Additive Utility Without Solvability on All Components

Christophe Gonzales
LAFORIA. Université Paris 6, Paris,
4, place Jussieu F-75252 Paris cedex 05, France
e-mail: gonzales@laforia.ibp.fr

Abstract

Standard theories of additive utility require solvability w.r.t. all components, which rules out applications where some of the variables are discrete. Possible relaxations of solvability are investigated, in 3-component spaces for the case of restricted solvability, and in n-component spaces for the case of unrestricted solvability. An example is given showing that the Thomsen condition—necessary for the existence of an additive representation—is not implied by the independence axiom when there are only 2 solvable components.

Keywords

objective function - utility function - additive conjoint measurement - analytic approach - independence - Thomsen condition.

1 Introduction

Since the results of [Debreu 60], [Luce and Tukey 64], and [Krantz et al. 71], conditions of existence of additive representations on Cartesian products have been extensively studied. However in the case of infinite sets, only sufficient conditions are known that are easily testable (see e.g. [Fishburn 66]; [Fishburn 71]; [Jaffray 74a]; [Jaffray 74b]; [Scott and Suppes 58] for hardly testable but necessary and sufficient conditions); as a matter of fact, the sufficient conditions given in the classical theorems of the literature are not necessary since they include either a connectedness assumption in the topological approach (see [Fishburn 70]; [Fuhrken and Richter 91], [Vind 91]; [Wakker 89]; [Wakker 93]; [Wakker 94]) or a solvability w.r.t. every component assumption in the algebraic one (see e.g. [Doignon and Falmagne 74]; [Fishburn 70]; [Krantz 64]; [Luce 66]; [Wakker 88]; [Wakker 91a]; [Wakker 91b]).

Throughout this paper, we weaken these nonnecessary assumptions by studying the problem of the existence of additive utilities in Cartesian products in which only two components satisfy solvability—either restricted or unrestricted.

This can be particularly useful for problems in which some components are discrete while others are continuous.

In section 2, we give and briefly discuss the definitions and axioms required in the representation theorems. They are mostly classical; however we introduce *the scaling axiom*, which is a part of the second order cancellation axiom, and so is necessary for the existence of any additive utility, and is easily testable. This axiom cannot be deduced from independence when restricted solvability holds w.r.t. only 2 components, and when these components have a short range compared to the one of the non solvable components.

In section 3, we first prove the existence of additive utilities in 3-component Cartesian products in which restricted solvability holds w.r.t. 2 components. In fact, we prove that the additive representation on the 2-component set can be extended to the 3-component Cartesian product. Then we extend this property to n-component sets when restricted solvability is replaced by unrestricted solvability. The representations in the former case are no longer unique up to strictly positive affine transformations.

In order to prove the additive representability on the 2-component spaces, we use the Thomsen condition, according to the classical theorems. But in those theorems, for 3 or more component spaces, the independence axiom implies the Thomsen condition. The problem we address in section 4 is to know if this is still the case here. To put it another way, is the Thomsen condition implied because there are three components or because all three components are solvable? We show that the second alternative is the right one.

All proofs are given in the appendix.

2 Definitions and Axioms

In this section we give the definitions and axioms needed in section 3. We consider a Cartesian product $X = \prod_{i=1}^{n} X_i$ ($n = 3$ in subsection 3.1). Given a binary preference relation $\succsim$ over the Cartesian product X, we introduce the indifference relation $x \sim y \Leftrightarrow [x \succsim y \text{ and } y \succsim x]$, the strict preference relation $x \succ y \Leftrightarrow [x \succsim y \text{ and } \mathrm{Not}(y \succsim x)]$, and $x \precsim y \Leftrightarrow y \succsim x$. We define $[x, y] = \{z \in X : x \precsim z \precsim y\}$.

First we introduce the classical axioms: the ordering axiom (1), and the independence axiom (2), which are necessary for the existence of any additive utility.

Axiom 1 (Ordering) *$\succsim$ is a weak order on X, i.e. $\succsim$ is complete (for any x,$y \in X$, $x \succsim y$ or $y \succsim x$) and transitive (for any x,y,$z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$).*

Axiom 2 (Independence w.r.t. the ith component) *For any x,$y \in X$, if $(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \succsim (y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n)$, then $(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \succsim (y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n)$.*

The independence axiom induces a natural ordering on the Cartesian product generated by any subset of components, i.e. for any set $N \subset \{1, 2, \ldots, n\}$ one can define the weak order $\succsim_N$ on $\prod_{i \in N} X_i$ as follows: for $a, b \in \prod_{i \in N} X_i$, $a \succsim_N b$ iff for some $p \in \prod_{i \notin N} X_i$, $(a, p) \succsim (b, p)$.

For 2-component spaces, we define the Thomsen condition as:

Axiom 3 (Thomsen condition) *For every $x_1, y_1, z_1 \in X_1$, $x_2, y_2, z_2 \in X_2$, if $(x_1, z_2) \sim_{12} (z_1, y_2)$ and $(z_1, x_2) \sim_{12} (y_1, z_2)$, then $(x_1, x_2) \sim_{12} (y_1, y_2)$.*

Now we introduce solvability—restricted and unrestricted. This is not a necessary condition for the additive representability, but a technicality used in the proofs of the representation theorems. In fact, solvability enables to structure the Cartesian product very properly. When less than 2 components are solvable, this structure is not strong enough to ensure that what [Krantz et al. 71] call the n-order cancellation axiom implies the $n+1$th order cancellation axiom, hence preventing the existence of easily testable conditions. Therefore solvability is supposed to hold w.r.t. only 2 components.

Axiom 4 (Restricted solvability w.r.t. the first two components) *For any $x_1, x_1' \in X_1$, $x_2, x_2' \in X_2$, $x_i \in X_i$, $i \in \{3, \ldots, n\}$, and $y \in X$: if $(x_1, x_2, \ldots, x_n) \precsim y \precsim (x_1', x_2, \ldots, x_n)$, then there exists $x_1'' \in X_1$ such that $y \sim (x_1'', x_2, \ldots, x_n)$. If $(x_1, x_2, x_3, \ldots, x_n) \precsim y \precsim (x_1, x_2', x_3 \ldots, x_n)$, then there exists $x_2'' \in X_2$ such that $y \sim (x_1, x_2'', x_3, \ldots, x_n)$.*

Axiom 5 (Unrestricted solvability w.r.t. the first 2 components) *For any $y \in X$ and $x_i \in X_i$, $i \neq 1$, there exists $z_1 \in X_1$ such that $y \sim (z_1, x_2, \ldots, x_n)$. For any $y \in X$ and $x_i \in X_i$, $i \neq 2$, there exists $z_2 \in X_2$ such that $y \sim (x_1, z_2, x_3, \ldots, x_n)$.*

In order to avoid trivial cases, we require that the solvable components affect the preferences, i.e. that they are essential.

Axiom 6 (Essentialness w.r.t. the solvable components) *If the i^{th} component is solvable (i.e. it satisfies either axiom 4 or axiom 5), then there exist x_i,$y_i \in X_i$ and $z \in \prod_{k \neq i} X_k$ such that $(x_i, z) \succ (y_i, z)$.*

Representing a weak order by a utility function is not possible if there are more indifference classes than there are real numbers. To avoid this possibility, the usual method is to have recourse to an Archimedean axiom. This is axiom 7. But before giving it, we must define standard sequences and over-standard sequences.

Definition 1 (Standard sequence w.r.t. the first component) *For any set N of consecutive integers (positive, negative, finite or infinite), a set $\{x_1^k : x_1^k \in X_1, k \in N\}$ is a standard sequence w.r.t. the 1st component iff Not$((x_1^0, x_2^0, \ldots, x_n^0) \sim (x_1^0, x_2^1, \ldots, x_n^1))$ and for all $k, k+1 \in N$, $(x_1^k, x_2^0, \ldots, x_n^0) \sim (x_1^{k+1}, x_2^1, \ldots, x_n^1)$.*

Definition 2 (Over-standard sequence w.r.t. the first component) *For any set N of consecutive integers, a set $\{x_1^k : x_1^k \in X_1, k \in N\}$ is an over-standard sequence w.r.t. the first component iff either $(x_1^0, x_2^0, \ldots, x_n^0) \prec (x_1^0, x_2^1,$*

$\ldots, x_n^1)$ *and for all* $k, k+1 \in N$, $(x_1^{k+1}, x_2^0, \ldots, x_n^0) \succsim (x_1^k, x_2^1, \ldots, x_n^1)$, *or* $(x_1^0, x_2^0, \ldots, x_n^0) \succ (x_1^0, x_2^1, \ldots, x_n^1)$ *and for all* $k, k+1 \in N$, $(x_1^{k+1}, x_2^0, \ldots, x_n^0) \precsim (x_1^k, x_2^1, \ldots, x_n^1)$.

Parallel definitions hold for the other components.

Note that a standard sequence is a special kind of over-standard sequence. We present the Archimedean axiom in terms of over-standard sequences instead of standard sequences (as in the literature) because one is likely to be able to build over-standard sequences w.r.t. non solvable components, when, due to the absence of solvability, standard sequences fail to exist.

Axiom 7 (Archimedean axiom w.r.t. i^{th} component)
Any strictly bounded over-standard sequence w.r.t. the i^{th} component is finite.

Now it is time to state and explain the scaling axiom. This is a part of the second order cancellation axiom; hence it is a necessary condition for the additive representability.

Axiom 8 (Scaling w.r.t. the third component) *Suppose that $a, b \in X_3$ and that, for any $x_1, y_1, z_1 \in X_1$ and $x_2, y_2, z_2 \in X_2$, $(x_1, z_2, a) \prec (y_1, z_2, b)$ and $(z_1, x_2, a) \prec (z_1, y_2, b)$. Then, if $(x_1, x_2, a) \sim (y_1, y_2, b)$ and $(x_1, z_2, a) \sim (z_1, y_2, b)$, then $(y_1, z_2, a) \sim (z_1, x_2, a)$.*

The first two indifference relations mean that the change of strength of preference from x_2 to z_2 in the plane $\{x_3 = a\}$ corresponds to that from y_1 to z_1 in the plane $\{x_3 = b\}$. But if an additive representation exists, these changes should not be plane dependent, i.e. the change from y_1 to z_1 in the plane $\{x_3 = b\}$ should equal that of y_1 to z_1 in the plane $\{x_3 = a\}$, and so the change from y_1 to z_1 in the plane $\{x_3 = a\}$ should be compensated by the change from z_2 to x_2, which corresponds to the third indifference relation in the scaling axiom. Hence the axiom just states that the scale of the preference strength is not plane dependent.

The usefulness of this axiom arises when restricted solvability holds w.r.t. the first two components, and the range of these components is so short that independence does not imply the second order cancellation axiom over the whole Cartesian product. This case cannot arise when restricted solvability holds w.r.t. every component because it is always possible to select a and b such that $(x_1, z_2, a) \sim (y_1, z_2, b)$. But if the third component is not solvable, then it is possible that independence does not imply the second order cancellation axiom, as is shown in the following example: let $X = [1,2] \times [1,2] \times \{1,2\}$ and $\succsim$ be represented by the following utility function: $u(x_1, x_2, x_3) = [\frac{7}{8}(x_1 + x_2)]^{x_3}$. $\succsim$ violates the scaling axiom because the scale of the utility is linear for $x_3 = 1$, i.e. $u(x_1, x_2, x_3) = \frac{7}{8}(x_1 + x_2)$, and quadratic for $x_3 = 2$, i.e. $u(x_1, x_2, x_3) = [\frac{7}{8}(x_1 + x_2)]^2$. This example is illustrated in figure 1, in which some indifference curves are drawn and the shadowed areas represent the elements in each plane that are indifferent to some elements of the other plane. The coordinates (x_1, x_2) of those elements are less than or equal to $4\sqrt{\frac{2}{7}} - 1 \approx 1.14$ in plane $x_3 = \{2\}$, and

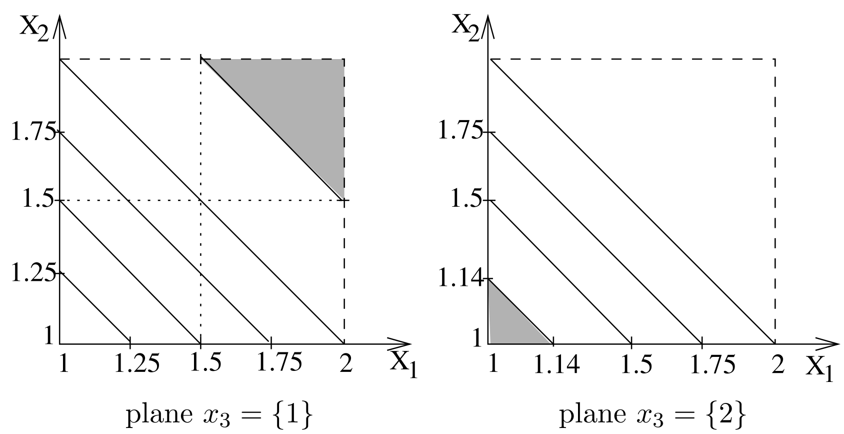

Figure 1: the inefficiency of the independence axiom

greater than or equal to 1.5 in plane $x_3 = \{1\}$. So the independence axiom, which is satisfied, cannot induce indifference relations between elements belonging to different planes—because if it were the case, these elements should have a common first or second coordinate—and so there remains degrees of freedom which enable some violations of the second order cancellation axiom. For instance, in the above example, $u(2, 1.5, 1) = u(1, 1, 2) = \frac{49}{16}$, $u(2, 2, 1) = u(4\sqrt{\frac{2}{7}} - 1, 1, 2) = \frac{7}{2}$ and $u(1, 2, 1) = \frac{21}{8} = 2.625 > u(4\sqrt{\frac{2}{7}} - 1, 1.5, 1) = \sqrt{\frac{7}{2}} - \frac{7}{16} \approx 1.433$.

Now, let us introduce the "overlap" relation: $\mathcal{O}$. It is useful for the uniqueness of the representations when restricted solvability holds w.r.t. the first two components—so we give the definition only for $X = X_1 \times X_2 \times X_3$. As a matter of fact, unlike the classical theorems, these representations are not unique up to positive linear transformations; for example, suppose that $X = [0, 1] \times [0, 1] \times \{0, 4, 9\}$ and that $\succsim$ is represented on X by $u(x_1, x_2, x_3) = x_1 + x_2 + x_3$. This utility function is additive, but $v(x_1, x_2, x_3) = \begin{cases} x_1 + x_2 & \text{if } x_3 = 0 \\ 2(x_1 + x_2) + 4 & \text{if } x_3 = 4 \\ 3(x_1 + x_2) + 9 & \text{if } x_3 = 9 \end{cases}$ is also an additive utility representing $\succsim$, and v is not an affine transform of u. This property comes from the fact that the solvable components can never compensate a change in the third component. Hence as long as $v(1, 1, 0) < v(0, 0, 4)$ and $v(1, 1, 4) < v(0, 0, 9)$, v does not need to be a positive linear transform of u to represent $\succsim$. On the contrary, if $(0, 0, 4) \precsim (1, 1, 0)$ and $(0, 0, 9) \precsim (1, 1, 4)$, the solvability enables compensations, and the uniqueness is up to a linear transformation. In terms of values of the utility function, this case corresponds to an "overlap" of the upper part of $X_1 \times X_2 \times \{0\}$ and the lower part of $X_1 \times X_2 \times \{4\}$, and of the upper part of $X_1 \times X_2 \times \{4\}$ and the lower part of $X_1 \times X_2 \times \{9\}$.

Definition 3 (Overlap function $\mathcal{O}$) *Let $\mathcal{P}(X_3)$ be the set of subsets of X_3. The overlap function $\mathcal{O} : X_3 \to \mathcal{P}(X_3)$ is defined as follows: $\mathcal{O}(x_3) = \{x \in X_3$*

such that there exist an integer n and a sequence $(y_3^i)_{1 \leq i \leq n}$ such that $y_3^0 = x_3$, $y_3^n = x$, and for any $i \in \{0, 1, \ldots, n-1\}$ there exists $(y_1^i, z_1^i, y_2^i, z_2^i) \in X_1^2 \times X_2^2$ such that $(y_1^i, y_2^i, y_3^{i+1}) \sim (z_1^i, z_2^i, y_3^i)$, and either for any $i \in \{0, 1, \ldots, n-1\}$, $y_3^{i+1} \prec_3 y_3^i$ or for any $i \in \{0, 1, \ldots, n-1\}$, $y_3^{i+1} \succ_3 y_3^i\}$.

The last condition of the above definition may seem restrictive, but, in fact, it is not, because, from any sequence $(y_3^i)_{1 \leq i \leq n}$ satisfying all the conditions above but the last one, by solvability w.r.t. the first components, it is always possible to extract a sequence satisfying also the last condition.

Definition 4 (Overlap relation $\mathcal{O}$)
For any $x_3, y_3 \in X_3$, $x_3 \mathcal{O} y_3 \Leftrightarrow o(x_3) \cap o(y_3) \neq \emptyset$.

Under the previous axioms, $\mathcal{O}$ is an equivalence relation.

3 Representation theorems

3.1 Restricted Solvability

In this subsection we suppose that $X = X_1 \times X_2 \times X_3$. The following theorem shows that under the classical axioms and the scaling axiom, the additive representability on the space where restricted solvability holds can be extended to the whole X. The representations are then unique up to stepwise positive linear transformations. In fact, we weaken the classical solvability assumption by enabling its violation by one component; in counterpart, we are obliged to assume the scaling axiom.

Theorem 1 (Representability under restricted solvability) *Suppose that $(X, \succsim)$ satisfies axioms 1 (ordering), 2 (independence w.r.t. the non solvable components), 4 (restricted solvability w.r.t. the first 2 components), 3 (Thomsen condition w.r.t. the solvable components), 6 (essentialness), 7 (Archimedean property w.r.t. every component) and 8 (scaling w.r.t. the third component). Then there exist real-valued functions u_1 on X_1, u_2 on X_2 and u_3 on X_3 such that:*

$$\text{for any } x, y \in X,\ x \precsim y \Leftrightarrow \sum_{i=1}^{3} u_i(x_i) \leq \sum_{i=1}^{3} u_i(y_i). \tag{1}$$

There also exist a set N of consecutive integers—finite or infinite—and a sequence of elements of X_3, say $(x_3^i)_{i \in N}$, such that, for any $x_3 \in X_3$, there exists $i \in N$ such that $x_3 \mathcal{O} x_3^i$, and, if $\text{Card}(N) > 1$, for any $i, i+1$ in N, $x_3^{i+1} \succ_3 x_3^i$ and $Not(x_3^i \mathcal{O} x_3^{i+1})$. If v_1, v_2, v_3 also satisfy (1), then there exist some constants $\alpha > 0$, α_1, α_2 and β_i, $i \in N$, such that:

$$\begin{cases} \text{for any } x \in X_1, \quad v_1(x_1) = \alpha \cdot u_1(x_1) + \alpha_1 \\ \text{for any } x \in X_2, \quad v_2(x_2) = \alpha \cdot u_2(x_2) + \alpha_2 \\ \text{for any } x_3 \in o(x_3^i),\ v_3(x_3) = \alpha \cdot u_3(x_3) + \beta_i \text{ where, for any } i, i+1 \in N, \\ \qquad \beta_{i+1} \geq \beta_i + \alpha \cdot [\max_{x_1, x_2}\{u_1(x_1) + u_2(x_2)\} + \max_{y_3 \in \mathcal{O}(x_3^i)} u_3(x_3)] \\ \qquad\qquad - \alpha \cdot [\min_{x_1, x_2}\{u_1(x_1) + u_2(x_2)\} + \min_{y_3 \in \mathcal{O}(x_3^{i+1})} u_3(x_3)] \\ \qquad \text{with equality only if either the } \min \text{ or the } \max \text{ is not reached.} \end{cases}$$

Moreover if $\text{Card}(N) > 1$*, then* u_1 *and* u_2 *are bounded.*

This theorem cannot be straightforwardly extended to the n-component case because, then, the structure induced by the solvable components is not always very strong, especially when the range of the solvable components is short compared to that of the non solvable components.

3.2 Unrestricted Solvability

Of course theorem 1 applies when restricted solvability is replaced by unrestricted solvability. But in this case, two improvements can be done: first, the scaling axiom always holds, and second, the theorem can be generalized to $X = \prod_{i=1}^{n} X_i$, with $n \geq 3$.

Theorem 2 (Representability under unrestricted solvability) *Suppose that* $X = \prod_{i=1}^{n} X_i$ *and that* $(X, \succsim)$ *satisfies axioms 1 (ordering), 2 (independence w.r.t. the non solvable components), 5 (unrestricted solvability w.r.t. the first* 2 *components), 3 (Thomsen condition w.r.t. the solvable components), 6 (essentialness) and 7 (Archimedean axiom w.r.t. every component). Then, there exist real-valued functions* u_i *on* X_i*,* $i \in \{1, \ldots, n\}$ *such that:*

$$\text{for any } x, y \in X,\ x \precsim y \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) \leq \sum_{i=1}^{n} u_i(y_i)$$

Moreover if $v_1, \ldots, v_n$ *also satisfy the equivalence above, then there exist some constants* $\alpha > 0$*,* β_i*,* $i \in \{1, \ldots, n\}$*, such that for any* i*,* $v_i = \alpha \cdot u_i + \beta_i$*.*

4 Thomsen Condition and Independence

In the theorems presented so far, the Thomsen condition is assumed to hold so that an additive representation is known to exist for $\succsim_{12}$. However, in [Fishburn 70] and [Krantz et al. 71], it is shown that, in 3 or more component spaces, when solvability holds w.r.t. every component, the Thomsen condition is implied by the independence axiom. The question that arises naturally is the following one: is this property still true with our weaker assumptions? To put it another way, is the Thomsen condition implied by independence just because there exists a third component or does this component need to be solvable? The question is important here because if the first alternative is right, then the Thomsen condition is not required in theorem 1 and theorem 2. Unfortunately, as is shown in this section, the second alternative is the right one. In fact, we prove that the following theorem is true:

Theorem 3 (Independence & Thomsen condition) *In* 3*-component Cartesian products, the Thomsen condition for* $\succsim_{12}$ *is not implied by independence w.r.t. all the components and solvability w.r.t. only* 2 *components.*

The proof consists in devising a general method for constructing a preference ordering $\succsim$ satisfying the assumptions in a Cartesian product $\Omega = \mathbb{R} \times \mathbb{R} \times \{z_0, z_1\}$, where z_0 and z_1 are arbitrary constants, and exhibiting a particular ordering that does not admit an additive representation. The approach we follow to define $\succsim$ is to construct one of its utility functions U on Ω by defining its indifference classes, or, more precisely, the indifference curves in plane $\{z = z_0\}$ and plane $\{z = z_1\}$. Of course, independence imposes some relations to hold between those planes. We first explain these constraints and then derive the construction of an example. In the latter, U not only satisfies the required conditions, but is also derivable.

Suppose that U exists. By independence,

$$\text{for any } x, x', y, y' \in \mathbb{R},\ (x, y, z_0) \sim (x', y', z_0) \Leftrightarrow (x, y, z_1) \sim (x', y', z_1).$$

This means that the indifference curves are the same in plane $\{z = z_0\}$ and plane $\{z = z_1\}$. Of course, even if their shape is the same in both planes, their values differ—otherwise one would have $U(x, y, z_0) = U(x, y, z_1)$, which, by independence, would be true for any couple (x, y), and so the third component would not be essential. This suggests that we construct two functions $V : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$, describing the indifference curves in plane $\{z = z_0\}$ and the transformation of the values of the indifference curves from plane $\{z = z_0\}$ to plane $\{z = z_1\}$ respectively. In mathematical terms, $U(x, y, z_0) = V(x, y)$ and $U(x, y, z_1) = \varphi \circ V(x, y)$, where $\varphi \circ V(x, y)$ stands for $\varphi(V(x, y))$. Constructing $\succsim$ on Ω can then be reduced to projecting the curves obtained by V onto plane $\{z = z_0\}$ and plane $\{z = z_1\}$ and to use φ to change the values assigned to the curves of plane $\{z = z_1\}$.

Ensuring that the independence axiom is not violated inside the planes is not difficult: it is sufficient that $V(x, y)$ strictly increases with x and y—i.e. $V(x, y) \geq V(x', y) \Leftrightarrow x \geq x'$ and $V(x, y) \geq V(x, y') \Leftrightarrow y \geq y'$—and that φ is strictly increasing. As a matter of fact, suppose these conditions hold. Then

$$\text{for any } x, x', y, y' \in \mathbb{R},\ (x, y, z_0) \succsim (x', y, z_0) \Leftrightarrow x \geq x' \Leftrightarrow (x, y', z_0) \succsim (x', y', z_0).$$

The same argument would apply if the roles of x and y had been exchanged. Since φ is strictly increasing, $V(x, y) \geq V(x', y') \Leftrightarrow \varphi \circ V(x, y) \geq \varphi \circ V(x', y')$, so the independence holds in both planes.

Now we must examine the constraints imposed by the independence axiom when both elements do not belong to the same plane, i.e. constraints imposed by relations similar to $(x, y, z_0) \succsim (x', y, z_1)$. We call these constraints "inter-plane independence constraints". They are explained in figure 2. Since the indifference curves are the same in both planes, we found it convenient to superpose them in the same drawing. To differentiate them, we drew the indifference curves of plane $\{z = z_0\}$ with bold lines, unlike the ones of $\{z = z_1\}$. V is strictly increasing with x and y, so "$V(x, y) = \text{constant}$" are decreasing curves—provided of course that they are continuous, which we suppose to be true—and hence can be written equivalently as "$y = \text{function}(x)$", where function is strictly decreasing. In figure 2 we assigned to each curve its function.

Suppose that $A = (x', y, z_0) \sim B = (x', y'', z_1)$. Then, by independence, $C = (x'', y, z_0) \sim D = (x'', y'', z_1)$. Suppose now that $F = (x, y', z_0) \sim A = (x', y, z_0) \sim G = (x'', y', z_1)$. Then still by independence, $E = (x, y'', z_0) \sim D = (x'', y'', z_1)$. Hence we must also have $E = (x, y'', z_0) \sim C = (x'', y, z_0)$.

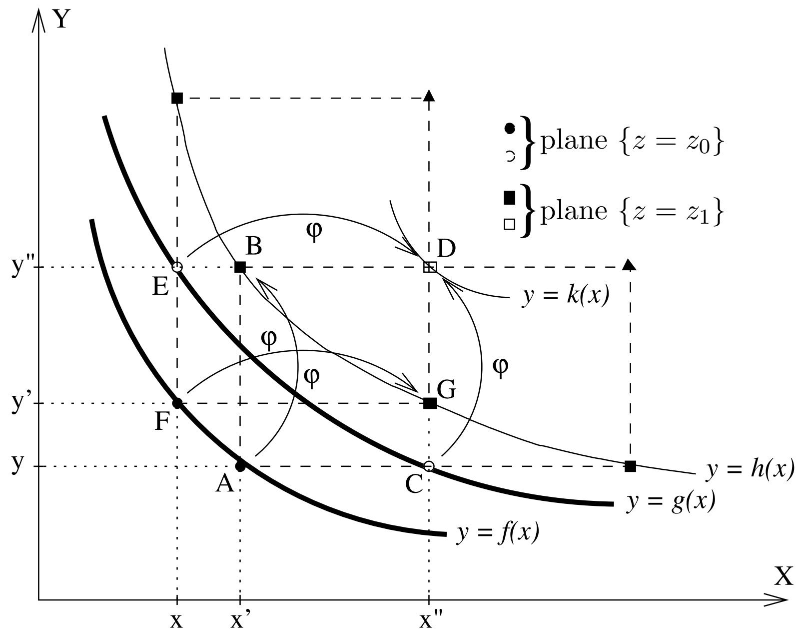

Figure 2: inter-plane constraints

Now, let us express this relation in terms of functions. Given an arbitrary point $C = (x'', y)$ and some known functions f and h, we define

$$\begin{cases} x'' \to y' = h(x'') \to x = f^{-1}(y') \\ y \to x' = f^{-1}(y) \to y'' = h(x'). \end{cases}$$

This determines two points on the curve of g because $y'' = g(x)$ and $x'' = g^{-1}(y)$, or, to put it another way, $h \circ f^{-1} \circ g(x'') = g \circ f^{-1} \circ h(x'')$. Hence independence inter planes implies that, for any x, $h \circ f^{-1} \circ g(x) = g \circ f^{-1} \circ h(x)$. This means that when constructing the example, if f and h are already known functions, then any function "inside" those two—i.e. any function whose indifference curve is between the indifference curves associated with f and h—is allowed to be chosen with a certain degree of freedom only on a small interval which corresponds to the interval $[CE]$. As for the degree of freedom, any curve will fit as long as independence holds inside the planes. Moreover, certain curves outside f and h—like the one at point D—are determined by the curves inside f and h. For instance, point D is determined by A, B, C and E, F, G. In fact this is the case for any outside curve because once the inside ones are chosen, locally near f and h, the outside curves—like k—are imposed. But then g and k can play the role taken previously by f and h, which impose another function deduced from h—which is "inside" g and k—and so on. By this process, we construct an infinite standard sequence, which, by the Archimedean axiom, implies that the whole space can be reached.

Now we have all the material needed to construct functions V and φ. For simplicity, our example uses the line $y = x$ as a symmetry axis. This is convenient because it implies some symmetry between the first two components. V describes indifference curves in $\mathbb{R} \times \mathbb{R}$; we call the latter $\mathcal{C}_\alpha$, using the following rule to evaluate α: the point of coordinates (α, α) belongs to the curve $\mathcal{C}_\alpha$. Moreover we impose on V to satisfy $V(x, x) = x$ for any $x \in \mathbb{R}$. Hence $\mathcal{C}_\alpha = \{(x, y) \in \mathbb{R}^2 : V(x, y) = \alpha\}$. To the curve $\mathcal{C}_\alpha$ we associate the function f_α, i.e. $\mathcal{C}_\alpha = \{(x, y) \in \mathbb{R}^2 : y = f_\alpha(x)\}$. Of course, there is a one to one mapping between f_α and $\mathcal{C}_\alpha$.

To start the construction, we have chosen as functions f and h of figure 2 functions f_0 and f_1. This means that $f_{\varphi(0)} = f_1$, or $\varphi(0) = 1$, or, more simply, that $(0, 0, z_1) \sim (1, 1, z_0)$. These functions can be taken arbitrarily—provided of course that they strictly decrease and do not intersect. Here we have chosen:

$$f_0(x) = \frac{-9 - 5x + 3\sqrt{9 + 2x + x^2}}{4} \tag{2}$$

$$f_1(x) = \frac{-5x + 3\sqrt{8 + x^2}}{4} \tag{3}$$

Note that f_0 and f_1 are continuous, strictly decreasing, and hence one to one, vary from $+\infty$ to $-\infty$ and the line $y = x$ is a symmetry axis.

Now we must construct the inside curves. For this purpose we use a two-step process. First we choose the "arbitrary" part of the utility function, i.e. for any $\alpha \in]0, 1[$, and any $x \in [Y_\alpha, X_\alpha]$, where X_α is such that $f_\alpha(X_\alpha) = f_0^{-1} \circ f_1(X_\alpha)$ and $Y_\alpha = f_\alpha(X_\alpha)$,

$$f_\alpha(x) = \frac{-9(1 - \alpha) - 5x + 3\sqrt{8 + (1 - \alpha)^2 + 2(1 - \alpha)x + x^2}}{4} \tag{4}$$

The value of X_α has been determined so that f_α is symmetric w.r.t. the line $y = x$; in practice, $X_\alpha \approx 1 + (3/\sqrt{2} - 1)\alpha$. Then inter-plane independence imposes the rest of the construction as seen in figure 2. This results in the following equation:

$$\text{for any } x \in \mathbb{R},\ f_\alpha \circ f_0^{-1} \circ f_1(x) = f_1 \circ f_0^{-1} \circ f_\alpha(x) \tag{5}$$

Note that equation (5) is satisfied for $\alpha = 0$ and $\alpha = 1$, and that (4) is not in conflict with (5) because f_α decreases on $[X_\alpha, Y_\alpha]$ and $f_\alpha(X_\alpha) = f_0^{-1} \circ f_1(X_\alpha)$. We present in figure 3 a summary of equation (5): if A belongs to $\mathcal{C}_\alpha$, then B must also belong to $\mathcal{C}_\alpha$, and conversely.

Curves $\mathcal{C}_\alpha$ defined by (4) and (5) satisfy all the conditions imposed previously. In particular, equation (5) extends the definition of $\mathcal{C}_\alpha$ over $\mathbb{R}$. The properties of these curves are described in the following lemma.

Lemma 1 (Properties of the inside curves) *Consider an arbitrary $\alpha \in]0, 1[$ and suppose that f_α is defined by (4) and (5). Then f_α is well defined on $\mathbb{R}$, is continuous, strictly decreases, $f_\alpha(\mathbb{R}) = \mathbb{R}$, and the line $y = x$ is a symmetry axis. Moreover, for any $\alpha, \beta \in [0, 1]$, $\alpha \leq \beta \Leftrightarrow f_\alpha(x) \leq f_\beta(x)$, for any $x \in \mathbb{R}$.*

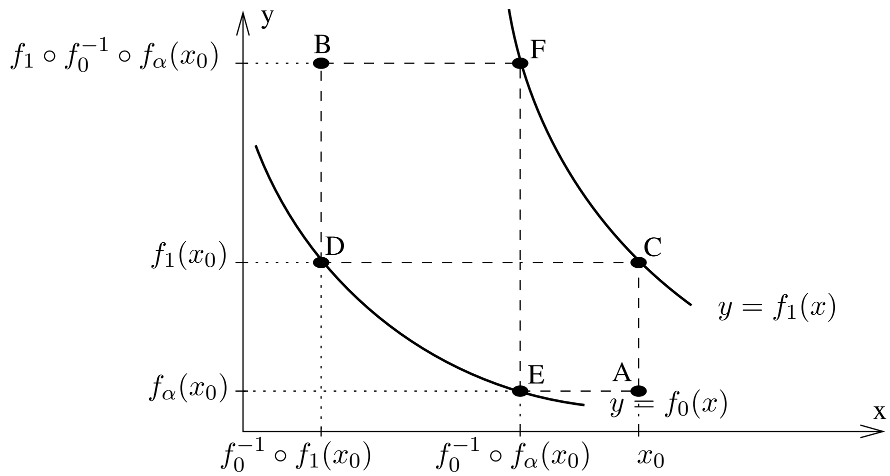

Figure 3: construction of the inside curves

Now that the construction of the inside curves is completed, there remains the one of the outside curves. For this purpose we use a two-step process again. First we describe how to construct them "locally" above f_1; this is equation (6). Second, we explain in (8) and (9) how this construction can be extended to the whole space.

Let us come back to figure 2. In this one, point D of plane $\{z = z_1\}$ is indifferent to points C and E of plane $\{z = z_0\}$. This means that $U(x'', y'', z_1) = U(x'', y, z_0)$, or, in terms of V and φ, $V(x'', y'') = \varphi \circ V(x'', y)$. But, because of inter-plane independence, we also know that $y'' = k(x'') = h \circ f^{-1} \circ g(x'')$. So we can deduce the following construction for our example:

$$\text{for any } x \in \mathbb{R},\ f_{\varphi(\alpha)}(x) = f_\alpha \circ f_0^{-1} \circ f_1(x) = f_1 \circ f_0^{-1} \circ f_\alpha(x) \qquad (6)$$

which corresponds in the following figure to: "if A and B belong to $\mathcal{C}_\alpha$, then E and G belong to $\mathcal{C}_{\varphi(\alpha)}$".

Properties of these curves are described in the following lemma:

Lemma 2 (Properties of $\mathbf{f}_{\boldsymbol{\varphi(\alpha)}}$) *Consider an arbitrary α in $[0,1]$, and suppose that $f_{\varphi(\alpha)}$ is defined by equation (6). Then $f_{\varphi(\alpha)}$ is well defined on $\mathbb{R}$, is continuous, strictly decreases, $f_{\varphi(\alpha)}(\mathbb{R}) = \mathbb{R}$, the line $y = x$ is a symmetry axis and, for any $\beta \in [0,1]$, $\alpha \leq \beta \Leftrightarrow f_{\varphi(\alpha)}(x) \leq f_{\varphi(\beta)}(x)$ for any $x \in \mathbb{R}$. Moreover*

$$\textit{for any } x \in \mathbb{R},\ f_{\varphi(\alpha)}(x) \circ f_{\varphi(0)}^{-1} \circ f_{\varphi(1)}(x) = f_{\varphi(1)} \circ f_{\varphi(0)}^{-1} \circ f_{\varphi(\alpha)}(x) \qquad (7)$$

Now it is time to give the global construction of the example. Equations (5) and (7) reveal that functions $f_{\varphi(\alpha)}$ and f_α have the same kind of inter-plane independence property. Hence $f_{\varphi^2(\alpha)}$—where φ^2 stands for $\varphi \circ \varphi$—can be defined from $f_{\varphi(\alpha)}$ in a similar way to that of $f_{\varphi(\alpha)}$ from f_α. This gives rise to

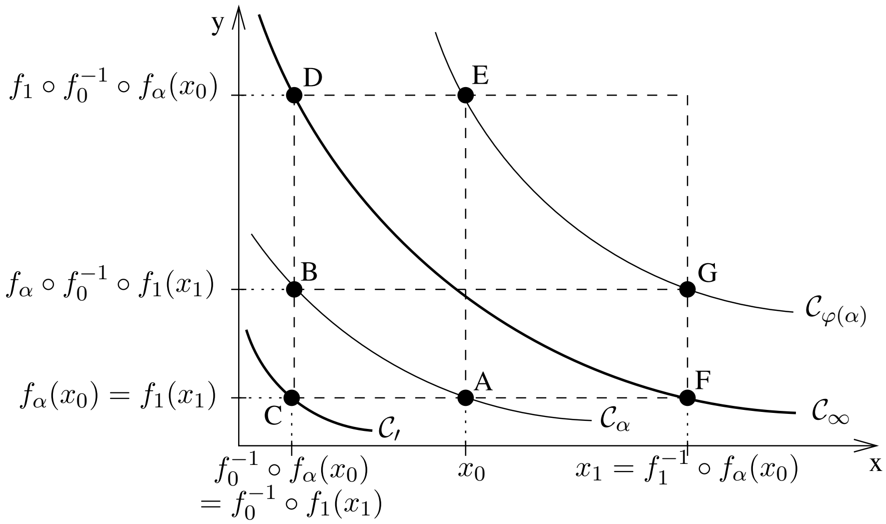

Figure 4: construction of the outside curves

equations (8) and (9), in which $\alpha \in [0,1]$ and $k \in \mathbb{N}$—φ^0 is supposed to be the identity on $\mathbb{R}$.

$$f_{\varphi^{k+1}(\alpha)}(x) \quad = \quad \begin{cases} f_{\varphi^k(\alpha)} \circ f^{-1}_{\varphi^k(0)} \circ f_{\varphi^k(1)}(x) \\ f_{\varphi^k(1)} \circ f^{-1}_{\varphi^k(0)} \circ f_{\varphi^k(\alpha)}(x) \end{cases} \tag{8}$$

$$f_{\varphi^{-k-1}(\alpha)}(x) \quad = \quad \begin{cases} f_{\varphi^{-k}(\alpha)} \circ f^{-1}_{\varphi^{-k}(0)} \circ f_{\varphi^{-k}(1)}(x) \\ f_{\varphi^{-k}(1)} \circ f^{-1}_{\varphi^{-k}(0)} \circ f_{\varphi^{-k}(\alpha)}(x) \end{cases} \tag{9}$$

The process of construction ensures that $f_{\varphi^{k+1}(\alpha)}$ and $f_{\varphi^{-k-1}(\alpha)}$ are well defined and continuous on $\mathbb{R}$, strictly decrease and admit $y = x$ as a symmetry axis, that $f_{\varphi^{k+1}(\alpha)}(\mathbb{R}) = \mathbb{R}$ and that $f_{\varphi^{-k-1}(\alpha)}(\mathbb{R}) = \mathbb{R}$. Moreover, if $\alpha, \beta \in [0,1]$, then $\alpha \leq \beta \Leftrightarrow f_{\varphi^k(\alpha)}(x) \leq f_{\varphi^k(\beta)}(x)$ for any $x \in \mathbb{R}$ and any integer k. Note that, by induction and since $\varphi(0) = 1$, it is easy to show that $f^{-1}_{\varphi^k(0)} \circ f_{\varphi^k(1)} = f_0^{-1} \circ f_1$.

The construction of the ordering is now completed, and there remains only to prove that it satisfies all the expected properties. This is done in the following theorem:

Lemma 3 (Properties of $\succsim$) *The binary relation $\succsim$ represented by functions $f_{\varphi^k(\alpha)}$ is a well defined weak order on Ω and satisfies independence and the Archimedean axiom. Moreover, the first two components are solvable.*

Till now the construction has been conducted on a very abstract level, and it is rather difficult to imagine the shape of the indifference curves. Hence we provide in figure 5 the drawing of some of them locally around the origin of the axes. Unlike what could be thought of from the figure, the curves are not

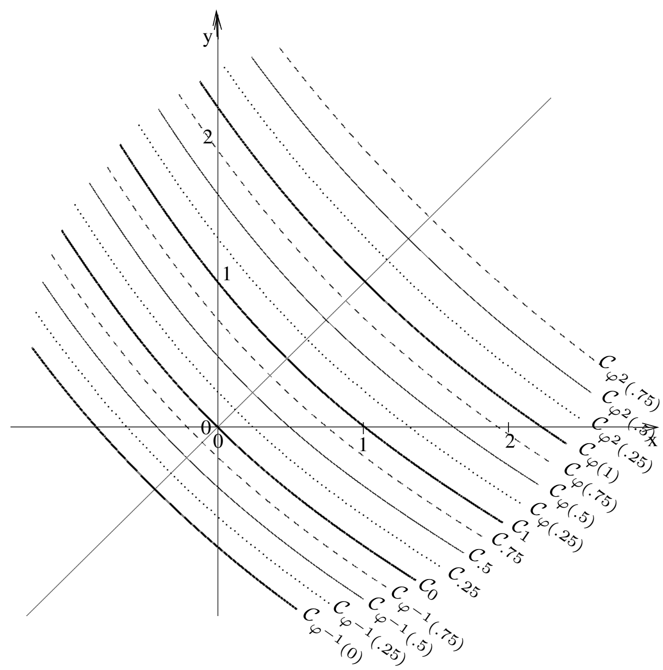

Figure 5: some indifference curves around the axes

deduced from other curves by translations — but it cannot be seen on the figure because the deviation from the translation is very small.

To conclude, it must be shown that the Thomsen condition does not hold everywhere in Ω. And as a matter of fact, if

$$\begin{aligned} x_1 &= \left(-23 - 15\sqrt{33} + \sqrt{7282 + 18\sqrt{33}}\right)/32, \\ x_2 &= \left(-17 + \sqrt{337}\right)/8, \\ y_1 &= \left(-5 + 3\sqrt{33}\right)/8 \\ y_2 &= \left(85 - 5\sqrt{337} + 3\sqrt{2}\sqrt{569 - 17\sqrt{337}}\right)/32, \end{aligned}$$

then, $U(x_2, .5) = U(x_1, y_1) = \frac{1}{3}$, $U(.5, y_1) = U(x_2, y_2) = 1$ and $U(.5, .5, z_0) = .5 < U(x_1, y_2) \approx .501088$. Hence there exists no additive representation of $\succsim$.

5 conclusion

In this paper, we studied theorems of existence of additive utility functions for spaces in which solvability does not hold w.r.t. every component. Such cases typically arise when some components are discrete while others are continuous. The scope of the paper concerns only preference spaces that are Cartesian products. Our main results are:

- Under restricted solvability w.r.t. 2 components:
  - In 3-component spaces, we proved the existence of additive utility functions under the classical necessary axioms (see [Krantz et al. 71] and [Fishburn 70]) and a new axiom called the scaling axiom.
  - Utilities are then unique up to stepwise positive affine transformations.
- Under Unrestricted solvability w.r.t. 2 components:
  - In n-component spaces, we proved the existence of additive utility functions under the classical axioms.
  - Utilities are then unique up to positive affine transformations.
- We proved that the Thomsen condition is not implied by independence when only two components of the Cartesian product are solvable.
- Our proofs use neither the algebraic nor the topological approach. In fact, they follow a new approach, which is called the analytic approach.

6 Appendix: Proofs

6.1 Proofs of section 3

In order to prove the main theorem we first introduce two lemmas. Lemma 4 shows that when two planes intersect (in terms of indifference relations), the common elements are always either the most preferred or the least preferred. Lemma 5 is the restriction of theorem 1 to the case in which the third component can take only two values.

Lemma 4 (Intersection of different planes) *Suppose that $(X, \succsim)$ satisfies axioms 1, 2, 4, and consider arbitrary elements $\underline{x_1}$ and $\overline{x_1}$ of X_1, $\underline{x_2}$ and $\overline{x_2}$ of X_2 and a,b of X_3 such that $a \prec_3 b$. Define $Y = [\underline{x_1}, \overline{x_1}] \times [\underline{x_2}, \overline{x_2}] \times \{a, b\}$ and $Z = \{(y_1, y_2, y_3) \in Y : \text{ there exists } (y_1', y_2', y_3') \in Y \text{ such that } (y_1, y_2, y_3) \sim (y_1', y_2', y_3') \text{ and } y_3' \neq y_3\}$. Then, for any $(y_1, y_2, a) \in Y \backslash Z$ and any $(y_1^0, y_2^0, y_3^0) \in Z$, $(y_1, y_2, a) \prec (y_1^0, y_2^0, y_3^0)$. And for any $(y_1, y_2, b) \in Y \backslash Z$, $(y_1^0, y_2^0, y_3^0) \prec (y_1, y_2, b)$.*

In the following proof, we first suppose the above lemma to be false; then, using restricted solvability, we show that this leads to a nonsense.

Proof of lemma 4: Consider two arbitrary elements $(y_1, y_2, a) \in Y \backslash Z$ and $(y_1^0, y_2^0, y_3^0) \in Z$, and suppose that $(y_1^0, y_2^0, y_3^0) \precsim (y_1, y_2, a)$. By hypothesis, there exists $(y_1^1, y_2^1, b) \in Z$ such that $(y_1^1, y_2^1, b) \sim (y_1^0, y_2^0, y_3^0) \precsim (y_1, y_2, a) \prec (y_1, y_2, b)$. If $y_2 \precsim_2 y_2^1$, $(y_1^1, y_2^1, b) \precsim (y_1, y_2, a) \prec (y_1, y_2^1, b)$, and by restricted solvability w.r.t. the first component, there exists $(y_1^2, y_2^1, b) \sim (y_1, y_2, a)$, and so $(y_1, y_2, a) \in Z$, which contradicts our hypothesis. So $y_2^1 \prec_2 y_2$; This implies that $(y_1^1, y_2^1, b) \prec (y_1^1, y_2, b)$. But then either $(y_1^1, y_2^1, b) \precsim (y_1, y_2, a) \precsim (y_1^1, y_2, b)$, which implies by restricted solvability w.r.t. the second component that there exists $(y_1^1, y_2^2, b) \sim (y_1, y_2, a)$, or $(y_1^1, y_2, b) \prec (y_1, y_2, a) \prec (y_1, y_2, b)$, which implies by restricted solvability w.r.t. the first component that there exists $(y_1^2, y_2, b) \sim (y_1, y_2, a)$. Hence the preference relation $(y_1^0, y_2^0, y_3^0) \precsim (y_1, y_2, a)$ always leads to a contradiction. A similar proof holds for $(y_1^0, y_2^0, y_3^0) \prec (y_1, y_2, b)$. ♦

Lemma 5 (Representability when Card($\mathbf{X_3}$) = 2) *Suppose that $X = X_1 \times X_2 \times \{a, b\}$ and $(X, \succsim)$ satisfies axioms 1, 2, 4, 3 (Thomsen condition w.r.t. the solvable components), 6, 7 and 8. Then there exist real-valued functions u_1 on X_1, u_2 on X_2 and u_3 on $\{a, b\}$ such that:*

$$\textit{for any } x, y \in X_1 \times X_2 \times \{a, b\},\ x \precsim y \Leftrightarrow \sum_{i=1}^{3} u_i(x_i) \leq \sum_{i=1}^{3} u_i(y_i).$$

If v_1, v_2, v_3 also satisfy the equivalence above, then there exist some constants $\alpha > 0$, α_1, α_2, α_3 and α_4 such that:

$$\left\{\begin{array}{l} \textit{for any } x \in X_1, \quad v_1(x_1) = \alpha \cdot u_1(x_1) + \alpha_1 \\ \textit{for any } x \in X_2, \quad v_2(x_2) = \alpha \cdot u_2(x_2) + \alpha_2 \\ \textit{if there exists } (x_1, x_2, a) \sim (y_1, y_2, b) \textit{ then } v_3 = \alpha \cdot u_3 + \alpha_3 \\ \textit{otherwise } v_3(a) = \alpha \cdot u_3(a) + \alpha_3 \textit{ and } v_3(b) = \alpha \cdot u_3(b) + \alpha_4 \\ \qquad \textit{where} \quad \alpha_4 \geq \alpha_3 \quad +\alpha \cdot [u_3(a) + \max_{x_1, x_2}\{u_1(x_1) + u_2(x_2)\}] \\ \qquad\qquad\qquad\qquad\quad -\alpha \cdot [u_3(b) + \min_{x_1, x_2}\{u_1(x_1) + u_2(x_2)\}] \\ \qquad \textit{with equality only if either the } \min \textit{ or the } \max \textit{ is not reached.} \end{array}\right.$$

The proof of this lemma consists in using classical axioms to show that an additive utility exists in $X_1 \times X_2$, and, then to extend this property on $X_1 \times X_2 \times \{a, b\}$ by using preference relations. The latter imply some necessary conditions to hold between $u_3(a)$ and $u_3(b)$. We show that these conditions are also sufficient for the existence of additive utilities.

Proof of lemma 5: According to the classical theorems, $\succsim_{12}$ is representable by an additive utility function $u_1 + u_2$. Hence, on $X_1 \times X_2 \times \{a\}$ and $X_1 \times X_2 \times \{b\}$, $x \precsim y \Leftrightarrow U(x) = \sum_{i=1}^{3} u_i(x_i) \leq U(y) = \sum_{i=1}^{3} u_i(y_i)$. Moreover, on these sets, U is cardinal. We are going to show in the sequel that it is possible to select a value for $u_3(b)$ such that U is a utility function over $X_1 \times X_2 \times \{a, b\}$.

First case: for any $(x_1, x_2), (x_1', x_2')$, $(x_1, x_2, a) \prec (x_1', x_2', b)$:

$u_1(X_1) + u_2(X_2)$ is bounded, where $u_i(X_i) = \{u_i(x_i),\ x_i \in X_i\}$. As a matter of fact, if its least upper bound were $+\infty$, it would be possible to create an infinite strictly increasing standard sequence in $X_1 \times X_2 \times \{a\}$, which would contradict the Archimedean axiom since it would be bounded by (x_1', x_2', b). The same argument applies for the greatest lower bound. Thus $\gamma = \sup_{x_1,x_2}\{u_1(x_1)+u_2(x_2)\}$ and $\delta = \inf_{x_1,x_2}\{u_1(x_1)+u_2(x_2)\}$ are finite. Now it is clear that $u_3(b) > u_3(a) + \gamma - \delta$ when γ and δ can be reached by some element of $X_1 \times X_2$, and $u_3(b) \geq u_3(a) + \gamma - \delta$ otherwise, are necessary and sufficient conditions for U to represent $\succsim$.

Second case: there exists $(x_1^0, x_1^1, x_2^0, x_2^1)$ such that $(x_1^0, x_2^0, a) \sim (x_1^1, x_2^1, b)$:

Clearly if there exists an additive utility, the following equation is true:

$$u_3(b) = u_3(a) + u_1(x_1^0) - u_1(x_1^1) + u_2(x_2^0) - u_2(x_2^1) \tag{10}$$

Now let $E = \{(x_1, x_2, x_3) :$ there exists $(y_1, y_2, y_3) \sim (x_1, x_2, x_3)$ and $y_3 \neq x_3\}$. By lemma 4, in order to prove that U is a utility over $X_1 \times X_2 \times \{a, b\}$, it is sufficient to prove it on E. For this purpose, consider two arbitrary elements (x_1^2, x_2^2, a) and (x_1^3, x_2^3, b) of E. By definition, there exist $(x_1^4, x_2^4, b) \sim (x_1^2, x_2^2, a)$ and $(x_1^5, x_2^5, a) \sim (x_1^3, x_2^3, b)$. Let $\underline{x_1} = \min_{\precsim_1}\{x_1^i,\ 0 \leq i \leq 5\}$, $\overline{x_1} = \max_{\precsim_1}\{x_1^i,\ 0 \leq i \leq 5\}$, $\underline{x_2} = \min_{\precsim_2}\{x_2^i,\ 0 \leq i \leq 5\}$ and $\overline{x_2} = \max_{\precsim_2}\{x_2^i,\ 0 \leq i \leq 5\}$. Call $A_1 = [\underline{x_1}, \overline{x_1}]$, $A_2 = [\underline{x_2}, \overline{x_2}]$, $A_3 = \{a, b\}$ and $A = A_1 \times A_2 \times A_3$. Let $B = \{(x_1, x_2, x_3) \in A :$ there exists $(y_1, y_2, y_3) \in A$, $(y_1, y_2, y_3) \sim (x_1, x_2, x_3)$ and $x_3 \neq y_3\}$. Proving that, given (10), U is a utility on B is sufficient to prove it on A and also on E because (x_1^2, x_2^2, a) and (x_1^3, x_2^3, b) are arbitrary elements of E. The advantage in introducing A is that it contains (x_1^1, x_2^1, b) and (x_1^0, x_2^0, a)—so that if U is a utility on A, (10) is satisfied—and that, on A, $u_3(b)$ can be redefined more conveniently.

Case 2.1: $(\underline{x_1}, \underline{x_2}, b) \precsim (\overline{x_1}, \underline{x_2}, a)$:

It is known that $(\underline{x_1}, \underline{x_2}, a) \prec (\underline{x_1}, \underline{x_2}, b)$; so, by restricted solvability w.r.t. the first component, there exists $a_1 \in A_1$ such that $(a_1, \underline{x_2}, a) \sim (\underline{x_1}, \underline{x_2}, b)$. Suppose that $u_3(b) = u_3(a) + u_1(a_1) - u_1(\underline{x_1})$. Then, by independence, for any $x_2 \in A_2$, $(a_1, x_2, a) \sim (\underline{x_1}, x_2, b)$, and $U(a_1, x_2, a) = U(\underline{x_1}, x_2, b)$.

Case 2.1.a: $(\overline{x_1}, \underline{x_2}, b) \precsim (\underline{x_1}, \overline{x_2}, b)$:

Consider an element $(x_1, \underline{x_2}, a) \in B$. By hypothesis, $(\underline{x_1}, \underline{x_2}, b) \sim (a_1, \underline{x_2}, a) \precsim (x_1, \underline{x_2}, a) \prec (x_1, \underline{x_2}, b) \precsim (\overline{x_1}, \underline{x_2}, b) \precsim (\underline{x_1}, \overline{x_2}, b) \sim (a_1, \overline{x_2}, a)$; so there exists $x_2 \in A_2$ such that $(x_1, \underline{x_2}, a) \sim (a_1, x_2, a)$, $x_1' \in A_1$ such that $(x_1', \underline{x_2}, b) \sim (x_1, \underline{x_2}, a)$, and $x_2' \in A_2$ such that $(x_1', \underline{x_2}, b) \sim (\underline{x_1}, x_2', b)$. But then, by transitivity of $\succsim$, $(\underline{x_1}, x_2', b) \sim (a_1, x_2, a)$. By the previous paragraph, $x_2' \sim_2 x_2$ and $U(\underline{x_1}, x_2', b) = U(a_1, x_2, a)$. Now, for any $(x_1, x_2, a) \in B$, consider $(x_1, \underline{x_2}, a)$; we know that there exists $(x_1', \underline{x_2}, b) \sim (x_1, \underline{x_2}, a)$ and that $u_1(x_1') + u_2(\underline{x_2}) + u_3(b) = u_1(x_1) + u_2(\underline{x_2}) + u_3(a)$. By independence, for any x_2, $(x_1', x_2, b) \sim (x_1, x_2, a)$ and of course $u_1(x_1') + u_2(x_2) + u_3(b) = u_1(x_1) + u_2(x_2) + u_3(a)$. Since U is already known to be a utility on $\{(x_1, x_2, a) \in B\}$ and $\{(x_1, x_2, b) \in B\}$, it is also a utility on B.

Case 2.1.b: $(\overline{x_1}, \underline{x_2}, b) \succ (\underline{x_1}, \overline{x_2}, b)$:

For any $(x_1, x_2, a) \in B$ such that $x_1 \precsim_1 a_1$, it is known that $(a_1, \underline{x_2}, a) \sim (\underline{x_1}, \underline{x_2}, b) \precsim (x_1, x_2, a) \precsim (a_1, x_2, a)$. Hence, by restricted solvability w.r.t. the 2nd component, there exists x_2' such that $(x_1, x_2, a) \sim (a_1, x_2', a)$; and, of course, $U(x_1, x_2, a) = U(a_1, x_2', a)$ since U is a utility function on $\{(x, y, a) \in B\}$. Similarly, for any (x_1, x_2, b) such that $(x_1, x_2, b) \precsim (\underline{x_1}, \overline{x_2}, b)$, there exists, x_2' such that $(x_1, x_2, b) \sim (\underline{x_1}, x_2', b)$, and their utilities are equal.

Suppose that there exists a_2 such that $(a_1, \overline{x_2}, a) \sim (a_2, \underline{x_2}, a)$. Then, obviously, $(\underline{x_1}, \underline{x_2}, b) \precsim (a_1, \overline{x_2}, a) \sim (a_2, \underline{x_2}, a) \sim (\underline{x_1}, \overline{x_2}, b) \precsim (a_1, \overline{x_2}, b) \sim (a_2, \underline{x_2}, b)$. Hence, by restricted solvability w.r.t. the 1st component, there exists y_1 such that $(a_2, \underline{x_2}, a) \sim (y_1, \underline{x_2}, b) \sim (\underline{x_1}, \overline{x_2}, b)$, and, of course, by the definition of U, the utility values of the last 3 triples are equal. Now, by independence, for any x_2, $(a_2, x_2, a) \sim (y_1, x_2, b)$, and their utilities are equal. Of course, for any $(x_1, x_2, a) \in B$ such that $a_1 \precsim_1 x_1 \precsim_1 a_2$, either $(a_1, x_2, a) \precsim (x_1, x_2, a) \precsim (a_1, \overline{x_2}, a)$, and , by restricted solvability w.r.t. the 2nd component, there exists x_2' such that $(a_1, x_2', a) \sim (x_1, x_2, a)$, or $(a_2, \underline{x_2}, a) \precsim (x_1, x_2, a) \precsim (a_2, \overline{x_2}, a)$, and, by restricted solvability w.r.t. the 2nd component, there exists x_2' such that $(a_2, x_2', a) \sim (x_1, x_2, a)$. In any case, by the definition of U, the utility values of the triples in the last indifferences are equal.

If there exist a_3, y_2 such that $(a_2, \overline{x_2}, a) \sim (a_3, \underline{x_2}, a) \sim (y_2, \underline{x_2}, b)$, then, the process described in the preceding paragraph can obviously be applied. By induction, one can create two sequences, (a_i) and (y_i), such that, for any i, $(a_i, \overline{x_2}, a) \sim (a_{i+1}, \underline{x_2}, a) \sim (y_i, \underline{x_2}, b)$. Note that those are standard sequences, and so, by the Archimedean axiom, they must be finite, say $i_{\max} = N$. Note that N can be equal to 1, that is, there exists no a_2 such that $(a_1, \overline{x_2}, a) \sim (a_2, \underline{x_2}, a)$. So, for any $(x_1, x_2, a) \sim (x_1', x_2', b)$ such that $x_1 \precsim_1 a_N$, $U(x_1, x_2, a) = U(x_1', x_2', b)$.

Now, there remains the case in which there exists no a_{N+1} such that $(a_N, \overline{x_2}, a) \sim (a_{N+1}, \underline{x_2}, a)$; in other terms, $(\overline{x_1}, \underline{x_2}, a) \prec (a_N, \overline{x_2}, a) \precsim (\overline{x_1}, \overline{x_2}, a)$. Hence, by restricted solvability w.r.t. the 2nd component, there exists α such that $(a_N, \overline{x_2}, a) \sim (\overline{x_1}, \alpha, a) \sim (y_{N-1}, \overline{x_2}, b)$, and their utilities are equal. But, then, $(a_N, \alpha, a) \sim (y_{N-1}, \alpha, b) \prec (a_N, \overline{x_2}, a) \sim (y_{N-1}, \overline{x_2}, b) \prec a_N, \overline{x_2}, b) \sim (\overline{x_1}, \alpha, b)$. By restricted solvability w.r.t. the 1st component, there exists y_N such that $(y_N, \alpha, b) \sim (\overline{x_1}, \alpha, a) \sim (y_{N-1}, \overline{x_2}, b)$, and their utility values are equal. Hence, by independence, for any x_2, $(y_N, x_2, b) \sim (\overline{x_1}, x_2, a)$, and their utility values are equal. Hence, it becomes obvious that U as defined is a utility function on B.

Case 2.2: $(\underline{x_1}, \overline{x_2}, a) \succsim (\underline{x_1}, \underline{x_2}, b)$:

By restricted solvability w.r.t. the 2nd component, there exists $a_2 \in A_2$ such that $(\underline{x_1}, a_2, b) \sim (\underline{x_1}, \overline{x_2}, a)$. It is shown as in case 2.1 that U is a utility on B.

Case 2.3: $(\overline{x_1}, \underline{x_2}, a) \prec (\underline{x_1}, \underline{x_2}, b)$ and $(\underline{x_1}, \overline{x_2}, a) \prec (\underline{x_1}, \underline{x_2}, b)$:

This gathers the remaining cases. $(\underline{x_1}, \underline{x_2}, b) \precsim (\overline{x_1}, \overline{x_2}, a)$. So $(\overline{x_1}, \underline{x_2}, a) \prec (\underline{x_1}, \underline{x_2}, b) \precsim (\overline{x_1}, \overline{x_2}, a) \prec (\overline{x_1}, \underline{x_2}, b)$. Therefore, for any (x_1, x_2, x_3) in B, there exists $y_1 \in A_1$ and $y_2 \in A_2$ such that $(x_1, x_2, x_3) \sim (y_1, \underline{x_2}, b) \sim (\overline{x_1}, y_2, a)$. In particular, there exists $a_2 \in A_2$ such that $(\overline{x_1}, a_2, a) \sim (\underline{x_1}, \underline{x_2}, b)$. If $u_3(b) = u_3(a) + u_1(\overline{x_1}) - u_1(\underline{x_1}) + u_2(a_2) - u_2(\underline{x_2})$, then $U(\overline{x_1}, a_2, a) = U(\underline{x_1}, \underline{x_2}, b)$.

Now consider two elements of B : $(\overline{x_1}, x_2, a) \sim (x_1, \underline{x_2}, b)$. By the scaling axiom, $[(\overline{x_1}, x_2, a) \sim (x_1, \underline{x_2}, b)$ and $(\overline{x_1}, a_2, a) \sim (\underline{x_1}, \underline{x_2}, b)] \Rightarrow (x_1, a_2, a) \sim (\underline{x_1}, x_2, a)$, and of course $U(x_1, a_2, a) = U(\underline{x_1}, x_2, a)$, which implies that $u_1(x_1) + u_2(a_2) = u_1(\underline{x_1}) + u_2(x_2)$. But $U(\overline{x_1}, x_2, a) = u_1(\overline{x_1}) + u_2(x_2) + u_3(a) = u_3(b) + u_1(\underline{x_1}) + u_2(\underline{x_2}) - u_2(a_2) + u_2(x_2) = u_3(b) + u_1(x_1) + u_2(a_2) + u_2(\underline{x_2}) - u_2(a_2) = U(x_1, \underline{x_2}, b)$. Since U is a utility on plane $\{(x_1, x_2, a) \in B\}$ and plane $\{(x_1, x_2, b) \in B\}$, it is also a utility on B.

So far we have proved the existence of an additive utility on $X_1 \times X_2 \times \{a, b\}$. It remains to prove the uniqueness property. But since the values given for $u_3(b)$ were always necessary and sufficient, the uniqueness property is obvious. ♦

The following proof is similar to the one above.

Proof of theorem 1: Let x_3^0, x_3^1 of X_3. By lemma 5, there exist real-valued functions u_1 on X_1, u_2 on X_2 and u_3 on $\{x_3^0, x_3^1\}$ such that $\succsim$ is represented by $U = u_1 + u_2 + u_3$. Without loss of generality, we suppose that $x_3^0 \precsim x_3^1$. We are to prove that this additive representation can be extended to $X_1 \times X_2 \times [x_3^0, x_3^1]$.

First case: there exists $(x_1^0, x_2^0, x_1^1, x_2^1)$ such that $(x_1^1, x_2^1, x_3^1) \sim (x_1^0, x_2^0, x_3^0)$:

A necessary condition for U to represent $\succsim$ is that $u_3(x_3^1) = u_3(x_3^0) + u_1(x_1^0) - u_1(x_1^1) + u_2(x_2^0) - u_2(x_2^1)$. Let us prove that it is also a sufficient condition. By restricted solvability and lemma 4, for any $x_3 \in [x_3^0, x_3^1]$, there exist (x_1, x_2), (x_1', x_2') and (x_1'', x_2'') such that $(x_1, x_2, x_3) \sim (x_1', x_2', x_3^0) \sim (x_1'', x_2'', x_3^1)$. So, by lemma 5, if $u_3(x_3) = u_3(x_3^0) + u_1(x_1') - u_1(x_1) + u_2(x_2') - u_2(x_2)$, $\succsim$ is representable by U on $X_1 \times X_2 \times \{x_3^0, x_3\}$ and if $u_3(x_3) = u_3(x_3^1) + u_1(x_1'') - u_1(x_1) + u_2(x_2'') - u_2(x_2)$, $\succsim$ is representable by U on $X_1 \times X_2 \times \{x_3, x_3^1\}$. But both values of $u_3(x_3)$ are equal because U represents $\succsim$ on $X_1 \times X_2 \times \{x_3^0, x_3^1\}$. Hence U represents $\succsim$ on $X_1 \times X_2 \times \{x_3^0, x_3, x_3^1\}$. Since x_3 is arbitrary, and, for any (x_1, x_2), by restricted solvability there exists (x_1', x_2') such that either $(x_1, x_2, x_3) \sim (x_1', x_2', x_3^0)$ or $(x_1, x_2, x_3) \sim (x_1', x_2', x_3^1)$, $\succsim$ is representable by an additive utility on $X_1 \times X_2 \times [x_3^0, x_3^1]$. Moreover, by the process of construction, this function is cardinal.

Second case: for any $(x_1^0, x_2^0, x_1^1, x_2^1)$, $(x_1^1, x_2^1, x_3^1) \succ (x_1^0, x_2^0, x_3^0)$:

Case 2.1: $x_3^1 \mathcal{O} x_3^0$:

In other words, there exists $(z_3^i)_{1 \leq i \leq n}$ such that, for any i in $\{0, ..., n-1\}$, $z_3^{i+1} \succ_3 z_3^i$ and there exist (z_1, z_2) and (z_1', z_2') such that $(z_1, z_2, z_3^{i+1}) \sim (z_1', z_2', z_3^i)$.

By case 1, by selecting an appropriate value for $u_3(z_3^{i+1})$ from the one of $u_3(z_3^i)$, U is a utility on $X_1 \times X_2 \times [z_3^i, z_3^{i+1}]$, and this function is cardinal. Now when comparing one element of $X_1 \times X_2 \times [z_3^{i-1}, z_3^i]$ and one element of $X_1 \times X_2 \times [z_3^i, z_3^{i+1}]$, there exists by lemma 4 an element of $X_1 \times X_2 \times \{z_3^i\}$ which is preferred to one element and not preferred to the other. Hence U is a utility function on $X_1 \times X_2 \times [z_3^{i-1}, z_3^{i+1}]$, and, by induction, on $X_1 \times X_2 \times [x_3^0, x_3^1]$.

Case 2.2: Not$(x_3^1 \mathcal{O} x_3^0)$:

Here, for any $x_3 \in \mathcal{O}(x_3^0)$, and any x_1, x_2, x_1^1, x_2^1, $(x_1, x_2, x_3) \prec (x_1^1, x_2^1, x_3^1)$. So, by lemma 5, $u_1(X_1)$ and $u_2(X_2)$ are bounded. By case 2.1, for any z_3 in $\mathcal{O}(x_3^0)$, one can select a value for $u_3(z_3)$ to extend U over $X_1 \times X_2 \times [x_3^0, z_3]$, and to the limit, over $X_1 \times X_2 \times \mathcal{O}(x_3^0)$. Now, by the Archimedean axiom, U has a least upper bound, say β_0, on this set. Similarly one can define an additive utility U^1 over $X_1 \times X_2 \times \mathcal{O}(x_3^1)$ such that $U^1 = u_1 + u_2 + u_3^1$. And U^1 is bounded below on $X_1 \times X_2 \times \mathcal{O}(x_3^1)$. Call α_1 this lower bound.

Case 2.2.a: there is no $x_3^2 \in [x_3^0, x_3^1]$ such that not$(x_3^2 \mathcal{O} x_3^0)$ and not$(x_3^2 \mathcal{O} x_3^1)$:

Define U on $X_1 \times X_2 \times \mathcal{O}(x_3^1)$ as $U(x_1, x_2, x_3) = U^1(x_1, x_2, x_3) + \gamma$ where γ is such that $\alpha_1 + \gamma > \beta_0$ when α_1 and β_0 are reached, and $\alpha_1 + \gamma \geq \beta_0$ otherwise. It is obvious that U is a utility over $X_1 \times X_2 \times [x_3^0, x_3^1]$, and that choosing γ as above is necessary and sufficient. So the representation is no more cardinal.

Case 2.2.b: there exists $x_3^2 \in [x_3^0, x_3^1]$ such that not$(x_3^2 \mathcal{O} x_3^0)$ and not$(x_3^2 \mathcal{O} x_3^1)$:

Following the above reasoning, there exists an additive utility $U^2 = u_1 + u_2 + u_3^2$ on $X_1 \times X_2 \times \mathcal{O}(X_3^2)$ and U^2 is bounded by α_2 and β_2. More generally, let $\mathcal{O}^\sim$ be the set of equivalence classes of $\mathcal{O}$, and consider $Z = \{\tilde{z}_3 \in \mathcal{O}^\sim :$ for any $z_3 \in \tilde{z}_3$, $z_3 \in [x_3^0, x_3^1]\}$. Suppose that Card(Z) is infinite; then it is possible to extract from Z an infinite sequence $(\tilde{z}_n^3)$ such that for any $z_3^n \in \tilde{z}_n^3$ and any $z_3^{n+1} \in \tilde{z}_{n+1}^3$, $z_3^{n+1} \succ_3 z_n^3$. From this sequence one can extract an infinite increasing over-standard sequence (z_3^n) which is bounded by (x_1^1, x_2^1, x_3^1). This contradicts the Archimedean axiom. Hence Card(Z) is a finite number N.

We already know that there exists an additive utility U^i on $X_1 \times X_2 \times \mathcal{O}(z_3^i)$ such that $U^i(x_1, x_2, x_3) = u_1(x_1) + u_2(x_2) + u_3^i(x_3)$ and U^i is bounded. Let α_i and β_i be its greatest lower bound and its least upper bound respectively. Now on $[z_3^i, z_3^{i+1}]$ it is possible to apply case 2.2.a; hence we can show inductively that U can be constructed over $X_1 \times X_2 \times [x_3^0, x_3^1]$.

Now consider an arbitrary element $x_3 \succ_3 x_3^1$. By a similar process, we construct U on $X_1 \times X_2 \times [x_3^1, x_3]$. But, by lemma 4, for any element of $X_1 \times X_2 \times [x_3^0, x_3^1]$ and $X_1 \times X_2 \times [x_3^1, x_3]$, there is always an element of $X_1 \times X_2 \times \{x_3^0\}$ which is preferred to one of them and less preferred to the other one. Hence U is a utility function on $X_1 \times X_2 \times [x_3^0, x_3]$. U can be extended on X.

By the previous paragraphs, x_3^0 is an arbitrary element of X_3, and, for any $x_3 \succsim_3 x_3^0$, there exists no infinite sequence (x_3^i) such that $x_3^0 \precsim_3 x_3^i \precsim_3 x_3$ and not$(x_3^i \mathcal{O} x_3^{i+1})$. Moreover, there exists an additive utility on $X_1 \times X_2 \times [x_3^0, x_3]$; so, for any integer n, any sequence (x_3^i) such that $n \leq u_3(x_3^i) \leq n+1$ and not$(x_3^i \mathcal{O} x_3^{i+1})$, is finite. Hence $\mathcal{O}^\sim$ is denumerable. So there exists a sequence (x_3^i) such that for any i, $x_3^{i+1} \succ_3 x_3^i$ and for any x_3 of X_3, there exists i such that $x_3 \mathcal{O} x_3^i$. This sequence is in fact created by taking one element in each indifference class of $\mathcal{O}$. The uniqueness of the additive representation is immediate. ♦

The principle of the following proof is to aggregate the non solvable components twice, using different aggregations. Then, using theorem 1, we show that additive representations exist for both aggregations, and that these are equal (up to a positive affine transformation). This equality implies some properties between the two utilities, which is shown to lead to an additive decomposability of the whole space X.

Proof of theorem 2: By unrestricted solvability, it is obvious that the scaling axiom always holds. First, suppose that $X = X_1 \times X_2 \times X_3$. Then theorem 1 can be applied, with $\text{card}(N) = 1$.

Suppose now that theorem 2 holds for n-dimensional Cartesian products. Let us prove that it is also true for $n+1$. $X = \prod_{i=1}^{n+1} X_i$. Let $Y_1 = X_1 \times X_{n+1}$ and $Y_2 = X_2 \times X_{n+1}$. Then $X = Y_1 \times X_2 \times \prod_{i=3}^{n} X_i = X_1 \times Y_2 \times \prod_{i=3}^{n} X_i$. Components on Y_1 and Y_2 are still solvable, so there exist real valued functions $u_1, u_2, \ldots, u_n$ on $Y_1, X_2, \ldots, X_n$ and $v_1, v_2, \ldots, v_n$ on $X_1, Y_2, \ldots, X_n$ respectively such that $U = \sum_{i=1}^{n} u_i$ and $V = \sum_{i=1}^{n} v_i$ represent $\succsim$ and are cardinal.

Now consider the set $\prod_{i=1}^{n} X_i \times \{x_{n+1}^0\}$ where $\{x_{n+1}^0\}$ is an arbitrary element of X_{n+1}. By the cardinal property of U and V, there exist some constants $\alpha_{x_{n+1}^0}$, β_{i,x_{n+1}^0} such that, for any $x_1, \ldots, x_n$, $v(x_1) = \alpha_{x_{n+1}^0} u_1(x_1, x_{n+1}^0) + \beta_{1,x_{n+1}^0}$, $v_2(x_2, x_{n+1}^0) = \alpha_{x_{n+1}^0} u_2(x_2) + \beta_{2,x_{n+1}^0}$ and $v_i = \alpha_{x_{n+1}^0} u_i + \beta_{i,x_{n+1}^0}$. Hence, for $i \geq 3$ and any x_{n+1}, x'_{n+1}, $v_i = \alpha_{x_{n+1}} u_i + \beta_{i,x_{n+1}} = \alpha_{x'_{n+1}} u_i + \beta_{i,x'_{n+1}}$. Since essentialness holds w.r.t. the i^{th} component, $\alpha_{x_{n+1}} = \alpha_{x'_{n+1}}$ and $\beta_{i,x_{n+1}} = \beta_{i,x'_{n+1}}$. Hence There exists constants $\alpha > 0, \beta_1(x_{n+1}), \beta_2(x_{n+1}), \beta_i$ such that $v(x_1) = \alpha u_1(x_1, x_{n+1}) + \beta_1(x_{n+1})$, $v_2(x_2, x_{n+1}) = \alpha u_2(x_2) + \beta_2(x_{n+1})$ and $v_i = \alpha u_i + \beta_i$.

Now $u_1(x_1, x_{n+1}) + \sum_{i=2}^{n} u_i(x_i) = \frac{1}{\alpha} v_1(x_1) - \frac{\beta_1(x_{n+1})}{\alpha} + \sum_{i=2}^{n} u_i(x_i)$. Hence an additive representation exists for $\prod_{i=1}^{n+1} X_i$. The uniqueness up to strictly positive linear transformations comes from the fact that if $\sum_{i=1}^{n+1} u_i$ is a utility representing $\succsim$ on X, $\sum_{i=1}^{n} u_i$ also represents $\succsim$ on $\prod_{i=1}^{n} X_i$, and that the uniqueness up to positive linear transformations is supposed to hold on this set. ♦

6.2 Proofs of Section 4

Proof of lemma 1: X_α is such that $f_\alpha(X_\alpha) = f_0^{-1} \circ f_1(X_\alpha)$. So, (4) does not conflict with (5). It is easy to see that, according to (4), f_α is continuously decreasing on $[Y_\alpha, X_\alpha]$, and, according to (2) and (3), f_0 and f_1 are continuously decreasing over $\mathbb{R}$. Now, it is not difficult to see that, for any $x \in \mathbb{R}$, $f_1 \circ f_o^{-1}(x) > x + 1$ and $f_o^{-1} \circ f_1(x) < x - 1$.

Hence, since $f_0^{-1} \circ f_1(X_\alpha) = Y_\alpha$, by (5), f_α is well defined on $]-\infty, X_\alpha]$, and moreover is continuously decreasing on this set. By symmetry of (2), (3) and (4) w.r.t. the line $y = x$, f_α is well defined on $\mathbb{R}$, is continuously decreasing, and is symmetric w.r.t. the line $y = x$. By the previous paragraph and (5), it is obvious that $f_\alpha(\mathbb{R}) = \mathbb{R}$.

Now, consider $\alpha, \beta \in [0, 1]$ such that $\alpha \leq \beta$. By equation (4)—and equations (2) and (3) if α or β is equal to 0 or 1—it is obvious that for any $x \in [Y_\beta, X_\alpha]$, $f_\alpha \leq f_\beta$. On $[Y_\alpha, Y_\beta]$, the inequality $f_\alpha(x) \leq f_\beta(x)$ must also hold, otherwise f_β would not be one to one. Now if there existed an x in $\mathbb{R}$

such that $f_\alpha(x) > f_\beta(x)$, then by repeated uses of (5), there would exist an $x' \in [Y_\alpha, X_\alpha]$ such that $f_\alpha(x') > f_\beta(x')$, which has been shown to be impossible. Conversely, if for any $x \in \mathbb{R}$, $f_\alpha(x) \leq f_\beta(x)$, then this is true in particular for any $x \in [Y_\beta, X_\alpha]$. But then by equation (4)—and (2) or (3)—$\alpha \leq \beta$. ♦

Proof of lemma 2: By (5), $f_{\varphi(\alpha)}$ is well defined on $\mathbb{R}$ for any $\alpha \in [0,1]$. Since f_α, f_0 and f_1 are continuous, strictly decrease, vary from $+\infty$ to $-\infty$ and are symmetric w.r.t. the line $y = x$, $f_{\varphi(\alpha)}$ has the same properties. By lemma 1, for any $\alpha, \beta \in [0,1]$, $\alpha \leq \beta \Rightarrow f_\alpha(x) \leq f_\beta(x)$ for any $x \in \mathbb{R}$, and it follows from the change of variable $y = f_1^{-1} \circ f_0(x)$, and the fact that $f_1^{-1} \circ f_0(x)$ varies from $-\infty$ to $+\infty$, that $\alpha \leq \beta \Rightarrow f_\alpha \circ f_0^{-1} \circ f_1(y) \leq f_\beta \circ f_0^{-1} \circ f_1(y)$ for any $y \in \mathbb{R}$.

$f_{\varphi(1)} = f_1 \circ f_0^{-1} \circ f_1$ and $f_{\varphi(0)} = f_1$ — since by hypothesis $\varphi(0) = 1$;
so $f_{\varphi(\alpha)} \circ f_{\varphi(0)}^{-1} \circ f_{\varphi(1)} = f_\alpha \circ f_0^{-1} \circ f_1 \circ f_1^{-1} \circ f_1 \circ f_0^{-1} \circ f_1 = f_\alpha \circ f_0^{-1} \circ f_1 \circ f_0^{-1} \circ f_1 = f_1 \circ f_0^{-1} \circ f_\alpha \circ f_0^{-1} \circ f_1 = f_1 \circ f_0^{-1} \circ f_1 \circ f_1^{-1} \circ f_\alpha \circ f_0^{-1} \circ f_1 = f_{\varphi(1)} \circ f_{\varphi(0)}^{-1} \circ f_{\varphi(\alpha)}$, hence proving that equation (6) holds. ♦

In order to prove lemma 3, we first introduce the two lemmas, namely lemma 6 and lemma 7. The first one shows that the whole space is well defined, i.e. that any point in $\mathbb{R} \times \mathbb{R}$ corresponds to a point $(x, f_{\varphi^k(\alpha)}(x))$, and that the curves $\mathcal{C}_\alpha$ never intersect. The second lemma shows that, not only the curves never intersect, but also the distance between two curves is never null.

Lemma 6 (The space is well defined) *Assume that, for any $\alpha \in [0,1]$, $\mathcal{C}_\alpha$ is given by (2), (3) or (4) and (5). Suppose that $\mathcal{C}_{\varphi^k(\alpha)}$ and $\mathcal{C}_{\varphi^{-k}(\alpha)}$ are given for any $k \in \mathbb{N}$ by (6) and (8), and (6) and (9) respectively. Then φ is well defined on $\mathbb{R}$ and $\mathcal{C}_\beta$ is well defined for any $\beta \in \mathbb{R}$. Moreover φ is one to one and strictly increases from $-\infty$ to $+\infty$, for any $(x,y) \in \mathbb{R}^2$, there exists $\alpha \in \mathbb{R}$ such that $(x,y) \in \mathcal{C}_\alpha$, and the following hold: for any $\alpha, \beta, x, x', y_1, y_2 \in \mathbb{R}$,*

$$\alpha \leq \beta \Leftrightarrow f_\alpha(x) \leq f_\beta(x) \text{ for any } x \in \mathbb{R}$$
$$\alpha, \beta \in \mathbb{R},\ y_1 = f_\alpha(x) = f_\beta(x')\ \&\ y_2 = f_{\varphi(\alpha)}(x) \Rightarrow y_2 = f_{\varphi(\beta)}(x').$$

The principle of this proof is to show by induction w.r.t. k that any point in the domain $\{(x,y) : f_{\varphi^k(0)} \leq y \leq f_{\varphi^k(1)}\}$ belongs to an indifference curve $\mathcal{C}_\alpha$. Then, it is shown that, when k varies from $-\infty$ to $+\infty$, the set above is $\mathbb{R} \times \mathbb{R}$. So, the space is well defined.

Proof of lemma 6: By (4) and (5), it is easily seen that every point in the domain $\{(x,y) : f_0(x) \leq y \leq f_1(x)\}$ belongs to an indifference curve $\mathcal{C}_\alpha$. Now suppose that for $k \geq 0$, every point in $\{(x,y) : f_{\varphi^k(0)}(x) \leq y \leq f_{\varphi^k(1)}(x)\}$ belongs to a curve $\mathcal{C}_{\varphi^k(\alpha)}$. Consider an arbitrary point (x_0, y_0) in $\{(x,y) : f_{\varphi^{k+1}(0)}(x) \leq y \leq f_{\varphi^{k+1}(1)}(x)$. By hypothesis, $f_{\varphi^{k+1}(0)}(x_0) \leq y_0 \leq f_{\varphi^{k+1}(1)}(x_0)$. So $y_1 = f_0 \circ f_1^{-1}(y_0)$is such that $f_{\varphi^k(0)}(x_0) \leq y_1 \leq f_{\varphi^k(1)}(x_0)$. By hypothesis, there exists a curve $\mathcal{C}_{\varphi^k(\alpha)}$ such that $(x_0, y_1) \in \mathcal{C}_{\varphi^k(\alpha)}$. So $y_1 = f_{\varphi^k(\alpha)}(x_0)$ and $y_0 = f_1 \circ f_0^{-1} \circ f_{\varphi^k(\alpha)}(x_0) = f_{\varphi^k(1)} \circ f_{\varphi^k(0)}^{-1} \circ f_{\varphi^k(\alpha)}(x_0)$. So any point in $\{(x,y) : f_{\varphi^{k+1}(0)}(x) \leq y \leq f_{\varphi^{k+1}(1)}(x)\}$ belongs to an indifference curve $\mathcal{C}_\beta$. A similar proof holds when k is negative.

Now we must extend this local property to the whole $\mathbb{R}^2$. Suppose that, for $k \geq 0$, $f_{\varphi^k(1)}(x) \geq -x+2+k$. Note that this is true for $k=0$—for which $f_{\varphi^k(1)}$ corresponds to f_1. $f_{\varphi^{k+1}(1)} = (f_1 \circ f_0^{-1}) \circ f_{\varphi^k}(1)$; But one can easily show that, for any $x \in \mathbb{R}$, $f_1 \circ f_0^{-1}(x) > x+1$; so, $f_{\varphi^{k+1}(1)}(x) \geq -x+2+k+1$.

Now, by induction, this must be true for any $k \geq 0$. So any point (x_0, y_0) in the $\{(x,y) : f_0(x) \leq y \leq -x+2+k\}$ is also in $\{f_0(x) \leq y \leq f_{\varphi^k(1)}(x)\}$. But we have seen in the previous paragraph that, then, there exists a curve $\mathcal{C}_\alpha$ containing (x_0, y_0). And $\lim_{k\to+\infty}\{f_0(x) \leq y \leq -x+2+k\} = \{f_0(x) \leq y\}$. So any point in the last set belongs to an indifference curve. A similar proof would show that any point in the set $\{y \leq f_0(x)\}$ belongs to an indifference curve.

So, any point of $\mathbb{R}^2$ belongs to a curve $\mathcal{C}_\alpha$. This is true in particular for any point on the line $y = x$. Hence $\mathcal{C}_\alpha$ is defined for any $\alpha \in \mathbb{R}$. The principle of construction guarantees that φ is defined over $\mathbb{R}$. Suppose now that α and β are real numbers such that $\varphi(\alpha) = \varphi(\beta)$. Then $f_{\varphi(\alpha)} = f_{\varphi(\beta)}$. But, by the previous paragraphs, there exist $k, k' \in \mathbb{N}$ and $\gamma, \delta \in [0,1]$ such that $\alpha = \varphi^k(\gamma)$ and $\beta = \varphi^{k'}(\delta)$. Then $f_{\varphi(\alpha)} = f_{\varphi^k(1)} \circ f^{-1}_{\varphi^k(0)} \circ f_{\varphi^k(\gamma)} = f_1 \circ f_0^{-1} \circ f_{\varphi^k(\gamma)}$ and $f_{\varphi(\beta)} = f_1 \circ f_0^{-1} \circ f_{\varphi^{k'}(\delta)}$. Since $f_1 \circ f_0^{-1}$ is one to one, $f_{\varphi^k(\gamma)} = f_{\varphi^{k'}(\delta)}$. Hence $\varphi^k(\gamma) = \varphi^{k'}(\delta)$ and so $\alpha = \beta$, which implies that φ is one to one.

It is already known that for any integer k, and for any $\alpha, \beta \in [0,1]$,

$$\alpha \leq \beta \Leftrightarrow f_{\varphi^k(\alpha)}(x) \leq f_{\varphi^k(\beta)}(x), \text{ for any } x \in \mathbb{R}$$

But by (8) and (9), $f_{\varphi^{k+1}(0)}(x) = f_{\varphi^k(1)}(x)$; so for any integers k, k',

$$\varphi^k(\alpha) \leq \varphi^{k'}(\beta) \Leftrightarrow f_{\varphi^k(\alpha)}(x) \leq f_{\varphi^{k'}(\beta)}(x) \text{ for any } x \in \mathbb{R}$$

So, for any $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta \Leftrightarrow f_\alpha(x) \leq f_\beta(x)$ for any $x \in \mathbb{R}$, and since $f_1 \circ f_0^{-1}$ is strictly increasing, $f_\alpha(x) \leq f_\beta(x) \Leftrightarrow f_1 \circ f_0^{-1} \circ f_\alpha(x) \leq f_1 \circ f_0^{-1} \circ f_\beta(x) \Leftrightarrow f_{\varphi(\alpha)}(x) \leq f_{\varphi(\beta)}(x)$. So φ also strictly increases.

Now, to complete the proof, suppose that $x, x', y_1, y_2, \alpha, \beta \in \mathbb{R}$ are such that $y_1 = f_\alpha(x) = f_\beta(x')$ and $y_2 = f_{\varphi(\alpha)}(x)$. Then $f_{\varphi(\alpha)}(x) = f_1 \circ f_0^{-1} f_\alpha(x) = f_1 \circ f_0^{-1} \circ f_\beta(x') = f_{\varphi(\beta)}(x')$. So $y_2 = f_{\varphi(\beta)}(x')$. ♦

Lemma 7 (Distance between $\mathbf{f_\alpha}$ and $\mathbf{f_\beta}$) *For any $\alpha, \beta \in [0,1]$, there exists a constant $m(\alpha,\beta)$ such that $m(\alpha,\beta) \leq f_\beta(x) - f_\alpha(x)$ for any $x \in \mathbb{R}$. Moreover, $\beta \geq \alpha \Leftrightarrow m(\alpha,\beta) \geq 0$.*

Proof of lemma 7: Let $m(\alpha,\beta) = \min_{x\in\mathbb{R}}\{f_\beta(x) - f_\alpha(x)\}$. If $\alpha = \beta$, then $m(\alpha,\beta) = 0$. Now, let us suppose that $\beta > \alpha$. By lemma 1, $\alpha \leq \beta \Leftrightarrow f_\alpha(x) \leq f_\beta(x)$ for any $x \in \mathbb{R}$. In Particular, this is true on any closed interval $[y,z]$; but then, $\min_{x\in[y,z]}\{f_\beta(x) - f_\alpha(x)\}$ is a strictly positive real number. So, if $m(\alpha,\beta) = 0$, then either $\lim_{x\to-\infty}(f_\alpha(x) - f_\beta(x)) = 0$ or $\lim_{x\to+\infty}(f_\alpha(x) - f_\beta(x)) = 0$. But both cases are impossible because $f_0(x)$ tends to $-3-2x$ (resp. $(-3-x)/2$) when x tends toward $-\infty$ (resp. $+\infty$), and $f_1(x)$ tends toward $-2x$ (resp. $-x/2$) when x tends toward $-\infty$ (resp. $+\infty$). Hence, if $\beta > \alpha$, then $m(\alpha,\beta) > 0$. Conversely, if $m(\alpha,\beta) > 0$, then $f_\beta(x) > f_\alpha(x)$, and so, by lemma 1), $\beta > \alpha$. ♦

In the following proof, it is mainly shown that independence as well as the Archimedean axiom. The former is shown by equation $\alpha \leq \beta \Leftrightarrow f_\alpha \leq f_\beta \Leftrightarrow \varphi(\alpha) \leq \varphi(\beta)$. The latter is shown by constructing standard sequences and showing that, if they are infinite, then the indifference curves involved can be arbitrarily far away from the origin of the axes.

Proof of lemma 3: In this proof, it will be shown successively that $\succsim$ is a weak order on Ω, that the first two components are solvable, that independence holds, as well as the Archimedean axiom.

First $\succsim$ is a well defined weak order on Ω because, by lemma 6, for any $(x, y) \in \mathbb{R}^2$, there exists $\alpha \in \mathbb{R}$ such that $(x, y) \in \mathcal{C}_\alpha$ and there exists $\beta \in \mathbb{R}$ such that $\varphi(\beta) = \alpha$, so that $(x, y) \in \mathcal{C}_{\varphi(\beta)}$.

For any $\alpha \in \mathbb{R}$, f_α is defined on $\mathbb{R}$ and is one to one. So for any $y \in \mathbb{R}$ (resp. $x \in \mathbb{R}$), there exists an $x \in \mathbb{R}$ (resp. $y \in \mathbb{R}$) such that $f_\alpha(x) = y$. This guarantees the solvability w.r.t. x (resp. y) in the plane $\{z = z_0\}$. φ being one to one, continuous and varying from $-\infty$ to $+\infty$, there exists $\beta \in \mathbb{R}$ such that $\varphi(\beta) = \alpha$; and since $U(x, y, z_1) = \varphi \circ U(x, y, z_0)$, the solvability w.r.t. x (resp. y) holds in the plane $\{z = z_1\}$. So the first two components are solvable.

Let $x_0, x_1, y_0, y_1 \in \mathbb{R}$ be such that $(x_0, y_0, z_0) \precsim (x_1, y_1, z_0)$. By lemma 6, there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha \leq \beta$, $(x_0, y_0) \in \mathcal{C}_\alpha$ and $(x_1, y_1) \in \mathcal{C}_\beta$. φ is strictly increasing and one to one so $\alpha \leq \beta \Leftrightarrow \varphi(\alpha) \leq \varphi(\beta) \Leftrightarrow U(x_0, y_0, z_0) = \alpha \leq U(x_1, y_1, z_0) = \beta \Leftrightarrow \varphi \circ U(x_0, y_0, z_0) = \varphi(\alpha) \leq \varphi \circ U(x_1, y_1, z_0) = \varphi(\beta) \Leftrightarrow (x_0, y_0, z_1) \precsim (x_1, y_1, z_1)$. Hence independence holds w.r.t. the third component.

Now let $x, y_0, y_1 \in \mathbb{R}$ be such that $(x, y_0, z_0) \precsim (x, y_1, z_0)$. There exist $\alpha, \beta \in \mathbb{R}$ such that $(x, y_0) \in \mathcal{C}_\alpha$ and $(x, y_1) \in \mathcal{C}_\beta$ with $\alpha \leq \beta$. By lemma 6, $\alpha \leq \beta \Leftrightarrow f_\alpha(x) \leq f_\beta(x)$ for any $x \in \mathbb{R}$. So $\alpha \leq \beta \Leftrightarrow y_0 \leq y_1$; thus for any $x' \in \mathbb{R}$, $(x', y_0, z_0) \precsim (x', y_1, z_0)$. By independence w.r.t. the third component, $(x, y_0, z_1) \precsim (x, y_1, z_1) \Leftrightarrow (x, y_0, z_0) \precsim (x, y_1, z_0) \Leftrightarrow [(x', y_0, z_0) \precsim (x', y_1, z_0)$ for any $x' \in \mathbb{R}] \Leftrightarrow [(x', y_0, z_1) \precsim (x', y_1, z_1)$ for any $x' \in \mathbb{R}]$. By symmetry between components x and y, $(x_0, y, z_1) \precsim (x_1, y, z_1) \Leftrightarrow (x_0, y, z_0) \precsim (x_1, y, z_0) \Leftrightarrow [(x_0, y', z_0 \precsim (x_1, y', z_0)$ for any $y' \in \mathbb{R}] \Leftrightarrow [(x_0, y', z_1) \precsim (x_1, y', z_1)$ for any $y' \in \mathbb{R}]$. Hence inside planes, independence holds w.r.t. the first two components.

Let $x, y_0, y_1 \in \mathbb{R}$ be such that $(x, y_0, z_0) \sim (x, y_1, z_1)$. Then there exists $\alpha \in \mathbb{R}$ such that $(x, y_1) \in \mathcal{C}_\alpha$ and $(x, y_0) \in \mathcal{C}_{\varphi(\alpha)}$. Thus, $y_1 = f_\alpha(x)$ while $y_0 = f_{\varphi(\alpha)}(x)$. But by lemma 6, for any $x' \in \mathbb{R}$, there exists $\beta \in \mathbb{R}$ such that $(x', y_1) \in \mathcal{C}_\beta$ and then $(x', y_0) \in \mathcal{C}\varphi(\beta)$. So, for any $x' \in \mathbb{R}$, $(x', y_0, z_0) \sim (x', y_1, z_1)$. Now suppose that $x, y_0, y_1 \in \mathbb{R}$ are such that $(x, y_0, z_0) \precsim (x, y_1, z_1)$. Then there exists y_2 such that $(x, y_0, z_0) \precsim (x, y_2, z_0) \sim (x, y_1, z_1)$. By the beginning of this paragraph and the previous one, for any $x' \in \mathbb{R}$, $(x', y_0, z_0) \precsim (x', y_2, z_0) \sim (x', y_1, z_1)$. By symmetry, $(x, y_0, z_1) \precsim (x, y_1, z_0) \Rightarrow (x', y_0, z_1) \precsim (x', y_1, z_0)$ for any $x' \in \mathbb{R}$. Hence independence holds w.r.t. the first component. And by symmetry between x and y, it also holds w.r.t. the second component.

As for the Archimedean axiom, consider a standard sequence w.r.t. the first component: $\{x_1^k : x_1^k \in \mathbb{R},\ k \in \mathbb{N},\ \text{Not}((x_1^0, x_2^0, z_0) \sim (x_1^0, x_2^1, z_0))$, and for all $k,\ k+1 \in \mathbb{N},\ (x_1^k, x_2^1, z_0) \sim (x_1^{k+1}, x_2^0, z_0)\}$. Note that by the solvability w.r.t. the second component, any standard sequence w.r.t. the first component can be transformed into a sequence like the one above. In the sequel we suppose that $(x_1^0, x_2^0, z_0) \prec (x_1^0, x_2^1, z_0)$; a similar proof would hold for the converse.

There exist α, β such that $x_2^0 = f_\alpha(x_1^0)$ and $x_2^1 = f_\beta(x_1^0)$; hence $x_2^1 = f_\beta \circ f_\alpha^{-1}(x_2^0)$. Similarly there exists γ_1 such that $x_2^1 = f_{\gamma_1}(x_1^1)$. But since $(x_1^0, x_2^1, z_0) \sim (x_1^1, x_2^0, z_0)$, the following equation is true:

$$f_{\gamma_1} = f_\beta \circ f_\alpha^{-1} \circ f_\beta$$

There exists γ_2 such that $x_2^1 = f_{\gamma_2}(x_1^2)$. But then since $(x_1^1, x_2^1, z_0) \sim (x_1^2, x_2^0, z_0)$, the following equation is true:

$$\begin{aligned} f_{\gamma_2} &= f_{\gamma_1} \circ f_\beta^{-1} \circ f_{\gamma_1} \\ f_{\gamma_2} &= (f_\beta \circ f_\alpha^{-1})^2 \circ f_\beta \end{aligned}$$

By induction, when examining x_1^k, the following equation would be found:

$$f_{\gamma_k} = (f_\beta \circ f_\alpha^{-1})^k \circ f_\beta$$

Now α (resp. β) can be written as $\varphi^n(\nu)$ (resp. $\varphi^m(\mu)$), with μ and ν in $[0,1]$. Then $f_{\gamma_k} = (f_\mu \circ (f_0^{-1} \circ f_1)^m \circ (f_0^{-1} \circ f_1)^{-n} \circ f_\nu^{-1})^k \circ f_\beta$ or, equivalently, $f_{\gamma_k} = (f_\mu \circ (f_0^{-1} \circ f_1)^{m-n} \circ f_\nu^{-1})^k \circ f_\mu \circ (f_0^{-1} \circ f_1)^m$.

If $m = n$, then $f_{\gamma_k} = [(f_\mu \circ f_\nu^{-1})]^k \circ f_\mu \circ (f_0^{-1} \circ f_1)^m$ and $\mu > \nu$. So, by lemma 7, $f_\mu \geq f_\nu + m(\nu, \mu)$, and so $f_{\gamma_k} \geq (\mathrm{Id} + m(\nu, \mu))^k \circ f_\mu \circ (f_0^{-1} \circ f_1)^m$. So, if k tends toward $+\infty$, then the standard sequence cannot be bounded.

If $m = n+1$, then $f_{\gamma_k} = [(f_\mu \circ f_0^{-1}) \circ (f_1 \circ f_\nu^{-1})]^k \circ f_\mu \circ (f_0^{-1} \circ f_1)^m$. By lemma 7, $f_\mu \geq f_0 + m(0, \mu)$ and $f_1 \geq f_\nu + m(\nu, 1)$. Hence $f_{\gamma_k} \geq (\mathrm{Id} + m(0, \mu) + m(\nu, 1))^k \circ f_\mu \circ (f_0^{-1} \circ f_1)^m$ where Id stands for the identity function. Since one cannot have $\mu = 0$ and $\nu = 1$ at the same time — otherwise, $\alpha = \beta$ — the standard sequence cannot be bounded when k tends toward $+\infty$.

Now, suppose that $m > n+1$. μ and ν belong to $[0,1]$; so, for any $x \in \mathbb{R}$, $f_0(x) \leq f_\mu(x), f_\nu(x) \leq f_1(x)$. Moreover $f_0^{-1} \circ f_1$ strictly increases. Hence, $f_{\gamma_k} > (f_0 \circ (f_0^{-1} \circ f_1)^{m-n} \circ f_1^{-1})^k \circ f_0 \circ (f_0^{-1} \circ f_1)^m = f_0 \circ (f_0^{-1} \circ f_1)^{k(m-n-1)+m}$. But, for any x, $f_0^{-1} \circ f_1^{-1}(x) > x - 3$ and $f_0(x) > -x$. So $f_{\gamma_k}(x) > g(x) = -x + 3[k(m-n-1)+m]$. The intersection of g with the line $y = x$ gives $x = \frac{3[k(m-n-1)+m]}{2}$. And f_{γ_k} is above g, so $\gamma_k > \frac{3[k(m-n-1)+m]}{2}$. Hence, $lim_{k \to +\infty} \gamma_k = +\infty$. And so the standard sequence cannot be bounded, which achieves the proof that the Archimedean axiom holds. ♦

References

[Debreu 60] Debreu, G. (1960): "Topological Methods in Cardinal Utility Theory," in *Mathematical Methods in the Social Sciences*, ed. by K. J. Arrow, S. Karlin, and P. Suppes, pp. 16–26. Stanford University Press, Stanford, CA.

[Doignon and Falmagne 74] Doignon, J. P., and J. C. Falmagne (1974): "Difference Measurement and Simple Scalability with Restricted Solvability," *Journal of Mathematical Psychology*, 11, 473–499.

[Fishburn 66] FISHBURN, P. C. (1966): "A Note on Recent Developments in Additive Utility Theories For Multiple Factor Situations," *Operations Research*, 14, 1143–1148.

[Fishburn 70] ——— (1970): *Utility Theory for Decision Making.* Wiley, NewYork.

[Fishburn 71] ——— (1971): "Additive Representations of Real Valued Functions on Subsets of Product Sets," *Journal of Mathematical Psychology*, 8, 382–388.

[Fuhrken and Richter 91] FUHRKEN, G., AND M. K. RICHTER (1991): "Additive Utility," *Economic Theory*, 1, 83–105.

[Jaffray 74a] JAFFRAY, J.-Y. (1974a): "Existence, Propriétés de Continuité, Additivité de Fonctions d'Utilité sur un Espace Partiellement ou Totalement Ordonné," Thèse d'Etat, Université Paris VI.

[Jaffray 74b] ——— (1974b): "On the Extension of Additive Utilities to Infinite Sets," *Journal of Mathematical Psychology*, 11(4), 431–452.

[Krantz 64] KRANTZ, D. H. (1964): "Conjoint Measurement: The Luce-Tukey Axiomatization and Some Extensions," *Journal of Mathematical Psychology*, 1, 248–277.

[Krantz et al. 71] KRANTZ, D. H., R. D. LUCE, P. SUPPES, AND A. TVERSKY (1971): *Foundations of measurement (Additive and Polynomial Representations)*, vol. 1. Academic Press, New York.

[Luce 66] LUCE, R. D. (1966): "Two extensions of Conjoint Measurement," *Journal of Mathematical Psychology*, 3, 348–370.

[Luce and Tukey 64] LUCE, R. D., AND J. W. TUKEY (1964): "Simultaneous Conjoint Measurement : A New Type of Fondamental Measurement," *Journal of Mathematical Psychology*, 1, 1–27.

[Scott and Suppes 58] SCOTT, D., AND P. SUPPES (1958): "Foundational Aspects of Theories of Measurement," *Journal of Symbolic Logic*, 23(2), 113–128.

[Vind 91] VIND, K. (1991): "Independent Preferences," *Journal of Mathematical Economics*, 20, 119–135.

[Wakker 88] WAKKER, P. P. (1988): "The Algebraic Versus the Topological Approach to Additive Representations," *Journal of Mathematical Psychology*, 32, 421–435.

[Wakker 89] ——— (1989): *Additive Representations of Preferences, A New Foundation of Decision Analysis.* Kluwer Academic Publishers, Dordrecht.

[Wakker 91a] ——— (1991a): "Additive Representation for Equally Spaced Structures," *Journal of Mathematical Psychology*, 35, 260–266, theoretical note.

[Wakker 91b] ——— (1991b): "Additive Representations on Rank-Ordered Sets. I. The Algebraic Approach," *Journal of Mathematical Psychology*, 35, 501–531.

[Wakker 93] ——— (1993): "Additive Representations on Rank-Ordered Sets. II. The Topological Approach," *Journal of Mathematical Economics*, 22, 1–26.

[Wakker 94] ——— (1994): "Mimeograph of the Nordic Research Course in Decision under Uncertainty," Copenhagen.