Additive Utilities When Some Components Are Solvable And Others Are Not

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Abstract

This paper presents positive and negative results concerning the existence of additive utilities on weakly ordered Cartesian products when some components are solvable and others are not. The classical theorems involving solvability can be derived when only 2 or 3 components are solvable, depending on whether the second order cancelation axiom or the independence axiom holds. Counterexamples show that our results cannot be significantly strengthened.

1 Introduction

Because additive utilities are computationally very attractive, the problem of their existence on weakly ordered Cartesian products has received many contributions. In the literature, theories for infinite sets follow either topological or algebraic approaches (see e.g. Fishburn (1970) and Fuhrken & Richter (1991) for the algebraic approach, Debreu (1960) and Wakker (1993) for the topological approach; and Wakker (1988) and Krantz, Luce, Suppes & Tversky (1990) for a comparison between both approaches). But in both cases, the structural conditions assumed are only sufficient and furthermore unnecessarily strong (see Jaffray (1974*a*) and Jaffray (1974*b*) for necessary and sufficient—but hardly testable—conditions). More precisely, the classical theorems assume the connectedness of the topological spaces in the former approach (see Debreu (1960), Wakker (1989) and Wakker (1994)), and solvability (Fishburn (1970)) or restricted solvability (see Luce (1966) and Krantz, Luce, Suppes & Tversky (1971)) with respect to every component of the Cartesian product in the latter approach . In this paper, we address the problem of the existence of additive utilities in cases where at least two components, but not necessarily all components, are solvable. This kind of problem may typically arise when some components (but not all) are discrete. Such structures can be found in medical decision making, for instance in problems involving life duration and money—which can be thought of as solvable—and states of health—which are generally not solvable.

Throughout the paper we study n-dimensional Cartesian products, with $n \geq 3$. In section 2, our main theorem shows that solvability need not be imposed on all components. In fact, it is sufficient to impose it on two components and derive additive representability there. It is shown that additive representability then extends to the other components—solvable or not. Note that the solvable structure includes cases in which topological connectedness holds w.r.t. 2 components and infinite standard sequences exist w.r.t. these components.

In section 3 we show that the results given in the previous section cannot be improved. More precisely, in order to derive an additive utility from the combination of the independence axiom and the Thomsen condition, it is shown that at least two solvable components are required; and with only the independence axiom, at least three components must be solvable. For both cases we present examples in which fewer solvable components lead to the non-existence of additive utilities.

In order to make this paper more comprehensible, all proofs are given in an appendix.

2 Representation Theorems

In this section we consider a Cartesian product $X = \prod_{i=1}^{n} X_i$. Given a binary relation $\succsim$ over the Cartesian product X, we standardly introduce the indifference relation $x \sim y \Leftrightarrow [x \succsim y$ and $y \succsim x]$, the strict preference relation $x \succ y \Leftrightarrow [x \succsim y$ and $\mathrm{Not}(y \succsim x)]$, and $x \precsim y \Leftrightarrow y \succsim x$. Without loss of generality, the solvable components are always the first components.

We assume the following axioms:

Axiom 1 (ordering) *$\succsim$ is a weak order on X, i.e. $\succsim$ is complete (for any x,$y \in X$, $x \succsim y$ or $y \succsim x$) and transitive (for any x,y,$z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$).*

Axiom 2 (independence) *For any $i \in \{1, 2, \ldots, n\}$ and any x,$y \in X$,*

$$\begin{array}{c}(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \succsim (y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n) \\ \Downarrow \\ (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \succsim (y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n).\end{array}$$

It is obvious that axioms 1 and 2 are necessary for the existence of an additive utility on X. The independence axiom induces a natural ordering on the Cartesian product generated by any subset of components in the following manner. For any set N included in $\{1, 2, \ldots, n\}$ one can define the weak order $\succsim_N$ on $\prod_{i \in N} X_i$ as follows: for $a, b \in \prod_{i \in N} X_i$, $a \succsim_N b$ iff for some $p \in \prod_{i \notin N} X_i$, $(a, p) \succsim (b, p)$.

Axiom 3 (solvability w.r.t. the ith component) *$[x \in X, y_j \in X_j$ for all $j \neq i] \Rightarrow [$there exists $z_i \in X_i$ such that $x \sim (y_1, \ldots, y_{i-1}, z_i, y_{i+1}, \ldots, y_n)]$.*

Theorem 1 *Assume that $(X, \succsim)$ is a weak order and that $\succsim$ satisfies the independence axiom, as well as solvability w.r.t. the first two components, and that there exists an additive utility representing $\succsim_{12}$. Then there exists an additive utility representing $\succsim$, i.e. there exist real valued functions u_i on X_i, $i = 1, \ldots, n$, such that*

$$\text{for any } x, y \in X, x \succsim y \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i). \tag{1}$$

Moreover, this utility is an interval scale, i.e. if $v_1, \ldots, v_n$ also satisfy (1) then there exist some constants $a > 0$ and $b_1, \ldots, b_n$ such that, for each $i \in \{1, \ldots, n\}$, $v_i = a \cdot u_i + b_i$.

Sketch of proof: In the solvable components space, it is a classical result that an additive utility exists. The principle is to extend this utility to spaces of greater dimension by adding the nonsolvable components one by one. When adding the ith component, select a reference point and assign any value for its utility; for any other point x_i, the idea is that the trade-off between x_i and the reference point can be compensated by a trade-off in the second component, leaving the first component unchanged. This gives the necessary value of $u_i(x_i)$; by independence, it is obvious that this value is independent of the first component. Now, a variation in the value of the second, third, ..., $i-1$th component can be compensated by a trade-off in the first component, which means that $u_i(x_i)$ is independent of the other components. Therefore, it is also sufficient for the existence of an additive utility. The uniqueness property is due to the uniqueness in the solvable components space. ♦

Theorem 1 states that the Cartesian product generated by the two solvable components is sufficiently rich to extend its additive representability to any Cartesian product including it. This suggests the principle of construction of such a utility: First set the nonsolvable components to some arbitrary values, and consider the subset Y of X thus generated. Construct the additive utility in Y—this is a classical construction. Then extend this construction by restoring—one at a time—the initial domains of the nonsolvable components.

Theorem 1 extends the classical theorems by weakening the strong assumption that every component be solvable. It is particularly useful for deriving additive utilities when some components of the Cartesian product belong to finite sets that cannot be extended to intervals in a meaningful way. In order to give more practical results, we give a few more axioms and derive some corollaries of theorem 1.

Axiom 4 (Thomsen condition w.r.t. the first 2 components) *For every* $x_1, y_1, z_1 \in X_1$, $x_2, y_2, z_2 \in X_2$, $[(x_1, z_2) \sim_{12} (z_1, y_2)$ *and* $(z_1, x_2) \sim_{12} (y_1, z_2)] \Rightarrow (x_1, x_2) \sim_{12} (y_1, y_2)$.

We now introduce the notion of a standard sequence, which we use in our following Archimedean axiom :

Definition 1 (standard sequence) *For any set* N *of consecutive integers (positive or negative, finite or infinite), a set* $\{x_1^k : x_1^k \in X_1, k \in N\}$ *is a standard sequence w.r.t. the first component iff Not*$((x_1^0, x_2^0, \ldots, x_n^0) \sim (x_1^0, x_2^1, \ldots, x_n^1))$ *and for all* $k, k+1 \in N$, $(x_1^k, x_2^0, \ldots, x_n^0) \sim (x_1^{k+1}, x_2^1, \ldots, x_n^1)$.

Axiom 5 (Archimedean axiom) *Any strictly bounded standard sequence w.r.t. the first component is finite.*

We now have all the material to deduce two practical corollaries.

Corollary 1 *Suppose* $(X, \succsim)$ *satisfies the weak ordering (axiom 1), independence (axiom 2), solvability w.r.t. 2 components (axiom 3), Thomsen condition w.r.t. the solvable components (axiom 4) and the Archimedean axiom (axiom 5). Then there exists an additive utility representing* $\succsim$*. Moreover, this utility is an interval scale.*

From an operational point of view, axiom 4 is more difficult to test than axiom 2. In the classical theorems for Cartesian products of dimensions greater than or equal to 3, this axiom does not appear because it is implied by the independence axiom and solvability. We will see next that the same result can be obtained when solvability holds w.r.t. 3 components.

Corollary 2 *Suppose $(X, \succsim)$ satisfies the weak ordering (axiom 1), independence (axiom 2), solvability w.r.t. three components (axiom 3) and the Archimedean axiom (axiom 5). Then there exists an additive utility representing for $\succsim$. Moreover, this utility is an interval scale.*

3 Negative Results : Counterexamples

In this section we present counterexamples showing that the results mentioned above cannot be significantly strengthened. More precisely, we first show that corollary 1 would not be true any more without solvability w.r.t. two components. Secondly, we show that one cannot drop the Thomsen condition if only two components are solvable.

3.1 Two Solvable Components Are Required In Corollary 1

In this subsection we give an example in which only one component is solvable, which satisfies axioms 1, 2, 4, 5, and which yet admits no additive utility.

Consider a two-dimensional Cartesian product $X = \mathbb{R} \times \{0, 2, 4, 6\}$ and a weak order $\succsim$ on X represented by the utility function $U : X \to \mathbb{R}$ defined as:

$$U(x,y) = \begin{cases} f(x) & \text{if } y = 0 \quad \text{where } f(x) = x \\ g(x) & \text{if } y = 2 \quad \text{where } g(x) = x + 2 \\ h(x) & \text{if } y = 4 \quad \text{where } h(x) = x + 4 \\ k(x) & \text{if } y = 6 \quad \text{where } k(x) = 5.5 + 2p + 0.5(z+1)^2 \\ & \qquad\qquad\quad \text{and } x = z + 2p,\ z \in [-1, 1[. \end{cases}$$

Since we defined $\succsim$ by one of its utility functions, $\succsim$ is a weak order; so axiom 1 holds. Independence is guaranteed by the fact that functions f, g, h and k are all strictly increasing—which means that $U(x,y) \geq U(x',y) \Leftrightarrow x \geq x' \Leftrightarrow U(x,y') \geq U(x',y')$—and that the graphs of the functions never intersect each other—which means that $U(x,y) \geq U(x,y') \Leftrightarrow y \geq y' \Leftrightarrow U(x',y) \geq U(x',y')$. The Archimedean axiom is satisfied for any standard sequence $\{x_1^k : x_1^k \in X_1,\ k \in N,\ \text{Not}((x_1^0, x_2^0) \sim (x_1^0, x_2^1))$ and for all $k, k+1 \in N$, $(x_1^k, x_2^0) \sim (x_1^{k+1}, x_2^1)\}$ for which $x_2^0, x_2^1 \in \{0, 2, 4\}$ because, on $\mathbb{R} \times \{0, 2, 4\}$, U is additive. Note that for any x, $x + 6 \leq k(x) \leq x + 6.5$, and so standard sequences in which x_2^0 or x_2^1 equal 6 have no accumulation point; hence we can conclude that the Archimedean axiom is satisfied.

The most difficult part is to show that axiom 4 (the Thomsen condition) holds. This is obviously satisfied on $\mathbb{R} \times \{0, 2, 4\}$ since the utility on this set is $U(x,y) = x + y$. Now, the Thomsen condition implies that $[(x, y') \sim (x', y)$ and $(x'', y) \sim (x, y'')] \Rightarrow (x', y'') \sim (x'', y')$. Replace y by 2, y' by 4 and y'' by 6. Then we obtain—see figure 1—$[A = (x, 4) \sim B = (x', 2)$ and $D = (x'', 2) \sim$

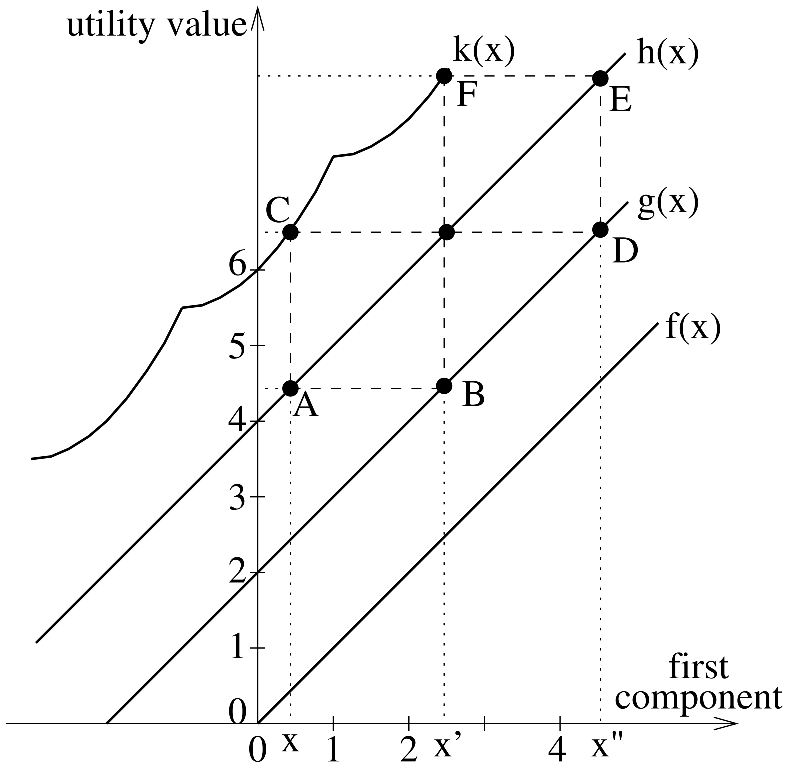

Figure 1: graph of the utility function U.

$C = (x, 6)] \Rightarrow [F = (x', 6) \sim E = (x'', 4)]$. Let us translate these indifferences in terms of functions: $A \sim B$ means that $h(x) = g(x')$, and so, since g is one to one, $x' = g^{-1} \circ h(x)$; similarly, $D \sim C$ is equivalent to $x'' = g^{-1} \circ k(x)$ and $F \sim E$ means that $h(x') = k(x'')$. Combining these relations, we obtain that, for any $x \in \mathbb{R}$, $k \circ g^{-1} \circ h(x) = h \circ g^{-1} \circ k(x)$. Replacing $h(x)$ and $g(x)$ by their values, one would obtain: $k(x+2) = k(x)+2$. Had we replaced y and y' by other values, we would have obtained the following equalities:

$$k \circ f^{-1} \circ g(x) = g \circ f^{-1} \circ k(x) \text{ and } k \circ f^{-1} \circ h(x) = h \circ f^{-1} \circ k(x), \text{ for any } x \in \mathbb{R}$$

which are equivalent respectively to

$$k(x+2) = k(x) + 2 \text{ and } k(x+4) = k(x) + 4, \text{ for any } x \in \mathbb{R}.$$

From our definition of the function k, it is obvious that all these equalities hold, and so does the Thomsen condition.

So axioms 1, 2, 4 and 5 hold, the first component is solvable, and yet there exists no additive utility because $(0, 6) \sim (2, 4)$, $(2, 0) \sim (0, 2)$, $(.5, 2) \sim (2.5, 0)$ and $(2.5, 4) \succ (.5, 6)$.

3.2 Three Solvable Are Components Required In Corollary 2

For three or more component spaces, under solvability assumptions, the Thomsen condition is implied by the independence axiom. The question that arises naturally is the following one: is this property still true with our weaker assumptions? To put it another way, is the Thomsen condition implied by the independence just because there exists a third component or does this component need to be solvable? The question is important because if the first alternative is right, then one does not need the Thomsen condition to derive the additive representability of $\succsim_{12}$ in corollary 1.

3.2.1 The Two-Component Case

The first case that arises is the one in which $\succsim$ is a preference relation in a two-dimensional Cartesian product. Then, solvability is supposed to hold w.r.t. all components. This case has been well studied in the literature. Examples are known for which independence holds but not the Thomsen condition, thus forbidding additive representability. For instance, let $\succsim$ on $\mathbb{R}^2$ be represented by the function $U : \mathbb{R}^2 \to \mathbb{R}$ such that $U(x, y) = x + y + \min\{x, y\}$. $\succsim$ satisfies independence but not the Thomsen condition since $(.2, .2) \sim (.5, .05)$, $(.7, .05) \sim (.2, .4)$ and $(.5, .4) \succ (.7, .2)$.

3.2.2 The Three-Component Case

In this subsection, our aim is to generalize the previous subsection to the case of 3-component Cartesian products. More precisely, we prove the following theorem:

Theorem 2 *On* 3*-component Cartesian products, the Thomsen condition for* $\succsim_{12}$ *is not implied by independence w.r.t. all the components and solvability w.r.t. only* 2 *components.*

The proof consists in devising a general method for constructing on a Cartesian product $\Omega = \mathbb{R} \times \mathbb{R} \times \{z_0, z_1\}$, where z_0 and z_1 are arbitrary constants, a preference ordering $\succsim$ satisfying the assumptions and exhibiting a particular one which does not admit an additive utility. We suppose in the sequel that $z_0 \prec_3 z_1$. The approach we follow to define $\succsim$ is to construct one of its utility functions U on Ω by defining the indifference classes of $\succsim$, or, more precisely, the indifference curves in the planes $\{z = z_0\}$ and $\{z = z_1\}$. Of course independence imposes some relations between those planes. We first explain these constraints and then construct an example.

Suppose that U exists, satisfying all the conditions described above. By independence,

$$\text{for any } x, x', y, y' \in \mathbb{R},\ (x, y, z_0) \sim (x', y', z_0) \Leftrightarrow (x, y, z_1) \sim (x', y', z_1).$$

This means that the indifference curves are the same in the planes $\{z = z_0\}$ and $\{z = z_1\}$. Of course, even if their shape is the same in both planes, their values differ—otherwise one would have $U(x, y, z_0) = U(x, y, z_1)$, which, by independence, would be true for any couple (x, y), and so z_0 would be indifferent to z_1. This suggests that we construct two functions $V : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$, describing the indifference curves in the plane $\{z = z_0\}$ and the transformation of the values of the indifference curves from the plane $\{z = z_0\}$ to the plane $\{z = z_1\}$ respectively. In mathematical terms, $U(x, y, z_0) = V(x, y)$ and $U(x, y, z_1) = \varphi \circ V(x, y)$. The construction of $\succsim$ on Ω can then be reduced to projecting the curves obtained by V onto the planes $\{z = z_0\}$ and $\{z = z_1\}$ and using φ to change the values associated with the curves of the plane $\{z = z_1\}$.

Ensuring that the independence axiom is not violated inside the planes is not difficult: it is sufficient that $V(x, y)$ strictly increases in x and y—i.e. $V(x, y) \geq V(x', y) \Leftrightarrow x \geq x'$ and $V(x, y) \geq V(x, y') \Leftrightarrow y \geq y'$—and that φ is strictly increasing. As a matter of fact, suppose these conditions hold. Then

$$\text{for any } x, x', y, y' \in \mathbb{R},\ (x, y, z_0) \succsim (x', y, z_0) \Leftrightarrow x \geq x' \Leftrightarrow (x, y', z_0) \succsim (x', y', z_0).$$

The same argument would apply if the roles of x and y had been exchanged. Since φ is strictly increasing, $V(x, y) \geq V(x', y') \Leftrightarrow \varphi \circ V(x, y) \geq \varphi \circ V(x', y')$, so independence holds in both planes.

Now we must examine the constraints imposed by the independence axiom when both elements do not belong to the same plane, i.e. constraints imposed by relations similar to $(x, y, z_0) \succsim (x', y, z_1)$. We call these constraints "inter-plane independence constraints". They are explained in figure 2. Since the indifference curves are the same in both planes, we found it convenient to superpose them in the same drawing. To differentiate them, we drew the indifference curves of the plane $\{z = z_0\}$ with bold lines, unlike that of $\{z = z_1\}$. V is strictly increasing in x and y, so "$V(x, y) =$ constant" are decreasing curves—provided of course that they are continuous, which we suppose to be true—and hence can be written equivalently as "$y =$ function(x)", where the function is strictly decreasing. In figure 2 we assigned to each curve its function.

Suppose that $A = (x', y, z_0) \sim B = (x', y'', z_1)$. Then, by independence, $C = (x'', y, z_0) \sim D = (x'', y'', z_1)$. Suppose now that $F = (x, y', z_0) \sim A = (x', y, z_0) \sim G = (x'', y', z_1)$. Then, still by independence, $E = (x, y'', z_0) \sim D = (x'', y'', z_1)$. Hence we must also have $E = (x, y'', z_0) \sim C = (x'', y, z_0)$. Now, let us express this relation in terms of functions. Given an arbitrary point $C = (x'', y)$ and some known functions f and h, we define $\begin{cases} x'' \to y' = h(x'') \to x = f^{-1}(y') \\ y \to x' = f^{-1}(y) \to y'' = h(x'). \end{cases}$
This determines two points on the curve $y = g(x)$ because $y'' = g(x)$ and $x'' = g^{-1}(y)$, or, to put it another way, $h \circ f^{-1} \circ g(x'') = g \circ f^{-1} \circ h(x'')$. Hence independence inter planes implies that for any x, $h \circ f^{-1} \circ g(x) = g \circ f^{-1} \circ h(x)$. This means that when constructing the example, if f and h are already known functions, then any function "inside" those two—i.e. any function whose

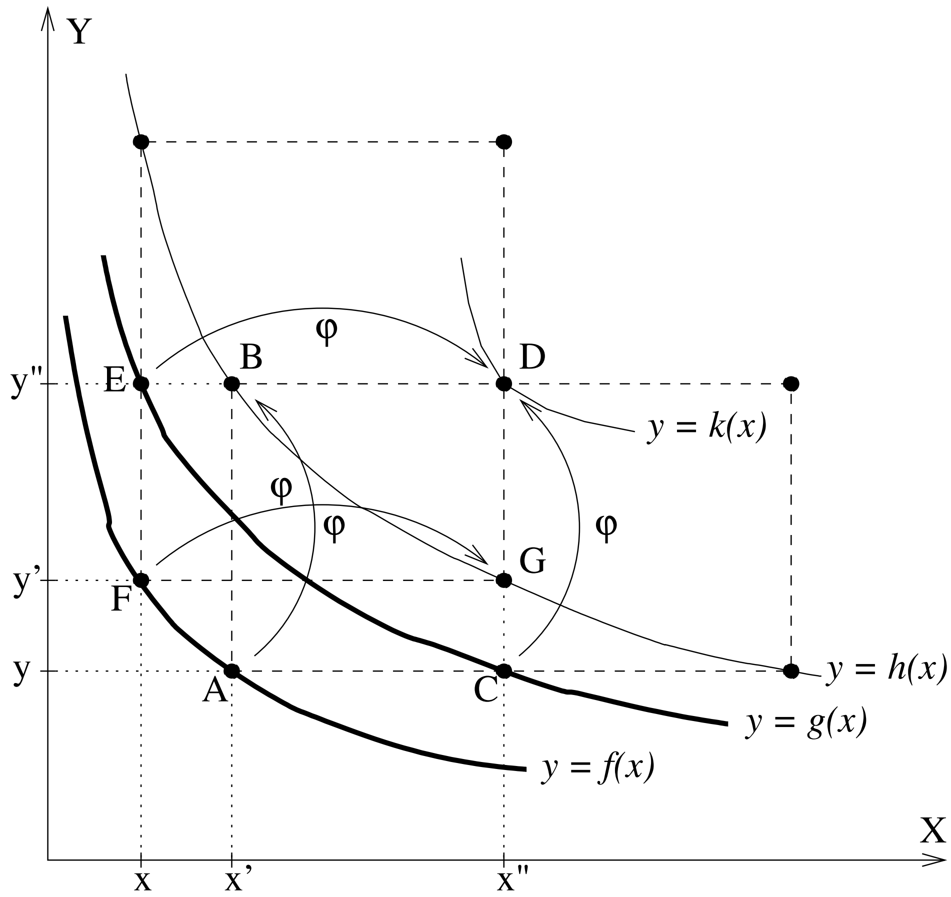

Figure 2: the inter-plane constraints.

indifference curve is between the indifference curves associated with f and h—is allowed to be chosen with a certain degree of freedom only on a small interval which corresponds to the interval $[CE]$. As for the degree of freedom, any curve will fit as long as independence holds inside the planes. Moreover, certain curves outside f and h—like the one at point D—are determined by the curves inside f and h. For instance, point D is determined by A, B, C and E, F, G. In fact this is the case for any outside curve because once the inside ones are chosen, locally near f and h, the outside curves—like k—are imposed. But then g and k can play the role taken previously by f and h, which impose another function deduced from h—which is "inside" g and k—and so on. By this process, we construct an infinite standard sequence, which, by the Archimedean axiom, implies that the whole space can be reached.

Now we have all the material needed to construct functions V and φ. For simplicity, our example uses the line $y = x$ as a symmetry axis. This is convenient because it implies some symmetry between the first two components. V describes indifference curves in $\mathbb{R} \times \mathbb{R}$; we call the latter $\mathcal{C}_\alpha$, using the following rule to evaluate α: the point of coordinates (α, α) belongs to the curve $\mathcal{C}_\alpha$. Moreover we impose on V to satisfy $V(x,x) = x$ for any $x \in \mathbb{R}$. Hence $\mathcal{C}_\alpha = \{(x,y) \in \mathbb{R}^2 : V(x,y) = \alpha\}$. To the curve $\mathcal{C}_\alpha$ we associate the function f_α, i.e. $\mathcal{C}_\alpha = \{(x,y) \in \mathbb{R}^2 : y = f_\alpha(x)\}$. Of course, there is a one to one mapping between f_α and $\mathcal{C}_\alpha$.

To start the construction, we have chosen as functions f and h of figure 2 the functions f_0 and f_1. This means that $f_{\varphi(0)} = f_1$, or $\varphi(0) = 1$, or, more simply, that $(0,0,z_1) \sim (1,1,z_0)$. These functions can be taken arbitrarily—provided of course that they are strictly decreasing and do not intersect. Here we have chosen:

$$f_0(x) = \begin{cases} -2x & \text{if } x \le 0 \\ -\frac{x}{2} & \text{if } x \ge 0 \end{cases} \tag{2}$$

$$f_1(x) = \begin{cases} -2x + 3 & \text{if } x \le 1 \\ \frac{3-x}{2} & \text{if } x \ge 1. \end{cases} \tag{3}$$

Note that f_0 and f_1 are continuous, strictly decreasing, and hence one to one, vary from $+\infty$ to $-\infty$ and the line $y = x$ is a symmetry axis.[1]

Now we must construct the inside curves. For this purpose we use a two-step process. First we choose the "arbitrary" part of the function to correspond with the utility given in the previous subsection:

$$\text{for any } \alpha \in]0,1[,\ f_\alpha(x) = \begin{cases} -2x + 3\alpha & \text{if } x \in [\frac{\alpha-1}{2}, \alpha] \\ \frac{3\alpha - x}{2} & \text{if } x \in [\alpha, 2\alpha+1]. \end{cases} \tag{4}$$

Then the inter-plane independence imposes the rest of the construction as seen in figure 2. This results in the following equation:

$$\text{for any } x \in \mathbb{R},\ f_\alpha \circ f_0^{-1} \circ f_1(x) = f_1 \circ f_0^{-1} \circ f_\alpha(x). \tag{5}$$

Note that Eq. (5) is satisfied for $\alpha = 0$ and $\alpha = 1$, and that (4) is not in conflict with (5) because f_α increases on $[\frac{\alpha-1}{2}, 2\alpha+1]$ and $f_0^{-1} \circ f_1(2\alpha+1) = \frac{\alpha-1}{2}$. We present in figure 3 a summary of Eq. (5): if A belongs to $\mathcal{C}_\alpha$, then B must also belong to $\mathcal{C}_\alpha$, and conversely.

Curves $\mathcal{C}_\alpha$ defined by (4) and (5) satisfy all the conditions imposed previously. In particular, Eq. (5) extends the definition of $\mathcal{C}_\alpha$ over $\mathbb{R}$. The properties of these curves are described in the following lemma.

Lemma 1 *Consider an arbitrary α in $]0,1[$, and suppose that f_α is defined by Eq. (4) and Eq. (5). Then f_α is well defined on $\mathbb{R}$, is continuous, strictly decreases, $f_\alpha(\mathbb{R}) = \mathbb{R}$, and the line $y = x$ is a symmetry axis. Moreover*

$$\text{for any } \alpha, \beta \in [0,1],\ \alpha \le \beta \Leftrightarrow f_\alpha(x) \le f_\beta(x), \text{ for any } x \in \mathbb{R}.$$

[1] The construction given below is very long, although its principle is quite simple. The only way to shorten it significantly seems to me to select an example in which all indifference curves, $\mathcal{C}_\alpha$, are translations of $\mathcal{C}_0$ w.r.t. the line $y = x$. However such an example seems very difficult to elicit; According to some of my experiments, such a curve $\mathcal{C}_0$, if tending asymptotically toward f_0 of equation 2, should be "close" to the hyperbole $y = f_0(x) = (-9 - 5x + 3\sqrt{9 + 2x + x^2})/4$.

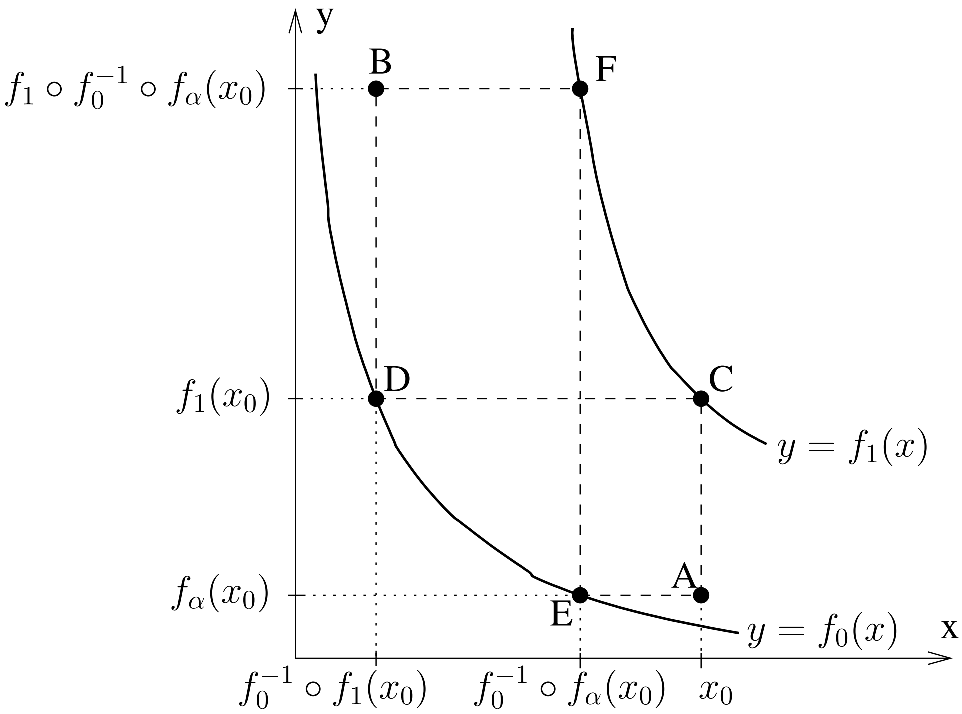

Figure 3: construction of the inside curves.

Now that the construction of the inside curves is completed, there remains the construction of the outside curves. For this purpose we use a two-step process again. First we describe how to construct them "locally" above f_1; this is Eq. (6). Second, we explain in (8) and (9) how this construction can be extended to the whole space.

Let us come back to figure 2. In this one, point D of the plane $\{z = z_1\}$ is indifferent to points C and E of the plane $\{z = z_0\}$. This means that $U(x'', y'', z_1) = U(x'', y, z_0)$, or, in terms of V and φ, $V(x'', y'') = \varphi \circ V(x'', y)$. But we also know that, by inter-plane independence, $y'' = k(x'') = h \circ f^{-1} \circ g(x'')$. So we can deduce the following construction for our example:

$$\text{for any } x \in \mathbb{R},\ f_{\varphi(\alpha)}(x) = f_\alpha \circ f_0^{-1} \circ f_1(x) = f_1 \circ f_0^{-1} \circ f_\alpha(x), \tag{6}$$

which corresponds in the following figure to: "if A and B belong to $\mathcal{C}_\alpha$, then E and G belong to $\mathcal{C}_{\varphi(\alpha)}$".

Properties of these curves are described in the following lemma:

Lemma 2 *Consider an arbitrary α in $[0,1]$, and suppose that $f_{\varphi(\alpha)}$ is defined by Eq. (6). Then $f_{\varphi(\alpha)}$ is well defined on $\mathbb{R}$, is continuous, strictly decreases, $f_{\varphi(\alpha)}(\mathbb{R}) = \mathbb{R}$, the line $y = x$ is a symmetry axis, and, for any $\beta \in [0,1]$, $\alpha \leq \beta \Leftrightarrow f_{\varphi(\alpha)}(x) \leq f_{\varphi(\beta)}(x)$ for any $x \in \mathbb{R}$. Moreover*

$$\text{for any } x \in \mathbb{R},\ f_{\varphi(\alpha)}(x) \circ f_{\varphi(0)}^{-1} \circ f_{\varphi(1)}(x) = f_{\varphi(1)} \circ f_{\varphi(0)}^{-1} \circ f_{\varphi(\alpha)}(x). \tag{7}$$

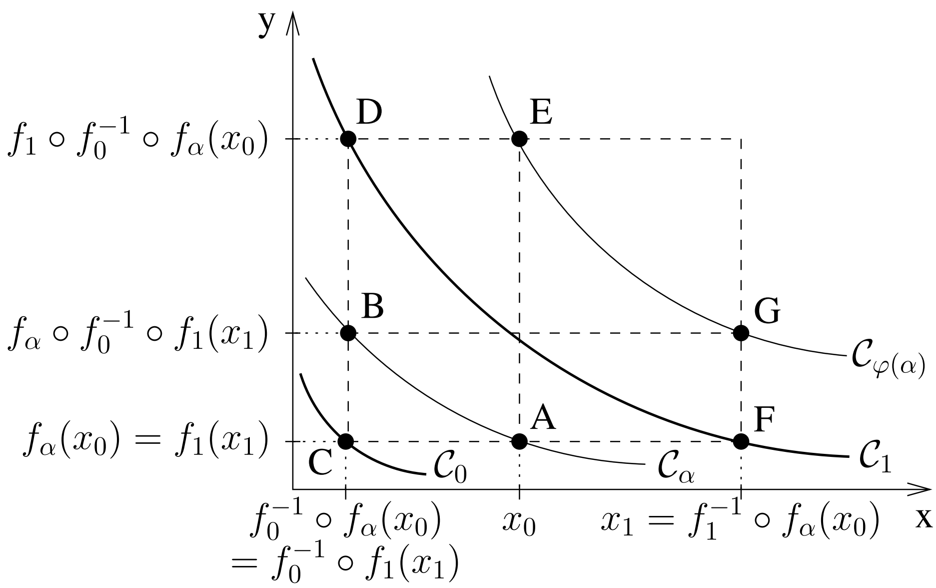

Figure 4: construction of the outside curves.

Now it is time to give the global construction of the example. Equations (5) and (7) reveal that functions $f_{\varphi(\alpha)}$ and f_α have the same kind of inter-plane independence property. Hence $f_{\varphi^2(\alpha)}$—where φ^2 stands for $\varphi \circ \varphi$—can be defined from $f_{\varphi(\alpha)}$ in a similar way to that of $f_{\varphi(\alpha)}$ from f_α. This gives rise to Eq. (8) and Eq. (9), in which $\alpha \in [0,1]$ and $k \in \mathbb{N}$—φ^0 is supposed to be the identity on $\mathbb{R}$.

$$f_{\varphi^{k+1}(\alpha)}(x) = \begin{cases} f_{\varphi^k(\alpha)} \circ f^{-1}_{\varphi^k(0)} \circ f_{\varphi^k(1)}(x) \\ f_{\varphi^k(1)} \circ f^{-1}_{\varphi^k(0)} \circ f_{\varphi^k(\alpha)}(x). \end{cases} \tag{8}$$

$$f_{\varphi^{-k-1}(\alpha)}(x) = \begin{cases} f_{\varphi^{-k}(\alpha)} \circ f^{-1}_{\varphi^{-k}(0)} \circ f_{\varphi^{-k}(1)}(x) \\ f_{\varphi^{-k}(1)} \circ f^{-1}_{\varphi^{-k}(0)} \circ f_{\varphi^{-k}(\alpha)}(x). \end{cases} \tag{9}$$

The process of construction ensures that $f_{\varphi^{k+1}(\alpha)}$ and $f_{\varphi^{-k-1}(\alpha)}$ are well defined and continuous on $\mathbb{R}$, strictly decrease and admit $y = x$ as a symmetry axis, that $f_{\varphi^{k+1}(\alpha)}(\mathbb{R}) = \mathbb{R}$ and that $f_{\varphi^{-k-1}(\alpha)}(\mathbb{R}) = \mathbb{R}$. Moreover, if $\alpha, \beta \in [0,1]$, then $\alpha \leq \beta \Leftrightarrow f_{\varphi^k(\alpha)}(x) \leq f_{\varphi^k(\beta)}(x)$ for any $x \in \mathbb{R}$ and any integer k.

Note that $f^{-1}_{\varphi^k(0)} \circ f_{\varphi^k(1)} = f_0^{-1} \circ f_1$; as a matter of fact, $f_1 = (f_1 \circ f_0^{-1})^0 \circ f_1$, by (6), $f_{\varphi(1)} = (f_1 \circ f_0^{-1}) \circ f_1$, and, for an arbitrary $k > 2$, if $f_{\varphi^k(1)} = (f_1 \circ f_0^{-1})^{k-1} \circ f_1$ and $f_{\varphi^{k-1}(1)} = (f_1 \circ f_0^{-1})^{k-2} \circ f_1$, then by (8), $f_{\varphi^{k+1}(1)} = f_{\varphi^k(1)} \circ f^{-1}_{\varphi^k(0)} \circ f_{\varphi^k(1)} = f_{\varphi^k(1)} \circ f^{-1}_{\varphi^{k-1}(1)} \circ f_{\varphi^k(1)} = (f_1 \circ f_0^{-1})^k \circ f_1 \circ f_1^{-1} \circ (f_0 \circ f_1^{-1})^{k-1} \circ (f_1 \circ f_0^{-1})^k \circ f_1 = (f_1 \circ f_0^{-1})^{k+1} \circ f_1$. So for any $k \in \mathbb{N}$, $f_{\varphi^k(1)} = (f_1 \circ f_0^{-1})^k \circ f_1$. But $\varphi(0) = 1$, so $f^{-1}_{\varphi^k(0)} \circ f_{\varphi^k(1)} = f_1^{-1} \circ [(f_1 \circ f_0^{-1})^{-1}]^{k-1} \circ (f_1 \circ f_0^{-1})^k \circ f_1 = f_0^{-1} \circ f_1$.

The construction of the ordering is now completed, and it remains only to prove that it satisfies all the expected properties. This is done in the following theorem:

Theorem 3 *The binary relation $\succsim$ represented by functions $f_{\varphi^k(\alpha)}$ is a well defined weak order on Ω and satisfies the independence and Archimedean axioms. Moreover, the first two components are solvable.*

Up till now, the construction has been conducted on a very abstract level, and it is rather difficult to imagine the shape of the indifference curves. Hence we provide in figure 5 the drawing of some of them, locally around the origin of the axes.

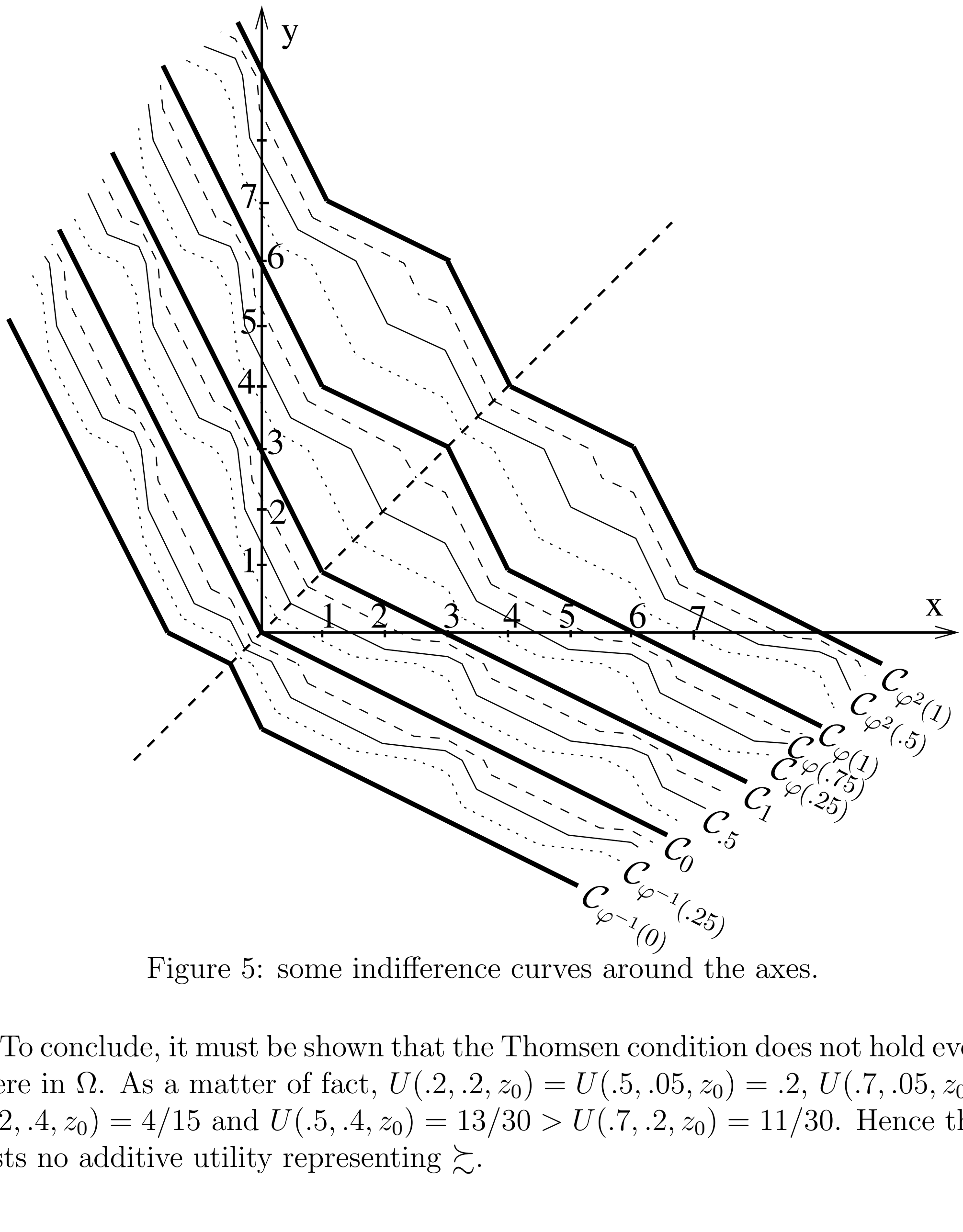

Figure 5: some indifference curves around the axes.

To conclude, it must be shown that the Thomsen condition does not hold everywhere in Ω. As a matter of fact, $U(.2, .2, z_0) = U(.5, .05, z_0) = .2$, $U(.7, .05, z_0) = U(.2, .4, z_0) = 4/15$ and $U(.5, .4, z_0) = 13/30 > U(.7, .2, z_0) = 11/30$. Hence there exists no additive utility representing $\succsim$.

3.2.3 The n-Component Case

So far we have proved that the combination of solvability w.r.t. 2 components and the independence axiom is not sufficient to derive the additive representability of $\succsim$ for three component spaces. But this result can be extended: suppose that $\Omega = \mathbb{R} \times \mathbb{R} \times \{z_0, z_1\} \times \{z_0, z_2\} \times \ldots \times \{z_0, z_p\}$ where z_0, z_1,…,z_p are arbitrary. Use the indifference curves defined in the preceding subsection and suppose that

$$U(x, y, z_{i_1}, z_{i_2}, \ldots, z_{i_p}) = \varphi^{i_1+i_2+\cdots+i_p} \circ U(x, y, z_0, \ldots, z_0),$$

where, for any $j \in \{1, \ldots, p\}$, $i_j \in \{0, j\}$.

First, consider an arbitrary $j \in \{1, \ldots, \frac{p(p+1)}{2}\}$ and $\alpha \in \mathbb{R}$, and define $\psi = \varphi^j$. $f_{\psi(0)} \circ f_0^{-1} \circ f_\alpha = f_{\varphi^j(0)} \circ f_0^{-1} \circ f_\alpha = (f_1 \circ f_0^{-1})^j \circ f_\alpha$. By repeated use of (5), we obtain $f_{\psi(0)} \circ f_0^{-1} \circ f_\alpha = f_\alpha \circ (f_0^{-1} \circ f_1)^j = f_\alpha \circ f_0^{-1} \circ f_{\psi(0)}$. This generalizes formula (5). $f_{\psi(\alpha)} = f_{\varphi^j(1)} \circ f^{-1}_{\varphi^j(0)} \circ f_{\varphi^{j-1}(\alpha)} = f_1 \circ f_0^{-1} \circ f_{\varphi^{j-1}(\alpha)}$ and so, by induction on j, $f_{\psi(\alpha)} = (f_1 \circ f_0^{-1})^j \circ f_\alpha = f_{\psi(0)} \circ f_0^{-1} \circ f_\alpha$, which generalizes formula (6). Similarly one can generalize (8) and (9) as follows: for any $k \in \mathbb{N}$,

$$\begin{aligned} f_{\psi^{k+1}(\alpha)}(x) &= f_{\psi^k(\alpha)} \circ f^{-1}_{\psi^k(0)} \circ f_{\psi^{k+1}(0)}(x) = f_{\psi^{k+1}(0)}(x) \circ f^{-1}_{\psi^k(0)} \circ f_{\psi^k(\alpha)} \\ f_{\psi^{-k-1}(\alpha)}(x) &= f_{\psi^{-k}(\alpha)} \circ f^{-1}_{\psi^{-k+1}(0)} \circ f_{\psi^{-k}(0)}(x) = f_{\psi^{-k}(0)}(x) \circ f^{-1}_{\psi^{-k+1}(0)} \circ f_{\psi^{-k}(\alpha)}. \end{aligned}$$

By a proof similar to the one of theorem 3, it is possible to prove that $\succsim$ represented by U is a weak order on Ω that satisfies the independence and Archimedean axioms, and that the first two components are solvable. And yet

$$\begin{cases} U(.2, .2, z_0, \ldots, z_0) = U(.5, .05, z_0, \ldots, z_0) = .2 \\ U(.7, .05, z_0, \ldots, z_0) = U(.2, .4, z_0, \ldots, z_0) = 4/15 \\ U(.5, .4, z_0, \ldots, z_0) = 13/30 > U(.7, .2, z_0, \ldots, z_0) = 11/30. \end{cases}$$

Hence there exists no additive utility representing $\succsim$.

4 Conclusion

Throughout, sufficient conditions for the existence of additive utilities have been given. It has been shown that the structural assumption usually used to ensure additive representability can be weakened to "unrestricted" solvability w.r.t. 2 components. However, this assumption drives additive utilities toward $+\infty$ and $-\infty$, which might raise some problems. For instance, components could be bounded (e.g. the mass of an object, or its speed). In the literature, the usual way to deal with such a problem is to substitute solvability (axiom 3) by restricted solvability (see axiom 6 below).

Axiom 6 (unrestricted solvability w.r.t. the ith component) *Let $x \in X$, $a_i, b_i \in X_i$ and $y_j \in X_j$ for all $j \neq i$, be such that $(y_1, \ldots, y_{i-1}, a_i, y_{i+1}, \ldots, y_n) \succsim x \succsim (y_1, \ldots, y_{i-1}, b_i, y_{i+1}, \ldots, y_n)$. Then, there exists $c_i \in X_i$ such that $x \sim (y_1, \ldots, y_{i-1}, c_i, y_{i+1}, \ldots, y_n)$.*

Therefore, it would be very appealing if our results could be extended to cases in which restricted solvability holds w.r.t. 2 components. However, this extension is not straightforward because, unlike unrestricted solvability, restricted solvability structures the space of preferences only locally. For instance, suppose that $X = [1,2]\times[1,2]\times 1,2$ and that $\succsim$ is represented by $u(x,y,z) = [\frac{7}{8}(x+y)]^z$. It is obvious that restricted solvability holds w.r.t. the first two components. It can be shown that independence, the Thomsen condition w.r.t. the first two components and the Archimedean axiom hold; but no additive utility exists because $u(2,1.5,1) = u(1,1,2) = \frac{49}{16}$, $u(2,2,1) = u(4\sqrt{\frac{2}{7}}-1,1,2) = \frac{7}{2}$ and $u(1,2,1) = \frac{21}{8} = 2.625 > u(4\sqrt{\frac{2}{7}}-1,1.5,1) = \sqrt{\frac{7}{2}} - \frac{7}{16} \approx 1.433$. This example as well as some sufficient conditions to deal with restricted solvability w.r.t. 2 components are studied in Gonzales (1995). However, the latter require more structural and cancellation axioms than the conditions presented here.

5 Acknowledgment

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6 Appendix: Proofs

Proof of theorem 1: Fix an arbitrary element of $\prod_{i=3}^n X_i$: $(x_3^0,\ldots,x_n^0)$. It is known that $(x_1,x_2) \precsim_{12} (y_1,y_2) \Leftrightarrow (x_1,x_2,x_3^0,\ldots,x_n^0) \precsim (y_1,y_2,x_3^0,\ldots,x_n^0)$. By hypothesis, $\succsim_{12}$ is representable by an additive utility; so there exist real valued functions u_1 and u_2 on X_1 and X_2 respectively such that:

$$(x_1,x_2,x_3^0,\ldots,x_n^0) \precsim (y_1,y_2,x_3^0,\ldots,x_n^0) \Leftrightarrow u_1(x_1)+u_2(x_2) \leq u_1(y_1)+u_2(y_2).$$

Now for $i = 3,4,\ldots,n$, let $u_i(x_i^0)$ be arbitrary real numbers. Clearly,

$$\begin{gathered}(x_1,x_2,x_3^0,\ldots,x_n^0) \precsim (y_1,y_2,x_3^0,\ldots,x_n^0)\\ \Leftrightarrow u_1(x_1)+u_2(x_2)+\textstyle\sum_{i=3}^n u_i(x_i^0) \leq u_1(y_1)+u_2(y_2)+\sum_{i=3}^n u_i(x_i^0).\end{gathered}$$

By solvability w.r.t. the first component, for any y_i of X_i, there exists an element $x_1(y_i)$ in X_1 such that $(x_1(y_i),x_2^0,\ldots,x_n^0) \sim (x_1^0,\ldots,x_{i-1}^0,y_i,x_{i+1}^0,\ldots,x_n^0)$. Define

$$u_i(y_i) = u_1(x_1(y_i)) - u_1(x_1^0) + u_i(x_i^0).$$

Note that $x_1(x_i^0) = x_1^0$, hence $u_i(x_i^0)$ is well defined. We are to show that functions u_i, defined as above, form an additive utility function.

Now suppose that in $\prod_{i=1}^k X_i \times \prod_{i=k+1}^n \{x_i^0\}$,

$$(x_1, \ldots, x_k, x_{k+1}^0, \ldots, x_n^0) \precsim (y_1, \ldots, y_k, x_{k+1}^0, \ldots, x_n^0)$$
$$\Leftrightarrow \sum_{i=1}^k u_i(x_i) + \sum_{i=k}^n u_i(x_i^0) \le \sum_{i=1}^k u_i(y_i) + \sum_{i=k}^n u_i(x_i^0).$$

Consider an arbitrary element $y = (y_1, \ldots, y_{k+1}, x_{k+2}^0, \ldots, x_n^0)$ of the set $\prod_{i=1}^{k+1} X_i \times \prod_{i=k+2}^n \{x_i^0\}$. By solvability w.r.t. the second component, there exists y_2' such that $(y_1, y_2, y_3, \ldots, y_k, x_{k+1}^0, \ldots, x_n^0) \sim (x_1^0, y_2', x_3^0, \ldots, x_n^0)$, and, from the equivalence above, $\sum_{i=1}^k u_i(y_i) = u_1(x_1^0) + u_2(y_2') + \sum_{i=3}^k u_i(x_i^0)$. Then, by the independence axiom, $y \sim (x_1^0, y_2', x_3^0, \ldots, x_k^0, y_{k+1}, x_{k+1}^0, \ldots, x_n^0)$. But, by the definition of u_i, there exists a point $x_1(y_{k+1})$ in X_1 such that $(x_1(y_{k+1}), x_2^0, \ldots, x_n^0) \sim (x_1^0, ..., x_k^0, y_{k+1}, x_{k+2}^0, ..., x_n^0)$ and $u_{k+1}(y_{k+1}) = u_1(x_1(y_{k+1})) - u_1(x_1^0) + u_{k+1}(x_{k+1}^0)$. Therefore, $y \sim (x_1(y_{k+1}), y_2', x_3^0, \ldots, x_n^0)$ and $\sum_{i=1}^{k+1} u_i(y_i) = u_1(x_1(y_1)) + u_2(y_2') + \sum_{i=3}^{k+1} u_i(x_i^0)$. So any element of $\prod_{i=1}^{k+1} X_i \times \prod_{i=k+2}^n \{x_i^0\}$ has an equivalent in $X_1 \times X_2 \times \prod_{i=3}^n \{x_i^0\}$, and their utilities are equal. So in $\prod_{i=1}^{k+1} X_i \times \prod_{i=k+2}^n \{x_i^0\}$,

$$(x_1, \ldots, x_{k+1}, x_{k+2}^0, \ldots, x_n^0) \precsim (y_1, \ldots, y_{k+1}, x_{k+2}^0, \ldots, x_n^0)$$
$$\Leftrightarrow \sum_{i=1}^{k+1} u_i(x_i) + \sum_{i=k+2}^n u_i(x_i^0) \le \sum_{i=1}^{k+1} u_i(y_i) + \sum_{i=k+2}^n u_i(x_i^0).$$

Since this property was true for $k = 2$, by induction it is also true for any $k \in \{2, \ldots, n\}$. So there exists an additive utility representing $\succsim$.

To complete the proof, let us show the uniqueness up to scale and location. First, u_1 and u_2 are interval scales. As a matter of fact, their existence, combined with solvability w.r.t. the first two components, ensure that on $X_1 \times X_2$, the Thomsen condition and the Archimedean axiom, and, more generally, all the hypotheses of the classical theorem hold. Using the latter, we conclude that the uniqueness is up to scale and location.

Suppose that v_i, $i = 1, 2, \ldots, n$, are functions having the same property as u_i, and consider any element y_k of X_k, for k in $\{3, \ldots, n\}$. Let $x_1(y_k)$ be such that $(x_1(y_k), x_2^0, \ldots, x_n^0) \sim (x_1^0, \ldots, x_{k-1}^0, y_k, x_{k+1}^0, \ldots, x_n^0)$. Then $v_k(y_k) = v_1(x_1(y_k)) - v_1(x_1^0) + v_k(x_k^0)$. By hypothesis, it is already known that there exists some constants $a > 0$, b_1, b_2 such that $v_1 = a \cdot u_1 + b_1$ and $v_2 = a \cdot u_2 + b_2$. So $v_k(y_k) = a \cdot u_1(x_1(y_k)) - a \cdot u_1(x_1^0) + v_k(x_k^0) = a \cdot u_k(y_k) + v_k(x_k^0) - a \cdot u_k(x_k^0)$. But x_k^0 is a constant so $v_k(x_k^0) - a \cdot u_k(x_k^0)$ is also a constant. Hence functions u_i are interval scales. ♦

Proof of corollaries 1 and 2: Obvious because axioms 4, 5 and 3 (solvability w.r.t. 3 components) guarantee the existence of additive utilities on $X_1 \times X_2$. ♦

Proof of lemma 1: By Eq. (2) and Eq.(3), we deduce the following equations:

$$f_0^{-1} \circ f_1(x) = \begin{cases} x - \frac{3}{2} & \text{if } x \in]-\infty, 1] \\ \frac{x-3}{4} & \text{if } x \in [1, 3] \\ x - 3 & \text{if } x \in [3, +\infty] \end{cases}$$

$$f_1 \circ f_0^{-1}(x) = \begin{cases} x + \frac{3}{2} & \text{if } x \in]-\infty, -\frac{1}{2}] \\ 4x + 3 & \text{if } x \in [-\frac{1}{2}, 0] \\ x + 3 & \text{if } x \in [0, +\infty]. \end{cases}$$

So by Eq. (5), for any α in $]0,1[$, $f_\alpha \circ f_0^{-1} \circ f_1(2\alpha+1) = f_\alpha(\frac{\alpha-1}{2})$. Hence, Eq. (4) does not contradict Eq. (5). Using (4) and (5), we define f_α over $[\frac{\alpha-4}{2}, \frac{\alpha-1}{2}]$ since $f_0^{-1} \circ f_1$ is obviously continuous and increasing, $f_0^{-1} \circ f_1(\frac{\alpha-1}{2}) = \frac{\alpha-4}{2}$ and $f_0^{-1} \circ f_1(2\alpha+1) = \frac{\alpha-1}{2}$. $f_0^{-1} \circ f_1$ and $f_1 \circ f_0^{-1}$ being continuous and strictly increasing over $[\frac{\alpha-1}{2}, 2\alpha+1]$, f_α is continuous and strictly decreases on $[\frac{\alpha-4}{2}, \frac{\alpha-1}{2}]$. By induction, we construct f_α on $]-\infty, 2\alpha+1]$ since it is easily shown that for any k in $\mathbb{N}$, $f_0^{-1} \circ f_1(\frac{\alpha-1-3k}{2}) = \frac{\alpha-4-3k}{2}$. Of course the construction by induction ensures that f_α is continuous and strictly decreasing.

Moreover for any k in $\mathbb{N}$, $f_1 \circ f_0^{-1} \circ f_\alpha(\frac{\alpha-1-3k}{2}) = 2\alpha + 1 + 3k$, which ensures that $\lim_{x\to-\infty} f_\alpha(x) = +\infty$. The line $y = x$ being a symmetry axis for f_0 and f_1 on $\mathbb{R}$, and for f_α on $[\frac{\alpha-1}{2}, 2\alpha+1]$, it is also a symmetry axis for f_α on $\mathbb{R}$. Hence f_α is well defined on $[-2\alpha-1, +\infty[$, continuous, strictly increasing and $\lim_{x\to+\infty} f_\alpha(x) = -\infty$.

Now, consider $\alpha, \beta \in [0,1]$ such that $\alpha \leq \beta$. By Eq. (4)—and Eq. (2) and Eq. (3) if α or β is equal to 0 or 1—it is obvious that for any $x \in [\frac{\beta-1}{2}, 2\alpha+1]$, $f_\alpha \leq f_\beta$. On $[\frac{\alpha-1}{2}, \frac{\beta-1}{2}]$, the inequality $f_\alpha(x) \leq f_\beta(x)$ must also hold, otherwise f_β would not be one to one. Now if there existed an x in $\mathbb{R}$ such that $f_\alpha(x) > f_\beta(x)$, then by repeated uses of (5), there would exist an $x' \in [\frac{\alpha-1}{2}, 2\alpha+1]$ such that $f_\alpha(x') > f_\beta(x')$, which has been shown to be impossible. Conversely, if for any $x \in \mathbb{R}$, $f_\alpha(x) \leq f_\beta(x)$, then this is true in particular for any $x \in [\frac{\beta-1}{2}, 2\alpha+1]$. But then by Eq. (4)—and (2) or (3)—$\alpha \leq \beta$. ♦

Proof of lemma 2: By (5), we know that $f_{\varphi(\alpha)}$ is well defined on $\mathbb{R}$ for any $\alpha \in [0,1]$. Since f_α, f_0 and f_1 are continuous, strictly decrease, vary from $+\infty$ to $-\infty$ and are symmetric w.r.t. the line $y = x$, $f_{\varphi(\alpha)}$ has the same properties. By lemma 1, for any $\alpha, \beta \in [0,1]$, $\alpha \leq \beta \Leftrightarrow f_\alpha(x) \leq f_\beta(x)$ for any $x \in \mathbb{R}$, and it follows from the change of variable $y = f_1^{-1} \circ f_0(x)$, and the fact that $f_1^{-1} \circ f_0(x)$ varies from $-\infty$ to $+\infty$, that $\alpha \leq \beta \Leftrightarrow f_\alpha \circ f_0^{-1} \circ f_1(y) \leq f_\beta \circ f_0^{-1} \circ f_1(y)$ for any $y \in \mathbb{R}$.

$f_{\varphi(1)} = f_1 \circ f_0^{-1} \circ f_1$ and $f_{\varphi(0)} = f_1$ — since by hypothesis $\varphi(0) = 1$;
so $f_{\varphi(\alpha)} \circ f_{\varphi(0)}^{-1} \circ f_{\varphi(1)} = f_\alpha \circ f_0^{-1} \circ f_1 \circ f_1^{-1} \circ f_1 \circ f_0^{-1} \circ f_1 = f_\alpha \circ f_0^{-1} \circ f_1 \circ f_0^{-1} \circ f_1 = f_1 \circ f_0^{-1} \circ f_\alpha \circ f_0^{-1} \circ f_1 = f_1 \circ f_0^{-1} \circ f_1 \circ f_1^{-1} \circ f_\alpha \circ f_0^{-1} \circ f_1 = f_{\varphi(1)} \circ f_{\varphi(0)}^{-1} \circ f_{\varphi(\alpha)}$, hence proving that Eq. (6) holds. ♦

In order to prove theorem 3, we first introduce two lemmas, namely lemmas 3 and 4.

Lemma 3 *Suppose that for any $\alpha \in [0,1]$, $\mathcal{C}_\alpha$ is given by (2), (3) or (4) and (5). Suppose that $\mathcal{C}_{\varphi^k(\alpha)}$ and $\mathcal{C}_{\varphi^{-k}(\alpha)}$ are given for any $k \in \mathbb{N}$ by (6) and (8), and (6) and (9) respectively. Then φ is well defined on $\mathbb{R}$ and $\mathcal{C}_\beta$ is well defined for any*

$\beta \in \mathbb{R}$. Moreover φ is one to one and strictly increases from $-\infty$ to $+\infty$; for any $(x,y) \in \mathbb{R}^2$, there exists $\alpha \in \mathbb{R}$ such that $(x,y) \in \mathcal{C}_\alpha$, and the following hold:

$$\text{for any } \alpha, \beta \in \mathbb{R},\ \alpha \leq \beta \Leftrightarrow f_\alpha(x) \leq f_\beta(x) \text{ for any } x \in \mathbb{R},$$
$$\text{for any } x, x', y_1, y_2, \alpha, \beta \in \mathbb{R},\ f_\alpha(x) = f_\beta(x') \Rightarrow f_{\varphi(\alpha)}(x) = f_{\varphi(\beta)}(x').$$

Proof of lemma 3: By (4), every point in the polyhedron defined by $\{y \leq f_1(x);\ y \geq f_0(x);\ y \leq 4x+3;\ y \geq \frac{x-3}{4}\}$ belongs to an indifference curve $\mathcal{C}_\alpha$. It is easily shown that any point in the domain $\{y \leq f_1(x);\ y \geq f_0(x);\ y \leq 4x+9+3;\ y \geq 4x+3\}$ is obtained by (5) from a point in the previous polyhedron; and, by induction, that, for any $k > 1$, any point in the polyhedron $\{y \leq f_1(x);\ y \geq f_0(x);\ y \leq 4x+9k+3;\ y \geq 4x+9k-6\}$ is obtained by formula (5) from a point in $\{y \leq f_1(x);\ y \geq f_0(x);\ y \leq 4x+9(k-1)+3;\ y \geq 4x+9(k-1)-6\}$. By using the symmetry axis $y = x$, we conclude that every point in the polyhedron $\{y \leq f_1(x);\ y \geq f_0(x)\}$ belongs to an indifference curve $\mathcal{C}_\alpha$.

Now suppose that, for $k \geq 0$, every point in the domain $\{y \leq f_{\varphi^k(1)}(x);\ y \geq f_{\varphi^k(0)}(x)\}$ belongs to a curve $\mathcal{C}_{\varphi^k(\alpha)}$. Consider an arbitrary point (x_0, y_0) in the domain defined by $\{y \leq f_{\varphi^{k+1}(1)}(x);\ y \geq f_{\varphi^{k+1}(0)}(x)\}$. By hypothesis, y_0 is such that $y_0 \leq f_{\varphi^{k+1}(1)}(x_0)$ and $y_0 \geq f_{\varphi^{k+1}(0)}(x_0)$. So $y_1 = f_0 \circ f_1^{-1}(y_0)$ is such that $f_{\varphi^k(0)}(x_0) \leq y_1 \leq f_{\varphi^k(1)}(x_0)$. And by hypothesis there exists a curve $\mathcal{C}_{\varphi^k(\alpha)}$ such that $(x_0, y_1) \in \mathcal{C}_{\varphi^k(\alpha)}$. So $y_1 = f_{\varphi^k(\alpha)}(x_0)$ and $y_0 = f_1 \circ f_0^{-1} \circ f_{\varphi^k(\alpha)}(x_0) = f_{\varphi^k(1)} \circ f_{\varphi^k(0)}^{-1} \circ f_{\varphi^k(\alpha)}(x_0)$. So any point in the domain defined by $\{f_{\varphi^{k+1}(0)}(x) \leq y \leq f_{\varphi^{k+1}(1)}(x)\}$ belongs to an indifference curve $\mathcal{C}_\beta$. The same kind of proof holds when k is negative. Now we must extend this local property to the whole $\mathbb{R}^2$.

Suppose that for $k \geq 0$,

$$f_{\varphi^k(1)}(x) \geq \begin{cases} 3(k+1) - 2x & \text{if } x \leq k+1 \\ \frac{3(k+1)-x}{2} & \text{otherwise.} \end{cases}$$

Note that this is true for $k = 0$—for which $f_{\varphi^k(1)}$ corresponds to f_1. $f_{\varphi^{k+1}}(1) = (f_1 \circ f_0^{-1})^{k+1} \circ f_1 = (f_1 \circ f_0^{-1}) \circ f_{\varphi^k}(1)$; so using the expression of $f_1 \circ f_0^{-1}$ given in the proof of lemma 1, we deduce that:

$$f_{\varphi^{k+1}(1)}(x) \geq \begin{cases} 3(k+2) - 2x & \text{if } x \leq k+1 \\ 3 + \frac{3(k+1)-x}{2} & \text{if } x \in [k+1, 3(k+1)] \\ 3 + 6(k+1) - 2x & \text{if } x \in [3(k+1), 3(k+1)+1] \\ \frac{3(k+2)-x}{2} & \text{if } x \geq 3(k+1)+1. \end{cases}$$

But in $[k+1, k+2]$, $\quad 3 + \frac{3(k+1)-x}{2} \geq -2x + 3(k+2)$,
in $[k+2, 3(k+1)]$, $\quad 3 + \frac{3(k+1)-x}{2} \geq \frac{3(k+2)-x}{2}$,
and in $[3(k+1), 3(k+1)+1]$, $\quad 3 + 6(k+1) - 2x \geq \frac{3(k+2)-x}{2}$.

So $f_{\varphi^{k+1}(1)}(x) \geq \begin{cases} 3(k+2) - 2x & \text{if } x \leq k+2 \\ \frac{3(k+2)-x}{2} & \text{otherwise.} \end{cases}$

Since this property is true for $k = 0$, by induction, it is true for any $k \geq 0$. So

any point (x_0, y_0) in the polyhedron defined by $\{f_0(x) \leq y \leq g_k(x)\}$ where
$g_k(x) = \begin{cases} 3(k+1) - 2x & \text{if } x \leq k+1 \\ \frac{3(k+1)-x}{2} & \text{otherwise} \end{cases}$
is also in the polyhedron $\{f_0(x) \leq y \leq f_{\varphi^k(1)}(x)\}$. But we have seen in the previous paragraph that then there exists a curve $\mathcal{C}_\alpha$ containing (x_0, y_0). And $\lim_{k\to+\infty}\{f_0(x) \leq y \leq g_k(x)\} = \{f_0(x) \leq y\}$. So any point in the last set belongs to an indifference curve. A similar proof would show that any point in the set $\{y \leq f_0(x)\}$ belongs to an indifference curve.

So any point of $\mathbb{R}^2$ belongs to a curve $\mathcal{C}_\alpha$. This is true in particular for any point on the line $y = x$. Hence $\mathcal{C}_\alpha$ is defined for any $\alpha \in \mathbb{R}$. The principle of construction guarantees that φ is defined over $\mathbb{R}$. Suppose now that α and β are real numbers such that $\varphi(\alpha) = \varphi(\beta)$. Then $f_{\varphi(\alpha)} = f_{\varphi(\beta)}$. But by the preceding paragraphs, there exist $k, k' \in \mathbb{N}$ and $\gamma, \delta \in [0,1]$ such that $\alpha = \varphi^k(\gamma)$ and $\beta = \varphi^{k'}(\delta)$. Then $f_{\varphi(\alpha)} = f_{\varphi^k(1)} \circ f^{-1}_{\varphi^k(0)} \circ f_{\varphi^k(\gamma)} = f_1 \circ f_0^{-1} \circ f_{\varphi^k(\gamma)}$ and $f_{\varphi(\beta)} = f_1 \circ f_0^{-1} \circ f_{\varphi^{k'}(\delta)}$. Since $f_1 \circ f_0^{-1}$ is one to one, $f_{\varphi^k(\gamma)} = f_{\varphi^{k'}(\delta)}$. Hence $\varphi^k(\gamma) = \varphi^{k'}(\delta)$ and so $\alpha = \beta$, which implies that φ is one to one.

It is already known that, for any integer k, and for any $\alpha, \beta \in [0,1]$,
$$\alpha \leq \beta \Leftrightarrow f_{\varphi^k(\alpha)}(x) \leq f_{\varphi^k(\beta)}(x), \text{ for any } x \in \mathbb{R}.$$
But by (8) and (9), $f_{\varphi^{k+1}(0)}(x) = f_{\varphi^k(1)}(x)$; so for any integers k, k',
$$\varphi^k(\alpha) \leq \varphi^{k'}(\beta) \Leftrightarrow f_{\varphi^k(\alpha)}(x) \leq f_{\varphi^{k'}(\beta)}(x) \text{ for any } x \in \mathbb{R}.$$
So for any $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta \Leftrightarrow f_\alpha(x) \leq f_\beta(x)$ for any $x \in \mathbb{R}$, and since $f_1 \circ f_0^{-1}$ is strictly increasing,
$$f_\alpha(x) \leq f_\beta(x) \Leftrightarrow f_1 \circ f_0^{-1} \circ f_\alpha(x) \leq f_1 \circ f_0^{-1} \circ f_\beta(x) \Leftrightarrow f_{\varphi(\alpha)}(x) \leq f_{\varphi(\beta)}(x).$$
So φ is strictly increasing.

Now, to complete the proof, suppose that $x, x', y_1, y_2, \alpha, \beta \in \mathbb{R}$ are such that $y_1 = f_\alpha(x) = f_\beta(x')$ and $y_2 = f_{\varphi(\alpha)}(x)$. Then $f_{\varphi(\alpha)}(x) = f_1 \circ f_0^{-1} f_\alpha(x) = f_1 \circ f_0^{-1} \circ f_\beta(x') = f_{\varphi(\beta)}(x')$. So $y_2 = f_{\varphi(\beta)}(x')$. ♦

Lemma 4 *For any $\alpha, \beta \in [0,1]$, there exists a constant $m(\alpha, \beta)$ such that $m(\alpha, \beta) \leq f_\beta(x) - f_\alpha(x)$ for any $x \in \mathbb{R}$. Moreover, $\alpha \leq \beta \Leftrightarrow m(\alpha, \beta) \geq 0$.*

Proof of lemma 4: The principle of construction of f_α, for $\alpha \in [0,1]$, is to create the function on a small interval with Eq. (4) and then to extend the definition on $\mathbb{R}$ thanks to (5). But if x is greater than or equal to 4, $f_\alpha(x) \leq -.5$ because $f_\alpha \leq f_1$ and $f_1(4) = -.5$. And, from the expression of $f_1 \circ f_0^{-1}$ given in the proof of lemma 1, $f_1 \circ f_0^{-1} \circ f_\alpha(x) = f_\alpha(x) + \frac{3}{2}$. So, in (5), the extension of f_α on $[4, +\infty[$ is made thanks to translations. This is similar for the extension on $]-\infty, -1.5]$ because $f_0 \leq f_\alpha$ and $f_0(-1.5) = 3$, which, following the expression of $f_0^{-1} \circ f_1$ given in the proof of lemma 1, implies translations.

Hence the difference $f_\beta(x) - f_\alpha(x)$ need only be calculated on $\{x \in [-1.5, 4]\}$. But in this set, by lemma 1, it is known that f_α and f_β are continuous. So, since [-1.5,4] is a closed interval, $f_\beta(x) - f_\alpha(x)$ has maximum and minimum values which are reached. Moreover, and again by lemma 1, $\alpha \leq \beta \Leftrightarrow f_\alpha(x) \leq f_\beta(x)$. Hence the minimum of $f_\beta(x) - f_\alpha(x)$ on [-1.5,4] is 0 if and only if $\alpha = \beta$. ♦

Proof of theorem 3: In this proof it will be shown successively that $\succsim$ is a weak order on Ω, that the first two components are solvable, that the independence axiom is satisfied and that the Archimedean axiom holds. First $\succsim$ is a well defined weak order on Ω because, by lemma 3, for any $(x,y) \in \mathbb{R}^2$, there exists $\alpha \in \mathbb{R}$ such that $(x,y) \in \mathcal{C}_\alpha$ and there exists $\beta \in \mathbb{R}$ such that $\varphi(\beta) = \alpha$, so that $(x,y) \in \mathcal{C}_{\varphi(\beta)}$.

For any $\alpha \in \mathbb{R}$, f_α is defined on $\mathbb{R}$ and is one to one. So, for any $y \in \mathbb{R}$ (resp. $x \in \mathbb{R}$), there exists an $x \in \mathbb{R}$ (resp. $y \in \mathbb{R}$) such that $f_\alpha(x) = y$. This guarantees solvability w.r.t. x (resp. y) in the plane $\{z = z_0\}$. φ being one to one, continuous and varying from $-\infty$ to $+\infty$, there exists $\beta \in \mathbb{R}$ such that $\varphi(\beta) = \alpha$; and since $U(x,y,z_1) = \varphi \circ U(x,y,z_0)$, solvability w.r.t. x (resp. y) holds in the plane $\{z = z_1\}$. So the first two components are solvable.

Let $x_0, x_1, y_0, y_1 \in \mathbb{R}$ be such that $(x_0,y_0,z_0) \precsim (x_1,y_1,z_0)$. By lemma 3, there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha \leq \beta$, $(x_0,y_0) \in \mathcal{C}_\alpha$ and $(x_1,y_1) \in \mathcal{C}_\beta$. φ is strictly increasing and one to one so $\alpha \leq \beta \Leftrightarrow \varphi(\alpha) \leq \varphi(\beta) \Leftrightarrow U(x_0,y_0,z_0) = \alpha \leq U(x_1,y_1,z_0) = \beta \Leftrightarrow \varphi \circ U(x_0,y_0,z_0) = \varphi(\alpha) \leq \varphi \circ U(x_1,y_1,z_0) = \varphi(\beta) \Leftrightarrow (x_0,y_0,z_1) \precsim (x_1,y_1,z_1)$. Hence independence holds w.r.t. the third component.

Now let $x, y_0, y_1 \in \mathbb{R}$ be such that $(x,y_0,z_0) \precsim (x,y_1,z_0)$. There exist $\alpha, \beta \in \mathbb{R}$ such that $(x,y_0) \in \mathcal{C}_\alpha$ and $(x,y_1) \in \mathcal{C}_\beta$ with $\alpha \leq \beta$. Then, by lemma 3, $\alpha \leq \beta \Leftrightarrow f_\alpha(x) \leq f_\beta(x)$ for any $x \in \mathbb{R}$. So $\alpha \leq \beta \Leftrightarrow y_0 \leq y_1$; thus for any $x' \in \mathbb{R}$, $(x',y_0,z_0) \precsim (x',y_1,z_0)$. By independence w.r.t. the third component, $(x,y_0,z_1) \precsim (x,y_1,z_1) \Leftrightarrow (x,y_0,z_0) \precsim (x,y_1,z_0) \Leftrightarrow [(x',y_0,z_0) \precsim (x',y_1,z_0)$ for any $x' \in \mathbb{R}] \Leftrightarrow [(x',y_0,z_1) \precsim (x',y_1,z_1)$ for any $x' \in \mathbb{R}]$. By symmetry between components x and y, we also have $(x_0,y,z_1) \precsim (x_1,y,z_1) \Leftrightarrow (x_0,y,z_0) \precsim (x_1,y,z_0) \Leftrightarrow [(x_0,y',z_0 \precsim (x_1,y',z_0)$ for any $y' \in \mathbb{R}] \Leftrightarrow [(x_0,y',z_1) \precsim (x_1,y',z_1)$ for any $y' \in \mathbb{R}]$. Hence inside the planes, independence holds w.r.t. the first and second components.

Let $x, y_0, y_1 \in \mathbb{R}$ be such that $(x,y_0,z_0) \sim (x,y_1,z_1)$. Then there exists $\alpha \in \mathbb{R}$ such that $(x,y_1) \in \mathcal{C}_\alpha$ and $(x,y_0) \in \mathcal{C}_{\varphi(\alpha)}$. Thus $y_1 = f_\alpha(x)$ while $y_0 = f_{\varphi(\alpha)}(x)$. But by lemma 3, for any $x' \in \mathbb{R}$, there exists $\beta \in \mathbb{R}$ such that $(x`,y_1) \in \mathcal{C}_\beta$ and then $(x',y_0) \in \mathcal{C}\varphi(\beta)$. So for any $x' \in \mathbb{R}$, $(x',y_0,z_0) \sim (x',y_1,z_1)$. Now suppose that $x, y_0, y_1 \in \mathbb{R}$ are such that $(x,y_0,z_0) \precsim (x,y_1,z_1)$. Then there exists y_2 such that $(x,y_0,z_0) \precsim (x,y_2,z_0) \sim (x,y_1,z_1)$. By the beginning of this paragraph and the previous one, for any $x' \in \mathbb{R}$, $(x',y_0,z_0) \precsim (x',y_2,z_0) \sim (x',y_1,z_1)$. By symmetry, the following must also be true: $(x,y_0,z_1) \precsim (x,y_1,z_0) \Rightarrow (x',y_0,z_1) \precsim (x',y_1,z_0)$ for any $x' \in \mathbb{R}$. Hence independence holds w.r.t. the first component. And by symmetry between x and y, it also holds w.r.t. the second component.

As for the Archimedean axiom, consider a standard sequence w.r.t. the first component: $\{x_1^k : x_1^k \in \mathbb{R}, k \in \mathbb{N}, \text{Not}((x_1^0,x_2^0,z_0) \sim (x_1^0,x_2^1,z_0))$ and for all k, $k+1 \in \mathbb{N}, (x_1^k,x_2^1,z_0) \sim (x_1^{k+1},x_2^0,z_0)\}$. Note that by solvability w.r.t. the second component, any standard sequence w.r.t. the first component can be transformed into a sequence like the one above. In the sequel we suppose that $(x_1^0,x_2^0,z_0) \prec (x_1^0,x_2^1,z_0)$; a similar proof would hold for the converse.

There exist α, β such that $x_2^0 = f_\alpha(x_1^0)$ and $x_2^1 = f_\beta(x_1^0)$, so that $x_2^1 = f_\beta \circ f_\alpha^{-1}(x_2^0)$. Similarly, there exists γ_1 such that $x_2^1 = f_{\gamma_1}(x_1^1)$. But since $(x_1^0,x_2^1,z_0) \sim$

(x_1^1, x_2^0, z_0), the following equation is true:

$$f_{\gamma_1} = f_\beta \circ f_\alpha^{-1} \circ f_\beta.$$

There exists γ_2 such that $x_2^1 = f_{\gamma_2}(x_1^2)$. But then, since $(x_1^1, x_2^1, z_0) \sim (x_1^2, x_2^0, z_0)$, the following equation is true:

$$f_{\gamma_2} = f_{\gamma_1} \circ f_\beta^{-1} \circ f_{\gamma_1} = (f_\beta \circ f_\alpha^{-1})^2 \circ f_\beta.$$

By induction, when examining x_1^k, the following equation would be found:

$$f_{\gamma_k} = (f_\beta \circ f_\alpha^{-1})^k \circ f_\beta.$$

Now α (resp. β) can be written as $\varphi^n(\nu)$ (resp. $\varphi^m(\mu)$), with μ and ν in $[0, 1]$. Then $f_{\gamma_k} = (f_\mu \circ (f_0^{-1} \circ f_1)^m \circ (f_0^{-1} \circ f_1)^{-n} \circ f_\nu^{-1})^k \circ f_\beta$ or, equivalently, $f_{\gamma_k} = (f_\mu \circ (f_0^{-1} \circ f_1)^{m-n} \circ f_\nu^{-1})^k \circ f_\mu \circ (f_0^{-1} \circ f_1)^m$.

μ and ν belong to $[0, 1]$; so, for any $x \in \mathbb{R}$, $f_0(x) \leq f_\mu(x) \leq f_1(x)$ and $f_0(x) \leq f_\nu(x) \leq f_1(x)$. Moreover $f_0^{-1} \circ f_1$ strictly increases. Hence, $f_{\gamma_k} > (f_0 \circ (f_0^{-1} \circ f_1)^{m-n} \circ f_1^{-1})^k \circ f_0 \circ (f_0^{-1} \circ f_1)^m = f_0 \circ (f_0^{-1} \circ f_1)^{k(m-n-1)+m}$. But $f_0^{-1} \circ f_1^{-1}(x) > x - 3$ and $f_0(x) > -x$ for any x. So $f_{\gamma_k}(x) > g(x) = -x + 3[k(m-n-1)+m]$ for any x. The intersection of g with the line $y = x$ gives $x = \frac{3[k(m-n-1)+m]}{2}$. And f_{γ_k} is above g, so $\gamma_k > \frac{3[k(m-n-1)+m]}{2}$. Hence if $m > n+1$, $lim_{k \to +\infty} \gamma_k = +\infty$. Therefore, the standard sequence cannot be bounded upward.

If $m = n+1$, then $f_{\gamma_k} = [(f_\mu \circ f_0^{-1}) \circ (f_1 \circ f_\nu^{-1})]^k \circ f_\mu \circ (f_0^{-1} \circ f_1)^m$. By lemma 4, $f_\mu \geq f_0 + m(0, \mu)$ and $f_1 \geq f_\nu + m(\nu, 1)$. Hence $f_{\gamma_k} \geq (\mathrm{Id} + m(0, \mu) + m(\nu, 1))^k \circ f_\mu \circ (f_0^{-1} \circ f_1)^m$ where Id stands for the identity function. The same reasoning as the one in the previous paragraph leads to the unboundedness of the standard sequence.

If $m = n$, then $f_{\gamma_k} = [(f_\mu \circ f_\nu^{-1})]^k \circ f_\mu \circ (f_0^{-1} \circ f_1)^m$. Here since $m = n$, $\mu > \nu$. So by lemma 4, $f_\mu \geq f_\nu + m(\nu, \mu)$, and so $f_{\gamma_k} \geq (\mathrm{Id} + m(\nu, \mu))^k \circ f_\mu \circ (f_0^{-1} \circ f_1)^m$. So there exists no upper bound, and this achieves the proof that the Archimedean axiom holds. ♦

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List of Figures

List of symbols

X, X_i some Cartesian products. $X = \prod_{i=1}^{n} X_i$.

$\succsim$ the relation "preferred or indifferent to". This is supposed to be a weak order on X.

$\sim$ the symmetric part of $\succsim$.

$\precsim$ the converse of $\succsim$.

$\succ$ the asymmetric part of $\succsim$.

$\prec$ the converse of $\succ$.

$\succsim_N$ the restriction of the preference relation $\succsim$ on the set $\prod_{i \in N} X_i$, with $N \subseteq \{1, \ldots, n\}$.

$\mathbb{R}$ the set of the real numbers.

$\mathbb{N}$ the set of the integers.

$\circ$ the binary operator such that $f \circ g(\cdot) = f(g(\cdot))$, where f and g are some functions.

Id the identity function.

f^{-1} the inverse of f, i.e. $f^{-1} \circ f = \mathrm{Id}$.

f^k $f^k = f \circ f \circ \cdots \circ f$ (k times).