{x_1, x_2 \vdash x_2}\,\text{var}}{x_1, x_2 \vdash x_1 x_2}\,\text{app}
$$

The additive λ-calculus as a **closed clone**

$$\begin{aligned}\Lambda(n) &\to \Lambda(n+1)\\ M &\mapsto x_1x_2\{W(M), \mathbf{pr}^{n+1}_{n+1}/x_1, x_2\}\end{aligned}$$

is invertible?

$$\dfrac{\dfrac{}{x_1, x_2 \vdash x_1}\,\mathrm{var} \qquad \dfrac{}{x_1, x_2 \vdash x_2}\,\mathrm{var}}{x_1, x_2 \vdash x_1x_2}\,\mathrm{app}$$

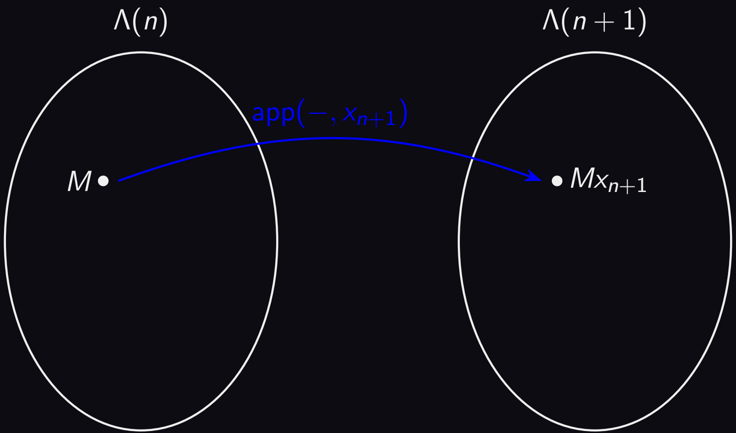
The additive λ-calculus as a **closed clone**

$$\Lambda(n) \to \Lambda(n+1)$$
$$M \mapsto Mx_{n+1}$$

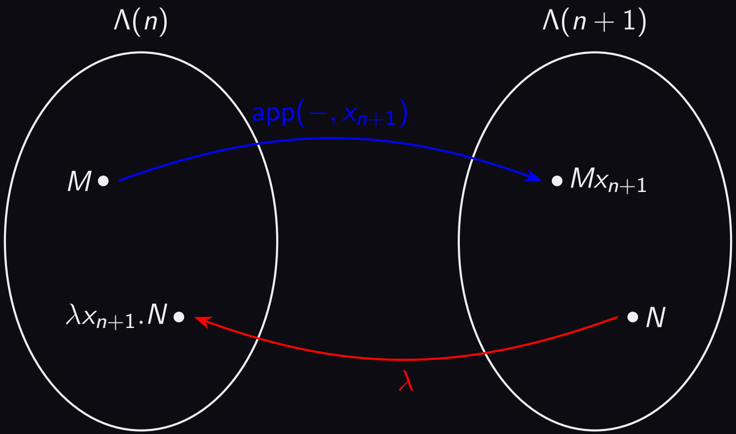
is invertible?

$$\dfrac{\dfrac{}{x_1, x_2 \vdash x_1}\,\mathrm{var} \qquad \dfrac{}{x_1, x_2 \vdash x_2}\,\mathrm{var}}{x_1, x_2 \vdash \color{red}{x_1 x_2}}\,\mathrm{app}$$

The additive λ-calculus as a **closed clone**



The additive λ-calculus as a **closed clone**



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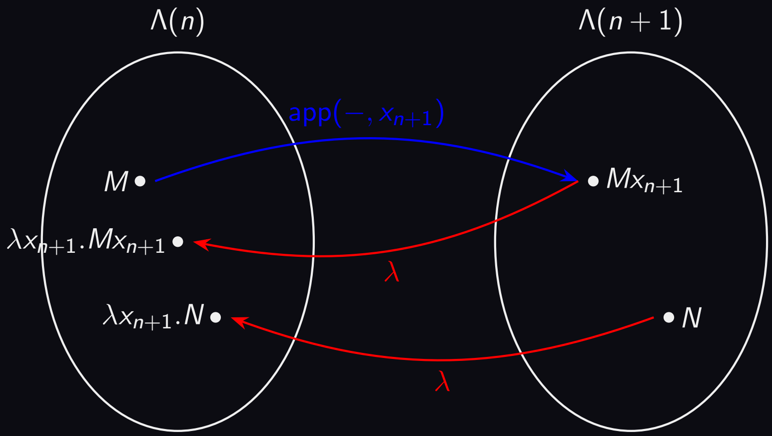
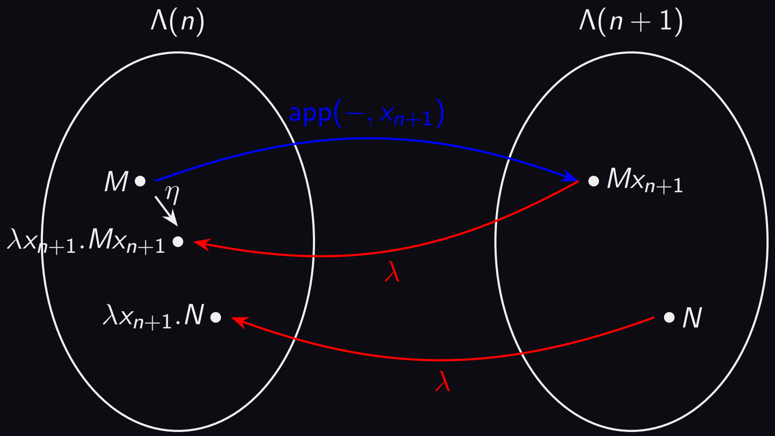
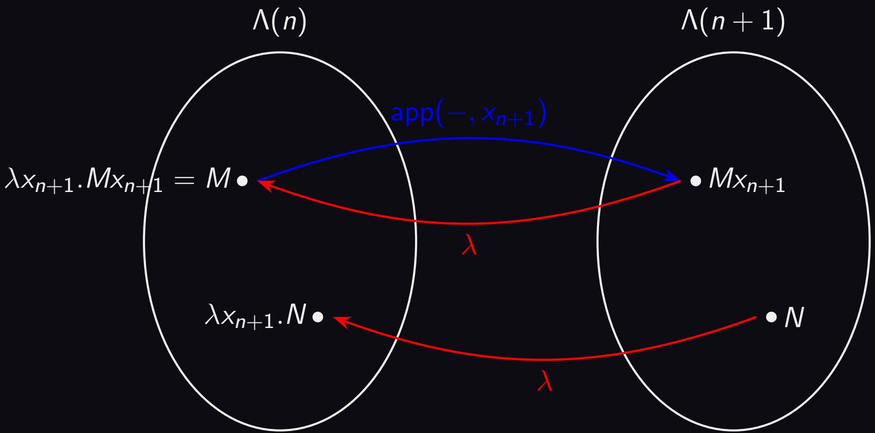


The additive λ-calculus as a **closed clone**



The additive λ-calculus as a **closed clone**



The untyped λ-calculus as a 2-dimensional structure (PhD project)

Recap

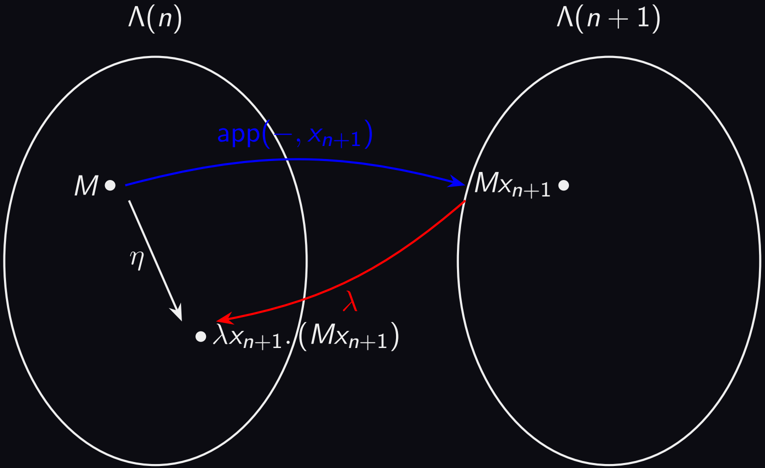
The untyped λ-calculus as a closed clone:

$$\Lambda(n) \xrightarrow{\sim} \Lambda(n+1)$$

The untyped λ-calculus as a 2-dimensional structure (i.e. closed Cat-clone):

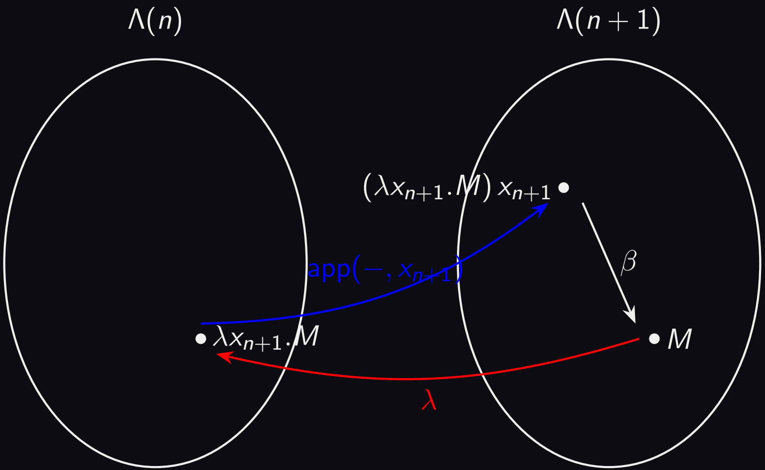
$$\Lambda(n) \underset{\lambda}{\overset{\mathrm{app}(-, x_{n+1})}{\underset{\longleftarrow}{\overset{\longrightarrow}{\dashv}}}} \Lambda(n+1)$$

What if...



$\Lambda(n)$
$\Lambda(n+1)$
$\mathsf{app}(-, x_{n+1})$
M
Mx_{n+1}
η
λ
$\lambda x_{n+1}.(Mx_{n+1})$

What if...



$\Lambda(n)$
$\Lambda(n+1)$
$(\lambda x_{n+1}.M)\,x_{n+1}$
$\mathrm{app}(-, x_{n+1})$
β
$\lambda x_{n+1}.M$
M
λ

What does it take to have this adjunction?

1. Each $\Lambda(n)$ has to be category (η and β must be morphisms).

$\boxed{\text{Idea of the solution}}$

One-hole contexts $\quad C ::= - \mid \lambda x.C \mid C\Lambda^- \mid \Lambda^- C$

The base reduction terms are $\beta_{x,M,N}$ and $\eta_{x,M}$.

Reduction terms $\quad r ::= C[\beta_{x,M,N}] \mid C[\eta_{x,M}]$

What does it take to have this adjunction?

1. Name the pairs of $\rightarrow_{\beta\eta}$ (i.e. the smallest compatible relation containing $\rightarrow_{\beta}$ and $\rightarrow_{\eta}$)
   E.g. $\eta_{x_{n+1},M} : M \rightarrow_{\beta\eta} \lambda x_{n+1}.(Mx_{n+1})$
   $\eta_{x_{n+1},M}\ x_{n+1} : Mx_{n+1} \rightarrow_{\beta\eta} (\lambda x_{n+1}.(Mx_{n+1}))\, x_{n+1}$
2. Assign an arity context $\rightsquigarrow$ reduction steps
   E.g. $x_1, \ldots, x_n \vdash \eta_{x_{n+1},M} : M \rightarrow_{\beta\eta} \lambda x_{n+1}.(Mx_{n+1})$

For each $n \in \mathbb{N}$, $\Lambda(n)$ is the category given by:

- $\mathrm{ob}(\Lambda(n)) := \{M \in \Lambda^{-} \mid x_1, \ldots, x_n \vdash M\}$
- For each object M, N, $\Lambda(n)(M, N)$ is the set of all the chains of reduction steps of the shape $x_1, \ldots, x_n \vdash r : M \to N$.
- Composition: concatenation
- Identities: the empty chain

What does it take to have this adjunction?

2. Coherence conditions to satisfy

$\boxed{\text{Idea of the solution}}$

1. Define a rewriting theory on the morphisms of $\Lambda(n)$.

$$\begin{tikzcd} M \arrow[rr, bend left, "r_1 \cdots r_n"] \arrow[rr, bend right, "r'_1 \cdots r'_k"'] \arrow[rr, phantom, "\Downarrow \rho"] & & N \end{tikzcd}$$

2. Consider the morphisms up to this rewriting theory.

Take-away messages

Take-away messages

Cartesian operads	Clones
Multiplicative, untyped λ-calculus	Additive, untyped λ-calculus
- The clone of the additive, untyped λ-calculus:
  - Homsets of terms
  - Multivariate substitution as the composition
  - Terms up to β-reduction and η-expansion $\rightsquigarrow$ **closed** clone
- The closed Cat-clone of the additive, untyped λ-calculus:
  - Homcategories of terms and chains of reduction steps
  - η-expansion as the unit and β-reduction as the counit

Thank you! Any questions?