Additive Utility Without Restricted Solvability on Every Components

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Abstract

It is shown that restricted solvability need not be required to hold w.r.t. every component for deriving the existence of additive utilities: additive representations can be shown to exist even when restricted solvability holds w.r.t. only two components. In such cases, the uniqueness property of these representations departs from the classical theory in that it is between ordinal and cardinal.

Keywords: additive conjoint measurement, solvability, utility.

1 Introduction

This paper studies the existence of additive utility functions on Cartesian products. Classical conditions ensuring this existence are still unsatisfactory, for they prevent the use of additive utilities in many applications (see, e.g., Gonzales (1996)) by assuming nonnecessary axioms, namely solvability w.r.t. every component in the algebraic approach (see Krantz (1964),Luce and Tukey (1964) and Fishburn (1970, chapter 5) for unrestricted solvability; see Luce (1966) and Krantz, Luce, Suppes and Tversky (1971, chapter 6) for restricted solvability) and connectedness w.r.t. every component in the topological approach (Debreu (1960); Wakker (1989, chapter 3); Wakker (1993)).

In this paper, some testable axioms are provided, that ensure the existence of additive representations when restricted solvability holds w.r.t. only two components. This generalizes the results of Gonzales (1996) in which existence theorems were stated when unrestricted solvability holds w.r.t. two components. As shown in Gonzales (1997), the results cannot be further extended to the case where restricted or unrestricted solvability holds w.r.t. only one component.

The paper is organized as follows. In section 2, the axioms used in the representation theorems are defined and explained. In particular, a slightly modified version of the classical Archimedean axiom is introduced as well as two nonclassical axioms, namely the *scaling axiom* and the *i-linkness axiom*. The former, which was already introduced in Gonzales (1995) for 3-dimensional Cartesian products, is a part of the second order cancellation axiom and, therefore, is known to be necessary, and the latter is a relaxation of restricted solvability. In section 3, two representation and uniqueness theorems are stated, assuming restricted solvability w.r.t. only 2 components. The first one concerns 3-dimensional Cartesian products, and the second is an extension to n-dimensional Cartesian products, the difference being in the use of the i-linkness axiom.

2 Definitions and Axioms

Throughout this paper, we consider a Cartesian product $X = \prod_{i=1}^n X_i$, $n \geq 3$. As usual, given a binary relation $\succsim$ over X, the indifference relation $x \sim y$ stands for $[x \succsim y$ and $y \succsim x]$, the strict preference relation $x \succ y$ for $[x \succsim y$ and Not$(y \succsim x)]$, and $x \precsim y \Leftrightarrow y \succsim x$. Now, let us see some necessary conditions for the existence of additive utilities representing $\succsim$.

2.1 Necessary conditions

An additive utility is a function $u : X \to \mathbb{R}$ such that, for all $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ in X,

$$x \succsim y \Leftrightarrow u(x) \geq u(y) \text{ and } u(x) = \sum_{i=1}^n u_i(x_i),\ u(y) = \sum_{i=1}^n u_i(y_i). \tag{1}$$

Clearly, the existence of u requires the following two axioms:

Axiom 1 (weak ordering) *$\succsim$ is a weak order of X, i.e. $\succsim$ is complete (for all x,$y \in X$, $x \succsim y$ or $y \succsim x$) and transitive (for all x,y,$z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$).*

Axiom 2 (independence) *For all $i \in \{1, \ldots, n\}$ and all x,$y \in X$,*
if $(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \succsim (y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n)$
then $(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \succsim (y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n)$.

Independence is also referred to as coordinate independence (see Wakker (1989, page 30)). For every set $N \subset \{1, \ldots, n\}$, define a weak order $\succsim_N$ on $\prod_{i \in N} X_i$ as follows: let a and b be two arbitrary elements of $\prod_{i \in N} X_i$, i.e., they are $|N|$-tuples $(a_i)_{i \in N}$ and $b = (b_i)_{i \in N}$. Then $a \succsim_N b$ if and only if $(a, p) \succsim (b, p)$ for some $(n - |N|)$-tuple $p = (p_j)_{j \in \{1,\ldots,n\} \setminus N} \in \prod_{i \notin N} X_i$, where (a, p) stands for the n-tuple $(x_1, \ldots, x_n)$ in which $x_i = a_i$ for all $i \in N$ and $x_i = p_i$ for all $i \notin N$. Independence implies that $\succsim_N$ does not depend on the choice of p. Of particular interest, $\succsim_{\{1,2\}}$ is the weak order induced by $\succsim$ on $X_1 \times X_2$, hereafter denoted by $\succsim_{12}$ for simplicity.

Axiom 3 (Thomsen condition w.r.t. the first 2 components) *For every $x_1, y_1, z_1 \in X_1$, $x_2, y_2, z_2 \in X_2$, if $(x_1, z_2) \sim_{12} (z_1, y_2)$ and $(z_1, x_2) \sim_{12} (y_1, z_2)$, then $(x_1, x_2) \sim_{12} (y_1, y_2)$.*

The above axioms can easily be tested empirically. Unfortunately, they are not sufficient to ensure additive representability. For instance, if $\succsim$ on $X = \{0, 1, 2, 3\}^2$ is represented by u, as defined in table 1, then $\succsim$ satisfies weak ordering; independence holds because $u(x_1, x_2)$ increases

$x_1 \backslash x_2$	0	1	2	3
0	0	3	6	9
1	6	9	12	18
2	15	21	24	27
3	24	30	33	36

Table 1: $u(x_1, x_2)$

with x_1 and x_2; there exist neither distinct $x_1, y_1, z_1 \in X_1$ nor distinct $x_2, y_2, z_2 \in X_2$ such that $(x_1, z_2) \sim (z_1, y_2)$ and $(z_1, x_2) \sim (y_1, z_2)$, so that axiom 3 trivially holds on X. However, no additive utility v represents $\succsim$ else $v_1(0) + v_2(3) = v_1(1) + v_2(1)$, $v_1(1) + v_2(0) = v_1(0) + v_2(2)$ and $v_1(2) + v_2(2) = v_1(3) + v_2(0)$, which would imply that $v_1(2) + v_2(3) = v_1(3) + v_2(1)$, or, equivalently, that $(2, 3) \sim (3, 1)$. And this is impossible since $u(2, 3) = 27 < u(3, 1) = 30$.

The above example highlighted a violation of the following axiom, which can be easily proved to be necessary for additive representability, for all integers $m > 0$:

Axiom 4 (mth-order cancellation axiom) *Consider $m+1$ elements $x^i = (x^i_1, \ldots, x^i_n) \in X$, $i \in \{1, \ldots, m+1\}$. Let $y^1, \ldots, y^{m+1} \in X$ be such that $(y^1_j, \ldots, y^{m+1}_j)$ is a permutation of $(x^1_j, \ldots, x^{m+1}_j)$ for every $j \in \{1, \ldots, n\}$. Then $[x^i \succsim y^i$ for all $i \neq m+1] \Rightarrow x^{m+1} \precsim y^{m+1}$.*

Independence and the Thomsen condition are special cases of cancellation axioms. Note that if the $(m+1)$st-order cancellation axiom holds, the mth-order one also holds. From a theoretical point of view, these axioms are useful since they are necessary for additive representability. But in practical situations, when their order increases, they make utility assessments difficult to perform because they require to test numerous preference relations. One possible way to circumvent this problem is to assume some (nonnecessary) structural conditions which enable to derive any cancellation axiom from independence and the Thomsen condition. In the classical framework, such structural conditions are connectedness of topological spaces (Debreu (1960) and Wakker (1989)), and restricted solvability w.r.t. every component (Krantz et al. (1971)).

2.2 Structural assumption to avoid testing high-order cancellation axioms

Hereafter we require that restricted solvability holds w.r.t. only two components: X_1 and X_2 (Gonzales (1997) showed that when restricted solvability holds only w.r.t. 1 component, the mth-order cancellation axiom does not necessarily imply the $(m+1)$st-order one).

Axiom 5 (restricted solvability w.r.t. the first two components) *For every $x_1, y_1 \in X_1$, $x_2, y_2 \in X_2$, $x_i \in X_i$, $i \in \{3, \ldots, n\}$, and every $z \in X$, if $(x_1, x_2, \ldots, x_n) \precsim z \precsim (y_1, x_2, \ldots, x_n)$, then there exists $t_1 \in X_1$ such that $z \sim (t_1, x_2, \ldots, x_n)$. Similarly, if $(x_1, x_2, x_3, \ldots, x_n) \precsim z \precsim (x_1, y_2, x_3, \ldots, x_n)$, then there exists $t_2 \in X_2$ such that $z \sim (x_1, t_2, x_3, \ldots, x_n)$.*

Usually, the structure induced by restricted solvability is strong enough to ensure that high-order cancellation axioms can be derived from low-order ones. However, this is true only when the solvable components play a role in the preference relation $\succsim$. Hence, the following axiom:

Axiom 6 (essentiality w.r.t. the first 2 components) *For every i in $\{1, 2\}$, there exist $x_i, y_i \in X_i$ and $z \in \prod_{k \neq i} X_k$ such that $(x_i, z) \succ (y_i, z)$.*

Under axioms 1, 5 and 6, all cancellation axioms can be induced from independence and the Thomsen condition in $(X_1 \times X_2, \succsim_{12})$. However, this property cannot be extended to the whole of $(X, \succsim)$, as shown in the next example, which is a variation of Gonzales (1997, subsection 3.4). Let m be an odd number greater than or equal to 3 and consider a preference relation $\succsim$ on $X = [0, \frac{1}{8}] \times [0, \frac{1}{8}] \times \{0, 1, \ldots, m^2\} \times \{0, 2(m-1), 2m\}$, representable by the utility function:

$$u(x_1, x_2, x_3, x_4) = \begin{cases} x_1 + x_2 + x_3 + x_4 & \text{if } x_4 < 2m, \\ x_1 + x_2 + x_3 + m^2 + 2m - 2.5 & \text{if } x_4 = 2m \text{ and } x_3 \text{ even}, \\ x_1 + x_2 + x_3 + m^2 + 2m - 3 & \text{if } x_4 = 2m \text{ and } x_3 \text{ odd}. \end{cases} \quad (2)$$

Restricted solvability, essentiality and the Thomsen condition hold w.r.t. the first two components because $\succsim_{12}$ is representable by $x_1 + x_2$. By definition of u, it is obvious that the first two components satisfy independence, and, more generally, it can be shown that lemma 1 holds.

Lemma 1 *Let m be an odd number ≥ 3, and $\succsim$ on $[0, \frac{1}{8}] \times [0, \frac{1}{8}] \times \{0, \ldots, m^2\} \times \{0, 2(m-1), 2m\}$ be represented by u as defined in (2). Then $\succsim$ satisfies the $(m+1)$st-order cancellation axiom.*

However, the $(m+2)$nd-order cancellation axiom does not hold because

$$\begin{aligned} (0, 0, i(m-1)+1, 0) &\sim (0, 0, (i-2)(m-1)+1, 2(m-1)), & i \in \{2, 4, \ldots, m+1\} \\ (0, 0, 1, 2m) &\sim (0, 0, (m+1)(m-1)+1, 2(m-1)), & \\ (0, 0, (i-2)(m-1), 2(m-1)) &\sim (0, 0, i(m-1), 0), & i \in \{2, 4, \ldots, m+1\} \\ (0, 0, (m+1)(m-1), 2(m-1)) &\prec (0, 0, 0, 2m). & \end{aligned}$$

Therefore restricted solvability w.r.t. only 2 components is not sufficient a structural assumption to avoid testing numerous cancellation axioms (when $|X| = 4$). Why? Simply because, in the above example, $\succsim$ is a lexicographic order, in which the nonsolvable components dominate the solvable ones. Thus the solvable components are too "weak" to propagate any structure to the whole space. This example suggests that trade-offs in the nonsolvable components should be compensated by trade-offs in the solvable ones. Hence the following definition:

Definition 1 (i-link) *Let $i \in \{3, \dots, n\}$. x_i and z_i are i-linked — henceforth denoted $x_i \mathcal{O}_i z_i$ — if there exists a sequence $(y_i^k)_{k=0}^p$ of elements of X_i such that $y_i^0 = x_i$, $y_i^p = z_i$ and such that, for every $k \in \{0, 1, \dots, p-1\}$, there exist $a^{k+1}, b^k \in \prod_{j=1}^{i-1} X_j$ such that $(a^{k+1}, y_i^{k+1}) \sim_{1 \dots i} (b^k, y_i^k)$.*

Note that relation $\mathcal{O}_i$ is an equivalence relation. It is illustrated in figure 1, in which links are represented by edges: let $\succsim$ be represented on $X = [0,1]^2 \times \{1,2,3,4\} \times \{0,78\}$ by utility $[2 + x_1 + x_2]^{x_3} + x_4$. Then, every x_3 and y_3 are 3-linked because $A = (1,1,1,0) \sim B = (0,0,2,0)$,

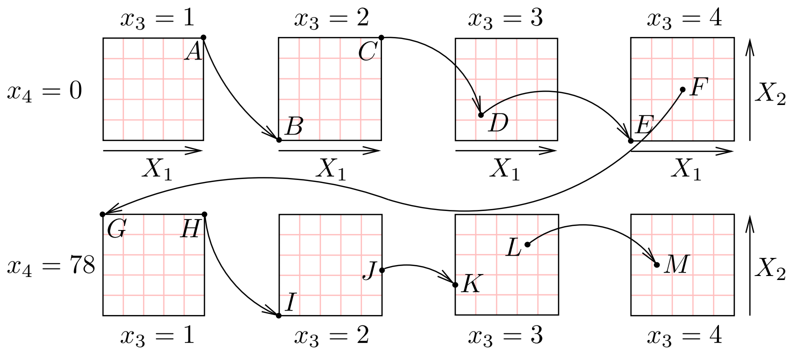

Figure 1: i-linkness.

$C = (1,1,2,0) \sim D = (\frac{1}{4}, 2\sqrt[3]{2} - \frac{9}{4}, 3, 0) \sim E = (0,0,4,0)$, or because $H = (1,1,1,1) \sim I = (0,0,2,1)$, $J = (1, 2\sqrt{3} - 3, 2, 1) \sim K = (0, \sqrt[3]{12} - 2, 3, 1)$, $L = (\frac{3}{4}, \frac{3}{4}, 3, 1) \sim M = (\frac{1}{4}, \sqrt[4]{\frac{343}{8}} - \frac{9}{4}, 4, 1)$. Similarly, $x_4 = 0$ and $x_4 = 78$ are 4-linked because $F = (\frac{1}{2}, \frac{1}{2}, 4, 0) \sim G = (0, 1, 1, 78)$.

Note that if independence holds, links w.r.t. the ith components do not depend on the values assigned to components $(i+1)$ to n. I-linkness propagates through links the structure induced by restricted solvability, to the whole space. In particular, the combination of i-linkness and the second order cancellation axiom implies all the other cancellation axioms.

Axiom 7 (I-linkness) *For every $x_i, z_i \in X_i$, $i \in \{3, \dots, n-1\}$, x_i and z_i are i-linked.*

2.3 Avoiding testing the second-order cancellation axiom

In the preceding subsection, it was mentioned that under i-linkness, essentiality and restricted solvability w.r.t. 2 components the second order cancellation axiom entails cancellation axioms of every order. In this case, it may be wondered whether independence and the Thomsen condition still imply the second order cancellation axiom. Unfortunately, it is no more true when restricted solvability w.r.t. only 2 components, as is shown by the following example.

Let $X = [1,2] \times [1,2] \times \{1,2\}$, and suppose that $\succsim$ is represented by:

$$u(x_1, x_2, x_3) = \begin{cases} x_1 + x_2 & \text{if } x_3 = 1, \\ \exp(x_1 + x_2) - 4 & \text{if } x_3 = 2. \end{cases} \tag{3}$$

It is easily shown that $\succsim$ is a weak order satisfying independence, and the Thomsen condition on $X_1 \times X_2$. However, the second order cancellation axiom cannot hold because $(2, \exp(2) - 6, 1) \sim$

$(1,1,2)$, $(\ln(8)-1,1,2) \sim (2,2,1)$ and Not$[(1,2,1) \sim (\ln(8)-1,\exp(2)-6,1)]$. This can be explained as follows: in spaces where restricted solvability holds, i.e., $[1,2]\times[1,2]\times\{1\}$ and $[1,2]\times[1,2]\times\{2\}$, cancellation axioms of every order are implied by independence and the Thomsen condition. So, a violation of a cancellation axiom can only occur when comparing some elements with distinct nonsolvable components. However, in this case, the Thomsen condition cannot be applied and, whenever $(x_1,x_2,1) \sim (y_1,y_2,2)$, x_1 and x_2 are greater than or equal to $\exp(2)-6 \approx 1.39$, and y_1 and y_2 are less than or equal to $\ln(8)-1 \approx 1.07$, so that independence can only be used through strict preference relations. Unfortunately, such relations are much weaker than those deduced by indifferences because they allow some degree of freedom — which induced the above violation of the cancellation axiom. This suggests determining the parts of X where independence can be used through indifference relations. Hence the following definition:

Definition 2 (directly-single-dimensionally-matching) *Let $a,b \in \prod_{j=3}^n X_j$. If there exist $x_1,y_1 \in X_1$ and $c_2 \in X_2$ such that $(x_1,c_2,a) \sim (y_1,c_2,b)$, or $x_2,y_2 \in X_2$ and $c_1 \in X_1$ such that $(c_1,x_2,a) \sim (c_1,y_2,b)$, then a and b are said to be directly-single-dimensionally-matched.*

To put it another way, the trade-off between x_1 and y_1 or the trade-off between x_2 and y_2 matches the trade-off between a and b; or a change from a to b or from b to a can be compensated by a change in only one solvable component.

Definition 2 is very appealing because if all the nonsolvable components are directly-single-dimensionally-matched, if independence holds, as well as the Thomsen condition, restricted solvability and essentiality w.r.t. at least two components, then every cancellation axiom holds. But it is still not general enough to enable a valuable generalization of the classical representation theorems because restricted solvability may hold while definition 2 does not (for instance when $\succsim$ is representable on $X=[0,1]\times[0,1]\times[0,3]$ by $x_1+x_2+x_3$). Therefore, the following generalization of directly-single-dimensionally-matching shall be introduced:

Definition 3 (single-dimensionally-matching) *Let $a,b \in \prod_{j=3}^n X_j$. a,b are single-dimensionally-matched if there exists a sequence $(y^k)_{k=0}^p$ of elements of $\prod_{j=3}^n X_j$ such that $y^0=a$, $y^p=b$, and y^k and y^{k+1} are directly-single-dimensionally-matched for every $k \in \{0,\dots,p-1\}$.*

A graphical interpretation of this definition is provided in figure 2: consider a preference relation $\succsim$ on $X=[0,10]\times[0,10]\times\{0,6,14,17\}$ representable by the utility function $x_1+x_2+x_3$. Directly-single-dimensionally-matching does not hold for $a=0$ and $b=17$. Yet, 0 and 17 are

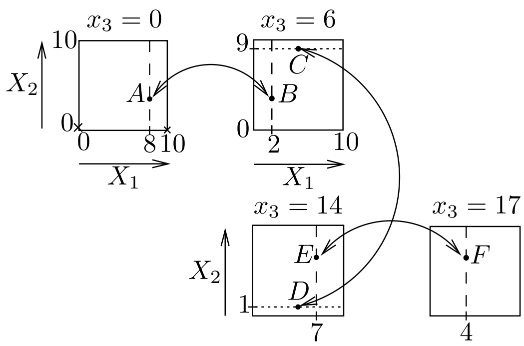

Figure 2: single-dimensionally-matching of 0 and 17.

single-dimensionally-matched because the sequence $y^0=0$, $y^1=6$, $y^2=14$ and $y^3=17$ satisfies the requirement of definition 3: indeed, $(8,x_2,0) \sim (2,x_2,6)$ for every x_2 (see A and B). Similarly, $(x_1,9,6) \sim (x_1,1,14)$ for every x_1 — $C \sim D$ — and $(7,x_2,14) \sim (4,x_2,17)$ — $E \sim F$.

Definition 3 can be viewed as a restriction of i-linkness and as a relaxation of restricted solvability. If restricted solvability w.r.t. components 3 to n is substituted by single-dimensionally-matching (and the classical Archimedean axiom is slightly modified as shown page 6), then independence and axiom 3 imply the second order cancellation axiom. But if there exist a and b not single-dimensionally-matched, must the latter be tested everywhere in X? No, because it can be shown that independence, axiom 3 and the following axiom (which is a strong weakening of the second order cancellation axiom) imply the second order cancellation axiom.

Axiom 8 (scaling axiom) *Let $a, b \in \prod_{j=3}^{n} X_j$. If a and b are not single-dimensionally-matched, then, for every $x_1, y_1, z_1 \in X_1$ and every $x_2, y_2, z_2 \in X_2$,*

$$[(z_1, x_2, a) \sim (x_1, z_2, b) \text{ and } (z_1, y_2, a) \sim (y_1, z_2, b)] \Rightarrow (x_1, y_2, a) \sim (y_1, x_2, a). \quad (4)$$

Equation (4) is a simple generalization of the Thomsen condition.

2.4 Archimedean property

Axioms 1 to 8 are still not sufficient to ensure representability: a well known counterexample is the lexicographic order in $\mathbb{R}^2$. This results from the fact that representability requires that $\succsim$ admits "fewer" indifference classes than there are real numbers. Usually an Archimedean axiom is added to prevent such cases to occur, but in the framework of this paper, this one is not very appealing because it is stated in terms of indifference relations, which may fail to exist due to the nonsolvability of some components. But it can be reformulated in a more flexible way, for example by replacing $\sim$'s by $\succsim$'s (thus allowing some degree of freedom):

Definition 4 (strong standard sequence w.r.t. the ith component) *For any set N of consecutive integers, a set $\{x_1^k$ such that $x_1^k \in X_1, k \in N\}$ is a strong standard sequence w.r.t. the 1st component if and only if either $(x_1^0, x_2^0, \ldots, x_n^0) \prec (x_1^0, x_2^1, \ldots, x_n^1)$ and $(x_1^{k+1}, x_2^0, \ldots, x_n^0) \succsim (x_1^k, x_2^1, \ldots, x_n^1)$ for all $k, k+1 \in N$, or $(x_1^0, x_2^0, \ldots, x_n^0) \succ (x_1^0, x_2^1, \ldots, x_n^1)$ and $(x_1^{k+1}, x_2^0, \ldots, x_n^0) \precsim (x_1^k, x_2^1, \ldots, x_n^1)$ for all $k, k+1 \in N$. The set $\{(x_2^0, \ldots, x_n^0), (x_2^1, \ldots, x_n^1)\}$ is called the mesh of the sequence. Parallel definitions hold w.r.t. the other components.*

Axiom 9 (Strengthened Archimedean axiom w.r.t. ith component) *Any bounded strong standard sequence w.r.t. the ith component is finite, i.e., if $\{x_i^k \in X_i, k \in N\}$ is a strong standard sequence with mesh $\{(x_1^0, \ldots, x_{i-1}^0, x_{i+1}^0, \ldots, x_n^0), (x_1^1, \ldots, x_{i-1}^1, x_{i+1}^1, \ldots, x_n^1)\}$ and if there exist $y, z \in X$ such that $y \precsim (x_1^0, \ldots, x_{i-1}^0, x_i^k, x_{i+1}^0, \ldots, x_n^0) \precsim z$ for all $k \in N$, then N is finite.*

3 Existence and uniqueness theorems

The following theorems extend the classical existence theorem of Krantz et al. (1971).

Theorem 1 *Let $\succsim$ be a binary relation on $X = X_1 \times X_2 \times X_3$. Suppose that $(X, \succsim)$ satisfies: i) weak ordering, independence, the Thomsen condition w.r.t. the first two components, the strengthened Archimedean axiom w.r.t. every component) and the scaling axiom; and ii) restricted solvability w.r.t. the first two components and essentiality w.r.t. the first two components. Then $\succsim$ is representable by an additive utility $u = \sum_{i=1}^{3} u_i$. Moreover, there exist a set N of consecutive integers—finite or infinite—and a sequence $(x_3^i)_{i \in N}$ of elements of X_3 such that:*

- *for all $x_3 \in X_3$, there exists $i \in N$ such that $x_3 \, \mathcal{O}_3 \, x_3^i$,*
- *for all $i, i+1$ in N (if $\text{Card}(N) > 1$), $x_3^{i+1} \succ_3 x_3^i$ and Not$(x_3^i \, \mathcal{O}_3 \, x_3^{i+1})$.*
- *if $\text{Card}(N) > 1$, then u_1 and u_2 are bounded.*

If $v = v_1 + v_2 + v_3$ also represents $\succsim$, then there exist some constants $\alpha > 0$, α_1, α_2 and β_i, $i \in N$, such that:

$$\begin{cases} \textit{for all } x_1 \in X_1, \quad v_1(x_1) = \alpha \cdot u_1(x_1) + \alpha_1 \\ \textit{for all } x_2 \in X_2, \quad v_2(x_2) = \alpha \cdot u_2(x_2) + \alpha_2 \\ \textit{for all } x_3 \, \mathcal{O}_3 \, x_3^i, \quad v_3(x_3) = \alpha \cdot u_3(x_3) + \beta_i \textit{ where} \\ \qquad \beta_{i+1} \geq \beta_i + \alpha \cdot [\sup_{x_1,x_2}\{u_1(x_1) + u_2(x_2)\} + \sup_{y_3 \, \mathcal{O}_3 \, x_3^i} u_3(y_3)] \\ \qquad\qquad - \alpha \cdot [\inf_{x_1,x_2}\{u_1(x_1) + u_2(x_2)\} + \inf_{y_3 \, \mathcal{O}_3 \, x_3^{i+1}} u_3(y_3)] \\ \qquad \textit{with equality only if the } \inf \textit{ and/or the } \sup \textit{ is not attained.} \end{cases}$$

This theorem simply states that additive representability on the space where restricted solvability holds, i.e., $X_1 \times X_2$, can be extended to the whole X. Moreover, additive utilities are cardinal within equivalence classes of $\mathcal{O}_3$. Note that properties i) are all necessary for additive representability whereas properties ii) are only structural. Note also that axiom 7 (i-linkness) is not required — it is useful only for spaces of dimension greater than 3. If restricted solvability holds w.r.t. the three components, then independence implies the second order cancellation axiom; therefore, in this case, the Thomsen condition and the scaling axiom need not be required.

The preceding theorem cannot be straightforwardly extended to $X = \prod_{i=1}^n X_i$. For instance, if $\succsim$ is representable on $X = [0,1]^2 \times \{0,1,2,3\}^2$ by $u(x_1,x_2,x_3,x_4) = x_1 + x_2 + v(x_3,x_4)$ where v is defined by table 1 on page 2, then the strengthened Archimedean axiom holds w.r.t. the first 2 components since $x_1 + x_2$ is additive, and w.r.t. the other components because x_3 and x_4 belong to a finite set; independence holds as well as the Thomsen condition, but violations of the second order cancellation axiom (see page 2) prevent additive representability. For this reason, the i-linkness axiom must be used in representation theorems. This leads to:

Theorem 2 *Suppose that $X = \prod_{i=1}^n X_i$ and that $(X, \succsim)$ satisfies: i) weak ordering, independence, scaling, the strengthened Archimedean axiom w.r.t. every component, the Thomsen condition on $X_1 \times X_2$; and ii) i-linkness, restricted solvability w.r.t. the first two components and essentiality w.r.t. the first two components. Then $\succsim$ is representable by an additive utility $u = \sum_{i=1}^n u_i$. Moreover, there exist a set N of consecutive integers—finite or infinite—and a sequence of elements of X_n, $(x_n^i)_{i \in N}$, such that:*

- *for all $x_n \in X_n$, there exists $i \in N$ such that $x_n \, \mathcal{O}_n \, x_n^i$,*
- *for all $i, i+1$ in N (if $\mathrm{Card}(N) > 1$), $x_n^{i+1} \succ_n x_n^i$ and Not($x_n^i \, \mathcal{O}_n \, x_n^{i+1}$).*
- *if $\mathrm{Card}(N) > 1$, then functions u_i, $i = 1, \ldots, n-1$, are bounded.*

If $v = v_1 + \cdots + v_n$ also represents $\succsim$, then there exist some constants $\alpha > 0$, $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$ and $\beta_i \in \mathbb{R}$, $i \in N$, such that:

$$\begin{cases} \textit{for all } x_i \in X_i, \ i < n, \ v_i(x_i) = \alpha \cdot u_i(x_i) + \alpha_i \\ \textit{for all } x_n \, \mathcal{O}_n \, x_n^i, \qquad v_n(x_n) = \alpha \cdot u_n(x_n) + \beta_i \textit{ where} \\ \qquad \beta_{i+1} \geq \beta_i + \alpha \cdot [\sup_{x_1,\ldots,x_{n-1}}\{\sum_{i=1}^{n-1} u_i(x_i)\} + \sup_{y_n \, \mathcal{O}_n \, x_n^i} u_n(y_n)] \\ \qquad\qquad - \alpha \cdot [\inf_{x_1,\ldots,x_{n-1}}\{\sum_{i=1}^{n-1} u_i(x_i)\} + \inf_{y_n \, \mathcal{O}_n \, x_n^{i+1}} u_n(y_n)] \\ \qquad \textit{with equality only if the } \inf \textit{ and/or the } \sup \textit{ is not attained.} \end{cases}$$

As for the preceding theorem, only properties i) are necessary for additive representability.

4 Appendix: Proofs

Proof of lemma 1: The proof is organized as follows: in step 1, it is shown that independence holds; in step 2, a condition on the third and fourth components is shown to be implied by a violation of the $(m+1)$st-order cancellation axiom by $\succsim$; this condition implies either the violation of this axiom by $\succsim_{12}$ in $X_1 \times X_2$, which is shown to be impossible in step 3, or its violation by $\succsim_{34}$ in $X_3 \times X_4$, which is shown in step 4 to amount to a violation of the $(m+1)$st-order cancellation axiom by another relation $\succsim^*$, which has been proved to be impossible.

first step: independence

$\succsim$ is representable by the utility function $u(x_1, x_2, x_3, x_4) = x_1 + x_2 + v(x_3, x_4)$, where:

$$v(x_3, x_4) = \begin{cases} x_3 + x_4 & \text{if } x_4 < 2m, \\ x_3 + m^2 + 2m - 2.5 & \text{if } x_4 = 2m \text{ and } x_3 \text{ even}, \\ x_3 + m^2 + 2m - 3 & \text{if } x_4 = 2m \text{ and } x_3 \text{ odd}; \end{cases}$$

So, clearly, independence holds w.r.t. the first two components. Let us show that independence also holds w.r.t. the third component. Suppose that, for some $x_1, x_2, x_3, x_4, y_1, y_2, y_4$,

$$(x_1, x_2, x_3, x_4) \succsim (y_1, y_2, x_3, y_4). \tag{5}$$

Then $x_4 \geq y_4$; indeed, in table 2, in which values of v are given for some x_3, x_4, figures increase

$x_3 \backslash x_4$	0	$2(m-1)$	$2m$
0	0	$2m-2$	$m^2+2m-2.5$
1	1	$2m-1$	m^2+2m-2
2	2	$2m$	$m^2+2m-.5$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
m^2-1	m^2-1	m^2+2m-3	$2m^2+2m-3.5$
m^2	m^2	m^2+2m-2	$2m^2+2m-3$

Table 2: Values of v in function of x_3 and x_4.

from left to right, and the difference between any two elements of a same row is greater than 1, so that if $y_4 > x_4$, then $u(y_1, y_2, x_3, y_4) \geq u(0, 0, x_3, y_4) = v(x_3, y_4) > v(x_3, x_4)+1 > v(x_3, x_4)+\frac{1}{4} = u(\frac{1}{8}, \frac{1}{8}, x_3, x_4) \geq u(x_1, x_2, x_3, x_4)$, which would contradict (5). Now, if $x_4 > y_4$, then, for the same reason, $u(x_1, x_2, y_3, x_4) > u(y_1, y_2, y_3, y_4)+\frac{3}{4}$ for all $y_3 \in X_3$, so that (5) implies $(x_1, x_2, y_3, x_4) \succsim (y_1, y_2, y_3, y_4)$ for all y_3. Since x_3 is arbitrary, (5) is equivalent to $(x_1, x_2, y_3, x_4) \succsim (y_1, y_2, y_3, y_4)$ for all y_3. If, on the contrary, $x_4 = y_4$, then (5) is equivalent to

$$\begin{array}{lll} x_1 + x_2 + x_3 + x_4 & \geq y_1 + y_2 + x_3 + x_4 & \text{if } x_4 < 2m, \\ x_1 + x_2 + x_3 + m^2 + 2m - 2.5 & \geq y_1 + y_2 + x_3 + m^2 + 2m - 2.5 & \text{if } x_4 = 2m \text{ and } x_3 \text{ even}, \\ x_1 + x_2 + x_3 + m^2 + 2m - 3 & \geq y_1 + y_2 + x_3 + m^2 + 2m - 3 & \text{if } x_4 = 2m \text{ and } x_3 \text{ odd}. \end{array}$$

For all $x_3 \in X_3$ and for all $x_4 \in X_4$, this is clearly equivalent to $x_1 + x_2 \geq y_1 + y_2$. This last inequality being independent of x_3, (5) is thus equivalent to $(x_1, x_2, y_3, x_4) \succsim (y_1, y_2, y_3, x_4)$ for all $y_3 \in X_3$. Therefore, independence holds w.r.t. the third component.

Independence also holds w.r.t. the fourth component: suppose that

$$(x_1, x_2, x_3, x_4) \succsim (y_1, y_2, y_3, x_4). \tag{6}$$

As in the preceding paragraph, since elements of table 2 increase from top to bottom, and since the difference between any two elements of a same column is greater than or equal to $\frac{1}{2}$, (6) implies that $x_3 \geq y_3$. The case $x_3 = y_3$ has been discussed in the preceding paragraph. If $x_3 > y_3$, then, for all y_4, $v(x_3, y_4) \geq v(y_3, y_4) + \frac{1}{2}$, so that, for all x_1, y_1, x_2, y_2, $u(x_1, x_2, x_3, y_4) \geq u(y_1, y_2, y_3, y_4)$, and, thus, independence holds w.r.t. the fourth component.

second step: condition on the third and fourth components

Let $(x_1^i, x_2^i, x_3^i, x_4^i)$ and $(y_1^i, y_2^i, y_3^i, y_4^i)$, $i \in \{1, \ldots, m+2\}$, be some elements of X such that

- for every $j \in \{1,2,3,4\}$, $(y_j^1, \ldots, y_j^{m+2})$ is a permutation of $(x_j^1, \ldots, x_j^{m+2})$,
- for every $k < m+2$, $(x_1^k, x_2^k, x_3^k, x_4^k) \succsim (y_1^k, y_2^k, y_3^k, y_4^k)$.

If those elements violate the $(m+1)$st-order cancellation axiom, then $(x_1^{m+2}, x_2^{m+2}, x_3^{m+2}, x_4^{m+2}) \succsim (y_1^{m+2}, y_2^{m+2}, y_3^{m+2}, y_4^{m+2})$ and there exists an index i in $\{1, \ldots, m+2\}$ such that $(x_1^i, x_2^i, x_3^i, x_4^i) \succ (y_1^i, y_2^i, y_3^i, y_4^i)$. Without loss of generality, suppose that $i = m+2$ (if it is not the case, just permute index i and $m+2$). Now, since independence holds, $\succsim_{34}$ is well defined and

$$\text{for all } k \in \{1, \ldots, m+2\},\ (x_3^k, x_4^k) \succsim_{34} (y_3^k, y_4^k), \tag{7}$$

else, since whenever two elements of table 2 are nonequal, their difference is greater than or equal to $\frac{1}{2}$, one would get $u(y_1, y_2, y_3^k, y_4^k) \geq u(0,0,y_3^k, y_4^k) = v(y_3^k, y_4^k) \geq v(x_3^k, x_4^k) + \frac{1}{2} = u(0,0,x_3^k,x_4^k) + \frac{1}{2} = u(\frac{1}{8}, \frac{1}{8}, x_3^k, x_4^k) + \frac{1}{4} \geq u(x_1, x_2, x_3^k, x_4^k) + \frac{1}{4}$, which would make $(x_1^k, x_2^k, x_3^k, x_4^k) \succsim (y_1^k, y_2^k, y_3^k, y_4^k)$ impossible.

third step: if $(x_3^k, x_4^k) \sim_{34} (y_3^k, y_4^k)$ for every $k \in \{1, \ldots, m+2\}$

Then, substituting (y_3^k, y_4^k) by (x_3^k, x_4^k) for all $k \in \{1, \ldots, m+2\}$, one gets $m+2$ elements $(x_1^k, x_2^k, x_3^k, x_4^k)$ of X and $m+2$ other elements $(y_1^k, y_2^k, x_3^k, x_4^k)$ such that

- for all $j \in \{1,2\}$, $(y_j^1, \ldots, y_j^{m+2})$ is a permutation of $(x_j^1, \ldots, x_j^{m+2})$,
- for all $k < m+2$, $(x_1^k, x_2^k, x_3^k, x_4^k) \succsim (y_1^k, y_2^k, x_3^k, x_4^k)$, and so $(x_1^k, x_2^k) \succsim_{12} (y_1^k, y_2^k)$,
- $(x_1^{m+2}, x_2^{m+2}, x_3^{m+2}, x_4^{m+2}) \succ (y_1^{m+2}, y_2^{m+2}, x_3^{m+2}, x_4^{m+2})$ and $(x_1^{m+2}, x_2^{m+2}) \succ_{12} (y_1^{m+2}, y_2^{m+2})$.

Therefore, $\succsim_{12}$ also violates the $(m+1)$st-order cancellation axiom. However, this is impossible because $\succsim_{12}$ is representable by the additive utility function $x_1 + x_2$, and because cancellation axioms of any order are necessary for additive representability.

fourth step: if there exists k such that $(x_3^k, x_4^k) \succ_{34} (y_3^k, y_4^k)$

By (7), the x_3^i's and x_4^i's are such that

- for all $j \in \{3,4\}$, $(y_j^1, \ldots, y_j^{m+2})$ is a permutation of $(x_j^1, \ldots, x_j^{m+2})$,
- for all $k < m+2$, $(x_3^k, x_4^k) \succsim_{34} (y_3^k, y_4^k)$,
- $(x_3^{m+2}, x_4^{m+2}) \succ_{34} (y_3^{m+2}, y_4^{m+2})$,

which is a violation of the $(m+1)$st-order cancellation axiom by $\succsim_{34}$.

In Gonzales (1997), it was showed that, for every odd number m greater than or equal to 3, if a preference relation $\succsim^*$ is representable on $Y = \mathbb{R} \times \{0, 2, 4, \ldots, 2m\}$ by the following utility:

$$w(x_1, x_2) = \begin{cases} x_1 + x_2 & if \quad x_2 < 2m \\ .5(x_1 \bmod 2)^2 + 2(x_1 \operatorname{div} 2) + m^2 + 2m - 2.5 & if \quad x_2 = 2m, \end{cases}$$

then the $(m+1)$st-order cancellation axiom holds. Here, $X_3 = \{0, \ldots, m^2\} \subset \mathbb{R}$ and $X_4 = \{0, 2(m-1), 2m\} \subset \{0, 2, 4, \ldots, 2m\}$, so that $X_3 \times X_4 \subset Y$. Moreover, for every $x_3, x_4 \in X_3 \times X_4$, $v(x_3, x_4) = w(x_3, x_4)$. Therefore, the violation of the $(m+1)$st-order cancellation axiom by $\succsim_{34}$ implies also its violation by $\succsim^*$ in Y. However, this is impossible according to Gonzales (1997, subsection 3.4). Thus, $\succsim$ necessarily satisfies the $(m+1)$st-order cancellation axiom, and lemma 1 holds. ∎

Lemma 2 *Let $X = X_1 \times X_2 \times X_3$ be such that $\succsim$ satisfies axioms 1 (weak ordering), 2 (independence) and 5 (restricted solvability w.r.t. the first 2 components). Then, whenever $(x_1, x_2, x_3) \precsim (y_1, y_2, y_3) \precsim (z_1, z_2, x_3)$, there exists $(a_1, a_2) \in X_1 \times X_2$ such that $(a_1, a_2, x_3) \sim (y_1, y_2, y_3)$.*

Proof of lemma 2: $\succsim_1$ and $\succsim_2$ are well defined by independence. Suppose that $(x_1, x_2, x_3) \precsim (y_1, y_2, y_3) \precsim (z_1, z_2, x_3)$. Then either $x_1 \precsim_1 z_1$ or $x_2 \precsim_2 z_2$.

If $x_1 \precsim_1 z_1$, then either $(x_1, z_2, x_3) \precsim (y_1, y_2, y_3)$, in which case $(x_1, z_2, x_3) \precsim (y_1, y_2, y_3) \precsim (z_1, z_2, x_3)$, which implies by axiom 5 that there exists a_1 such that $(a_1, z_2, x_3) \sim (y_1, y_2, y_3)$, or $(x_1, z_2, x_3) \succ (y_1, y_2, y_3)$, in which case $(x_1, x_2, x_3) \precsim (y_1, y_2, y_3) \prec (x_1, z_2, x_3)$, which implies by axiom 5 that there exists a_2 such that $(x_1, a_2, x_3) \sim (y_1, y_2, y_3)$. A similar proof holds when $x_2 \precsim_2 z_2$. ■

Lemma 3 *Let $X = X_1 \times X_2 \times \{a, b\}$ be such that $(X, \succsim)$ satisfies axioms 1 (weak ordering), 2 (independence), and 5 (restricted solvability w.r.t. the first 2 components), and be such that $a \prec_3 b$. Define $Y = \{(y_1, y_2, y_3) \in X : \text{there exists } (x_1, x_2, x_3) \in X \text{ such that } (x_1, x_2, x_3) \sim (y_1, y_2, y_3)$ and $y_3 \neq x_3\}$. If $Y \neq \emptyset$ and $X \backslash Y \neq \emptyset$ then, for all $(y_1, y_2, y_3) \in Y$ and all $(x_1, x_2, a), (z_1, z_2, b) \in X \backslash Y$, $(x_1, x_2, a) \prec (y_1, y_2, y_3) \prec (z_1, z_2, b)$.*

Proof of lemma 3: If $Y = \emptyset$ or $X \backslash Y = \emptyset$ then lemma 3 trivially holds; so suppose that $Y \neq \emptyset$ and $X \backslash Y \neq \emptyset$. Let $(x_1, x_2, a) \in X \backslash Y$ and $(y_1, y_2, y_3) \in Y$ be such that $(y_1, y_2, y_3) \precsim (x_1, x_2, a)$. Since $(y_1, y_2, y_3) \in Y$, there exists $(z_1, z_2, b) \in Y$ such that $(z_1, z_2, b) \sim (y_1, y_2, y_3)$. Therefore, $(z_1, z_2, b) \precsim (x_1, x_2, a)$. But by definition of $\prec_3$ and independence, and since $a \prec_3 b$, $(x_1, x_2, a) \prec (x_1, x_2, b)$; so, the following preference relation holds: $(z_1, z_2, b) \precsim (x_1, x_2, a) \prec (x_1, x_2, b)$. But then, by lemma 2, there exist b_1, b_2 such that $(x_1, x_2, a) \sim (b_1, b_2, b)$, hence contradicting the assumption that $(x_1, x_2, a) \in X \backslash Y$. A similar proof holds for $(y_1, y_2, y_3) \prec (z_1, z_2, b)$. ■

Lemma 4 *Let $X = X_1 \times X_2 \times \{a, b\}$. If $(X, \succsim)$ satisfies axiom 1 (ordering), 2 (independence), 3 (Thomsen condition w.r.t. the first 2 components), 5 (restricted solvability w.r.t. the first 2 components), 6 (essentiality w.r.t. the first two components), 9 (strengthened Archimedean axiom) and 8 (scaling), then $\succsim$ is representable by an additive utility u. If v is another additive utility representing $\succsim$ and if $a \precsim_3 b$ then there exist some constants $\alpha > 0$, α_1, α_2, α_3, α_4 such that:*

$$\begin{cases} \textit{for all } x_1 \in X_1, \ \ v_1(x_1) = \alpha \cdot u_1(x_1) + \alpha_1 \\ \textit{for all } x_2 \in X_2, \ \ v_2(x_2) = \alpha \cdot u_2(x_2) + \alpha_2 \\ \textit{if there exist } \ x_1, y_1 \in X_1 \textit{ and } x_2, y_2 \in X_2 \textit{ such that } (x_1, x_2, a) \sim (y_1, y_2, b) \\ \quad \textit{then} \ \ v_3(x_3) = \alpha \cdot u_3(x_3) + \alpha_3, \textit{ for all } x_3 \in \{a, b\} \\ \quad \textit{else} \ \ v_3(a) = \alpha \cdot u_3(a) + \alpha_3 \textit{ and } v_3(b) = \alpha \cdot u_3(b) + \alpha_4 \\ \qquad\qquad \textit{where } \alpha_4 \geq \alpha_3 + \alpha \cdot [u_3(a) + \sup_{x_1, x_2}\{u_1(x_1) + u_2(x_2)\}] \\ \qquad\qquad\qquad - \alpha \cdot [u_3(b) + \inf_{x_1, x_2}\{u_1(x_1) + u_2(x_2)\}] \\ \qquad\qquad \textit{with equality only if the } \inf \textit{ and/or the } \sup \textit{ is not attained.} \end{cases}$$

Proof of lemma 4: The proof is constructive: in a first step, the existence of an additive utility representing $\succsim_{12}$ on $X_1 \times X_2$ is proved; then, using the definition of $\succsim_{12}$, it is shown that $\succsim$ is representable by an additive utility on $X_1 \times X_2 \times \{a\}$ and on $X_1 \times X_2 \times \{b\}$. In the second step, this utility is extended to represent $\succsim$ on X. For this purpose, we show that if an additive utility, say u, exists, then some relationships between $u_3(a)$ and $u_3(b)$ must hold; then, these relationships are shown to be not only necessary but also sufficient for additive representability.

first step: $\succsim_{12}$ is representable by an additive utility

Independence holds, so $\succsim_{12}$ is well defined. According to the hypotheses of lemma 4, $(X_1 \times X_2, \succsim_{12})$ satisfies the conditions of (the classical) theorem 13 of Krantz et al. (1971, page 302) which ensures that $\succsim_{12}$ is representable by an additive utility u, unique up to scale and location.

According to the definition of $\succsim_{12}$ given on page 2 and by independence w.r.t. the third component, $(x_1, x_2) \succsim_{12} (y_1, y_2) \Leftrightarrow (x_1, x_2, a) \succsim (y_1, y_2, a) \Leftrightarrow (x_1, x_2, b) \succsim (y_1, y_2, b)$. So, for all arbitrary real numbers $u_3(a)$ and $u_3(b)$, and all $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$, the following hold:

$$\begin{aligned}(x_1, x_2, a) \succsim (y_1, y_2, a) &\Leftrightarrow u_1(x_1) + u_2(x_2) + u_3(a) \geq u_1(y_1) + u_2(y_2) + u_3(a),\\ (x_1, x_2, b) \succsim (y_1, y_2, b) &\Leftrightarrow u_1(x_1) + u_2(x_2) + u_3(b) \geq u_1(y_1) + u_2(y_2) + u_3(b),\end{aligned}$$

and, moreover, u_1, u_2, u_3 on $\{a\}$ and u_3 on $\{b\}$, are unique up to scale and location. Now, in order to get an additive utility on X, it is sufficient (and necessary) to choose $u_3(a)$ and $u_3(b)$ such that, for all $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$ and all $x_3, y_3 \in \{a, b\}$ such that $x_3 \neq y_3$,

$$(x_1, x_2, x_3) \succsim (y_1, y_2, y_3) \Leftrightarrow u_1(x_1) + u_2(x_2) + u_3(x_3) \geq u_1(y_1) + u_2(y_2) + u_3(y_3). \tag{8}$$

The remainder of the proof consists in showing that such $u_3(a)$ and $u_3(b)$ exist. Without loss of generality, we will assume hereafter that $a \prec_3 b$.

second step: extension of u to represent $\succsim$ on the whole X

First case: if there exists no $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$ such that $(x_1, x_2, a) \sim (y_1, y_2, b)$:

By lemma 2, for all $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$, $(x_1, x_2, a) \prec (y_1, y_2, b)$. Suppose now that u_1 is unbounded from above; then, for every real number r, there exists an element $x_1(r) \in X_1$ such that $u_1(x_1(r)) \geq r$. By hypothesis of lemma 4, essentiality holds w.r.t. the second component, so there exist $x_2^0, x_2^1 \in X_2$ such that $x_2^1 \succ_2 x_2^0$. Accordingly, for every $x_1^0 \in X_1$, $(x_1^0, x_2^0, a) \prec (x_1^0, x_2^1, a)$. But then, if $r = u_2(x_2^1) - u_2(x_2^0) + u_1(x_1^0)$, we know that there exists $x_1^1 = x_1(r)$ in X_1 such that $u_1(x_1^1) \geq r = u_2(x_2^1) - u_2(x_2^0) + u_1(x_1^0)$, or, equivalently, such that $u_1(x_1^1) + u_2(x_2^0) \geq u_1(x_1^0) + u_2(x_2^1)$. u being a utility function on $X_1 \times X_2 \times \{a\}$, the last inequality is equivalent to $(x_1^0, x_2^1, a) \precsim (x_1^1, x_2^0, a)$. X being a Cartesian product, (x_1^1, x_2^1, a) belongs to X. Iterating the above process with $r = u_2(x_2^1) - u_2(x_2^0) + u_1(x_1^1)$, one can find an element $x_1^2 \in X_1$ such that $(x_1^1, x_2^1, a) \precsim (x_1^2, x_2^0, a)$. By induction, one can find an infinite sequence $(x_1^k)_{k \in \mathbb{N}}$ such that $x_2^1 \succ_2 x_2^0$ and $(x_1^k, x_2^1, a) \precsim (x_1^{k+1}, x_2^0, a)$ for all $k \in \mathbb{N}$.

Therefore, by definition 4, $(x_1^k)_{k \in \mathbb{N}}$ is a strong standard sequence, infinite and bounded by (x_1^0, x_2^0, a) and by (x_1^0, x_2^0, b) (by the conclusion of the preceding paragraph). But this is impossible because it would contradict the strengthened Archimedean axiom (which holds according to the hypotheses of lemma 4). Therefore, u_1 cannot be unbounded from above. Similarly, it can be shown that u_1 is bounded from below. So u_1 is bounded. By symmetry, u_2 is also bounded.

Now let $\gamma = \sup_{x_1, x_2}\{u_1(x_1) + u_2(x_2)\}$ and $\delta = \inf_{x_1, x_2}\{u_1(x_1) + u_2(x_2)\}$. Suppose that $u_3(b) < u_3(a) + \gamma - \delta$; let $\xi \in \mathbb{R}_+^*$ be such that $u_3(b) = u_3(a) + \gamma - \delta - \xi$. By definition of the inf and the sup, for every $\epsilon \in \mathbb{R}_+^*$, there exist $x_1^\gamma(\epsilon), x_2^\gamma(\epsilon), x_1^\delta(\epsilon), x_2^\delta(\epsilon)$ such that $u_1(x_1^\gamma(\epsilon)) + u_2(x_2^\gamma(\epsilon)) \geq \gamma - \epsilon$ and $u_1(x_1^\delta(\epsilon)) + u_2(x_2^\delta(\epsilon)) \leq \delta + \epsilon$. Now, for $\epsilon < \xi/2$, $u_1(x_1^\delta(\epsilon)) + u_2(x_2^\delta(\epsilon)) + u_3(b) \leq u_3(b) + \delta + \epsilon < u_3(b) + \delta + \xi/2 = u_3(a) + \gamma - \xi/2 \leq u_3(a) + \gamma - \epsilon \leq u_1(x_1^\gamma(\epsilon)) + u_2(x_2^\gamma(\epsilon)) + u_3(a)$. Therefore u cannot represent $\succsim$ because $(x_1^\gamma(\epsilon), x_2^\gamma(\epsilon), a) \prec (x_1^\delta(\epsilon), x_2^\delta(\epsilon), b)$ by the first paragraph. So, a necessary condition for u to represent $\succsim$ is that:

$$u_3(a) + \gamma - \delta \leq u_3(b). \tag{9}$$

If both γ and δ are attained, then there exist $x_1^\gamma, x_1^\delta \in X_1$ and $x_2^\gamma, x_2^\delta \in X_2$, such that $u_1(x_1^\gamma) + u_2(x_2^\gamma) = \gamma$ and $u_1(x_1^\delta) + u_2(x_2^\delta) = \delta$. To represent $\succsim$, $u_3(b)$ cannot be equal to $u_3(a) + \gamma - \delta$ else $(x_1^\gamma, x_2^\gamma, a) \prec (x_1^\delta, x_2^\delta, b)$ and $u_1(x_1^\gamma) + u_2(x_2^\gamma) + u_3(a) = u_1(x_1^\delta) + u_2(x_2^\delta) + u_3(b)$. So, in this case, a necessary condition is that

$$u_3(a) + \gamma - \delta < u_3(b). \tag{10}$$

It is also a sufficient condition because, by definition of sup and inf, (10) implies that, for all x_1, x_2, y_1, y_2, $u_1(x_1) + u_2(x_2) + u_3(a) \le u_1(x_1^\gamma) + u_2(x_2^\gamma) + u_3(a) < u_1(x_1^\delta) + u_2(x_2^\delta) + u_3(b) \le u_1(y_1) + u_2(y_2) + u_3(b)$, and because $(x_1, x_2, a) \precsim (x_1^\gamma, x_2^\gamma, a) \prec (x_1^\delta, x_2^\delta, b) \precsim (y_1, y_2, b)$ since u, as defined by (10), is a utility function on $X_1 \times X_2 \times \{a\}$ and on $X_1 \times X_2 \times \{b\}$.

If γ is not attained then, for all x_1, x_2, (9) implies that $u_1(x_1) + u_2(x_2) + u_3(a) < \gamma + u_3(a) \le \delta + u_3(b) \le u_1(y_1) + u_2(y_2) + u_3(b)$, and so u represents $\succsim$ on X. And similarly if δ is not attained.

To summarize, a necessary and sufficient condition for u to represent $\succsim$ on X is that

- $u_3(a) + \gamma - \delta < u_3(b)$ if both γ and δ are attained,
- $u_3(a) + \gamma - \delta \le u_3(b)$ if at least one of the bounds, γ or δ, is not attained.

Second case: there exists $(x_1^0, x_1^1, x_2^0, x_2^1)$ such that $(x_1^0, x_2^0, a) \sim (x_1^1, x_2^1, b)$:

first substep: bounding the working space

In the following, for any finite set $A_1 \subset X_1$, $\min_{\succsim_1} A_1$ denotes an element x_1 of A_1 such that $x_1 \precsim_1 y_1$ for all $y_1 \in A_1$. Parallel definitions hold for $\max_{\succsim_1}$, $\min_{\succsim_2}$, and $\max_{\succsim_2}$. Moreover, for all $x_i, y_i \in X_i$, $[x_i, y_i]$ denotes the set $\{z_i \in X_i : x_i \precsim_i z_i \precsim_i y_i\}$.

Clearly a necessary condition for the existence of an additive utility is the following equation:

$$u_3(b) = u_3(a) + u_1(x_1^0) - u_1(x_1^1) + u_2(x_2^0) - u_2(x_2^1). \tag{11}$$

If a and b are single-dimensionally-matched in X, then either i) there exist z_1^1, z_1^2 such that $z_1^2 \succ_1 z_1^1$ and $(z_1^2, z_2, a) \sim (z_1^1, z_2, b)$ and, by essentiality, there also exist z_2^1, z_2^2 such that $z_2^2 \succ_2 z_2^1$, or ii) there exist z_2^1, z_2^2 such that $z_2^2 \succ_2 z_2^1$ and $(z_1, z_2^2, a) \sim (z_1, z_2^1, b)$ and, by essentiality, there also exist z_1^1, z_1^2 such that $z_1^2 \succ_1 z_1^1$. Else, by essentiality, there exist z_1^1, z_1^2, z_2^1 and z_2^2 such that $z_1^2 \succ_1 z_1^1$ and $z_2^2 \succ_2 z_2^1$. Now, if, for any couple of arbitrary elements of X, (x_1^2, x_2^2, a) and (x_1^3, x_2^3, b), there exist real numbers $u_3(a)$ and $u_3(b)$ such that u represents $\succsim$ on $Z = Z_1 \times Z_2 \times \{a, b\}$, where

$$\begin{array}{lll} Z_1 = [\underline{z_1}, \overline{z_1}], & \underline{z_1} = \min_{\succsim_1}\{x_1^0, x_1^1, x_1^2, x_1^3, z_1^1, z_1^2\}, & \overline{z_1} = \max_{\succsim_1}\{x_1^0, x_1^1, x_1^2, x_1^3, z_1^1, z_1^2\}, \\ Z_2 = [\underline{z_2}, \overline{z_2}], & \underline{z_2} = \min_{\succsim_2}\{x_2^0, x_2^1, x_2^2, x_2^3, z_2^1, z_2^2\}, & \overline{z_2} = \max_{\succsim_2}\{x_2^0, x_2^1, x_2^2, x_2^3, z_2^1, z_2^2\}, \end{array}$$

then u, as defined by (11), represents $\succsim$ on X; indeed, since (x_1^0, x_2^0, a) and (x_1^1, x_2^1, b) belong to Z, u represents $\succsim$ on Z only if (11) holds, so that the real numbers $u_3(a)$ and $u_3(b)$ do not depend on (x_1^2, x_2^2, a) and (x_1^3, x_2^3, b); consequently, u is well defined on X. Moreover, since (x_1^2, x_2^2, a) and (x_1^3, x_2^3, b) are arbitrary, u represents $\succsim$ on X. z_1^1, z_1^2, z_2^1 and z_2^2 are needed i) to ensure that $\overline{z_1} \succ_1 \underline{z_1}$ and $\overline{z_2} \succ_2 \underline{z_2}$, so that standard sequences of mesh $(\overline{z_2}, \underline{z_2})$ can be constructed in case 2.1 and case 2.2; and ii) to ensure that case 2.3 occurs only when a and b are not single-dimensionally-matched.

Now, to prove that u, as defined by (11), represents $\succsim$ on X, it is sufficient to prove that it is representing on Z for any couple (x_1^2, x_2^2, a), (x_1^3, x_2^3, b).

second substep: restricting the working space

Let (x_1^2, x_2^2, a) and (x_1^3, x_2^3, b) be arbitrary elements of X, and let Z be defined as in the preceding substep. Let $Y = \{(x_1, x_2, x_3) \in Z :$ there exists $(y_1, y_2, y_3) \in Z$ such that $(y_1, y_2, y_3) \sim (x_1, x_2, x_3)$ and $y_3 \neq x_3\}$. Now, suppose that there exist real numbers $u_3(a)$ and $u_3(b)$ such that u represents $\succsim$ on Y; then, since $(x_1^0, x_2^0, a), (x_1^1, x_2^1, b) \in Y$, (11) holds, so that $u_3(a)$ and $u_3(b)$ still do not depend on the choice of (x_1^2, x_2^2, a) and (x_1^3, x_2^3, b). Moreover, u also represents $\succsim$ on Z: indeed, we already know that it represents $\succsim$ on $X_1 \times X_2 \times \{a\}$ and on $X_1 \times X_2 \times \{b\}$; so, it only remains to prove that (8) — see page 11 — holds for all $(x_1, x_2, x_3) \in Z$ and all $(y_1, y_2, y_3) \in Z$ such that $x_3 \neq y_3$. Two subcases shall be examined:

first subcase: at least one element, (x_1, x_2, x_3) or (y_1, y_2, y_3), belongs to Y:

If $(x_1, x_2, x_3) \in Y$, then, by definition of Y, there exists $(z_1, z_2, y_3) \in Y$ such that $(z_1, z_2, y_3) \sim (x_1, x_2, x_3)$. But $u_1(x_1) + u_2(x_2) + u_3(x_3) = u_1(z_1) + u_2(z_2) + u_3(y_3)$ because, by hypothesis, u represents $\succsim$ on Y. But u is also representing on $X_1 \times X_2 \times \{y_3\}$, so $(z_1, z_2, y_3) \succsim (y_1, y_2, y_3) \Leftrightarrow u_1(z_1) + u_2(z_2) + u_3(y_3) \geq u_1(y_1) + u_2(y_2) + u_3(y_3)$. Therefore, $(x_1, x_2, x_3) \succsim (y_1, y_2, y_3) \Leftrightarrow u_1(x_1)+u_2(x_2)+u_3(x_3) \geq u_1(y_1)+u_2(y_2)+u_3(y_3)$. A similar proof would hold if $(y_1, y_2, y_3) \in Y$.

second subcase: $(x_1, x_2, x_3), (y_1, y_2, y_3) \in Z \backslash Y$*:*

$Y \neq \emptyset$ and $Z\backslash Y \neq \emptyset$ because $(x_1^0, x_2^0, a) \in Y$ and $(x_1, x_2, x_3) \in Z\backslash Y$. So, by lemma 3, (x_3, y_3) equals (b, a) and $(x_1, x_2, b) \succ (x_1^1, x_2^1, b) \sim (x_1^0, x_2^0, a) \succ (y_1, y_2, a)$. Since u represents $\succsim$ on $X_1 \times X_2 \times \{b\}$, $u_1(x_1) + u_2(x_2) + u_3(b) > u_1(x_1^1) + u_2(x_2^1) + u_3(b)$; since u represents $\succsim$ on Y, $u_1(x_1^1) + u_2(x_2^1) + u_3(b) = u_1(x_1^0) + u_2(x_2^0) + u_3(a)$; since u represents $\succsim$ on $X_1 \times X_2 \times \{a\}$, $u_1(x_1^0)+u_2(x_2^0)+u_3(a) > u_1(y_1)+u_2(y_2)+u_3(a)$. By transitivity of $\geq$, $u_1(x_1)+u_2(x_2)+u_3(x_3) \geq u_1(y_1) + u_2(y_2) + u_3(y_3)$.

Therefore, in all cases, if u, as defined in (11), represents $\succsim$ on Y, then it also represents $\succsim$ on Z, and consequently on X. So, in the remainder of the proof it is sufficient to show that u represents $\succsim$ on Y, for all (x_1^2, x_2^2, a) and (x_1^3, x_2^3, b).

third substep: proof that u represents $\succsim$ *on Y*

Case 2.1: if $(\underline{z_1}, \underline{z_2}, b) \precsim (\overline{z_1}, \underline{z_2}, a)$: (12)

This case will be studied in 3 subsubsteps. Figure 3 shows the different elements of Z examined in each subsubstep.

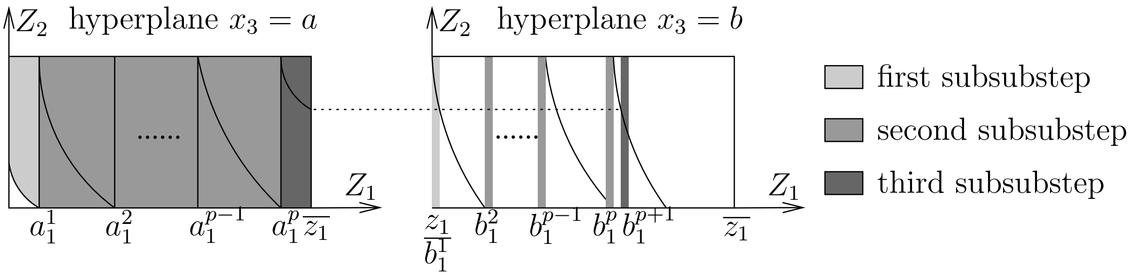

Figure 3: Areas generated by the subsubsteps of case 2.1

first subsubstep: generating the ▢ *area*

Since $a \prec_3 b$, $(\underline{z_1}, \underline{z_2}, a) \prec (\underline{z_1}, \underline{z_2}, b)$, which, by (12), implies that $(\underline{z_1}, \underline{z_2}, a) \prec (\underline{z_1}, \underline{z_2}, b) \precsim (\overline{z_1}, \underline{z_2}, a)$. So, by axiom 5, there exists $a_1^1 \in Z_1$ such that

$$(a_1^1, \underline{z_2}, a) \sim (\underline{z_1}, \underline{z_2}, b). \tag{13}$$

Assume that $u_3(b) = u_3(a)+u_1(a_1^1)-u_1(\underline{z_1})$. This is clearly necessary for additive representability. Now, by independence, for all $z_2 \in Z_2$,

$$(a_1^1, z_2, a) \sim (\underline{z_1}, z_2, b), \text{ and } u_1(a_1^1) + u_2(z_2) + u_3(a) = u_1(\underline{z_1}) + u_2(z_2) + u_3(b). \tag{14}$$

Consider any $(z_1, z_2, a) \in Y$ such that $z_1 \precsim_1 a_1^1$. By definition of $\underline{z_1}$ and $\underline{z_2}$, $(z_1, z_2, a) \succsim (\underline{z_1}, \underline{z_2}, b)$. By (13) and since $z_1 \precsim_1 a_1^1$, $(a_1^1, \underline{z_2}, a) \precsim (z_1, z_2, a) \precsim (a_1^1, z_2, a)$, which implies by axiom 5 that there exists $y_2 \in Z_2$ such that $(z_1, z_2, a) \sim (a_1^1, y_2, a)$. u representing $\succsim$ on $Z_1 \times Z_2 \times \{a\}$, the last indifference relation is equivalent to $u_1(z_1) + u_2(z_2) + u_3(a) = u_1(a_1^1) + u_2(y_2) + u_3(a)$. Now, by (14) and transitivity of $\succsim$ and $=$,

$$(z_1, z_2, a) \sim (\underline{z_1}, y_2, b) \text{ and } u_1(z_1) + u_2(z_2) + u_3(a) = u_1(\underline{z_1}) + u_2(y_2) + u_3(b). \tag{15}$$

So, to summarize, for all $(z_1, z_2, a) \in Y$ such that $z_1 \precsim_1 a_1^1$, there exists $y_2 \in Z$ such that (15) holds. This corresponds to the area in figure 3.

second subsubstep: generating the area by induction

Suppose that, for $k \geq 1$, sequences $(a_1^1, \ldots, a_1^k)$ and $(b_1^1, \ldots, b_1^k)$ exist such that

- for all r such that $2 \leq r \leq k$, $(a_1^{r-1}, \overline{z_2}, a) \sim (a_1^r, \underline{z_2}, a)$ and $u_1(a_1^{r-1}) + u_2(\overline{z_2}) + u_3(a) = u_1(a_1^r) + u_2(\underline{z_2}) + u_3(a)$; (16)
- for all r such that $1 \leq r \leq k$, $(a_1^r, \underline{z_2}, a) \sim (b_1^r, \underline{z_2}, b)$ and $u_1(a_1^r) + u_2(\underline{z_2}) + u_3(a) = u_1(b_1^r) + u_2(\underline{z_2}) + u_3(b)$; (17)
- for all $(z_1, z_2, a) \in Y$ such that $z_1 \precsim_1 a_1^k$, there exist $y_1 \in Z_1, y_2 \in Z_2$ such that $(z_1, z_2, a) \sim (y_1, y_2, b)$ and $u(z_1) + u_2(z_2) + u_3(a) = u_1(y_1) + u_2(y_2) + u_3(b)$. (18)

Note that, by defining $b_1^1 = \underline{z_1}$, the existence of sequences (a_1^r) and (b_1^r) has been shown for $k = 1$ in the first subsubstep (since (16) trivially holds because there exists no r such that $2 \leq r \leq 1$). Now, suppose that there exists $a_1^{k+1} \in Z_1$ such that

$$(a_1^k, \overline{z_2}, a) \sim (a_1^{k+1}, \underline{z_2}, a), \tag{19}$$

otherwise go to the third subsubstep. As u represents $\succsim$ on $Z_1 \times Z_2 \times \{a\}$,

$$u_1(a_1^k) + u_2(\overline{z_2}) + u_3(a) = u_1(a_1^{k+1}) + u_2(\underline{z_2}) + u_3(a). \tag{20}$$

By independence, (17), (19), and since $\underline{z_2} \prec_2 \overline{z_2}$ and $a \prec_3 b$, $(b_1^k, \underline{z_2}, b) \prec (b_1^k, \overline{z_2}, b) \sim (a_1^k, \overline{z_2}, a) \prec (a_1^k, \overline{z_2}, b) \sim (a_1^{k+1}, \underline{z_2}, b)$, which implies by axiom 5 that there exists b_1^{k+1} such that $(b_1^k, \overline{z_2}, b) \sim (b_1^{k+1}, \underline{z_2}, b)$. As u represents $\succsim$ on $Z_1 \times Z_2 \times \{b\}$, $u_1(b_1^k) + u_2(\overline{z_2}) + u_3(b) = u_1(b_1^{k+1}) + u_2(\underline{z_2}) + u_3(b)$. Combining (17), (19), (20) and the last two equations, one gets

$$(a_1^{k+1}, \underline{z_2}, a) \sim (b_1^{k+1}, \underline{z_2}, b) \text{ and } u_1(a_1^{k+1}) + u_2(\underline{z_2}) + u_3(a) = u_1(b_1^{k+1}) + u_2(\underline{z_2}) + u_3(b). \tag{21}$$

To summarize, (16) and (17), already satisfied by sequences $(a_1^r)_{r=1}^k$ and $(b_1^r)_{r=1}^k$, are also satisfied by a_1^{k+1} and b_1^{k+1}. Now, let us show that (18) also holds w.r.t. a_1^{k+1}. Consider any $(z_1, z_2, a) \in Y$ such that $a_1^k \precsim_1 z_1 \precsim_1 a_1^{k+1}$. Then two subcases can occur:

first subcase: if $(z_1, z_2, a) \precsim (a_1^k, \overline{z_2}, a)$ then, since $a_1^k \precsim_1 z_1$, $(a_1^k, z_2, a) \precsim (z_1, z_2, a) \precsim (a_1^k, \overline{z_2}, a)$, which implies by axiom 5 that there exists $y_2 \in Z_2$ such that

$$(z_1, z_2, a) \sim (a_1^k, y_2, a). \tag{22}$$

But, since u represents $\succsim$ in $Z_1 \times Z_2 \times \{a\}$, $u(z_1) + u_2(z_2) + u_3(a) = u_1(a_1^k) + u_2(y_2) + u_3(a)$. Now, combining (22) with (17), and, next, the last equality with (17), one gets

$$(z_1, z_2, a) \sim (b_1^k, y_2, b) \text{ and } u(z_1) + u_2(z_2) + u_3(a) = u_1(b_1^k) + u_2(y_2) + u_3(b).$$

second subcase: if $(z_1, z_2, a) \succ (a_1^k, \overline{z_2}, a)$ then, by (19) and since $z_1 \precsim_1 a_1^{k+1}$, $(a_1^k, \overline{z_2}, a) \sim (a_1^{k+1}, \underline{z_2}, a) \prec (z_1, z_2, a) \precsim (a_1^{k+1}, z_2, a)$, which implies by axiom 5 that there exists $y_2 \in Z_2$ such that

$$(z_1, z_2, a) \sim (a_1^{k+1}, y_2, a). \tag{23}$$

But, since u represents $\succsim$ in $Z_1 \times Z_2 \times \{a\}$, $u(z_1) + u_2(z_2) + u_3(a) = u_1(a_1^{k+1}) + u_2(y_2) + u_3(a)$. Now, combining (23) with (21), and, next, the last equality with (21), one gets

$$(z_1, z_2, a) \sim (b_1^{k+1}, y_2, b) \text{ and } u(z_1) + u_2(z_2) + u_3(a) = u_1(b_1^{k+1}) + u_2(y_2) + u_3(b).$$

To summarize, for every $(z_1, z_2, a) \in Y$ such that $a_1^k \precsim_1 z_1 \precsim_1 a_1^{k+1}$, there exists $y_2 \in Z_2$ such that either i) $(z_1, z_2, a) \sim (b_1^k, y_2, b)$ and $u_1(z_1) + u_2(z_2) + u_3(a) = u_1(b_1^k) + u_2(y_2) + u_3(b)$, or ii) $(z_1, z_2, a) \sim (b_1^{k+1}, y_2, b)$ and $u_1(z_1) + u_2(z_2) + u_3(a) = u_1(b_1^{k+1}) + u_2(y_2) + u_3(b)$. Combined with (18), this leads to: for every $(z_1, z_2, a) \in Y$ such that $z_1 \precsim_1 a_1^{k+1}$, there exist $y_1 \in Z_1$ and $y_2 \in Z_2$ such that $(z_1, z_2, a) \sim (y_1, y_2, b)$ and $u_1(z_1) + u_2(z_2) + u_3(a) = u_1(y_1) + u_2(y_2) + u_3(b)$.

Now, to complete this subsubstep, we show that sequences (a_1^r) and (b_1^r) cannot be infinite. By definition of $\underline{z_2}$ and $\overline{z_2}$, and by (16), $\overline{z_2} \succ_2 \underline{z_2}$ and $(a_1^r, \overline{z_2}) \sim_{12} (a_1^{r+1}, \underline{z_2})$ for all r. Therefore, (a_1^r) is a standard sequence w.r.t. the first component, bounded in Z by $\underline{z_1}$ and $\overline{z_1}$; so, by the strengthened Archimedean axiom (axiom 9), it is necessarily finite. Similarly for (b_1^r).

third subsubstep: generating the area

When we reach this subsubstep, sequence (a_1^r) has been shown to be finite. Let p be the last index of the sequence, i.e., the index such that $(a_1^p, \overline{z_2}, a) \succ (\overline{z_1}, \underline{z_2}, a)$.

Consider any element $(z_1, z_2, a) \in Y$ such that $a_1^p \precsim_1 z_1 \precsim_1 \overline{z_1}$. Two subcases can occur:

first subcase: if $(z_1, z_2, a) \precsim (a_1^p, \overline{z_2}, a)$, then, since $a_1^p \precsim_1 z_1$, $(a_1^p, z_2, a) \precsim (z_1, z_2, a) \precsim (a_1^p, \overline{z_2}, a)$, which implies by axiom 5 that there exists $y_2 \in Z_2$ such that $(z_1, z_2, a) \sim (a_1^p, y_2, a)$. But, since u represents $\succsim$ in $Z_1 \times Z_2 \times \{a\}$, $u_1(z_1) + u_2(z_2) + u_3(a) = u_1(a_1^p) + u_2(y_2) + u_3(a)$. Now, combining these equations with the fact that $(a_1^p, \underline{z_2}, a) \sim (b_1^p, \underline{z_2}, b)$ and $u_1(a_1^p) + u_2(\underline{z_2}) + u_3(a) = u_1(b_1^p) + u_2(\underline{z_2}) + u_3(b)$ — as was shown in the first two subsubsteps — one gets: $(z_1, z_2, a) \sim (b_1^p, y_2, b)$ and $u(z_1) + u_2(z_2) + u_3(a) = u_1(b_1^p) + u_2(y_2) + u_3(b)$.

second subcase: if $(z_1, z_2, a) \succ (a_1^p, \overline{z_2}, a)$ then, since$(a_1^p, \overline{z_2}, a) \succ (\overline{z_1}, \underline{z_2}, a)$ and since $a_1^p \precsim_1 \overline{z_1}$, $(\overline{z_1}, \underline{z_2}, a) \prec (a_1^p, \overline{z_2}, a) \precsim (\overline{z_1}, \overline{z_2}, a)$, which implies by axiom 5 that there exists $y_2 \in Z_2$ such that $(a_1^p, \overline{z_2}, a) \sim (\overline{z_1}, y_2, a)$. Since u represents $\succsim$ on $Z_1 \times Z_2 \times \{a\}$, $u_1(a_1^p) + u_2(\overline{z_2}) + u_3(a) = u_1(\overline{z_1}) + u_2(y_2) + u_3(a)$.

Combining these equations with $y_2 \precsim_2 \overline{z_2}$ and $a \prec_3 b$, one gets: $(b_1^p, y_2, b) \precsim (b_1^p, \overline{z_2}, b) \sim (a_1^p, \overline{z_2}, a) \prec (a_1^p, \overline{z_2}, b) \sim (\overline{z_1}, y_2, b)$, which implies by axiom 5 that there exists b_1^{p+1} such that $(b_1^p, \overline{z_2}, b) \sim (b_1^{p+1}, y_2, b)$. As u represents $\succsim$ on $Z_1 \times Z_2 \times \{b\}$, $u_1(b_1^p) + u_2(\overline{z_2}) + u_3(b) = u_1(b_1^{p+1}) + u_2(y_2) + u_3(b)$. Combining these equations, one gets

$$(\overline{z_1}, y_2, a) \sim (b_1^{p+1}, y_2, b) \text{ and } u_1(\overline{z_1}) + u_2(y_2) + u_3(a) = u_1(b_1^{p+1}) + u_2(y_2) + u_3(b). \tag{24}$$

Now, let us come back to (z_1, z_2, a). By hypothesis, $(z_1, z_2, a) \succ (a_1^p, \overline{z_2}, a) \sim (\overline{z_1}, y_2, a)$; and, by definition of $\overline{z_1}$, $(z_1, z_2, a) \precsim (\overline{z_1}, z_2, a)$; consequently, $(\overline{z_1}, y_2, a) \prec (z_1, z_2, a) \precsim (\overline{z_1}, z_2, a)$, which implies by axiom 5 that there exists $t_2 \in Z_2$ such that $(z_1, z_2, a) \sim (\overline{z_1}, t_2, a)$; and, since u represents $\succsim$ on $Z_1 \times Z_2 \times \{a\}$, $u_1(z_1) + u_2(z_2) + u_3(a) = u_1(\overline{z_1}) + u_2(t_2) + u_3(a)$. Now, combining these equations with (24), one gets: $(z_1, z_2, a) \sim (b_1^{p+1}, t_2, b)$ and $u_1(z_1) + u_2(z_2) + u_3(a) = u_1(b_1^{p+1}) + u_2(t_2) + u_3(b)$.

Therefore, to summarize, for every element $(z_1, z_2, a) \in Y$ such that $a_1^p \precsim_1 z_1 \precsim_1 \overline{z_1}$, there exists $y_2 \in Z_2$ such that either i) $(z_1, z_2, a) \sim (b_1^p, y_2, b)$ and $u(z_1) + u_2(z_2) + u_3(a) = u_1(b_1^p) + u_2(y_2) + u_3(b)$, or ii) $(z_1, z_2, a) \sim (b_1^{p+1}, y_2, b)$ and $u(z_1) + u_2(z_2) + u_3(a) = u_1(b_1^{p+1}) + u_2(y_2) + u_3(b)$.

fourth subsubstep: conclusion of case 2.1

At this step, we have shown that, for every $(z_1, z_2, a) \in Y$ such that $z_1 \precsim_1 \overline{z_1}$ — which means actually every $(z_1, z_2, a) \in Y$ — there exist $y_1 \in Z_1$ and $y_2 \in Z_2$ such that

$$(z_1, z_2, a) \sim (y_1, y_2, b) \text{ and } u(z_1) + u_2(z_2) + u_3(a) = u_1(y_1) + u_2(y_2) + u_3(b). \tag{25}$$

We will prove in this subsubstep, that it is sufficient for u to represent $\succsim$ on Y. Indeed, consider two elements $(z_1, z_2, a), (t_1, t_2, b) \in Y$. By (25), there exist $y_1 \in Z_1$ and $y_2 \in Z_2$ such that $(z_1, z_2, a) \sim (y_1, y_2, b)$ and such that $u_1(z_1) + u_2(z_2) + u_3(a) = u_1(y_1) + u_2(y_2) + u_3(b)$. Now, since

u represents $\succsim$ on $Z_1 \times Z_2 \times \{b\}$, $u_1(y_1)+u_2(y_2)+u_3(b) \geq u_1(t_1)+u_2(t_2)+u_3(b) \Leftrightarrow (y_1, y_2, b) \succsim (t_1, t_2, b)$ and $u_1(y_1) + u_2(y_2) + u_3(b) \leq u_1(t_1) + u_2(t_2) + u_3(b) \Leftrightarrow (y_1, y_2, b) \precsim (t_1, t_2, b)$. So, by transitivity of $\succsim$ and $\geq$, $(z_1, z_2, a) \succsim (t_1, t_2, b) \Leftrightarrow u_1(z_1)+u_2(z_2)+u_3(a) \geq u_1(t_1)+u_2(t_2)+u_3(b)$ and $u_1(z_1) + u_2(z_2) + u_3(a) \leq u_1(t_1) + u_2(t_2) + u_3(b) \Leftrightarrow (y_1, y_2, a) \precsim (t_1, t_2, b)$.

Therefore, u, as defined by $u_3(b) = u_3(a) + u_1(a_1^1) - u_1(\underline{z_1})$, represents $\succsim$ on Y. And since (x_1^0, x_2^0, a) and (x_1^1, x_2^1, b) belong to Y, u is as defined by (11).

Case 2.2: if $(\underline{z_1}, \underline{z_2}, b) \precsim (\underline{z_1}, \overline{z_2}, a)$: (26)

Using a symmetric proof of case 2.1 — symmetric w.r.t. components one and two — it is easily shown that u, as defined by (11), is a utility function on Y.

Case 2.3: if $(\overline{z_1}, \underline{z_2}, a) \prec (\underline{z_1}, \underline{z_2}, b)$ and $(\underline{z_1}, \overline{z_2}, a) \prec (\underline{z_1}, \underline{z_2}, b)$: (27)

Y is nonempty, so $(\underline{z_1}, \underline{z_2}, b) \precsim (\overline{z_1}, \overline{z_2}, a)$ and, by (27) and independence,

$$(\overline{z_1}, \underline{z_2}, a) \prec (\underline{z_1}, \underline{z_2}, b) \precsim (\overline{z_1}, \overline{z_2}, a) \prec (\overline{z_1}, \underline{z_2}, b). \tag{28}$$

For all $(z_1, z_2, z_3) \in Y$, there exists $(t_1, t_2, t_3) \in Z$ such that $(z_1, z_2, z_3) \sim (t_1, t_2, t_3)$ and such that $z_3 \neq t_3$; so, $(\underline{z_1}, \underline{z_2}, b) \precsim (z_1, z_2, z_3) \precsim (\overline{z_1}, \overline{z_2}, a)$, which implies by (28) and by axiom 5 that there exist $y_1 \in Z_1$ and $y_2 \in Z_2$ such that

$$(z_1, z_2, z_3) \sim (y_1, \underline{z_2}, b) \sim (\overline{z_1}, y_2, a). \tag{29}$$

In particular, there exists $a_2 \in Z_2$ such that

$$(\overline{z_1}, a_2, a) \sim (\underline{z_1}, \underline{z_2}, b). \tag{30}$$

So, a necessary condition for u to represent $\succsim$ on Y is clearly that

$$u_3(b) = u_3(a) + u_1(\overline{z_1}) - u_1(\underline{z_1}) + u_2(a_2) - u_2(\underline{z_2}). \tag{31}$$

Now let (z_1, z_2, a) be an arbitrary element of Y. By (29), there exist $y_1 \in Z_1$ and $y_2 \in Z_2$ such that $(z_1, z_2, a) \sim (y_1, \underline{z_2}, b) \sim (\overline{z_1}, y_2, a)$. Now, by (30) and the scaling axiom, $(y_1, \underline{z_2}, b) \sim (\overline{z_1}, y_2, a)$ and $(\overline{z_1}, a_2, a) \sim (\underline{z_1}, \underline{z_2}, b)$ imply that $(\underline{z_1}, y_2, a) \sim (y_1, a_2, a)$. Indeed equation (4) of the scaling axiom can be used because, by definition of $\underline{z_1}$, $\overline{z_1}$, $\underline{z_2}$ and $\overline{z_2}$, when case 2.3 occurs, a and b are not single-dimensionally-matched in X. But u, as defined by (31), represents $\succsim$ in $Z_1 \times Z_2 \times \{a\}$; so, $u_1(\underline{z_1})+u_2(y_2)+u_3(a) = u_1(y_1)+u_2(a_2)+u_3(a)$. Now, this equality and (31) imply that $u_1(y_1) + u_2(\underline{z_2}) + u_3(b) = u_1(\overline{z_1}) + u_2(y_2) + u_3(a)$. By (29) and since u represents $\succsim$ on $Z_1 \times Z_2 \times \{a\}$, $u_1(z_1) + u_2(z_2) + u_3(a) = u_1(\overline{z_1}) + u_2(y_2) + u_3(a)$.

Therefore, to summarize, for all $(z_1, z_2, a) \in Y$, there exists $y_1 \in Z_1$ such that $(z_1, z_2, a) \sim (y_1, \underline{z_2}, b)$ and $u_1(z_1) + u_2(z_2) + u_3(a) = u_1(y_1) + u_2(\underline{z_2}) + u_3(b)$. Consequently, u, as defined by (31), represents $\succsim$ on Y; and since (x_1^0, x_2^0, a) and (x_1^1, x_2^1, b) belong to Y, (11) holds.

So far, we proved the existence of an additive utility on $X_1 \times X_2 \times \{a, b\}$. It remains to show the uniqueness property. Since u_1 and u_2 are unique up to scale and location, (11), being necessary and sufficient for u to represent $\succsim$, ensures that u_3 is unique up to scale and location. ■

Lemma 5 *Let $X = X_1 \times X_2 \times X_3$. Suppose that $(X, \succsim)$ satisfies axiom 1 (weak ordering), axiom 2 (independence) and axiom 5 (restricted solvability w.r.t. the first two components). Let $x_3, z_3 \in X_3$ be such that x_3 and z_3 are 3-linked, i.e., $x_3 \mathcal{O}_3 z_3$. Then, x_3 and z_3 can be 3-linked by a strictly monotonic sequence, i.e., by a sequence $(y_3^k)_{k=0}^p$ of elements of X_3 such that*

• *$y_3^0 = x_3$, $y_3^p = z_3$,*
• *for every $k \in \{0,\dots,p-1\}$, there exist $a^{k+1}, b^k \in X_1 \times X_2$ such that $(b^k, y_3^k) \sim (a^{k+1}, y_3^{k+1})$,*
• *either $y_3^{k+1} \succ_3 y_3^k$ for all $k \in \{0,\dots,p-1\}$, or $y_3^{k+1} \prec_3 y_3^k$ for all $k \in \{0,\dots,p-1\}$.*
Moreover, if x_3 and z_3 are single-dimensionally-matched, then they can be single-dimensionally-matched by a strictly monotonic sequence.

Proof of lemma 5: Suppose that x_3 and z_3 are 3-linked. Without loss of generality, suppose that $x_3 \precsim_3 z_3$ — since, as was mentioned in section 2, $x_3\ \mathcal{O}_3\ z_3 \Leftrightarrow z_3\ \mathcal{O}_3\ x_3$. By definition of i-linkness, there exists a finite sequence $(t_3^k)_{k=0}^r$ such that $t_3^0 = x_3$, $t_3^r = z_3$, and for all $k \in \{0,\dots,r-1\}$, there exist $a^{k+1}, b^k \in X_1 \times X_2$ such that $(b^k, t_3^k) \sim (a^{k+1}, t_3^{k+1})$. If, moreover, x_3 and z_3 are single-dimensionally-matched, then a^{k+1} and b^k have one component in common.

First step: extracting a 3-linking sequence constituted by elements "between" x_3 and z_3

If $t_3^{k+1} \succ_3 t_3^k$ for all $k \in \{0,\dots,r-1\}$, then lemma 5 is proved. Otherwise, extract from (t_3^k) sequence $(s_3^k)_{k=0}^q$ constituted by all the elements "between" x_3 and z_3, i.e., such that $z_3 \succsim_3 s_3^k \succsim_3 x_3$ for all k. Let $f(\cdot)$ be such that $s_3^k = t_3^{f(k)}$ for all k. We will show that, for all $k \in \{0,\dots,q-1\}$, there exist $c^{k+1}, d^k \in X_1 \times X_2$ such that $(d^k, s_3^k) \sim (c^{k+1}, s_3^{k+1})$. If $f(k+1) = f(k)+1$, $s_3^k = t_3^{f(k)}$ and $s_3^{k+1} = t_3^{f(k)+1}$, so $(b^{f(k)}, s_3^k) \sim (a^{f(k)+1}, s_3^{k+1})$. If x_3 and z_3 are single-dimensionally-matched, then $b^{f(k)}$ and $a^{f(k)+1}$ have one component in common, so s_3^k and s_3^{k+1} are directly-single-dimensionally-matched.

Otherwise, suppose that some t_3^i, for $i \in \{f(k)+1,\dots,f(k+1)-1\}$, are such that $t_3^i \succ_3 z_3$ and others are such that $t_3^i \prec_3 x_3$; then, there exists $i_1 \in \{f(k)+1,\dots,f(k+1)-1\}$ such that either $t_3^{i_1} \succ_3 z_3$ and $t_3^{i_1+1} \prec_3 x_3$ or such that $t_3^{i_1} \prec_3 x_3$ and $t_3^{i_1+1} \succ_3 z_3$. Consider the first case (the other one can be proved similarly). Then $(b^{i_1}, x_3) \precsim (b^{i_1}, z_3) \prec (b^{i_1}, t_3^{i_1}) \sim (a^{i_1+1}, t_3^{i_1+1}) \prec (a^{i_1+1}, x_3) \precsim (a^{i_1+1}, z_3)$, which implies, by lemma 2, that there exist a, b such that $(a, x_3) \sim (b, z_3) \sim (b^{i_1}, t_3^{i_1})$, and so sequence $(s_3^0 = x_3, s_3^1 = z_3)$ is monotonic and 3-linking. Moreover, if x_3 and z_3 are single-dimensionally-matched, then a^{i_1+1} and b^{i_1} have one component in common, so a and b have also one component in common and x_3^0 and x_3^1 are directly-single-dimensionally-matched.

If, on the other hand, there do not exist some t_3^i, for $i \in \{f(k)+1,\dots,f(k+1)-1\}$, such that $t_3^i \succ_3 z_3$ and others such that $t_3^i \prec_3 x_3$, then either $t_3^i \succ_3 z_3$ for all $i \in \{f(k)+1,\dots,f(k+1)-1\}$, or $t_3^i \prec_3 x_3$ for all $i \in \{f(k)+1,\dots,f(k+1)-1\}$. Consider the first case (the other one can be proved similarly); then, if $t_3^{f(k)} \precsim_3 t_3^{f(k+1)}$, $(a^{f(k)+1}, t_3^{f(k+1)}) \precsim (b^{f(k)}, t_3^{f(k)}) \sim (a^{f(k)+1}, t_3^{f(k)+1}) \precsim (b^{f(k)}, t_3^{f(k+1)})$, which implies, by lemma 2, that there exists c^{k+1} such that $(b^{f(k)}, t_3^{f(k)}) \sim (c^{k+1}, t_3^{f(k+1)})$, or, equivalently, such that $(b^{f(k)}, s_3^k) \sim (c^{k+1}, s_3^{k+1})$. if, on the contrary, $t_3^{f(k)} \succsim_3 t_3^{f(k+1)}$, then $(b^{f(k+1)-1}, t_3^{f(k)}) \precsim (b^{f(k+1)-1}, t_3^{f(k+1)-1}) \sim (a^{f(k+1)}, t_3^{f(k+1)}) \precsim (a^{f(k+1)}, t_3^{f(k)})$, which implies, by lemma 2, that there exists d^k such that $(d^k, t_3^{f(k)}) \sim (a^{f(k+1)}, t_3^{f(k+1)})$, or, equivalently, such that $(d^k, s_3^k) \sim (a^{f(k+1)}, s_3^{k+1})$. Moreover, if x_3 and z_3 are single-dimensionally-matched, then $a^{f(k)+1}$ and $b^{f(k)}$ (resp. $b^{f(k+1)-1}$ and $a^{f(k+1)}$) have one component in common, so c^{k+1} and $b^{f(k)}$ (resp. d^k and $a^{f(k+1)}$) have also one component in common, hence resulting in s_3^k and s_3^{k+1} being directly-single-dimensionally-matched.

So, $(s_3^k)_{k=0}^q$ 3-links or single-dimensionally-matches x_3 and z_3. Suppose that, for all $k \in \{1,\dots,q-1\}$, $s_3^k \neq x_3$ and $s_3^k \neq z_3$ — otherwise, just extract the smallest subsequence such that this property holds. Next, we will show that one can extract from $(s_3^k)_{k=0}^q$ a strictly increasing sequence 3-linking or single-dimensionally-matching x_3 and z_3.

Second step: extracting a strictly increasing sequence 3-linking x_3 and z_3

If $s_3^{k+1} \succ_3 s_3^k$ for all $k \in \{0,\dots,q-1\}$, then lemma 5 is proved. Otherwise, there exists k_1 in $\{0,\dots,q-2\}$ such that $s_3^{k_1+1} \precsim_3 s_3^{k_1}$. Let k_2 be the smallest index in $\{0,\dots,q-1\}$ such that $k_2 \geq$

k_1 and $s_3^{k_2} \succ_3 s_3^{k_1}$. Let $(h_3^k)_{k=0}^p$ be the sequence $(s_3^0, \ldots, s_3^{k_1}, s_3^{k_2}, \ldots, s_3^q)$, and $g(\cdot)$ be such that $h_3^k = s_3^{g(k)}$. Clearly, for all $k < k_1 - 1$, $(d^k, s_3^k) \sim (c^{k+1}, s_3^{k+1})$, so $(d^k, h_3^k) \sim (c^{k+1}, h_3^{k+1})$; similarly, for all $k \geq k_1 + 1$, $(d^{g(k)}, s_3^{g(k)}) \sim (c^{g(k+1)}, s_3^{g(k+1)})$, so that $(d^{g(k)}, h_3^k) \sim (c^{g(k+1)}, h_3^{k+1})$. Moreover, if x_3 and z_3 are single-dimensionally-matched, h_3^k and h_3^{k+1} are directly-single-dimensionally-matched for all $k < k_1 - 1$ and all $k \geq k_1 + 1$. Now, by definition of k_2, $k_2 > k_1 + 1$, $s_3^{k_2-1} \prec_3 s_3^{k_1}$, so $(c^{k_2}, s_3^{k_1}) \prec (c^{k_2}, s_3^{k_2}) \sim (d^{k_2-1}, s_3^{k_2-1}) \precsim (d^{k_2-1}, s_3^{k_1})$, which implies, by lemma 2, that there exists i^{k_1} such that $(i^{k_1}, s_3^{k_1}) \sim (c^{k_2}, s_3^{k_2})$. Therefore, $(i^{k_1}, h_3^{k_1}) \sim (c^{k_2}, h_3^{k_1+1})$, and so sequence (h_3^k) 3-links x_3 and z_3. Moreover, if x_3 and z_3 are single-dimensionally-matched, then c^{k_2} and d^{k_2-1} have one component in common and so c^{k_2} and i^{k_1} have also one component in common, hence $h_3^{k_1}$ and $h_3^{k_1+1}$ are directly-single-dimensionally-matched.

If (h_3^k) is strictly increasing, then lemma 5 holds; otherwise, repeat the same process with (h_3^k) playing the role previously taken by (s_3^k); by construction, $\text{Card}(h_3^k) \leq \text{Card}(s_3^k) - 1$, so at most q iterations are needed to extract a strictly increasing sequence 3-linking or single-dimensionally-matching x_3 and z_3. ■

Proof of theorem 1: The proof is organized in two steps. First, it is shown that, for all x_3^0, x_3^1, $\succsim$ is representable on $X_1 \times X_2 \times [x_3^0, x_3^1]$ by an additive utility u. In step 2, u is extended to represent $\succsim$ on X and the set of equivalence classes of $\mathcal{O}_3$ is shown to be denumerable.

We know that there exist real-valued functions u_1 on X_1 and u_2 on X_2 such that $\succsim$ is represented on $X_1 \times X_2$ by $u_1 + u_2$. Now consider some arbitrary elements x_3^0, x_3^1 of X_3. Without loss of generality, suppose that $x_3^0 \prec_3 x_3^1$. We will prove in the first step that $u_1 + u_2$ can be extended to represent $\succsim$ on $X_1 \times X_2 \times [x_3^0, x_3^1]$ — remind that $[x_3^0, x_3^1] = \{x_3 : x_3^0 \precsim_3 x_3 \precsim_3 x_3^1\}$.

First step: generating $\boldsymbol{u}$ on $\boldsymbol{X_1 \times X_2 \times [x_3^0, x_3^1]}$

First case: if there exists $(x_1^0, x_2^0, x_1^1, x_2^1)$ such that $(x_1^1, x_2^1, x_3^1) \sim (x_1^0, x_2^0, x_3^0)$: (32)

<u>*first substep: generating u on $X_1 \times X_2 \times \{x_3^0, x_3^1\}$*</u>

A necessary and sufficient condition for u to represent $\succsim$ on $X_1 \times X_2 \times \{x_3^0, x_3^1\}$ is that

$$u_3(x_3^1) = u_3(x_3^0) + u_1(x_1^0) - u_1(x_1^1) + u_2(x_2^0) - u_2(x_2^1). \tag{33}$$

Indeed, if x_3^0 and x_3^1 are not single-dimensionally-matched in X, then they are neither single-dimensionally-matched in $X_1 \times X_2 \times \{x_3^0, x_3^1\}$. Thus, case 2.3 of lemma 4 can be applied, which proves that an additive utility exists. If, on the contrary, x_3^0 and x_3^1 are single-dimensionally-matched in X, then, by lemma 5, there exists a sequence $(y_3^k)_{k=0}^p$ such that $y_3^0 = x_3^0$, $y_3^p = x_3^1$, and such that $y_3^i \prec_3 y_3^{i+1}$, and y_3^i and y_3^{i+1} are directly-single-dimensionally-matched for all $i \in \{0, \ldots, p-1\}$. Hence, by cases 2.1 and/or 2.2 of lemma 4, $\succsim$ is representable by additive utilities $u_1^i + u_2^i + u_3^i$, unique up to scale and location, on every space $X_1 \times X_2 \times \{y_3^i, y_3^{i+1}\}$. Applying proper scalings and adding proper constants, it is not restrictive to assume that $u_1^i = u_1$ and $u_2^i = u_2$ for all $i \in \{0, \ldots, p\}$, and that $u_3^i(y_3^{i+1}) = u_3^{i+1}(y_3^{i+1})$ for all $i \in \{0, \ldots, p-1\}$.

Now, let $u_3 : \{y_3^i : i \in \{0, \ldots, p\}\} \to \mathbb{R}$ be defined as $u_3(y_3^i) = u_3^i(y_3^i)$ for all i. Then $u_1 + u_2 + u_3$ is an additive utility, unique up to scale and location, representing $\succsim$ on $X_1 \times X_2 \times \{y_3^i : i \in \{0, \ldots, p\}\}$. Indeed, compare two elements (x_1, x_2, y_3^i) and (y_1, y_2, y_3^{i+k}), $k \geq 0$. If $k \leq 1$, then by definition of u_3, it is trivial that u preserves the ordering. If $k > 1$, then if $(x_1, x_2, y_3^i) \prec (z_1, z_2, y_3^{i+1})$ for all $(z_1, z_2) \in X_1 \times X_2$, then, since u represents $\succsim$ on $X_1 \times X_2 \times \{y_3^i, y_3^{i+1}\}$,

$$u_1(x_1) + u_2(x_2) + u_3(y_3^i) \leq \inf_{(z_1, z_2) \in X_1 \times X_2} u_1(z_1) + u_2(z_2) + u_3(y_3^{i+1}), \tag{34}$$

with equality only if the inf is not attained. $(y_3^j)_{j=0}^p$ is a strictly increasing sequence and u represents $\succsim$ on $X_1 \times X_2 \times \{y_3^j, y_3^{j+1}\}$ for all $j \in \{0, \ldots, p-1\}$, hence, by induction, $u_3(y_3^{i+1}) <$

$u_3(y_3^{i+k})$. Consequently, by (34), $u_1(x_1) + u_2(x_2) + u_3(y_3^i) < u_1(y_1) + u_2(y_2) + u_3(y_3^{i+k})$, and of course $(x_1, x_2, y_3^i) \prec (y_1, y_2, y_3^{i+1}) \prec (y_1, y_2, y_3^{i+k})$ since (y_3^i) is a strictly increasing sequence.

If, on the contrary, there exists $(z_1, z_2) \in X_1 \times X_2$ such that $(x_1, x_2, y_3^i) \sim (z_1, z_2, y_3^{i+1})$, then, since u represents $\succsim$ on $X_1 \times X_2 \times \{y_3^i, y_3^{i+1}\}$, $u_1(x_1) + u_2(x_2) + u_3(y_3^i) = u_1(z_1) + u_2(z_2) + u_3(y_3^{i+1})$. Now, it remains to compare (z_1, z_2, y_3^{i+1}) and (y_1, y_2, y_3^{i+k}). By induction on the above process, it is clear that u represents $\succsim$ on $X_1 \times X_2 \times \{y_3^i : i \in \{0, \ldots, p\}\}$, hence *a fortiori* on $X_1 \times X_2 \times \{x_3^0, x_3^1\}$.

Now, consider an arbitrary element $x_3 \in [x_3^0, x_3^1]$. Then, by (32), $(x_1^1, x_2^1, x_3) \precsim (x_1^1, x_2^1, x_3^1) \sim (x_1^0, x_2^0, x_3^0) \precsim (x_1^0, x_2^0, x_3)$. So, by lemma 2, that there exist x_1, x_2 such that

$$(x_1, x_2, x_3) \sim (x_1^0, x_2^0, x_3^0) \sim (x_1^1, x_2^1, x_3^1). \tag{35}$$

<u>*second substep: generating u on $X_1 \times X_2 \times \{x_3^0, x_3, x_3^1\}$*</u>

Using a proof similar to the first substep, $\succsim$ is representable by an additive utility function $v_1 + v_2 + v_3$ on $X_1 \times X_2 \times \{x_3^0, x_3\}$, and v satisfies

$$v_1(x_1) + v_2(x_2) + v_3(x_3) = v_1(x_1^0) + v_2(x_2^0) + v_3(x_3^0), \tag{36}$$

and, moreover, v_1, v_2 and v_3 are unique up to scale and location. Therefore, since $v_1 + v_2$ and $u_1 + u_2$ both represent $\succsim_{12}$ on $X_1 \times X_2$, it is not restrictive to assume that $v_1 = u_1$, $v_2 = u_2$, and $v_3(x_3^0) = u_3(x_3^0)$. Consequently, a necessary and sufficient condition for u to represent $\succsim$ on $X_1 \times X_2 \times \{x_3^0, x_3\}$ and on $X_1 \times X_2 \times \{x_3^0, x_3^1\}$ is that

$$\begin{array}{rcl} u_3(x_3^1) & = & u_3(x_3^0) + u_1(x_1^0) - u_1(x_1^1) + u_2(x_2^0) - u_2(x_2^1), \\ u_3(x_3) & = & u_3(x_3^0) + u_1(x_1^0) - u_1(x_1) + u_2(x_2^0) - u_2(x_2). \end{array} \tag{37}$$

Using a similar proof, it can be easily shown that it is also a necessary and sufficient condition for u to represent $\succsim$ on $X_1 \times X_2 \times \{x_3, x_3^1\}$. Note that (35) and (37) ensure that the values of $u_3(x_3)$ inferred from the application of the first substep on $X_1 \times X_2 \times \{x_3^0, x_3\}$ and on $X_1 \times X_2 \times \{x_3, x_3^1\}$ are identical.

Now, it is clear that (37) is necessary and sufficient for u to represent $\succsim$ on $X_1 \times X_2 \times \{x_3^0, x_3, x_3^1\}$ because, when comparing any two elements $(y_1, y_2, y_3), (z_1, z_2, z_3) \in X_1 \times X_2 \times \{x_3^0, x_3, x_3^1\}$, (y_3, z_3) belong either to $\{x_3, x_3^1\}^2$, to $\{x_3^0, x_3\}^2$ or to $\{x_3, x_3^1\}^2$.

<u>*third substep: generating u on $X_1 \times X_2 \times [x_3^0, x_3^1]$*</u>

Consider two elements $y_3, z_3 \in [x_3^0, x_3^1]$. We have shown at the beginning of this case that there exist (s_1, s_2, y_3) and (t_1, t_2, z_3) such that $(s_1, s_2, y_3) \sim (t_1, t_2, z_3) \sim (x_1^0, x_2^0, x_3^0) \sim (x_1^1, x_2^1, x_3^1)$. By the preceding substep, a necessary and sufficient condition for u to represent $\succsim$ on $X_1 \times X_2 \times \{x_3^0, y_3, x_3^1\}$ and on $X_1 \times X_2 \times \{x_3^0, z_3, x_3^1\}$, is that

$$\begin{array}{l} u_3(x_3^1) = u_3(x_3^0) + u_1(x_1^0) - u_1(x_1^1) + u_2(x_2^0) - u_2(x_2^1), \\ u_3(y_3) = u_3(x_3^0) + u_1(x_1^0) - u_1(s_1) + u_2(x_2^0) - u_2(s_2), \\ u_3(z_3) = u_3(x_3^0) + u_1(x_1^0) - u_1(t_1) + u_2(x_2^0) - u_2(t_2). \end{array} \tag{38}$$

We will show next that it is also sufficient for u to represent $\succsim$ on $X_1 \times X_2 \times \{x_3^0, y_3, z_3, x_3^1\}$. According to the preceding paragraph, there remains to show that u is still representing when comparing an element of $X_1 \times X_2 \times \{y_3\}$ with one of $X_1 \times X_2 \times \{z_3\}$. So, consider two arbitrary elements (y_1, y_2, y_3) and (z_1, z_2, z_3). Either i) $(y_1, y_2, y_3) \precsim (x_1^0, x_2^0, x_3^0)$, in which case, since $y_3 \succsim_3 x_3^0$, $(y_1, y_2, x_3^0) \precsim (y_1, y_2, y_3) \precsim (x_1^0, x_2^0, x_3^0)$ and, by lemma 2, there exist a_1, a_2 such that $(y_1, y_2, y_3) \sim (a_1, a_2, x_3^0)$; or ii) $(y_1, y_2, y_3) \succ (x_1^0, x_2^0, x_3^0) \sim (x_1^1, x_2^1, x_3^1)$, in which case $(x_1^1, x_2^1, x_3^1) \prec (y_1, y_2, y_3) \precsim (y_1, y_2, x_3^1)$ and, by lemma 2, there exist a_1, a_2 such that

$(y_1, y_2, y_3) \sim (a_1, a_2, x_3^1)$. Now suppose that i) holds (case ii) is similar). Then, since u is representing on $X_1 \times X_2 \times \{x_3^0, y_3, x_3^1\}$, $u_1(y_1)+u_2(y_2)+u_3(y_3) = u_1(a_1)+u_2(a_2)+u_3(x_3^0)$. Since u is also representing on $X_1 \times X_2 \times \{x_3^0, z_3, x_3^1\}$, $(a_1, a_2, x_3^0) \succsim (z_1, z_2, z_3) \Leftrightarrow u_1(a_1)+u_2(a_2)+u_3(x_3^0) \geq u_1(z_1) + u_2(z_2) + u_3(z_3)$ and $(a_1, a_2, x_3^0) \precsim (z_1, z_2, z_3) \Leftrightarrow u_1(a_1) + u_2(a_2) + u_3(x_3^0) \leq u_1(z_1) + u_2(z_2)+u_3(z_3)$. So $(y_1, y_2, y_3) \succsim (z_1, z_2, z_3) \Leftrightarrow u_1(y_1)+u_2(y_2)+u_3(y_3) \geq u_1(z_1)+u_2(z_2)+u_3(z_3)$ and $(y_1, y_2, y_3) \precsim (z_1, z_2, z_3) \Leftrightarrow u_1(y_1) + u_2(y_2) + u_3(y_3) \leq u_1(z_1) + u_2(z_2) + u_3(z_3)$.

So (38) is necessary and sufficient for u to represent $\succsim$ on $X_1 \times X_2 \times \{x_3^0, y_3, z_3, x_3^1\}$. Now, note that in (38), $u_3(y_3)$ was not defined in function of $u_3(z_3)$, and conversely; moreover, y_3 and z_3 were arbitrary; consequently, a necessary and sufficient condition for u to represent $\succsim$ on $X_1 \times X_2 \times [x_3^0, x_3^1]$ is that, for all $y_3 \in [x_3^0, x_3^1]$, $u_3(y_3)$ is defined by:

$$u_3(y_3) = u_3(x_3^0) + u_1(x_1^0) - u_1(s_1) + u_2(x_2^0) - u_2(s_2) \tag{39}$$

when $(s_1, s_2, y_3) \sim (x_1^0, x_2^0, x_3^0)$.

Second case: if Not$[(x_1^1, x_2^1, x_3^1) \sim (x_1^0, x_2^0, x_3^0)]$ for every $(x_1^0, x_2^0, x_1^1, x_2^1)$:

Then $(x_1^1, x_2^1, x_3^1) \succ (x_1^0, x_2^0, x_3^0)$ else, since $x_3^0 \prec x_3^1$, $(x_1^1, x_2^1, x_3^1) \precsim (x_1^0, x_2^0, x_3^0) \prec (x_1^0, x_2^0, x_3^1)$ and, by lemma 2, there would exist a_1, a_2 such that $(x_1^0, x_2^0, x_3^0) \sim (a_1, a_2, x_3^1)$.

Case 2.1: if $x_3^1 \, \mathcal{O}_3 \, x_3^0$ (i.e., x_3^0 and x_3^1 are 3-linked (see definition 1 on page 4)):

By lemma 5, there exists a finite sequence $(z_3^i)_{i=1}^p$ such that i) $z_3^0 = x_3^0$, $z_3^p = x_3^1$; ii) $z_3^{i+1} \succ_3 z_3^i$ for all i in $\{0, \ldots, p-1\}$; and iii) there exist (z_1^i, z_2^i) and (y_1^{i+1}, y_2^{i+1}) such that $(y_1^{i+1}, y_2^{i+1}, z_3^{i+1}) \sim (z_1^i, z_2^i, z_3^i)$. Applying the first case on $X_1 \times X_2 \times [z_3^i, z_3^{i+1}]$, i.e., selecting an appropriate value for $u_3(z_3^{i+1})$ from that of $u_3(z_3^i)$, as in (39), u is a utility on $X_1 \times X_2 \times [z_3^i, z_3^{i+1}]$ for all i; and, since u_1, u_2 and $u_3(z_3^i)$ are unique up to positive affine transforms, u is also unique up to scale and location on each $X_1 \times X_2 \times [z_3^i, z_3^{i+1}]$.

Now, consider an arbitrary element $(x_1, x_2, x_3) \in X_1 \times X_2 \times [z_3^0, z_3^1]$ and an arbitrary element $(y_1, y_2, y_3) \in X_1 \times X_2 \times [z_3^1, z_3^2]$; if there exists (a_1, a_2, z_3^1) such that $(x_1, x_2, x_3) \sim (a_1, a_2, z_3^1)$ or $(y_1, y_2, y_3) \sim (a_1, a_2, z_3^1)$, then, since u represents $\succsim$ on $X_1 \times X_2 \times [z_3^0, z_3^1]$ and on $X_1 \times X_2 \times [z_3^1, z_3^2]$, $(x_1, x_2, x_3) \succsim (y_1, y_2, y_3) \Leftrightarrow u_1(x_1)+u_2(x_2)+u_3(x_3) \geq u_1(y_1)+u_2(y_2)+u_3(y_3)$ and $(x_1, x_2, x_3) \precsim (y_1, y_2, y_3) \Leftrightarrow u_1(x_1) + u_2(x_2) + u_3(x_3) \leq u_1(y_1) + u_2(y_2) + u_3(y_3)$; else according to lemma 3, $(x_1, x_2, x_3) \prec (x_1, x_2, z_3^1) \prec (y_1, y_2, y_3)$, and, since u represents $\succsim$ on $X_1 \times X_2 \times [z_3^0, z_3^1]$ and on $X_1 \times X_2 \times [z_3^1, z_3^2]$, $(x_1, x_2, x_3) \prec (y_1, y_2, y_3) \Leftrightarrow u_1(x_1)+u_2(x_2)+u_3(x_3) < u_1(y_1)+u_2(y_2)+u_3(y_3)$.

Therefore, u represents $\succsim$ on $X_1 \times X_2 \times [z_3^0, z_3^2]$. By induction, using the process described above, it is easily shown that u, as extended above, represents $\succsim$ on $X_1 \times X_2 \times [x_3^0, x_3^1]$. Moreover, due to the equality in (39), u is unique up to scale and location.

Case 2.2: if Not$(x_3^1 \, \mathcal{O}_3 \, x_3^0)$: (40)

Lemma 4 on $X_1 \times X_2 \times \{x_3^0, x_3^1\}$ implies that $u_1(X_1)$ and $u_2(X_2)$ are bounded (here lemma 4 can be applied directly because x_3^0 and x_3^1 are not single-dimensionally-matched in X). For every $x_3, y_3 \in X_3$ such that $x_3 \, \mathcal{O}_3 \, y_3 \, \mathcal{O}_3 \, x_3^0$ and $y_3 \succsim_3 x_3 \succsim_3 x_3^0$, by the first case and case 2.1, there exists a unique up to scale and location additive utility u (resp. u') representing $\succsim$ on $X_1 \times X_2 \times [x_3^0, x_3]$ (resp. on $X_1 \times X_2 \times [x_3^0, y_3]$). u' represents $\succsim$ on $X_1 \times X_2 \times [x_3^0, y_3]$, so the restriction of u' on $X_1 \times X_2 \times [x_3^0, x_3]$ is an affine transform of u. By multiplying and adding proper constants to u', the restriction of u' on $X_1 \times X_2 \times [x_3^0, x_3]$ is equal to u. So, u has been extended to represent $\succsim$ on $X_1 \times X_2 \times [x_3^0, y_3]$. Repeating this process, one can extend u to represent $\succsim$ on $X_1 \times X_2 \times [x_3^0, z_3]$ for all $z_3 \succsim_3 y_3$, and so on. Hence, there exists an additive utility, unique up to scale and location, representing $\succsim$ on $X_1 \times X_2 \times \{x_3 : x_3 \mathcal{O}_3 x_3^0$ and $x_3 \succsim_3 x_3^0\}$.

Now, let us prove that there exists $\beta^0 \in \mathbb{R}$ such that, for every $x_3 \in X_3$ such that $x_3 \mathcal{O}_3 x_3^0$ and $x_3 \succsim_3 x_3^0$, $u_3(x_3) \leq \beta^0$. Assume the contrary; then, for every $r \in \mathbb{R}$, there exists $x_3(r)$ such that $x_3(r) \, \mathcal{O}_3 \, x_3^0$ and $u_3(x_3(r)) \geq r$. The second component being essential, there exist $x_2, y_2 \in X_2$ such that $x_2 \succ_2 y_2$. Now, for $y_3^0 = x_3^0$, if $r = u_3(y_3^0) + u_2(x_2) - u_2(y_2)$, there exists $y_3^1 = x_3(r)$ such that $u_3(y_3^1) \geq u_3(y_3^0) + u_2(x_2) - u_2(y_2)$, or, equivalently, such that $(y_1, y_2, y_3^1) \succsim (y_1, x_2, y_3^0)$ for all $y_1 \in X_1$. By induction, for all $i \in \mathbb{N}$, if $r = u_3(y_3^i) + u_2(x_2) - u_2(y_2)$, there exists y_3^{i+1} such that $(y_1, y_2, y_3^{i+1}) \succsim (y_1, x_2, y_3^i)$. (y_3^{i+1}) is an infinite increasing strong standard sequence, bounded by y_3^0 and x_3^1 (since Not$(x_3^1 \, \mathcal{O}_3 \, x_3^0)$), which is impossible according to the strengthened Archimedean axiom. So, there exists $\beta^0 \in \mathbb{R}$ such that $u_3(x_3) \leq \beta^0$ for all $x_3 \in X_3$ such that $x_3 \, \mathcal{O}_3 \, x_3^0$.

Similarly, one can define an additive utility u^1 over $X_1 \times X_2 \times \{x_3 : x_3 \, \mathcal{O}_3 \, x_3^1$ and $x_3 \precsim_3 x_3^1\}$, and u^1 has a finite greatest lower bound, say α^1, on this set. Since, by the first case and case 2.1, u^1 is unique up to scale and location, and since $u_1 + u_2$ and $u_1^1 + u_2^1$ both represent $\succsim_{12}$ on $X_1 \times X_2$, $u_1^1 + u_2^1$ is an affine transform of $u_1 + u_2$; so, it is not restrictive to suppose that $u_1^1 = u_1$ and $u_2^1 = u_2$. Now, we will show that it is possible to define $u_3(x_3)$ for all $x_3 \in [x_3^0, x_3^1]$ so that u is representing on $X_1 \times X_2 \times [x_3^0, x_3^1]$. Clearly, it is now sufficient to show that it is possible to define $u_3(x_3)$ for all $x_3 \in [x_3^0, x_3^1]$ such that Not$(x_3 \mathcal{O}_3 x_3^0)$ so that u is representing on $X_1 \times X_2 \times [x_3^0, x_3^1]$; and a necessary condition to get this result is that $u_3(\cdot) = u_3^1(\cdot) +$ a constant γ.

Case 2.2.a: if there is no $x_3^2 \in [x_3^0, x_3^1]$ such that Not$(x_3^2 \, \mathcal{O}_3 \, x_3^0)$ and Not$(x_3^2 \, \mathcal{O}_3 \, x_3^1)$:

Then, for all $x_3 \in [x_3^0, x_3^1]$, either $x_3 \mathcal{O}_3 x_3^0$ or $x_3 \mathcal{O}_3 x_3^1$. Now, by lemma 3, it is known that, for all $(x_1, x_2, x_3) \in X_1 \times X_2 \times \{x_3 : x_3 \mathcal{O}_3 x_3^0$ and $x_3 \succsim_3 x_3^0\}$, and for all $(y_1, y_2, y_3) \in X_1 \times X_2 \times \{y_3 : y_3 \, \mathcal{O}_3 \, x_3^1$ and $y_3 \precsim_3 x_3^1\}$, $(x_1, x_2, x_3) \prec (y_1, y_2, y_3)$. But we already know that

$$u_1(x_1) + u_2(x_2) + u_3(x_3) \leq \sup_{z_1 \in X_1}\{u_1(z_1)\} + \sup_{z_2 \in X_2}\{u_2(z_2)\} + \sup_{z_3 \, \mathcal{O}_3 \, x_3^0}\{u_3(z_3)\},$$
$$u_1(y_1) + u_2(y_2) + u_3^1(y_3) \geq \inf_{z_1 \in X_1}\{u_1(z_1)\} + \inf_{z_2 \in X_2}\{u_2(z_2)\} + \inf_{z_3 \, \mathcal{O}_3 \, x_3^1}\{u_3^1(z_3)\}.$$

Therefore, if all the sup's and inf's are attained, then, clearly, a necessary and sufficient condition to get $u_1(x_1) + u_2(x_2) + u_3(x_3) < u_1(y_1) + u_2(y_2) + u_3(y_3)$ for all $(x_1, x_2, x_3) \in X_1 \times X_2 \times \{x_3 : x_3 \, \mathcal{O}_3 \, x_3^0$ and $x_3 \succsim_3 x_3^0\}$ and all $(y_1, y_2, y_3) \in X_1 \times X_2 \times \{y_3 : y_3 \, \mathcal{O}_3 \, x_3^1$ and $y_3 \precsim_3 x_3^1\}$, is to add to u_3^1 a constant γ such that $\sup_{z_1 \in X_1}\{u_1(z_1)\} + \sup_{z_2 \in X_2}\{u_2(z_2)\} + \sup_{z_3 \, \mathcal{O}_3 \, x_3^0}\{u_3(z_3)\} < \inf_{z_1 \in X_1}\{u_1(z_1)\} + \inf_{z_2 \in X_2}\{u_2(z_2)\} + \inf_{z_3 \, \mathcal{O}_3 \, x_3^1}\{u_3(z_3)\}$. So, adding any constant γ such that the last inequality holds ensures that u represents $\succsim$ on $X_1 \times X_2 \times [x_3^0, x_3^1]$. Similarly, if at least one sup or inf is not attained, then, to get $u_1(x_1) + u_2(x_2) + u_3(x_3) < u_1(y_1) + u_2(y_2) + u_3(y_3)$ for all $(x_1, x_2, x_3) \in X_1 \times X_2 \times \{x_3 : x_3 \mathcal{O}_3 x_3^0$ and $x_3 \succsim_3 x_3^0\}$ and all $(y_1, y_2, y_3) \in X_1 \times X_2 \times \{y_3 : y_3 \mathcal{O}_3 x_3^1$ and $y_3 \precsim_3 x_3^1\}$, it is necessary and sufficient that $\sup_{z_1 \in X_1}\{u_1(z_1)\} + \sup_{z_2 \in X_2}\{u_2(z_2)\} + \sup_{z_3 \, \mathcal{O}_3 \, x_3^0}\{u_3(z_3)\} \leq \inf_{z_1 \in X_1}\{u_1(z_1)\} + \inf_{z_2 \in X_2}\{u_2(z_2)\} + \inf_{z_3 \, \mathcal{O}_3 \, x_3^1}\{u_3(z_3)\}$. So, to summarize, there exists an additive utility u on $X_1 \times X_2 \times [x_3^0, x_3^1]$, and the uniqueness property of theorem 1 clearly holds.

Case 2.2.b: there exists $x_3^2 \in [x_3^0, x_3^1]$ such that Not$(x_3^2 \, \mathcal{O}_3 \, x_3^0)$ and Not$(x_3^2 \, \mathcal{O}_3 \, x_3^1)$:

Let $\mathcal{O}_3^{\sim}$ be the set of equivalence classes of $\mathcal{O}_3$, and consider $Z = \{\tilde{z}_3 \in \mathcal{O}_3^{\sim}$: there exists $z_3 \in \tilde{z}_3$ such that $z_3 \in [x_3^0, x_3^1]\}$. Suppose that Card(Z) is infinite; then it is possible to extract from Z an infinite sequence $(\tilde{z}_3^p)$ such that either $z_3^{p+1} \succ_3 z_3^p$ for all $(z_3^p, z_3^{p+1}) \in \tilde{z}_3^p \times \tilde{z}_3^{p+1}$, or such that $z_3^{p+1} \prec_3 z_3^p$ for all $(z_3^p, z_3^{p+1}) \in \tilde{z}_3^p \times \tilde{z}_3^{p+1}$. Indeed, construct the sequence as follows: let $\tilde{z}_3^0 = \tilde{x}_3^0$. If there exists an element $\tilde{z}_3 \in Z$ such that there exists no $\tilde{y}_3 \in Z$ such that $z_3^0 \prec_3 y_3 \prec_3 z_3$, then let $\tilde{z}_3^1 = \tilde{z}_3$. Repeat the same process to define $\tilde{z}_3^2, \tilde{z}_3^3$, etc. Two cases can occur: either the process can be repeated infinitely, in which case the strictly monotonic infinite sequence mentioned above has been constructed, or there exists an index p such that

$$\text{for all } \tilde{z}_3 \in Z \text{ such that } z_3^p \prec_3 z_3, \text{ there exists } \tilde{y}_3 \in Z \text{ such that } z_3^p \prec_3 y_3 \prec_3 z_3. \quad (41)$$

In this case, construct a strictly decreasing infinite sequence $(\tilde{t}_3^p)$ as follows: take any such $\tilde{z}_3$ and define $\tilde{t}_3^0 = \tilde{z}_3$. By (41), there exists $\tilde{y}_3 \in Z$ such that $z_3^p \prec_3 y_3 \prec_3 t_3^0$. Let $\tilde{t}_3^1 = \tilde{y}_3$. But $\tilde{y}_3$ is such that $z_3^p \prec_3 y_3$, so (41) can be applied again with y_3 playing the role previously taken by z_3. Therefore, $\tilde{t}_3^2$ can be constructed. (41) ensures that the process of construction will never stop, hence resulting in the existence of a strictly decreasing infinite sequence.

Now, suppose that an infinite strictly increasing sequence $(\tilde{z}_3^p)$ has been constructed (a similar proof would for a strictly decreasing sequence). The second component being essential, there exist $x_2, y_2 \in X_2$ such that $x_2 \succ_2 y_2$. Let x_1 be an arbitrary element of X_1. By definition of sequence $(\tilde{z}_3^p)$, Not$(z_3^{p+1}\ \mathcal{O}_3\ z_3^p))$, so $(x_1, y_2, z_3^{p+1}) \succ (x_1, x_2, z_3^p)$; hence (z_3^p) is an infinite increasing strong standard sequence bounded by x_3^0 and x_3^1, which contradicts the strengthened Archimedean axiom. Therefore, Card(Z) is a finite number, say N, and so $Z = \{\tilde{z}_3^1, \ldots, \tilde{z}_3^N\}$, $\tilde{z}_3^1 = \tilde{x}_3^0$, $\tilde{z}_3^N = \tilde{x}_3^1$.

We know that there exists an additive utility u^i, bounded and unique up to scale and location, on $X_1 \times X_2 \times \{x_3 : x_3\ \mathcal{O}_3\ z_3^i\}$ such that $u^i(x_1, x_2, x_3) = u_1(x_1) + u_2(x_2) + u_3^i(x_3)$. Now it can be shown inductively, using case 2.2.a, that u can be extended to represent $\succsim$ on $X_1 \times X_2 \times \{x_3 : x_3 \mathcal{O}_3 z_3^1$ or $x_3 \mathcal{O}_3 z_3^2$ or $\ldots$ or $x_3 \mathcal{O}_3 z_3^{i+1}\}$. So, u can be extended to represent $\succsim$ on $X_1 \times X_2 \times [x_3^0, x_3^1]$. As for the uniqueness property, inside an equivalence class of $\mathcal{O}_3$, elements satisfy the first case or case 2.1, which implies a uniqueness up to scale and location, and between two consecutive equivalence classes, case 2.2.a holds. Hence, the uniqueness property of theorem 1 holds.

Second step: generating u on $X_1 \times X_2 \times X_3$

Suppose that u has been constructed on $X_1 \times X_2 \times [x_3^0, x_3^1]$. Take an arbitrary element $x_3 \succ_3 x_3^1$. Similarly to the first step, there exists an additive utility, $u_1 + u_2 + u_3^1$, representing $\succsim$ on $X_1 \times X_2 \times [x_3^1, x_3]$. Using again a process similar to the first step, one can prove that if $u_3^1(\cdot) = u_3(\cdot) + u_3^1(x_3^1) - u_3(x_3^1)$, u can be extended to represent $\succsim$ on $X_1 \times X_2 \times [x_3^0, x_3]$. Using this property repeatedly, u can be extended to represent $\succsim$ on $X_1 \times X_2 \times [x_3^0, x_3]$ for all $x_3 \succsim_3 x_3^0$. The construction of u by extension ensures that we never question what was previously constructed. So, u can be extended to represent $\succsim$ on $X_1 \times X_2 \times \{x_3 : x_3 \succsim_3 x_3^0\}$. The process works fine because there exists an additive utility on $X_1 \times X_2 \times [x_3^0, x_3]$ for every x_3, which is true because the number of equivalence classes of $\mathcal{O}_3$ is finite inside $[x_3^0, x_3]$. Similarly, for all $y_3 \prec_3 x_3^0$, u can be extended to represent $\succsim$ on $X_1 \times X_2 \times [y_3, x_3^0]$; hence, u can be extended to represent $\succsim$ on $X_1 \times X_2 \times \{y_3 : y_3 \precsim_3 x_3^0\}$. By lemma 3, u represents $\succsim$ on X.

In the process of construction, we started with an arbitrary element x_3^0 of X_3, and we showed that for all $x_3 \succsim_3 x_3^0$, there exists no infinite sequence (x_3^i) such that $x_3^0 \precsim_3 x_3^i \precsim x_3$ and Not$(x_3^i\ \mathcal{O}_3\ x_3^{i+1})$. We also showed that an additive utility existed; so for every integer p, every sequence (x_3^i) such that $p \leq u_3(x_3^i) \leq p+1$ and Not$(x_3^i\ \mathcal{O}_3\ x_3^{i+1})$, is finite. Hence $\mathcal{O}_3^{\sim}$ is denumerable and there exists a sequence (x_3^i) such that for all i, $x_3^{i+1} \succ_3 x_3^i$ and for all x_3 of X_3, there exists i such that $x_3\ \mathcal{O}_3\ x_3^i$. This sequence is in fact created by taking one element in each indifference class of $\mathcal{O}_3$. The uniqueness of the additive representation is immediate. ■

Lemma 6 *Let $\succsim$ be a weak order on $\prod_{i=1}^n X_i$, satisfying axioms 2, 3, 5, 6, 7, 8, and 9. Let $k \in \{3, \ldots, n\}$. If $k = n$, let $X_{3,k} = \prod_{i=3}^n X_i$ and assume that for all $x_n, y_n \in X_n$, $x_n\ \mathcal{O}_n\ y_n$, else let $(a_{k+1}, \ldots, a_n)$ be an arbitrary element of $\prod_{i=k+1}^n X_i$ and let $X_{3,k} = \prod_{i=3}^k X_i \times \{a_{k+1}\} \times \cdots \times \{a_n\}$. Then $\succsim$ is a weak order satisfying axioms 2, 3, 5, 6, 7, 8, 9, on the 3-dimensional Cartesian product $Y = X_1 \times X_2 \times X_{3,k}$.*

Proof of lemma 6: According to Wakker (1989, pp.30–31), axiom 2 holds on Y. Moreover, by their very definitions, axioms 3, 5, 6, clearly hold on Y.

Let us prove by induction on k that axiom 7 holds on $Y:$ when k is equal to 3, there is nothing to prove. Suppose now that axiom 7 holds on sets $Y' = X_1 \times X_2 \times X_{3,k'}$ for all $k' < k$, that is, by independence, for all $x_{3,k'}, z_{3,k'} \in Y'$, there exists a sequence $(y^l_{3,k'})_{l=0}^q \in X_{3,k'}$ such that $y^0_{3,k'} = x_{3,k'}$, $y^q_{3,k'} = z_{3,k'}$, and such that for all $l \in \{0,\dots,q-1\}$, there exist $c^{l+1}, d^l \in X_1 \times X_2$ such that $(c^{l+1}, y^{l+1}_{3,k'}) \sim (d^l, y^l_{3,k'})$. Consider two arbitrary elements of $X_{3,k}$, say $(x_3,\dots,x_k,a_{k+1},\dots,a_n)$ and $(z_3,\dots,z_k,a_{k+1},\dots,a_n)$. By hypothesis, x_k and z_k are k-linked. Therefore, there exists a sequence $(y^l_k)_{l=0}^p$ in X_k such that $y^0_k = x_k$, $y^p_k = z_k$, and such that for all $l \in \{0,\dots,p-1\}$, there exist $a^{l+1} = (a^{l+1}_1,\dots,a^{l+1}_{k-1})$ and $b^l = (b^l_1,\dots,b^l_{k-1})$ in $\prod_{j=1}^{k-1} X_j$ such that $(a^{l+1}, y^{l+1}_k) \sim_{1\dots k} (b^l, y^l_k)$, or equivalently such that $(a^{l+1}, y^{l+1}_k, a_{k+1},\dots,a_n) \sim (b^l, y^l_k, a_{k+1},\dots,a_n)$. Note that, by independence, for all $l \in \{0,\dots,p-1\}$, $(a^{l+1}_3,\dots,a^{l+1}_{k-1}, y^{l+1}_k, a_{k+1},\dots,a_n)$ and $(b^{l+1}_3, b^{l+1}_{k-1}, y^{l+1}_k, a_{k+1}, \dots,a_n)$ belong to $X_{3,k-1}$. Therefore, by hypothesis of induction and independence, for all $l \in \{0,\dots,p-1\}$, there exists a sequence $(w^{m,l}_{3,k-1})_{m=0}^{q(l)}$ such that $w^{0,l}_{3,k-1} = (a^{l+1}_3,\dots,a^{l+1}_{k-1}, y^{l+1}_k, a_{k+1}, \dots,a_n)$, $w^{q(l),l}_{3,k-1} = (b^{l+1}_3,\dots,b^{l+1}_{k-1}, y^{l+1}_k, a_{k+1},\dots,a_n)$ and such that, for all $m \in \{0,\dots,q(l)-1\}$, there exist $c^{m+1,l}$, $d^{m,l} \in X_1 \times X_2$ such that $(c^{m+1,l}, w^{m+1,l}_{3,k-1}) \sim (d^{m,l}, w^{m,l}_{3,k-1})$. Similarly, there exists two sequences, say $(w^{m,-1}_{3,k-1})_{m=0}^{q(-1)}$ and $(w^{m,p+1}_{3,k-1})_{m=0}^{q(p+1)}$, $(3,k-1)$-linking respectively $(x_3,\dots,x_{k-1},x_k,a_{k+1},\dots,a_n)$ and $(b^0_1,\dots,b^0_{k-1},x_k,\ a_{k+1},\dots,a_n)$, and $(z_3,\dots,z_{k-1},z_k,a_{k+1}, \dots,a_n)$ and $(a^p_3,\dots,a^p_{k-1},z_k,a_{k+1},\ \dots,a_n)$. Then, the sequence $\left((w^{m,l}_{3,k})_{m=0}^{q(l)}\right)_{l=-1}^{p+1}$, formed by appending the sequences $(w^{m,l}_{3,k})_{m=0}^{q(l)}$ for all l, $(3,k)$-links $x_{3,k}$ and $z_{3,k}$. Hence axiom 7 holds on Y.

Let us now prove that axiom 9 holds on Y. Let $x = (x_1,x_2,x_{3,k})$ and $z = (z_1,z_2,z_{3,k})$ be two elements of Y and assume that there exists an infinite increasing strong standard sequence w.r.t. $X_{3,k}$, say $(w^l_{3,k})$, with mesh $(x^0_1,x^0_2) \prec (x^1_1,x^1_2) \in X_1 \times X_2$, and such that for all l, $x \precsim (x^0_1,x^0_2,w^l_{3,k}) \precsim z$. According to the preceding paragraph, $x_{3,k}$ and $z_{3,k}$ are $(3,k)$-linked, hence there exists a sequence $(y^m_{3,k})_{m=0}^p$ of elements of $X_{3,k}$ such that $y^0_{3,k} = x_{3,k}$, $y^p_{3,k} = z_{3,k}$, and such that for all $m \in \{0,\dots,p-1\}$, there exist $a^{m+1} = (a^{m+1}_1, a^{m+1}_2)$, $b^m = (b^m_1,b^m_2) \in X_1 \times X_2$ such that $(a^{m+1}, y^{m+1}_{3,k}) \sim (b^m, y^m_{3,k})$. For convenience, let $a^0 = (x_1,x_2)$ and $b^p = (z_1,z_2)$. Indeed, if it were not true, the sequence $(t^m_{3,k})_{m=-1}^{p+1}$ defined as $t^{-1}_{3,k} = x_{3,k}$, $t^{p+1}_{3,k} = z_{3,k}$, $t^i_{3,k} = y^i_{3,k}$ for all $i \in \{0,\dots,p\}$, $(a^{-1}_1,a^{-1}_2) = (x_1,x_2)$ and $(b^{-1}_1,b^{-1}_2) = (a^0_1,a^0_2)$, $(b^{p+1}_1,b^{p+1}_2) = (z_1,z_2)$ and $(a^{p+1}_1,a^{p+1}_2) = (b^p_1,b^p_2)$, and $(a^{m+1},t^{m+1}_{3,k}) \sim (b^m,t^m_{3,k})$ for all $m \in \{0,\dots,p-1\}$, could be used instead. Thus, for all l, there obviously exists $m \in \{0,\dots,p\}$ such that either i) $(b^m,y^m_{3,k}) \precsim (x^0_1,x^0_2,w^l_{3,k}) \precsim (a^m,y^m_{3,k})$; or ii) $(a^m,y^m_{3,k}) \precsim (x^0_1,x^0_2,w^l_{3,k}) \precsim (b^m,y^m_{3,k})$. Since $(w^l_{3,k})$ is infinite, there exists an index m such that either i) or ii) holds for an infinite number of l's. Consider such an index m and assume for convenience that i) holds for all l's in an infinite set L (the proof is similar for ii)).

If $a^m_1 \precsim b^m_1$, then for all $l \in L$, $(b^m_1,b^m_2,y^m_{3,k}) \precsim (x^0_1,x^0_2,w^l_{3,k}) \precsim (b^m_1,a^m_2,y^m_{3,k})$. So, by restricted solvability w.r.t. the second component, for all $l \in L$, there exists $c^l_2 \in X_2$ such that $(x^0_1,x^0_2,w^l_{3,k}) \sim (b^m_1,c^l_2,y^m_{3,k})$. Let us show that $(c^l_2)_{l\in L}$ is an infinite strong standard sequence w.r.t. the second component. By hypothesis, for all $l \in L$,

$$(b^m_1,c^l_2,y^m_{3,k}) \sim (x^0_1,x^0_2,w^l_{3,k}) \prec (x^1_1,x^1_2,w^l_{3,k}) \precsim (x^0_1,x^0_2,w^{l+1}_{3,k}) \sim (b^m_1,c^{l+1}_2,y^m_{3,k}). \qquad (42)$$

By essentiality w.r.t. X_1, there exists $d_1 \in X_1$ such that $\text{Not}(d_1 \sim_1 b^m_1)$. Suppose now that $d_1 \succ_1 b^m_1$ (a similar proof holds when $d_1 \prec_1 b^m_1$). Then, by equation (42),

- either there exists l such $(b^m_1,c^l_2,y^m_{3,k}) \prec (d_1,c^l_2,y^m_{3,k}) \precsim (x^1_1,x^1_2,w^l_{3,k})$, and $(c^l_2)_{l\in L}$ is a strong standard sequence of mesh $\{(b^m_1,y^m_{3,k}),(d_1,y^m_{3,k})\}$. To prove this it is sufficient to show that $\forall l' \in L$, $(b^m_1,c^{l'}_2,y^m_{3,k}) \prec (d_1,c^{l'}_2,y^m_{3,k}) \precsim (x^1_1,x^1_2,w^{l'}_{3,k})$. But if the third order cancellation axiom holds, then for all l' we get :

$$\underbrace{\begin{array}{c}(x_1^0, x_2^0, w_{3,k}^l) \sim (b_1^m, c_2^l, y_{3,k}^m) \\ (b_1^m, c_2^{l'}, y_{3,k}^m) \sim (x_1^0, x_2^0, w_{3,k}^{l'}) \\ (d_1, c_2^l, y_{3,k}^m) \precsim (x_1^1, x_2^1, w_{3,k}^l)\end{array}}_{\Downarrow} \\ (x_1^1, x_2^1, w_{3,k}^{l'}) \succsim (d_1, c_2^{l'}, y_{3,k}^m). \tag{43}$$

Let us show now that the third order cancellation axiom holds. Let $(f_{3,n}^p)$ be a sequence single-dimensionally matching $w_{3,k}^l$ and $y_{3,k}^m$ in X if such a sequence exists, else sequence $(w_{3,k}^l, y_{3,k}^m)$. Similarly, for $l' \neq l$, let $(g_{3,n}^s)$ (resp. $(h_{3,n}^t)$) be a sequence single-dimensionally matching in X $w_{3,k}^{l'}$ and $y_{3,k}^m$ (resp. $w_{3,k}^{l'}$ and $w_{3,k}^l$) if such a sequence exists, or else sequence $(w_{3,k}^{l'}, y_{3,k}^m)$ (resp. $(w_{3,k}^{l'}, w_{3,k}^l)$). Let $Z_{3,n} = \{f_{3,n}^p\} \cup \{g_{3,n}^s\} \cup \{h_{3,n}^t\}$. Then $\succsim$ is a weak order on $X_1 \times X_2 \times Z_{3,n}$ satisfying the independence axiom. We already know that restricted solvability, essentiality and the Thomsen condition hold on X_1 and X_2. The strengthened Archimedean axiom holds as well on X_1 and X_2, and on $Z_{3,n}$ since the latter is finite. The scaling axiom holds because when two elements of $Z_{3,n}$ are single-dimensionally matched in X, they are also single-dimensionally matched in $Z_{3,n}$, so that when they are not single-dimensionally matched in $Z_{3,n}$, they are neither in X and equation (4) of the scaling axiom holds. In conclusion, $(X_1 \times X_2 \times Z_{3,n}, \succsim)$ satisfies all the requirements of theorem 1, and $\succsim$ is representable by an additive utility on this set. Hence the third order cancellation axiom holds on this set, which contains $X_1 \times X_2 \times \{y_{3,k}^m, w_{3,k}^l, w_{3,k}^{l'}\}$. Thus (43) holds and $(c_2^l)_{l \in L}$ is a strong standard sequence.

- or there exists l such that $(b_1^m, c_2^l, y_{3,k}^m) \prec (x_1^1, x_2^1, w_{3,k}^l) \precsim (d_1, c_2^l, y_{3,k}^m)$, and by restricted solvability w.r.t. X_1, there exists $e_1 \in X_1$ such that $(x_1^1, x_2^1, w_{3,k}^l) \sim (e_1, c_2^l, y_{3,k}^m)$. Thus $(c_2^l)_{l \in L}$ is a strong standard sequence of mesh $\{(b_1^m, y_{3,k}^m), (e_1, y_{3,k}^m)\}$. The proof to show that the last independence relation does not depend on the value of l is similar to that of the "either" part.

But both cases are impossible according to the strengthened Archimedean axiom w.r.t. the second component.

If $a_2^m \precsim b_2^m$, then by symmetry w.r.t. the first component, there would exist an infinite strong standard sequence w.r.t. the first component "between" $(b^m, y_{3,k}^m)$ and $(a^m, y_{3,k}^m)$, which is impossible. Assume now that $b_1^m \prec a_1^m$ and that $b_2^m \prec a_2^m$. If there exists an infinite set L' such that for all $l \in L'$, $(a_1^m, b_2^m, y_{3,k}^m) \precsim (x_1^0, x_2^0, w_{3,k}^l) \precsim (a_1^m, a_2^m, y_{3,k}^m)$, then, similarly to the preceding paragraph, there exists a bounded infinite strong standard sequence w.r.t. the second component, else there exists an infinite set L'' such that for all $l \in L''$, $(b_1^m, b_2^m, y_{3,k}^m) \precsim (x_1^0, x_2^0, w_{3,k}^l) \precsim (a_1^m, b_2^m, y_{3,k}^m)$, and there exists a bounded infinite strong standard sequence w.r.t. the first component. Consequently, the existence of $(w_{3,k}^l)$ implies the existence of infinite bounded strong standard sequences w.r.t. X_1 or X_2, which is impossible by hypothesis, and axiom 9 holds on Y.

Now, let us prove that axiom 8 holds on Y. Let $x = (x_3, \ldots, x_k, a_{k+1}, \ldots, a_n)$ and $y = (y_3, \ldots, y_k, a_{k+1}, \ldots, a_n)$ be two arbitrary elements of $X_{3,k}$. Assume that, in X, x and y are not single-dimensionnally-matched. Then, by axiom 8 on X, equation (4) (see page 6) holds. If, on the contrary, x and y are single-dimensionnally-matched, then, by definition, their last components, say x_n and y_n, are n-linked. So $x, y \in Z = \prod_{i=3}^{n-1} X_i \times \{z_n : z_n \; \mathcal{O}_n \; x_n\}$. But according to the preceding paragraphs, on $X_1 \times X_2 \times Z$, $\succsim$ satisfies axioms 2, 3, 5, 6, 7 and 9. Moreover, it trivially satisfies axiom 8 on $X_1 \times X_2 \times Z$ since single-dimensionnally-matching implies i-linkness. So, by theorem 1, there exists an additive utility function representing $\succsim$ on $X_1 \times X_2 \times Z$. Equation (4) is obviously necessary for additive representability. Consequently, the scaling axiom holds w.r.t. x and y, and, *a fortiori*, on Y. ■

Proof of theorem 2: Let $(w_1, \ldots, w_n)$ be an arbitrary element of X. Hereafter, for all i, j in $\{1, \ldots, n\}$, $i < j$, $X_{i,j}$ denotes the Cartesian product $\prod_{k=i}^{j} X_k \times \prod_{k=j+1}^{n} \{w_k\}$. By lemma 6, $(X_{1,3}, \succsim)$ satisfies the assumptions of theorem 1, hence there exist real-valued functions, u_1, u_2, u_3, such that $(x_1, x_2, x_3) \succsim_{123} (y_1, y_2, y_3) \Leftrightarrow \sum_{i=1}^{3} u_i(x_i) \geq \sum_{i=1}^{3} u_i(y_i)$, and, moreover, by axiom 7, u_1, u_2, u_3 are unique up to scale and location.

First step: generating u on $X_{1,n-1}$:

Consider $X_{1,4} = \prod_{i=1}^{4} X_i \times \prod_{k=5}^{n} \{w_k\}$. Aggregate the last $(n-2)$ components; then $X_{1,4} = X_1 \times X_2 \times X_{3,4}$. Then, by lemma 6 and theorem 1, there exist real-valued functions $v_1, v_2, v_{3,4}$ such that $(x_1, x_2, x_3, x_4) \succsim_{1\ldots4} (y_1, y_2, y_3, y_4) \Leftrightarrow v_1(x_1) + v_2(x_2) + v_{3,4}(x_3, x_4) \geq v_1(y_1) + v_2(y_2) + v_{3,4}(y_3, y_4)$. Now, if n, the dimension of X, is strictly greater than 4 — else see the second step —, by axiom 7, v_1, v_2, $v_{3,4}$ are unique up to scale and location. But, for a fixed value of the fourth component, say x_4^0, $(x_1, x_2, x_3, x_4^0) \succsim_{1\ldots4} (y_1, y_2, y_3, x_4^0) \Leftrightarrow (x_1, x_2, x_3) \succsim_{1\ldots3} (y_1, y_2, y_3) \Leftrightarrow v_1(x_1) + v_2(x_2) + v_{3,4}(x_3, x_4^0) \geq v_1(y_1) + v_2(y_2) + v_{3,4}(y_3, x_4^0)$. Hence there exist constants $\alpha, \beta_1, \beta_2, \beta_{3,4}(x_4^0)$ such that, for all $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$, $v_1(x_1) = \alpha\, u_1(x_1) + \beta_1$, $v_2(x_2) = \alpha\, u_2(x_2) + \beta_2$ and $v_{3,4}(x_3, x_4^0) = \alpha\, u_3(x_3) + \beta_{3,4}(x_4^0)$. But, by independence, this is true for every value x_4^0. Thus, $v_{3,4}(x_3, x_4)$ is, in fact, the sum of two functions of one component each. Therefore, if $u_4(\cdot) = \beta_{3,4}(\cdot)$, $(x_1, x_2, x_3, x_4) \succsim_{1\ldots4} (y_1, y_2, y_3, y_4) \Leftrightarrow \sum_{i=1}^{4} u_i(x_i) \geq \sum_{i=1}^{4} u_i(y_i)$, and, of course, u_1, u_2, u_3, u_4, are unique up to scale and location. By induction, it can be proved that there exist real-valued functions $u_1, \ldots, u_{n-1}$, unique up to scale and location, and such that $(x_1, \ldots, x_{n-1}) \succsim_{1\ldots n-1} (y_1, \ldots, y_{n-1}) \Leftrightarrow \sum_{i=1}^{n-1} u_i(x_i) \geq \sum_{i=1}^{n-1} u_i(y_i)$.

Second step: generating u_n:

Select an arbitrary x_n^0 in X_n. Let $Y = \prod_{i=1}^{n-1} X_i \times \{x_n : x_n\, \mathcal{O}_n\, x_n^0\}$. On Y, $x_n\, \mathcal{O}_n\, y_n$ for every $x_n, y_n \in X_n$. So, the process described above can be applied and there exists a real-valued function u_n on $\{x_n : x_n \mathcal{O}_n x_n^0\}$, unique up to scale and location, such that $x \succsim y \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i)$ for all $x, y \in Y$. Of course, this is true for any x_n^0. So, on each equivalence class of $\mathcal{O}_n$, there exists an additive utility representing $\succsim$, unique up to scale and location.

If $\mathcal{O}_n$ has only one equivalence class, then u represents $\succsim$ on X. Otherwise, aggregate the non solvable components: let $X_{3,n} = \prod_{i=3}^{n} X_i$. Let us now show that axiom 9 holds on $X_1 \times X_2 \times X_{3,n}$. Assume that there exist $z, t \in X$ and an infinite increasing strong standard sequence $(y_{3,n}^i)_{i \geq 0}$ of mesh $\{(x_1^0, x_2^0), (x_1^1, x_2^1)\}$, such that for all i, $z \precsim (x_1^0, x_2^0, y_{3,n}^i) \precsim t$ (the proof is similar when the sequence is decreasing). If there existed an equivalence class of $\mathcal{O}_n$ containing an infinite number of elements of $(y_{3,n}^i)$, then this would contradict lemma 6 since within this class, all elements are i-linked. So the existence of $(y_{3,n}^i)$ implies the existence of an infinite number of equivalence classes of $\mathcal{O}_{3,n}$ "between" the classes containing $(z_3, \ldots, z_n)$ and $(t_3, \ldots, t_n)$. One can now extract from $(y_{3,n}^i)$ an infinite increasing strong standard sequence of mesh $\{(x_1^0, x_2^0), (x_1^1, x_2^1)\}$, say $(r_{3,n}^i)$, such that Not$(r_{3,n}^i\, \mathcal{O}_{3,n}\, r_{3,n}^j)$ for all i, j. Let (r_n^i) be the sequence of the nth components of $(r_{3,n}^i)$ in X. Then, by definition, Not$(r_n^i\, \mathcal{O}_n\, r_n^j)$ for all $i \neq j$. But since $(r_{3,n}^i)$ is increasing, (r_n^i) is also increasing. Moreover, by definition of $\mathcal{O}_n$, for all $(a_1, \ldots, a_{n-1}) \succ_{1\ldots n-1} (b_1, \ldots, b_{n-1})$ (such elements exist by essentiality), we have $(b_1, \ldots, b_{n-1}, r_n^{i+1}) \succ (a_1, \ldots, a_{n-1}, r_n^i)$. Hence (r_n^i) is an increasing strong standard sequence w.r.t. the nth component, bounded by z and t, which is impossible by hypothesis. So the strengthened Archimedean axiom holds on $X_{3,n}$.

Moreover, on $X_1 \times X_2 \times X_{3,n}$, axioms 2, 3, 5, 6, and 8, hold. And by the i-linkness axiom (axiom 7), two elements of $X_{3,n}$, say $x_{3,n}$ and $y_{3,n}$, are $(3, n)$-linked only if $x_n\, \mathcal{O}_n\, y_n$, where x_n

and y_n denote the nth component of $x_{3,n}$ and $y_{3,n}$ respectively. So, by theorem 1, there exists an additive utility $u_1 + u_2 + u_{3,n}$ on $X_1 \times X_2 \times X_{3,n}$, and u_1 and u_2 are unique up to scale and location, as well as $u_{3,n}$ on each equivalence class of $\mathcal{O}_n$. But then, on each equivalence class of $\mathcal{O}_n$, we already know that $\sum_{i=1}^n u_i$ exists representing $\succsim$. Therefore, on every equivalence class, there exists a constant such that $u_{3,n}(\cdot) = \sum_{i=3}^n u_i(\cdot) + \text{constant}$; Aggregating the constant with u_n, one gets $u_{3,n}(\cdot) = \sum_{i=3}^{n-1} u_i(\cdot) + [u_n(\cdot) + \text{constant}]$. The constant need not be the same in each equivalence class since u_n was defined separately on each equivalence class. Therefore, there exists an additive utility u representing $\succsim$ on X. Moreover, the set of equivalence classes of $\mathcal{O}_n$ is at most denumerable.

The problem that remains is the uniqueness of u. We already know that $u_1, \ldots, u_{n-1}$ are unique up to scale and location, as well as u_n on each equivalence class of $\mathcal{O}_n$. Now, suppose that there exists another additive utility: $\sum_{i=1}^n v_i$. By theorem 1, there exist a set, say N, of consecutive integers, and a sequence of elements of $X_{3,n}$, $(x_{3,n}^i)_{i \in N}$ such that, for all $i, i+1$ in N, $x_{3,n}^{i+1} \succ_{3,n} x_{3,n}^i$ and Not$(x_{3,n}^i \; \mathcal{O}_n \; x_{3,n}^{i+1})$, and, for all $x_{3,n} \in X_{3,n}$ there exists $i \in N$ such that $x_{3,n} \; \mathcal{O}_n \; x_{3,n}^i$. Moreover, for all $x_{3,n}$ such that $x_{3,n} \; \mathcal{O}_n \; x_{3,n}^i$, $v_{3,n}(x_{3,n}) = \alpha u_{3,n}(x_{3,n}) + \beta_i$, where, for all $i, i+1 \in N$, $\beta_{i+1} \geq \beta_i + \alpha[\sup_{x_1,x_2}\{u_1(x_1) + u_2(x_2)\} + \sup_{y_{3,n} \, \mathcal{O}_n \, x_{3,n}^i} u_{3,n}(y_{3,n})] - \alpha[\inf_{x_1,x_2}\{u_1(x_1) + u_2(x_2)\} + \inf_{y_{3,n} \, \mathcal{O}_n \, x_{3,n}^{i+1}} u_{3,n}(y_{3,n})]$, with equality only if the inf and/or the sup is not attained. The uniqueness of u_n follows from that of $u_3, \ldots, u_{n-1}$. ■

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