The λ-calculus as a 2-Dimensional Operad

LIS Laboratory

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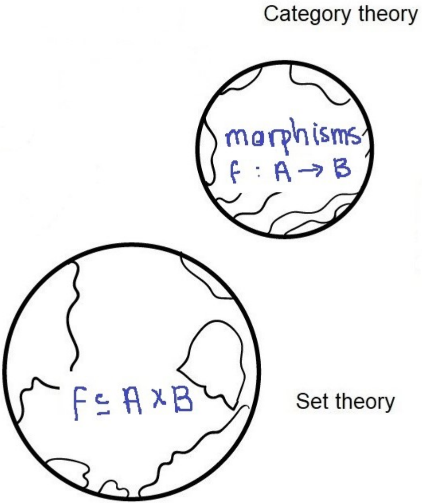
Generalizing currying

Currying

$f : X \times Y \to Z$ $\qquad$ $g : X \to (Y \to Z)$

Uncurrying

Generalizing currying



Generalizing currying

Generalizing using *cartesian closed categories* (ccc)

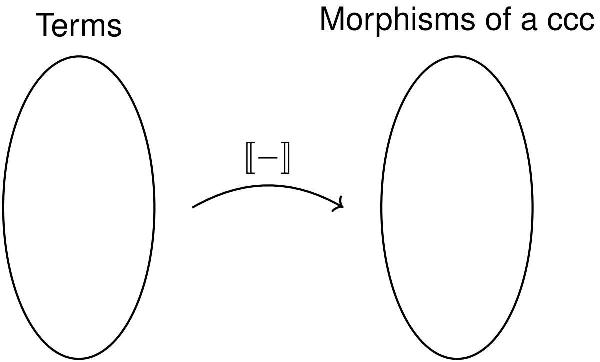
Cartesian product	Binary product
Set of functions	Exponential
One-point set	Terminal object

Generalizing currying

Generalization in a ccc : $\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, Z^Y)$

Proof : Apply the soundness theorem

The soundness theorem



If two terms are $\beta\eta$-Equal,
then their images by $[\![-]\!]$ are equal.

Plan

- General context
  - Terms and their types
  - $=_{\beta\eta}$-Equality
  - The typing relation
  - An interpretation
  - How to construct a semantics of types and terms from an interpretation
- From soundness to the research project

Terms and their types

Variables x, y, z, ... are terms

0 is a term

$\lambda x.\lambda y.x + y$ is a term

Types $\quad A ::= G \mid \text{unit} \mid A \times A \mid A \to A$

where G is a ground type

Terms and their types

Closed terms

Terms whose variables are all λ-bound

$\lambda x : A.\lambda y : A.x + y$ is a closed term

versus

$\lambda x : A.x + y$

The typing relation

Typing relation

The typing relation $\vdash$ relates a list of (variable,type)-pairs to a (term,type)-pair.

$$x : \mathbb{N}, y : \mathbb{N} \vdash x : \mathbb{N}$$

$$x : \mathbb{N} \vdash \lambda y : \mathbb{N}.x : \mathbb{N} \to \mathbb{N}$$

$$\vdash \lambda x : \mathbb{N}.\lambda y : \mathbb{N}.x : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})$$

$\beta\eta$-Equality

$$((\lambda x : \mathbb{N}.\lambda y : \mathbb{N}.x + y)\ 3)\ 1$$

$$\beta\big\downarrow$$

$$(\lambda y : \mathbb{N}.3 + y)\ 1$$

$$\beta\big\downarrow$$

$$3 + 1$$

$$\beta\big\downarrow^{\star}$$

$$4$$

$=_{\beta\eta}$-Equality

Let t be a term whose type is functional. Denote its type by $A \to B$.

$$t \overset{\eta}{\longrightarrow} \lambda x : A.(t\ x)$$

$=_{\beta\eta}$-Equality

$\rightarrow$ **Relates terms** which have the same type

$\rightarrow$ Smallest **congruence relation** which contains β**-convertible terms** and η**-convertible terms**.

$=_{\beta\eta}$-Equality

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Example (Some β-convertible terms)

Simplified β-conversion

$$\frac{\Gamma, x : A \vdash t : A' \qquad \Gamma \vdash t' : A}{(\lambda x : A.t)\ t' =_{\beta\eta} t[t'/x]}$$

where Γ is a list of (variable,type)-pairs

Interpretation

Interpretation

Let $\mathcal{C}$ be a ccc. An interpretation is a function

$$[\![-]\!]_{Gnd} : Gnd \to \mathrm{obj}\,\mathcal{C}$$

Semantics of types and terms

①

$$
\begin{aligned}
[\![-]\!] : & T(Gnd) \to \mathrm{obj}\,\mathcal{C} \\
& A \mapsto \begin{cases} [\![A]\!]_{Gnd} & \text{if } A \text{ is a ground type} \\ \top & \text{if } A = \text{unit} \\ [\![A_1]\!] \times [\![A_2]\!] & \text{if } \exists A_1, A_2 \text{ s.t. } A = A_1 \times A_2 \\ [\![A_2]\!]^{[\![A_1]\!]} & \text{if } \exists A_1, A_2 \text{ s.t. } A = A_1 \to A_2 \end{cases}
\end{aligned}
$$

Semantics of types and terms

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$$A \mapsto \begin{cases} [\![A]\!]_{Gnd} & \text{if } A \text{ is a ground type} \\ \top & \text{if } A = \text{unit} \\ [\![A_1]\!] \times [\![A_2]\!] & \text{if } \exists A_1, A_2 \text{ s.t. } A = A_1 \times A_2 \\ [\![A_2]\!]^{[\![A_1]\!]} & \text{if } \exists A_1, A_2 \text{ s.t. } A = A_1 \to A_2 \end{cases}$$

② **The semantics of types and terms**

$$[\![-]\!] : \Gamma \vdash t : A \mapsto [\![\Gamma \vdash t : A]\!]$$

where $[\![\Gamma \vdash t : A]\!] \in \mathcal{C}([\![\Gamma]\!], [\![A]\!])$

Back to soundness

Soundness

$$t =_{\beta\eta} t' \Rightarrow [\![\Gamma \vdash t : A]\!] = [\![\Gamma \vdash t' : A]\!]$$

Goal : construct

- a ccc
- a semantics of types and terms

which give us the converse

Back to soundness

The ccc **F** of types + terms

- Objects : types

Back to soundness

The ccc **F** of types + terms

- Objects : types
- Let *A* and *B* be two types.

$$\mathbf{F}(A, B) = \{t \mid t \text{ is a closed term and } \vdash t : A \to B\} / =_{\beta\eta}$$

Back to soundness

$[\![-]\!]$ is the semantics of types and terms associated to

$$\begin{aligned} o_{Gnd} : & Gnd \to \text{obj } \mathbf{F} \\ & G \mapsto G \end{aligned}$$

Research project

$$\boxed{[\![\Gamma \vdash t : A]\!] = [\![\Gamma \vdash t' : A]\!] \Leftrightarrow t =_{\beta\eta} t'}$$

Step 1 :

New approaches

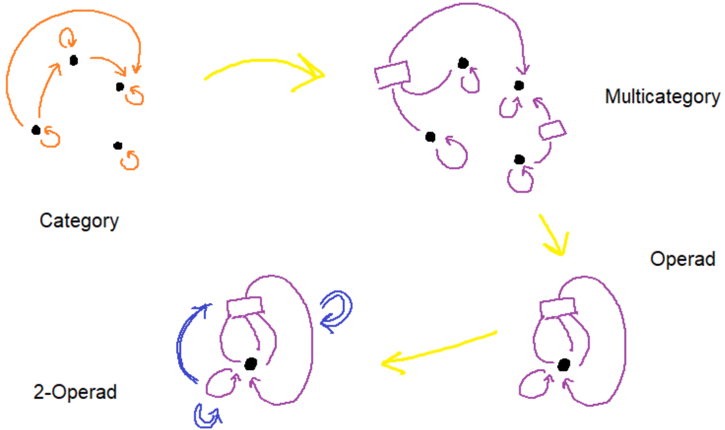
Multicategorical → To encompass more terms

2-dimensional → To encompass reductions

Goal : $[\![\alpha : t \to t']\!] = [\![\beta : t \to t']\!] \Leftrightarrow \alpha =_{ST} \beta$

Step 2 : Formalization

A multicategorical, 2-dim. structure



Summary

Characterization of equality between programs/terms

↓ Research project

Characterization of equality between program executions/reductions

↙ ↘

Multicategorical approach

2-dimensional approach

Thank you !