Multiattribute Utility Theory

Mohammed Abdellaoui
GREGHEC, HEC
Jouy en Josas, France

Christophe Gonzales
LIP6 - université Paris 6
Paris, France

January 5, 2013

Important decisions for both individuals and organizations often take into account multiple objectives. From the purchase of a family car to the choice of the most appropriate localization of a nuclear plant, decision makers' choices depend on many different objectives. Most often, the multiobjective nature of important decisions is revealed by assertions like "we are willing to pay a little bit more to gain the comfort or prestige of brand A instead of that of brand B" in the case of a car purchase; or "we agree to increase a little the access time to the airport if, in return, the possibilities of its future extension are increased as well, or if this can reduce noise pollution for residents" in the case of the localization of a new airport. These statements involve *tradeoffs* between different objectives of the decision maker. These tradeoffs result either from an introspective consideration performed by the decision maker herself or from an explicit decision aiding process in which the decision maker expresses her will to make *coherent tradeoffs* in order to make the "best" possible decision.

The first attempts of multiple objective decision aiding date back to the 60's with the works by, e.g., Raiffa and Edwards [Rai69, Edw71], which gave birth to *Decision Analysis*. In these works, the decision maker's preferences are represented numerically on the set of all possible choices using a numerical function called a *utility function* (or "utility" for short). The key idea of this approach lies in the fact that, after a utility function has been elicited (i.e., constructed) in a simple decision context, it can be used to assign "scores" or utilities to all *potential actions* (i.e., the possible choices) that the decision maker faces. These scores can thus be used to rank the actions from the least desirable to the most desirable one (and conversely).

However, the very fact that such scores can be constructed requires two different kinds of conditions to hold. The first one concerns "coherence conditions" that must be satisfied by the decision maker's preferences for the latter to be numerically representable by a utility function. The second condition concerns other constraints that must be satisfied in order for the initial multiobjective utility function to be decomposable as a "simple combination" of mono-objective utility functions (these are also called *multiattribute* and *single-attribute* utility functions respectively). The limited cognitive abilities of decision makers make it necessary to use such decompositions for constructing their utility functions. Indeed, each individual having her own preferences, each decision maker has her own utility function. To elicit utility, the analyst usually asks the decision maker a series of simple choice questions. The presence of more than two attributes is however cognitively more demanding (Andersen et al. [AAW86] provide an example in which alternatives are represented by 25 attributes). When multiattribute utilities can be decomposed into simple combinations of single-attribute utility functions, the tradeoffs used for their elicitation only need to involve a small set of differing attributes and, hence, they remain cognitively easy to assess.

The aim of this chapter is to study the most commonly used decompositions. More precisely, we will address in Sections 2 and 3 the additive decomposition of utility functions, the difference between these two sections being in the informations available to the decision maker when she actually makes her decision: in Section 2, she knows precisely which consequence results from each possible choice. On the other hand, in Section 3, when the decision maker makes her decisions, she does not know yet with certainty the precise consequence resulting from her choice. Finally, Section 4 will address the very construction of multiattribute utility functions and especially the most recent techniques on this matter will be presented.

1 Introduction

1.1 Utility functions

From a mathematical point of view, modeling preferences is a trivial task. As an example, assume a decision maker has some preferences over a set of choices $X = \{\text{eat some lamb, eat some duck, eat an apple pie, eat some carpaccio}\}$, that is, for each pair of elements x, y of X, she can either i) judge these elements incomparable (for instance, it may be difficult to express a definite preference for duck against the apple pie as one is a main course whereas the other is a dessert); or ii) assess a preference for one over the other, or an indifference between both meals x and y. Mathematically, this amounts to represent the decision maker's preferences by a binary relation $\succsim$ defined on $X \times X$. $x \succsim y$ then simply means that "either the decision maker prefers x to y or she is indifferent between both elements". Thus, two elements being incomparable translates into (Not($x \succsim y$) and Not($y \succsim x$)). The decision maker preferring x at least as much as y corresponds to $x \succsim y$, and a strict preference for x over y can be expressed as ($x \succsim y$ and Not($y \succsim x$)), which is generally denoted by $x \succ y$. Finally, when the decision maker is indifferent between x and y, i.e., when she likes x as much as y and conversely, then we have ($x \succsim y$ and $y \succsim x$), which is usually denoted by $x \sim y$.

However, in practice, manipulating directly relation $\succsim$ for decision aiding tasks is often neither easy nor efficient. For instance, storing in extension the set S of all pairs (x, y) such that $x \succsim y$ may be impossible in complex situations due to the huge number of such pairs. Moreover, searching S, e.g., for determining the most preferred elements, can be very time consuming, unless some structure intrinsic to S is exploited. This explains why, in practice, instead of using directly $\succsim$ for decision aiding, preferences are often first represented numerically through so-called *utility functions* —or utilities for short— and the latter are used for decision aiding. The idea underlying utility functions is quite simple: these are functions $u : X \mapsto \mathbb{R}$ attaching to each object of X a real number such that the higher the preferred the object. More formally, this amounts to:

$$\text{for all } x, y \in X, \quad x \succsim y \Leftrightarrow u(x) \geq u(y). \tag{1}$$

1.2 Decision under certainty, uncertainty and risk

In general, it is admitted that the decision maker's preferences on the set of possible alternatives is related to her preferences on the possible consequences of her choices. As an illustration, in [Sav54, page14], Savage presents the following example: your are cooking an omelet. You already broke five eggs in a plate. There remains a sixth egg to be broken and you must decide what you should do with it: i) break the egg in the plate already containing the other five eggs; ii) break this additional egg into another plate to check it before mixing it with the other eggs; iii) do not use this egg. How should you decide which of these options is the best one? Simply by analyzing what are the consequences of each decision. Thus, if the egg is safe to eat, option i) will result in a bigger omelet, but if it is unfit for consumption, the other five eggs will be wasted. Choosing option ii), if the egg is OK, then you will dirty unnecessarily a plate, and so on. By analyzing the consequences of each alternative, it is thus possible to estimate which is the best option.

As shown in the preceding example, each alternative can have several consequences, depending on the state of the egg. In Decision Theory's technical jargon, these uncertain factors —here the state of the egg— are called *events* and, as in probability theory, elementary events play a very special role and are called *states of nature*. For each state of nature (e.g., good egg or bad egg) the choice of any alternative (options i), ii) or iii)) results in one and only one consequence. Thus, alternatives can be described as sets of pairs (state of nature,consequence). This is what is usually called an *act* in Decision Theory. More formally, let A be the set of all possible alternatives, let X be the set of all possible consequences and E be the set of the states of nature. Then, an act is a function $E \mapsto X$ which, to any state of nature $e \in E$, assigns a unique consequence in X. Thus, act f corresponding to the choice of option i) is such that $f(\text{good egg})$ = "big omelet" and $f(\text{bad egg})$ = "five eggs wasted".

Let us come back to utility functions. We have already seen that such functions represent the decision maker's preferences. Since, from a cognitive point of view, alternatives can be described by acts, to the decision maker's preference relation over alternatives corresponds a preference relation over acts (see Savage [Sav54] and von Neumann-Morgenstern [vNM44] for a deeper technical discussion on this matter). Hence, let $\mathcal{A}$ denote the set of acts and $\succsim_{\mathcal{A}}$ be the preference relation over the set of acts. A utility function representing $\succsim_{\mathcal{A}}$ is thus some function $U : \mathcal{A} \mapsto \mathbb{R}$ such that $\text{act}_1 \succsim_{\mathcal{A}} \text{act}_2 \Leftrightarrow U(\text{act}_1) \geq U(\text{act}_2)$.

Of course, the decision maker's preferences over acts reveal both her preferences over consequences —she would probably prefer a big omelet rather than wasting five eggs— and her *belief* in the plausibility of occurrence of the events. Thus, if the decision maker is obsessed by use-by dates, then the pair (bad egg, five wasted eggs) will probably be only marginally be taken into account in the evaluation of option i), whereas it will be of greater importance if the decision maker is often inattentive. Utility function U must thus not only take into account the decision maker's preferences on consequences, but also the plausibility of the possible events. Now, this is possible only by taking into account the decision maker's knowledge about these events. To different types of knowledge will correspond different decision models for U. The three most important ones are certainly:

- *decision making under certainty*: whatever the state of nature that obtains, an act always results in the same consequence. This can be the case, for instance, when a decision maker chooses a given menu rather than another one in a restaurant: here, the consequences are entirely determined by the chosen menu.

  Let $\succsim_{\mathcal{A}}$ denote the preference relation over acts and $\succsim$ be that over the consequences. Assume that $\succsim_{\mathcal{A}}$ and $\succsim$ be represented by utility functions $U : \mathcal{A} \mapsto \mathbb{R}$ and $u : X \mapsto \mathbb{R}$ respectively. Call x_{act} the consequence of a given act. Then, choice under certainty amounts to assert that: for all $\text{act} \in \mathcal{A},\ U(\text{act}) = u(x_{\text{act}})$.

- *decision making under risk*: here, the alternatives can have several consequences, depending on which event actually obtains. Moreover, it is assumed that there exists an "objective" probability distribution over the events. This is the case, for instance, when a decision maker chooses or not to play games like a national lottery: the probabilities of winning as well as the resulting gain are known in advance.

  The expected utility model described below is the standard tool for decision making under risk. It was axiomatized by von Neumann and Morgenstern [vNM44]. Since to each event are assigned a probability and a consequence, there exists an objective probability that a given consequence obtains. Thus, acts can be represented as finite sets of pairs (probability of a consequence,consequence). These sets are called *lotteries*. Assume that an act corresponds to lottery $(x^1, p_1; \ldots ; x^n, p_n)$, that is, this act has consequence x^1 with probability p_1, x^2 with probability p_2, and so on. Then von Neumann-Morgenstern axiomatics implies the existence of a function U such that $U(\text{act}) = \sum_{i=1}^{n} p_i u(x^i)$, where u is the restriction of U to the set of consequences.

- *decision making under uncertainty*: this is a situation quite similar to the preceding one. However, in this case, the existence of a probability distribution over the events is not assumed but is rather derived from a set of axioms defining the *rationality* of the decision maker [Sav54]. This applies to situations, e.g., where you decide or not to bet on soccer games: the result of the games are not known at the time the decision is made. Moreover, the objectivist approach to probabilities cannot be applied since no infinite sequence of soccer games is available to estimate the probabilities of the possible events. Hence, in decision making under uncertainty, probabilities are subjective, i.e., they are estimated by the decision maker.

In this model, the decision maker assigns to each state of nature a (subjective) probability p_i of occurrence and the utility of a given act is, as in von Neumann-Morgenstern's model, $U(\text{act}) = \sum_{i=1}^{n} p_i u(x^i)$.

In the remainder of this chapter, we will consider various situations in which one or the other of these models can be applied, and we will focus our attention on the utility functions over the consequences, i.e., u.

1.3 Multiattribute utility functions

In practical situations, decision makers have multiple contradictory objectives in mind when making their decisions. This leads to describe the possible consequences using various *attributes*, that is, the set of consequences is a multidimensional space. Thus, a decision maker wishing to buy a new car may have as a choice set $X = \{$Opel Corsa, Renault Clio, Peugeot 206$\}$, but if the *choice criteria* (the attributes) are, among others, the engine size, the brand, and the price of the car, then set X can also be described as $X = \{$(1.2L; Opel; 11400 €), (1.2L; Renault; 11150 €), (1.1L; Peugeot;11600 €)$\}$. Any utility function over this set thus satisfies the following equation:

$$\text{for all } x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in X, \quad x \succsim y \Leftrightarrow u(x_1, x_2, x_3) \geq u(y_1, y_2, y_3).$$

This is precisely what is called a *multiattribute* utility function.

Of course, the meaning of the attributes of relation $\succsim$ heavily depends on the domain of application. For instance, in [Wak89, Ble96], Wakker (1989, p.28) and Bleichrodt cite, among others, the following domains:

- in consumer theory, the attributes represent the amounts of some commodity and, for any $x, y \in X$, $x \succsim y$ means that, from the decision maker's point of view, commodity bundle x is at least as good as y;

- in producer theory, $x \in X$ is a vector of inputs and $x \succsim y$ means that x provides at least as much output as y. The utility function is then called a "production function";

- in welfare theory, x is an allocation or a social situation. Each attribute represents the wealth of an agent or a player, and $x \succsim y$ means that the wealth of group x is greater than or equal to that of group y;

- in medical decision making, especially in QALYs theory (*Quality Adjusted Life Years*), the first attribute represents the level of quality of life that can be expected after undergoing some medical treatment, and the second one corresponds to the expected number of years living at this level of quality of life.

Of course this list is not exhaustive and, to each new situation, there exists an appropriate set of attributes. In [KR93], Keeney and Raiffa show how these attributes can be exhibited in practice (the so-called "structuring of objectives").

1.4 Decompositions of utility functions

When the utility function over the consequences u is known, it is very easy to exploit it using a computer: it is sufficient to apply the formulas given by von Neumann-Morgenstern or Savage. Simple optimization software can then determine the best actions that the decision maker should take. However, in practice, the effective construction of function u raises numerous problems. Indeed, although the construction of single-attribute utility functions is generally quite easy, that of multiattribute utility functions is usually very hard to perform due to the cognitive limitations of decision makers. Hence, the usual requirement that they be decomposable as a simple combinations of single-attribute more easily constructed utility functions. Consider for instance the case of someone wishing to buy a desktop computer. The attributes of interest are the brand of the computer, its processor, the storage capacity of its hard drive, the size of its LCD display, its memory amount and, of course, its price. One can easily understand why the decision maker should not have too much trouble comparing tuples (Dell; core duo 2GHz; 120GO; 17"; 2GO; 700 €) and (Apple; core duo 2GHz; 120GO; 17"; 2GO; 700 €) as these computers differ only by their brand. On the contrary, from a cognitive point of view, it is much more difficult to compare (Dell; 3GHz; 120GO; 24"; 1GO; 800 €) with (Apple; core duo 1.8GHz; 200GO; 19"; 2GO; 600 €) as these computers have very different features.

This explains why it is usually not possible to construct directly a utility function representing the decision maker's preferences. Rather, it is more efficient to construct a special form of this function the construction of which will be cognitively more "feasible".

Several such forms have been studied in the literature, the main ones being described below. In this list, X_i denotes the set of possible values for the ith attribute and it is assumed that $X \subseteq \prod_{i=1}^{n} X_i$. The axiomatizations guaranteeing the existence of these various forms differ depending on whether the decision problem is one of decision under certainty or decision under risk/uncertainty with an expected utility (EU) criterion $U(\cdot) = \sum_j p_j u(x^j)$ (as in von Neumann-Morgenstern's and Savage's models). Hence, for each item of the list, the context of application is explicitly mentioned.

1. the additive decomposition: there exist some functions $u_i : X_i \mapsto \mathbb{R}$ such that $u(x_1, \ldots, x_n) = \sum_{i=1}^{n} u_i(x_i)$. Here are some references about this decomposition: [Fis70, chapters 4 and 5], [KLST71, chapter 6], [KR93,

chapter 3], [LT64], [Deb60] and [Wak89, chapter 3] for decision making under certainty; and [Fis70, chapter 11] and [KR93, chapters 5 and 6] for the EU context.

2. the multiplicative decomposition: there exist some functions $u_i : X_i \mapsto \mathbb{R}$ such that $u(x_1, \ldots, x_n) = \prod_{i=1}^n u_i(x_i)$. This decomposition is closely related to the preceding one as it can be derived from it using a logarithmic transformation (assuming the u_i's are such that $u_i \geq 0$).

3. the multilinear decomposition (it is also called polynomial or multiplicative-additive): there exist functions $u_i : X_i \mapsto \mathbb{R}$ and, for every $j \in J$, where J is the set of subsets of $\{1, \ldots, n\}$, there exist some $\pi_j \in \mathbb{R}$ such that $u(x_1, \ldots, x_n) = \sum_{j \in J} \pi_j \prod_{k \in j} u_k(x_k)$. This decomposition is described in [KLST71, chapter 7], [Fis75], [Bel87] and [FR91] for decision making under certainty; and in [KR93, chapters 5 and 6] and [Far81] for the EU situations.

4. the decomposable structure: there exist functions $u_i : X_i \mapsto \mathbb{R}$ and some function $F : \mathbb{R}^n \mapsto \mathbb{R}$ such that $u(x_1, \ldots, x_n) = F(u_1(x_1), \ldots, u_n(x_n))$. [BP02] and [KLST71, chapter 7] study this representation under certainty. This structure is more general than the preceding ones but it has a major drawback: the uniqueness of both the u_i's and F cannot be guaranteed. As we shall see, this can raise some problems during the construction phase of the utility functions.

5. the additive nontransitive decomposition: there exist functions $v_i : X_i \times X_i \mapsto \mathbb{R}$ such that $x \succsim y \Leftrightarrow \sum_{i=1}^n v_i(x_i, y_i) \geq 0$. See [Fis91] and [BP02, BP04] for decision making under certainty; and [Nak90] for cases in which a generalization of the EU criterion is applied. Among the additive nontransitive functions lies the special case of the additive difference model: there exist functions $u_i : X_i \mapsto \mathbb{R}$ and some functions $F_i : \mathbb{R} \mapsto \mathbb{R}$ such that $x \succsim y \Leftrightarrow \sum_{i=1}^n F_i(u_i(x_i) - u_i(y_i)) \geq 0$. See [Tve69], [Fis92] and [BP02] for decision making under certainty.

In the remainder of this chapter, we will concentrate on models 1 (decomposition under certainty) and 3 (decomposition under risk/uncertainty). Let us now see the price to pay for guaranteeing that such decompositions actually represent the decision maker's preferences.

2 Decomposition under certainty

In this section, we consider situations in which every act has precisely one consequence, which is furthermore independent of the state of nature that obtains. In addition, we assume that the set of consequences X is the Cartesian product of the attributes X_i's. In other words, $X = \prod_{i=1}^n X_i$. For instance, in the "car" example mentioned in the introduction, we would have $X_1 = \{1.1\text{L}; 1.2\text{L}\}$, $X_2 = \{\text{Opel}, \text{Renault}, \text{Peugeot}\}$, $X_3 = \{11400\,€, 11150\,€, 11600\,€\}$ and $X =$

$X_1 \times X_2 \times X_3$. Note that this implies that, from a cognitive point of view, we do not preclude the existence of cars like (1.1L; Opel; 11600 €), even if such cars do not actually exist. We will see later how to relax, at least partially, this restriction. Note however that it is not possible to cope with arbitrary subsets of Cartesian product $\prod_{i=1}^{n} X_i$: this is a price to pay to have decomposable utility functions.

The rest of this section is devoted to the additive decomposability of function u. In the first subsection, such decomposability is studied in the case where $X = X_1 \times X_2$. Then, we consider the case where the set of consequences X is a Cartesian product of more than two attributes and, finally, special cases where $X \subsetneq \prod_{i=1}^{n} X_i$. For each case, our aim is to present some conditions that must be satisfied by the decision maker's preference relation $\succsim$ over the set of consequences in order to prove the existence of some functions $u_i : X_i \mapsto \mathbb{R}$ such that:

$$\text{a) for all } x, y \in \prod_{i=1}^{n} X_i,\ x \succsim y \Leftrightarrow u(x) \geq u(y) \text{ and}$$

$$\text{b) for all } (x_1, \ldots, x_n) \in \prod_{i=1}^{n} X_i,\ u(x_1, \ldots, x_n) = \sum_{i=1}^{n} u_i(x_i).$$

Of special interest, we will see that functions u_i's are unique up to very particular transformations. This will prove useful for constructing the u_i's (the so-called *elicitation* process).

2.1 Additive decomposition in 2-dimensional spaces

In this subsection, we consider decision problems in which the possible consequences of every act can be described by two attributes, i.e., $X = X_1 \times X_2$. First, let us see some necessary conditions for the existence of functions u_1 and u_2 such that:

$$\text{for all } x, y \in X_1 \times X_2,\ x \succsim y \Leftrightarrow u_1(x_1) + u_2(x_2) \geq u_1(y_1) + u_2(y_2). \quad (2)$$

The most obvious necessary condition for (2) to hold is that $\succsim$ can be represented by a utility function —not necessarily additive— $u : X_1 \times X_2 \mapsto \mathbb{R}$. In [Deb54], Debreu gave some necessary and sufficient conditions to ensure this. Among them is the completeness of $\succsim$, that is, for every pair of consequences x and y, either $x \succsim y$ or $y \succsim x$. Indeed, if $\succsim$ is representable by a utility function u, then $u(x)$ and $u(y)$ are real numbers and, consequently, either $u(x) \geq u(y)$ or $u(y) \geq u(x)$ which, by (1) implies that either $x \succsim y$ or $y \succsim x$. Similarly, $\geq$ being a transitive relation, $\succsim$ must be transitive as well, i.e., if $x \succsim y$ and $y \succsim z$ then $x \succsim z$. As a conclusion, in order to be representable by a utility function, $\succsim$ must be a weak order (this is of course a necessary condition, but it is not actually sufficient, see [Deb54]):

Definition 1 (Weak ordering): *A weak order $\succsim$ is a binary relation that is transitive ($[x \succsim y$ and $y \succsim z] \Rightarrow x \succsim z$) and complete (for all $x, y \in X$, either $x \succsim y$ or $y \succsim x$).*

Let us now see some properties specific to additive utilities. Assume there exists $u = u_1 + u_2$ representing $\succsim$. Then, for every $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$,

$$\begin{aligned}(x_1, x_2) \succsim (y_1, x_2) &\Leftrightarrow u_1(x_1) + u_2(x_2) \geq u_1(y_1) + u_2(x_2) \\ &\Leftrightarrow u_1(x_1) \geq u_1(y_1) \\ &\Leftrightarrow u_1(x_1) + u_2(y_2) \geq u_1(y_1) + u_2(y_2) \\ &\Leftrightarrow (x_1, y_2) \succsim (y_1, y_2).\end{aligned}$$

This property expresses some independence among the attributes: in her preferences, the decision maker takes into account the attributes separately, that is, there is no synergy effect between them. This leads to the following axiom:

Axiom 1 (independence): *For all $x_1, y_1 \in X_1$ and for all $x_2, y_2 \in X_2$,*
$(x_1, x_2) \succsim (y_1, x_2) \Leftrightarrow (x_1, y_2) \succsim (y_1, y_2)$,
$(x_1, x_2) \succsim (x_1, y_2) \Leftrightarrow (y_1, x_2) \succsim (y_1, y_2)$.

Let us represent $\succsim$'s indifference curves in the outcome space $X_1 \times X_2$, that is, curves the points of which are all judged indifferent. If $X_1 = X_2 = \mathbb{R}$ then the independence axiom simply states that if a point (an outcome), say A, is preferred to another one on the same vertical line, say C, then for all pairs of points (B, D) such that $ABCD$ is a rectangle (see Figure 1), B must also be preferred to D. Similarly, if B is preferred to A, then, for all pairs (C, D) such that $ABCD$ is a rectangle, D must be preferred to C.

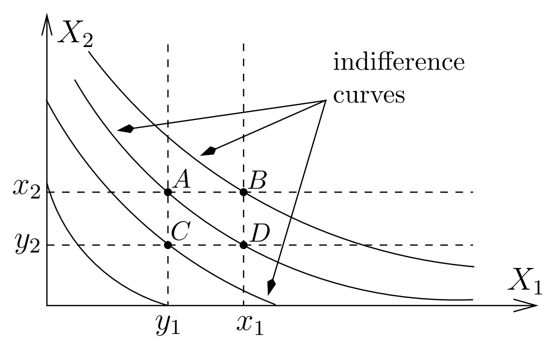

Figure 1: $\succsim$'s indifference curves and the independence axiom

The independence axiom is of utmost importance for the additive decomposability. To grab a strong understanding of this axiom, it may be worth working in a space slightly different from $X_1 \times X_2$: let $u(x_1, x_2) = u_1(x_1) + u_2(x_2)$ be an additive utility function representing $\succsim$. After assigning a given value x_2 to X_2, u does only depend on X_1. Denoting this function from X_1 to $\mathbb{R}$ by $u_{[x_2]}$, we have $u_{[x_2]}(x_1) = u(x_1, x_2)$ for all $x_1 \in X_1$. Now, we can represent $u_{[x_2]}$ in the classical space $X_1 \times \mathbb{R}$ (see Figure 2). u's additive decomposition implies that:

$$\text{for all } x_1 \in X_1, \text{ for all } x_2, y_2 \in X_2, \ u(x_1, x_2) - u(x_1, y_2) = u_2(x_2) - u_2(y_2).$$

Remark that this value does not depend on x_1. This translates on Figure 2 as: the graph of any function $u_{[x_2]}$, $x_2 \in X_2$, can be deduced from that of any $u_{[y_2]}$, $y_2 \in X_2$, by a vertical translation. Conversely, if the graphs of functions $u_{[x_2]}$, $x_2 \in X_2$, can be deduced from one another by vertical translation, u is additively decomposable. Indeed, assume that the graph of $u_{[x_2]}$ is derived by a vertical translation from that of $u_{[x_2^0]}$, for a given value $x_2^0 \in X_2$. Then, for all $x_1 \in X_1$, $u(x_1, x_2) = u(x_1, x_2^0) + \text{constant } h(x_2)$. But, then, as x_2^0 is fixed, $u(x_1, x_2^0)$ only depends on x_1, hence $u(x_1, x_2)$ is the sum of a function of x_1, i.e., $u(x_1, x_2^0)$, and a function of x_2, i.e., $h(x_2)$. The following proposition summarizes the above discussion:

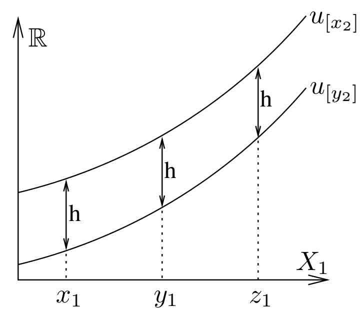

Figure 2: Additive utilities in two-dimensional spaces

Proposition 1 (additive decomposability): *Let $\succsim$ be a preference relation on $X_1 \times X_2$ representable by a utility function u. Then u is additive if and only if, for all $x_2, y_2 \in X_2$, the graph of function $u_{[x_2]}$ in space $X_1 \times \mathbb{R}$ can be deduced from that of $u_{[y_2]}$ by a vertical translation.*

Now, let us come back to the independence axiom: $(x_1, x_2) \succsim (x_1, y_2) \Leftrightarrow (y_1, x_2) \succsim (y_1, y_2)$ can be translated in terms of utility functions as $u_{[x_2]}(x_1) \geq u_{[y_2]}(x_1) \Leftrightarrow u_{[x_2]}(y_1) \geq u_{[y_2]}(y_1)$. This simply means that if $u_{[x_2]}$'s graph is "above" that of $u_{[y_2]}$ for a given point $x_1 \in X_1$, then the same holds for all the other points of X_1. But then, if the graphs of the $u_{[.]}$'s are sufficiently close to each other, any even slight variation of height between two graphs — which would rule out u's additive decomposition— would inevitably result in the intersection of at least two graphs, which would violate the independence axiom.

Under some structural conditions, it can be shown that, when the outcome set X has at least three attributes, the $u_{[x_2]}$'s graphs are always "so close to each other" that the independence axiom is almost sufficient by itself to induce u's additive decomposability. Unfortunately, this is not the case when $X = X_1 \times X_2$ and other necessary conditions such as the Thomsen condition are needed: assume again that u is additive. Then, for all $x_1, y_1, z_1 \in X_1$ and for all $x_2, y_2, z_2 \in X_2$,

$$\begin{array}{l}(x_1, z_2) \sim (z_1, y_2) \Leftrightarrow u_1(x_1) + u_2(z_2) = u_1(z_1) + u_2(y_2) \\ (z_1, x_2) \sim (y_1, z_2) \Leftrightarrow u_1(z_1) + u_2(x_2) = u_1(y_1) + u_2(z_2)\end{array}$$

Summing both equalities on the right hand side of $\Leftrightarrow$, we get:

$$u_1(x_1) + u_2(z_2) + u_1(z_1) + u_2(x_2) = u_1(z_1) + u_2(y_2) + u_1(y_1) + u_2(z_2).$$

Cancelling out the terms belonging to both sides of the equality, we get $u_1(x_1) + u_2(x_2) = u_1(y_1) + u_2(y_2)$ and, consequently, $(x_1, x_2) \sim (y_1, y_2)$ since u is a utility function. Hence the following necessary condition for the additive decomposability:

Axiom 2 (Thomsen condition): *For all $x_1, y_1, z_1 \in X_1$, for all $x_2, y_2, z_2 \in X_2$, $[(x_1, z_2) \sim (z_1, y_2)$ and $(z_1, x_2) \sim (y_1, z_2)] \Rightarrow (x_1, x_2) \sim (y_1, y_2)$.*

When we can exhibit sufficiently many indifferent ($\sim$) elements in X, the combination of independence and the Thomsen condition is sufficiently strong to imply that the vertical distances between any two $u_{[\cdot]}$'s graphs are constant, hence that u is additive. The Thomsen condition can be illustrated graphically using indifference curves: it simply states that if $A \sim B$ and $C \sim D$ then $E \sim F$.

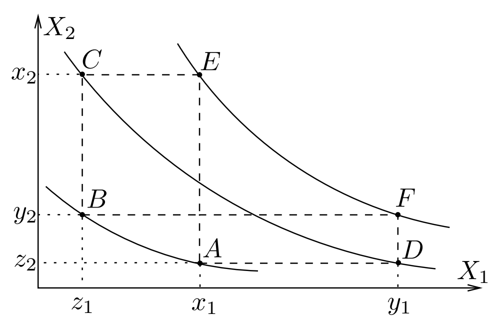

Figure 3: Thomsen condition

There still remains one important problem to fix in order to guarantee the additive decomposability: $\succsim$ must not have "many more" indifference curves that there are real numbers, else it cannot be represented by a utility function. Indeed, by definition, all the points lying on a same indifference curve are indifferent among each other and, consequently, they have the same utility, i.e., the same real number is assigned to all of them. But if there exist much more indifference curves than there exist real numbers, how can we assign to each indifference curve a different real number? The following Archimedean axiom will prevent this kind of situation to occur. Assume that $\succsim$ is representable by an additive utility function u. Let (x_1^0, x_2^0) and (x_1^0, x_2^1) be two arbitrary elements of X such that:

$$(x_1^0, x_2^0) \prec (x_1^0, x_2^1).$$

If there exists $x_1^1 \in X_1$ such that $(x_1^1, x_2^0) \sim (x_1^0, x_2^1)$ then, in terms of utility functions, this indifference is equivalent to:

$$u_1(x_1^1) = u_1(x_1^0) + (u_2(x_2^1) - u_2(x_2^0)).$$

Let $\alpha = u_2(x_2^1) - u_2(x_2^0)$. Since u represents $\succsim$, we must have $\alpha > 0$. Moreover, as by hypothesis u is additive, we know that the independence axiom holds. Hence, as X is a Cartesian product, (x_1^1, x_2^1) belongs to X and satisfies:

$$(x_1^1, x_2^0) \prec (x_1^1, x_2^1).$$

We can then iterate this process: if there exists $x_1^2 \in X_1$ such that $(x_1^2, x_2^0) \sim (x_1^1, x_2^1)$ then:

$$u_1(x_1^2) = u_1(x_1^1) + \alpha = u_1(x_1^0) + 2\alpha.$$

By induction, this creates a sequence $\{x_1^0, x_1^1, \ldots, x_1^k\}$ called a *standard sequence* such that $u_1(x_1^k) = u_1(x_1^0) + k\alpha$. So, as $\alpha > 0$, when k tends toward $+\infty$, $u_1(x_1^k)$ must also tend toward $+\infty$. Hence, if there existed $z \in X$ such that, for any k, $(x_1^k, x_2^0) \prec z$, then the utility of z would be equal to $+\infty$, which is of course impossible. As a consequence, the next axiom is necessary for the additive decomposability.

Definition 2 (standard sequence w.r.t. the 1st attribute): *For any set N of consecutive integers*[1]*, a set $\{x_1^k \in X_1, k \in N\}$ is a standard sequence w.r.t. the first attribute if and only if Not($(x_1^0, x_2^0) \sim (x_1^0, x_2^1)$) and $(x_1^k, x_2^1) \sim (x_1^{k+1}, x_2^0)$ for all $k, k+1 \in N$. $\{x_2^0; x_2^1\}$ is called the mesh of the sequence.*
A similar definition holds for standard sequences w.r.t. the other attribute.

Axiom 3 (Archimedean): *Any bounded standard sequence is finite: if (x_1^k) is a standard sequence of mesh $\{x_2^0; x_2^1\}$ such that there exist $y, z \in X$ such that $z \precsim (x_1^k, x_2^0) \precsim y$ for all $k \in N$, then sequence (x_1^k) is finite.*

Figure 4 shows the graphical interpretation of this property: the construction of the standard sequence starts at the point on lower left corner of the figure. Moving vertically from that point, when we reach the horizontal dotted line we have increased the utility by $\alpha > 0$. Now, moving down along the indifference curves (represented by solid curves on the figure) does not change the value of the utility. Consequently, the sequence of actions (vertical move, move along indifference curves) defines a sequence of points (x_1^k) the utility of which always increases by α: this is a standard sequence.

Of course, the Archimedean axiom is useful only if standard sequences can be constructed. One consequence is that there must exist some points such that $x_2^1 \succ x_2^0$ and $x_1^1 \succ x_1^0$. Hence the following axiom must be used in conjunction with the Archimedean axiom:

[1] No restriction is imposed on N: it may be finite or infinite and its integers may be positive or negative.

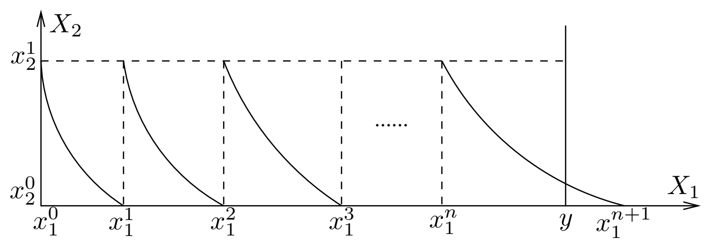

Figure 4: The Archimedean condition

Axiom 4 (essentiality): *X_1 is essential if and only if there exist $a_1, b_1 \in X_1$ and $x_2 \in X_2$ such that $(a_1, x_2) \succ (b_1, x_2)$. A similar axiom holds for the other attribute.*

The Archimedean axiom and the Thomsen condition are very powerful to structure the consequence space. However, they have a major drawback: to be useful, they require indifferences between many points of X. When such indifferences do not exist, these axioms become useless and the additive decomposability cannot be proved to hold. For instance, when $X = \mathbb{R} \times \{0, 2, 4, 6\}$ and $\succsim$ is representable on X by the following utility function:

$$u(x_1, x_2) = \begin{cases} x_1 + x_2 & \text{if} \quad x_2 \leq 4 \\ 0,5(x_1 \bmod 2)^2 + \lfloor x_1/2 \rfloor + 6,5 & \text{if} \quad x_2 = 6, \end{cases}$$

there are not enough indifferences in X and, although the independence axiom holds, it can be shown that the Thomsen condition does not. Similarly, if $X = [0, 2] \times \mathbb{N}$ and if $\succsim$ satisfies the following properties:

$\succsim$ is representable by $u(x_1, x_2) = x_1 + 2^{x_2}$ on $[0, 2] \times \mathbb{N}^*$,
$\succsim$ is representable by $u(x_1, x_2) = x_1$ on $[0, 1] \times \{0\}$,
$(x_1, 0) \succ (y_1, y_2)$ for all $x_1, y_1 \in [0, 2]$ and for all $y_2 \neq 0$,

then the Archimedean axiom is utterly useless as it is impossible to construct standard sequences with more than two elements. In this very example, it can be shown that there exists no additive utility representing $\succsim$. So, to enable the Archimedean axiom and the Thomsen condition to strongly structure the outcome space, the following additional axiom is traditionally required in the literature. It will induce the existence of a huge amount of indifferences within set X.

Axiom 5 ((restricted) solvability w.r.t. the first attribute):
For all $y_1^0, y_1^1 \in X_1$, for all $y_2 \in X_2$ and for all $x \in X$, if $(y_1^0, y_2) \precsim x \precsim (y_1^1, y_2)$, then there exists $z_1 \in X_1$ such that $x \sim (z_1, y_2)$. A similar axiom holds for the other attribute.

In two-dimensional spaces $X_1 \times X_2$, the graphical interpretation of restricted solvability is quite simple, as shown on Figure 5: if points (y_1^0, y_2) and (y_1^1, y_2)

lie on each side of the indifference containing point x, then the horizontal line passing through (y_1^0, y_2) and (y_1^1, y_2) intersects the indifference curve (of course this intersection belongs to X).

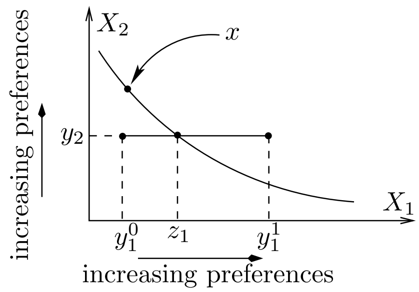

Figure 5: Restricted solvability

The combination of all the axioms presented so far is sufficient to ensure the additive representability of relation $\succsim$, as is shown by the following proposition [KLST71, chapter 6]:

Proposition 2 (existence and unicity of additive utilities):
Let $X = X_1 \times X_2$ be an outcome set, and let $\succsim$ be a binary relation on $X \times X$ satisfying restricted solvability and essentiality w.r.t. X_1 and X_2. Then, the following statements are equivalent:

1. *$\succsim$ is a weak order satisfying the Thomsen condition and, for each attribute, the independence axiom and the Archimedean axiom;*
2. *there exists an additive utility $u = u_1 + u_2$ representing $\succsim$. Moreover, this utility is unique up to scale and location. In other words, if there exists another additive utility $v = v_1 + v_2$ representing $\succsim$, then there exist $\alpha > 0$ and $\beta_1, \beta_2 \in \mathbb{R}$ such that $v_1(\cdot) = \alpha u_1(\cdot) + \beta_1$ and $v_2(\cdot) = \alpha u_2(\cdot) + \beta_2$.*

Assertion 2 implying Assertion 1 has been shown previously. As for $1 \Rightarrow 2$, the intuition of the proposition can explained using Figure 6: start from an arbitrary point $x^0 = (x_1^0, x_2^0) \in X$. Assign utility value 0 to this point. By essentiality, there exists $x_2^1 \succ x_2^0$. Without loss of generality, assign utility value 1 to (x_1^0, x_2^1). Using restricted solvability and the Archimedean axiom, construct standard sequence (x_1^k) and assign $u_1(x_1^k) = k$. Similarly, construct a vertical standard sequence of mesh $\{x_1^0; x_1^1\}$, say (x_2^r), and assign $u_2(x_2^r) = r$. The Thomsen condition guarantees that what has just been constructed is actually coherent since, if the decision maker is indifferent between A and B and between C and D, then she must also be indifferent between E and F. Fortunately, the utility assignment process used so far guarantees that the same values have been assigned to both E and F. More generally, the construction process ensures that the values assigned to all the points on the grid $\{(x_1^k, x_2^r)\}$ actually forms a utility function representing $\succsim$. Now, either this grid corresponds to the whole set X

and we just constructed an additive utility function on X, or there exist points in X that do not belong to this grid. In this case, the model can be refined by doubling the set of points on the grid: generally, the idea is to find a point $(x_1^{1/2}, x_2^{1/2})$ such that, in standard sequences of mesh $\{x_2^0; x_2^{1/2}\}$ and $\{x_1^0; x_1^{1/2}\}$, every other element corresponds to an element of (x_1^k) and (x_2^r) defined above. It is then obvious that $u_1(x_1^{1/2}) = u_2(x_2^{1/2}) = 1/2$. The process is iterated until a utility function is defined on the whole space X. This technique is used in particular in [Wak89].

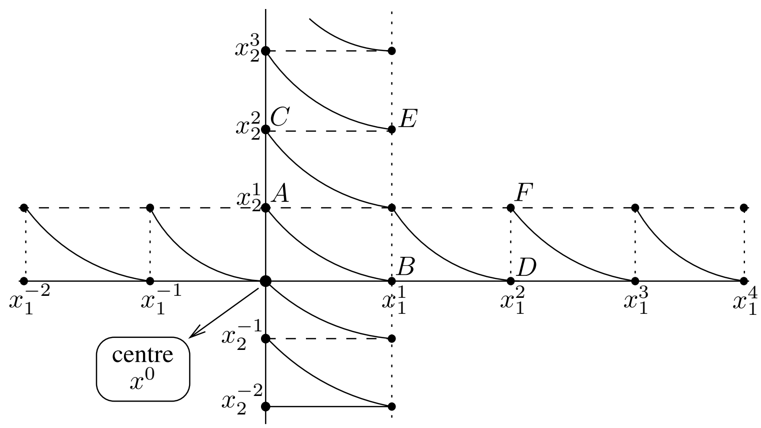

Figure 6: Intuitions behind Proposition 2

2.2 Extension to more general outcome sets

In this subsection, we will briefly see two extensions of the additive decomposability results presented so far: first, we will consider outcome sets that are still Cartesian products but that are described by more than two attributes; then, we will briefly address the case of subsets of Cartesian products.

Additive decomposability for n-dimensional Cartesian products, $n \geq 3$, is not fundamentally different from that in 2-dimensional spaces. The main difference lies in the fact that the graphs of functions $u_{[\cdot]}$, which were not necessarily very close to each other in dimension 2, are now very close due to the combined effects of independence and restricted solvability in n-dimensional spaces. As a consequence, the Thomsen condition, which was primarily used to ensure that the vertical distances between pairs of $u_{[\cdot]}$'s graphs could not vary significantly, is no more needed. The other axioms seen so far are still used and just require slight modifications to be adapted to the higher dimension of X. Only the independence axiom can be extended in several ways:

Axiom 6 (independence a.k.a. coordinate independence):
For all i, for all $z_i, t_i \in X_i$ and for all $x_j, y_j \in X_j$, $j \neq i$,

$$(x_1, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_n) \succsim (y_1, \ldots, y_{i-1}, z_i, y_{i+1}, \ldots, y_n)$$
$$\Leftrightarrow (x_1, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, x_n) \succsim (y_1, \ldots, y_{i-1}, t_i, y_{i+1}, \ldots, y_n).$$

or

Axiom 7 (weak separability):
For all i, for all $z_i, t_i \in X_i$ and for all $x_j, y_j \in X_j$, $j \neq i$,
$$(x_1, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_n) \succsim (x_1, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, x_n)$$
$$\Leftrightarrow (y_1, \ldots, y_{i-1}, z_i, y_{i+1}, \ldots, y_n) \succsim (y_1, \ldots, y_{i-1}, t_i, y_{i+1}, \ldots, y_n).$$

Axiom 6 obviously implies Axiom 7. On the other hand, the converse is false and Axiom 7 is too weak to induce by itself the existence of additive utilities. Hence, we should rather extend the independence axiom of the preceding subsection by Axiom 6. As we shall see later, weak separability can nevertheless also be used in some representation theorems.

In the context of n-dimensional spaces, we shall introduce new notations to simplify the formulas we need to manipulate. So let X_J denote the set of attributes the indices of which belong to $J \subset N = \{1, ..., n\}$. Let $x_J y$ denote the consequence in X with attributes' values x_j for $j \in J$ and y_k for $k \in N - J$. By abuse of notation, when $J = \{j\}$, we will write $x_j y$ instead of $x_J y$. Coordinate independence can thus be stated as:

Axiom 6 (independence):

For all i, for all $z_i, t_i \in X_i$ and for all $x, y \in X$, $z_i x \succsim z_i y \Leftrightarrow t_i x \succsim t_i y$.

Proposition 2 of the preceding subsection can now be extended to n-dimensional spaces by the following proposition:

Proposition 3 (existence and unicity of additive utilities):
Let $X = \prod_{i=1}^n X_i$, $n \geq 3$, be an outcome set and let $\succsim$ be a binary relation on $X \times X$ satisfying essentiality and restricted solvability w.r.t. every attribute. Then the following statements are equivalent:

1. *$\succsim$ is a weak order satisfying, for every attribute, independence (Axiom 6) and the Archimedean axiom;*

2. *there exists an additive utility function $u = \sum_{i=1}^n u_i$ representing $\succsim$ on X. Moreover, this utility is unique up to scale and location. In other words, if there exists another additive utility $v = \sum_{i=1}^n v_i$ representing $\succsim$, then there exist $\alpha > 0$ and $\beta_i \in \mathbb{R}$, $i \in \{1, \ldots, n\}$, such that $v_i(\cdot) = \alpha u_i(\cdot) + \beta_i$ for all $i \in \{1, \ldots, n\}$.*

This proposition, the proof of which can be found in [KLST71], chapter 6, is restrictive in two respects: first, the assumption that restricted solvability holds w.r.t. every attribute may be questionable in some practical situations. This is the case for instance when some attributes are naturally defined over continuums (like, e.g., money or time) while others are defined only over discrete sets (e.g., the number of rooms in a flat or some qualitative attributes like the job of a human being). For such cases, there exist some extensions of the above proposition requiring restricted solvability only w.r.t. a small number of attributes [Gon00, Gon03] or even substituting restricted solvability by "lighter" axioms requiring some density properties [Nak02]. Note however that these extensions are more difficult to use in practice than the above proposition. This is the price to pay to have theorems not requiring much structural conditions.

The second restriction imposed by Proposition 3 is the fact that X must necessarily be the Cartesian product of the X_i's: when X is only a subset of this Cartesian product, the axioms used so far can be significantly less powerful and can thus be unable to ensure the additive representability. For instance, without solvability, we already saw that the Archimedean axiom can become utterly useless if X does not contain sufficiently many pairs of indifferent elements to ensure that lengthy standard sequences can be constructed. When X is only a subset of a Cartesian product, it can have an "exotic" shape that prevents the existence of any long standard sequences, even when restricted solvability holds. Such a case is mentioned in [Wak93] where X has the shape of an Eiffel tower lying at a 45 degrees angle. Hence, when X is a subset of a Cartesian product, additive decomposability requires additional structural conditions on $(X, \succsim)$.

There are very few articles on this matter. First because we can often think of X as a Cartesian product even if, in reality, this is not precisely the case. Indeed, X corresponds to the very set of outcomes that the decision maker can imagine, not to the set of outcomes that are actually possible. And the decision maker can cognitively imagine outcomes that may be far from possible in the real world. Second, the additive decomposability on subsets of Cartesian products requires axioms that are much harder to use and to test than those presented so far. In addition, these axioms often have no real meaning in terms of preferences but rather are technical axioms only needed to complete mathematical proofs. See for instance the next proposition, due to Chateauneuf and Wakker [CW93]. Before giving it, however, we need a last additional notion. By the independence axiom (Axiom 6), or even by weak separability (Axiom 7), for every i,

$$\begin{aligned} &(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \succsim (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \\ \Leftrightarrow\ &(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n) \succsim (y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n). \end{aligned}$$

Since this preference should be satisfied for whatever $x_j, y_j \in X_j$, $j \neq i$, this means that, when the decision maker compares two outcomes, she only uses the attributes that differ from one outcome to the other. Hence, we can define for every i a new preference relation $\succsim_i$ such that $x_i \succsim_i y_i$ is equivalent to the above preference.

Proposition 4 (additive representability on open spaces):
Let $X \subset \prod_{i=1}^n X_i$. Let $\succsim$ be a weak order on X. Assume that the X_i's are endowed with the order topology w.r.t. $\succsim_i$. Assume that X is endowed with the product topology and that it is open. Moreover, assume that $\succsim$ is continuous over X and that the following sets are connected:

1. *$int(X)$, the interior of X;*

2. *all the sets of the form* $\{x \in int(X) : x_i = s_i\}$ *for all* i, s_i*;*

3. *all the equivalence classes of* $int(X)$ *w.r.t.* $\sim$.

Then, if $\succsim$ *is representable by an additive utility function on any Cartesian product included in* X*, then* $\succsim$ *is also representable by an additive utility on* X.

As we can see, the interpretation in terms of preferences of the hypotheses of this proposition is not easy. The key idea behind this proposition is to construct an additive utility on a "small" Cartesian product and, then, to extend this construction on another Cartesian product in the "neighborhood" of the first one, and to iterate this process. $\text{int}(X)$'s connexity hypothesis ensures for instance that this iterative construction process will result in an additive utility function defined over the whole of X.

In [CW93, Seg94], Chateauneuf et al. and Segal propose other representation theorems on even more general subsets. Here again, the axioms used in these theorems are rather technical and are not prone to a simple interpretation in terms of preferences. Nevertheless, there exist some subsets of Cartesian products in which the existence of additive utilities can be simply derived from that on full Cartesian products. This is the case, for instance, of rank dependent ordered sets, i.e., sets in which tuples $(x_1, \ldots, x_n)$ have the following property: all their attributes belong to the same set X_1 and there exists a weak order $\succsim'$ over X_1 such that $x_1 \succsim' x_2 \succsim' \cdots \succsim' x_n$ [Wak91].

3 Decompositions under uncertainty

The preceding section concerned situations where each act had a unique consequence, known with certainty. In this section, we address uncertain situations where each act has $m > 1$ possible consequences depending on the state of nature that obtains. Thus, the act having consequence x^i when event E_i occurs is now denoted by $(x^1, E_1; ...; x^m, E_m)$, where $\{E_1, ..., E_m\}$ is a partition of the set of states of nature considered by the decision maker. Recall that the expected utility criterion for decision under risk (see von Neumann and Morgenstern [vNM44]) assumes that the probabilities of the events are known (objectively) whereas Savage's subjective expected utility criterion [Sav54] allows to assign to each event a subjective probability that reflects the decision maker's beliefs. When the set of the states of nature is endowed with a probability measure, act $(x^1, E_1; ...; x^m, E_m)$ induces a lottery $(x^1, p_1; ...; x^m, p_m)$, where p_i denotes the probability of event E_i. Note that in Savage's axiomatics, acts can also have infinite support. Finally, in both of these expected utility axiomatics, consequences can be qualitative but also quantitative, unidimensional but also multidimensional.

In the remainder of this section, we will consider that the set of consequences X is equal to the Cartesian product $\prod_{i=1}^{n} X_i$, as in Section 2. The set of lotteries $(x^1, p_1; \ldots; x^m, p_m)$ over X is now denoted by $\mathbb{P}$ and is assumed to be endowed with the usual preference relation $\succsim$. Indifference relation $\sim$ and the strict preference relation $\succ$ are defined as before. The expected utility criterion requires the existence of a utility function $u : X \to \mathbb{R}$, defined up to scale and location, such that:

$$\text{for all } P, Q \in \mathbb{P},\ P \succsim Q \Longleftrightarrow E(u, P) \geq E(u, Q).$$

where $E(u, P)$ and $E(u, Q)$ denote the mathematical expectations of the utilities of lotteries P and Q respectively.

Similarly to the certain case, in practice, the construction of multiattribute utility function u raises numerous problems. For instance, consider the case of a decision maker having to make a decision involving h possible consequences $x^1, ..., x^h$. In theory, using the expected utility criterion, each consequence may be assigned a utility value as follows: assume that x^0 and x^* represent the least and most preferred consequences respectively. As u is defined up to scale and location, we can set without loss of generality $u(x^0) = 0$ and $u(x^*) = 1$. Now, for each consequence x^i, asking a simple question to the decision maker, it is possible to determine probability p_i such that she is indifferent between receiving a gain of x^i with certainty and obtaining the lottery ticket providing consequence x^* with probability p_i and consequence x^0 with probability $1 - p_i$. According to the expected utility criterion, this indifference implies that $u(x^i) = p_i$ for every $i = 1, ..., h$.

Due to the cognitive limitations of decision makers, it is clearly impossible to use this kind of elicitation method when the number of attributes is high. Moreover, even when the latter stays relatively small, the combinatorial nature of X can induce a large set of consequences which, again, prevents the above elicitation method to be usable. Hence, in practice, analysts need decomposing u in single-attribute utility functions much easier and more intuitive to elicit (see [Pol67, Kee68, KR93, vWE93] among others). Of course, as in the certain case, under uncertainty, utility function u being additively decomposable requires that additional constraints on the decision maker's preferences be satisfied. In this direction, Miyamoto and Wakker have proposed a decomposition approach based on models generalizing the classical expected utility model [MW96].

3.1 Decomposition in 2-dimensional spaces

The additivity of von Neumann-Morgenstern utility function requires an independence notion more general than in the certain case. Indeed, assuming preferences can be modeled using the expected utility criterion, if $u = u_1 + u_2$, with $u_i : X_i \mapsto \mathbb{R}$ for $i = 1, 2$, then, for any $x_1, x_1', y_1, y_1' \in X_1$, $x_2, z_2 \in X_2$,

$$((x_1, x_2), \frac{1}{2}; (x_1', x_2), \frac{1}{2}) \succsim ((y_1, x_2), \frac{1}{2}; (y_1', x_2), \frac{1}{2})$$
$$\Updownarrow$$
$$\frac{1}{2}u(x_1, x_2) + \frac{1}{2}u(x_1', x_2) \geq \frac{1}{2}u(y_1, x_2) + \frac{1}{2}u(y_1', x_2)$$
$$\Updownarrow$$
$$\frac{1}{2}u(x_1, z_2) + \frac{1}{2}u(x_1', z_2) \geq \frac{1}{2}u(y_1, z_2) + \frac{1}{2}u(y_1', z_2)$$
$$\Updownarrow$$
$$((x_1, z_2), \frac{1}{2}; (x_1', z_2), \frac{1}{2}) \succsim ((y_1, z_2), \frac{1}{2}; (y_1', z_2), \frac{1}{2}).$$

The above equivalences show that preferences over lotteries differing only on attribute X_1 do not depend on their common level on attribute X_2. In such a case, attribute X_1 is said to be *utility independent* from attribute X_2. A similar reasoning implies that, for every $x_1, z_1 \in X_1$, $x_2, x_2', y_2, y_2' \in X_2$,

$$((x_1, x_2), \frac{1}{2}; (x_1, x_2'), \frac{1}{2}) \succsim ((x_1, y_2), \frac{1}{2}; (x_1, y_2'), \frac{1}{2})$$
$$\Updownarrow$$
$$((z_1, x_2), \frac{1}{2}; (z_1, x_2'), \frac{1}{2}) \succsim ((z_1, y_2), \frac{1}{2}; (z_1, y_2'), \frac{1}{2}).$$

In this case, attribute X_2 is said to be utility independent from attribute X_1. When X_1 is in addition utility independent from X_2, both attributes are said to satisfy *mutual utility independence*. Note that the independence axiom under certainty (Axiom 1) is a special case of mutual utility independence in which probability $\frac{1}{2}$ is substituted by probability 1.

Under expected utility, utility independence of attribute X_1 from attribute X_2 implies that, for any two $x_2, x_2' \in X_2$, utility functions $u(., x_2)$ and $u(., x_2')$ represent the same preferences over X_1. They are therefore identical up to scale and location. In other words, $u(., x_2) = \alpha u(., x_2') + \beta$, where $\alpha > 0$ and $\beta \in \mathbb{R}$ depend only on the given consequences x_2 and x_2'. Assuming that x_2 varies and that x_2' is fixed at a given level x_2^0, we can write more specifically that:

$$\text{for all } (x_1, x_2) \in X_1 \times X_2,\ u(x_1, x_2) = \alpha(x_2)u(x_1, x_2^0) + \beta(x_2) \qquad (3)$$

where $\alpha(.) > 0$ and $\beta(.) \in \mathbb{R}$ depend implicitly on consequence level x_2^0. Similarly, if attribute X_2 is utility independent from attribute X_1, then, for any consequence level x_1^0, we have that:

$$\text{for all } (x_1, x_2) \in X_1 \times X_2,\ u(x_1, x_2) = \gamma(x_1)u(x_1^0, x_2) + \delta(x_1) \qquad (4)$$

where $\gamma(.) > 0$ and $\delta(.) \in \mathbb{R}$ depend implicitly on consequence level x_1^0.

Assume now that $u(x_1^0, x_2^0) = 0$. By Equations (3) and (4), $\beta(x_2) = u(x_1^0, x_2)$, $\delta(x_1) = u(x_1, x_2^0)$ and:

$$u(x_1, x_2^0)[\alpha(x_2) - 1] = u(x_1^0, x_2)[\gamma(x_1) - 1].$$

This equation obviously holds when $x_1 = x_1^0$ or when $x_2 = x_2^0$. Otherwise, i.e., when both $x_1 \neq x_1^0$ and $x_2 \neq x_2^0$, we get the following equality:

$$\frac{\alpha(x_2) - 1}{u(x_1^0, x_2)} = \frac{\gamma(x_1) - 1}{u(x_1, x_2^0)} = k$$

where k is a constant which is independent of variables x_1 and x_2. Hence, it can be deduced that $\alpha(x_2) = ku(x_1^0, x_2) + 1$. Substituting in Equation (3), we get:

$$\forall (x_1, x_2) \in X_1 \times X_2,\ u(x_1, x_2) = u(x_1, x_2^0) + u(x_1^0, x_2) + ku(x_1, x_2^0)u(x_1^0, x_2) \quad (5)$$

where $u(\cdot, x_2^0)$ and $u(x_1^0, \cdot)$ are single-attribute utility functions. Constant k represents a factor of interaction between attributes X_1 and X_2. As shown in [KR93, page 240], the sign of this constant precises explicitly the nature of this interaction. Thus, when $u(\cdot, x_2^0)$ and $u(x_1^0, \cdot)$ are functions increasing in x_1 and x_2 respectively, a positive (resp. negative) k means that attributes X_1 and X_2 are complementary (resp. substitutable). The following Proposition introduces the above multilinear decomposition in a slightly different manner substituting $u(x_1, x_2^0)$ and $u(x_1^0, x_2)$ by $k_1u_1(x_1)$ and $k_2u_2(x_2)$ respectively, k_1 and k_2 being *scaling constants* depending implicitly on the consequences used for normalizing functions $u_i(.)$, $i = 1, 2$ (see [Fis65] and [KR93, pages 234–235]).

Proposition 5: *Assume that X_1 and X_2 are mutually utility independent. Then utility function u can be decomposed using the following multilinear form:*

$$\forall (x_1, x_2) \in X_1 \times X_2,\ u(x_1, x_2) = k_1u_1(x_1) + k_2u_2(x_2) + kk_1k_2u_1(x_1)u_2(x_2).$$

where:

- $u_i(.)$ *is a single-attribute utility function normalized by* $u_i(x_i^0) = 0$ *and* $u_i(x_i^*) = 1$, $i = 1, 2$, *for* x_1^* *and* x_2^* *such that* $(x_1^*, x_2^0) \succ (x_1^0, x_2^0)$ *and* $(x_1^0, x_2^*) \succ (x_1^0, x_2^0)$.
- $k_1 = u(x_1^*, x_2^0) > 0$, $k_2 = u(x_1^0, x_2^*) > 0$ *and* $k_1 + k_2 + kk_1k_2 = 1$.

As shown above, mutual utility independence is not sufficient to induce the additive decomposition of u. The latter actually requires in addition that constant k be equal to 0. Let us now see a sufficient condition which, when combined with mutual utility independence, results in the additive decomposability of u. Assume that there *exist* some consequences $x_1, x_1' \in X_1$ and $x_2, x_2' \in X_2$ such that:

$$((x_1, x_2), \frac{1}{2}; (x_1', x_2'), \frac{1}{2}) \sim ((x_1, x_2'), \frac{1}{2}; (x_1', x_2), \frac{1}{2}). \quad (6)$$

Translating this indifference in terms of expected utilities, and cancelling out the terms appearing on both sides of the resulting equality, we get:

$$k[u(x_1, x_2^0) - u(x_1', x_2^0)][u(x_1^0, x_2) - u(x_1^0, x_2')] = 0.$$

If Not$[(x_1, x_2^0) \sim (x_1', x_2^0)]$ and Not$[(x_1^0, x_2) \sim (x_1^0, x_2')]$, then k is constrained to be equal to 0.

When $k \neq 0$, the multilinear decomposition (5) can be rewritten as:

$$v(x_1, x_2) = v(x_1, x_2^0)v(x_1^0, x_2)$$

where $v(x_1, x_2) = 1 + ku(x_1, x_2)$. This shows that mutual utility independence actually induces a *multiplicative* decomposition of utility function u.

Using scaling constants k_i as in Proposition 5, this model can also be written as:

$$1 + ku(x_1, x_2) = \prod_{i=1}^{2}[1 + kk_i u_i(x_i)].$$

Now, since scaling constants k_1 and k_2 belong to the unit interval and since $1 + k_1 = \prod_{i=1}^{2}[1 + kk_i]$, constant $k = [1 - (k_1 + k_2)]/k_1 k_2$ lies necessarily between -1 and 0 for $k_1 + k_2 > 1$ and is greater than 0 for $k_1 + k_2 < 1$.

An extension of the cases in which indifference (6) holds induces a new condition called *additive independence*. This new condition is sufficient to guarantee the additive decomposition of utility function u.

Definition 3: *Attributes X_1 and X_2 are said to be additively independent if indifference (6) holds for any consequences $x_1, x_1' \in X_1$ and $x_2, x_2' \in X_2$.*

Substituting consequence (x_1', x_2') by the reference level consequence (x_1^0, x_2^0) in indifference (6), we obtain the following indifference:

$$((x_1, x_2), \frac{1}{2}; (x_1^0, x_2^0), \frac{1}{2}) \sim ((x_1, x_2^0), \frac{1}{2}; (x_1^0, x_2), \frac{1}{2}).$$

Set $u(x_1^0, x_2^0) = 0$. Then, the translation of the above indifference in terms of expected utilities results in the equality below:

$$\text{for all } (x_1, x_2) \in X_1 \times X_2,\ u(x_1, x_2) = u(x_1, x_2^0) + u(x_1^0, x_2) \tag{7}$$

The following Proposition simply rewrites Equation (7) in a more additive manner by introducing scaling constants.

Proposition 6: *Assume that attributes X_1 and X_2 are additively independent. Then utility function u can be written as:*

$$\textit{for all } (x_1, x_2) \in X_1 \times X_2,\ u(x_1, x_2) = k_1 u_1(x_1) + k_2 u_2(x_2),$$

where:

- $u_i(.)$ *is a single-attribute utility function normalized by* $u_i(x_i^0) = 0$ *and* $u_i(x_i^*) = 1$, $i = 1, 2$, *for* x_1^* *and* x_2^* *such that* $(x_1^*, x_2^0) \succ (x_1^0, x_2^0)$ *and* $(x_1^0, x_2^*) \succ (x_1^0, x_2^0)$.
- $k_1 = u(x_1^*, x_2^0) > 0$, $k_2 = u(x_1^0, x_2^*) > 0$ *and* $k_1 + k_2 = 1$.

As can be seen above, the very fact that, in a decision problem, the consequences are described by several attributes raises the problem of choosing the appropriate decomposition of the utility function. Most often, the analyst must check with the decision maker whether mutual utility independence holds among the attributes. For this purpose, a simple approach consists of verifying whether the certainty equivalent w.r.t. a given attribute X_i of a lottery with two equiprobable consequences having the same value of X_i depends or not on the common level assigned to attribute X_i. More precisely, assume that $X_i = [x_i^0, x_i^*]$ for $i = 1, 2$. In order to check whether attribute X_1 is utility independent from attribute X_2, it is sufficient to choose three equidistant levels $\overline{x}_2, \overline{x}'_2, \overline{x}''_2$ in $[x_2^0, x_2^*]$ and to determine the certainty equivalents of lotteries $((x_1^*, a), \frac{1}{2}; (x_1^1, a), \frac{1}{2})$, $a = \overline{x}_2, \overline{x}'_2, \overline{x}''_2$. Identical certainty equivalents (up to some reasonable errors) lead to assume that attribute X_1 is actually utility independent from attribute X_2. Utility independence of X_2 w.r.t. X_1 can be tested using a similar approach in which the roles of both attributes are exchanged.

In situations where it is reasonable to assume that the appropriate model is additively decomposable, it is possible to directly check additive independence. To do so, it is sufficient to fix three or four equidistant consequences in each of the intervals $X_i = [x_i^0, x_i^*]$, $i = 1, 2$, and to check condition (6) for the elements of the resulting Cartesian product.

3.2 Extension of the 2-dimensional decomposition

The decompositions of von Neumann-Morgenstern utility functions with more than two attributes result from quite simple extensions of the concepts and tools developed for the two-dimensional case. We just need introducing some convenient notations to address the n-dimensional case.

Let us first recall that, if $J \subset N = \{1, ..., n\}$, $x_J y$ stands for the consequence in X having coordinates x_j for $j \in J$ and coordinates y_j for $j \in N - J$. Moreover, when $J = \{j\}$, to simplify the notation, we write $x_j y$ instead of $x_J y$. In addition, $x_i x_j y$ means that the ith and jth coordinates of y have been substituted by x_i and x_j respectively. Finally, x_J denotes the (sub-)consequence constituted only by coordinates x_j, with $j \in J$.

Definition 4: *The set of attributes* X_J, $J \subset N$, *is said to be* utility independent *if for all* $x, x', y, y', t, z \in X$

$$(x_J t, \frac{1}{2}; y_J t, \frac{1}{2}) \succsim (x'_J t, \frac{1}{2}; y'_J t, \frac{1}{2}) \Leftrightarrow (x_J z, \frac{1}{2}; y_J z, \frac{1}{2}) \succsim (x'_J z, \frac{1}{2}; y'_J z, \frac{1}{2}). \quad (8)$$

There is mutual utility independence in the attributes of X *if* X_J *is* utility independent *for every* $J \subset N$.

Under certainty, the independence axiom (Axiom 6) is a particular case of utility independence in which equivalence (8) above becomes $x_J t \succsim x'_J t \Longleftrightarrow x_J z \succsim x'_J z$. Note that it is easy to show that, when u is additively decomposable, equivalence (8) holds for every $J \subset N$.

Under expected utility, for a given J, utility independence of X_J implies that, for any two distinct consequences t and z of X, utility functions $u(., t_{-J})$ and $u(., z_{-J})$ represent the same preferences. As in the two-attribute case, it can be deduced that these utilities are identical up to scale and location. Assuming that t_{-J} varies and that z_{-J} is set to a given reference level x^0_{-J}, the following can be written:

$$\text{for all } x \in X,\ u(x) = \alpha_J(x_{-J})u(x_J x^0) + \beta_J(x_{-J})$$

where $\alpha_J(.) > 0$ and $\beta_J(.) \in \mathbb{R}$ depend implicitly on the reference level consequence x^0_{-J}.

In cases where mutual utility independence holds, a similar reasoning to that of the two-attribute case leads to the following decomposition:

$$\text{for all } x \in X,\ u(x) = u(x_1 x^0) + \textstyle\sum_{j=2}^{n} \prod_{i=1}^{j-1} [ku(x_i x^0) + 1]u(x_j x^0) \qquad (9)$$

where k is a constant playing a role similar to that in (5). When this constant is equal to 0, the above equation results in an additive decomposition:

$$\text{for all } x \in X,\ u(x) = \textstyle\sum_{j=1}^{n} u(x_j x^0)$$

As in the two-attribute case, when $k \neq 0$ ($\sum_i k_i \neq 1$), Equation (9) can be rewritten as follows:

$$v(x) = \textstyle\prod_{j=1}^{n} v(x_j x^0)$$

where $v(x_j y) = 1 + ku(x_j y)$ for every $x_j \in X_j$, $j = 1, ..., n$, and $y \in X$. Scaling constants $k_1, ..., k_n$ can also be emphasized by substituting $u(x_j x^0)$ by $k_j u_j(x_j)$ for every $j = 1, ..., n$. Hence the (equivalent) multiplicative decomposition:

$$ku(x) + 1 = \textstyle\prod_{j=1}^{n} [kk_j u_j(x_j) + 1] \qquad (10)$$

where $u_j(x_j^0) = 0$ and $u_j(x_j^*) = 1$, $j = 1, \ldots, n$.

As an illustration, in the three-attribute case X_1, X_2, X_3, the decomposition of the utility function implied by (10) reduces to the following equality:

$$\begin{aligned} u(x_1, x_2, x_3) = {} & k_1 u_1(x_1) + k_2 u_2(x_2) + k_3 u_3(x_3) + kk_1 k_2 u_1(x_1) u_2(x_2) \\ & + kk_1 k_3 u_1(x_1) u_3(x_3) + kk_2 k_3 u_2(x_2) u_3(x_3) \\ & + k^2 k_1 k_2 k_3 u_1(x_1) u_2(x_2) u_3(x_3) \end{aligned}$$

where, as in the two-attribute case, u_1, u_2 and u_3 are single-attribute utility functions and $k_1 + k_2 + k_3 + kk_1k_2k_3 + k^2k_1k_2k_3 = 1$. Now, when mutual independence is substituted by utility independence of X_J, for $J = \{1\}, \{2\}, \{3\}$, the resulting decomposition of the utility function is much richer. As a matter of fact, it can be shown that coefficient k, which represents the interaction among the attributes, is substituted by some specific interaction coefficients k_{12}, k_{13}, k_{23} and k_{123}:

$$\begin{aligned}u(x_1, x_2, x_3) = {} & k_1 u_1(x_1) + k_2 u_2(x_2) + k_3 u_3(x_3) + k_{12} k_1 k_2 u_1(x_1) u_2(x_2) \\ & + k_{13} k_1 k_3 u_1(x_1) u_3(x_3) + k_{23} k_2 k_3 u_2(x_2) u_3(x_3) \\ & + k_{123} k_1 k_2 k_3 u_1(x_1) u_2(x_2) u_3(x_3).\end{aligned}$$

The relative complexity of the above decomposition justifies why Keeney and Raiffa [KR93, p.298] and other authors suggest to limit the set of admissible decompositions to the multiplicative and additive forms when $n \geq 4$. The following proposition generalizes Proposition 5 [Fis65].

Proposition 7: *Assume mutual utility independence. Then utility function u can be decomposed as in Equation (9).*

When mutual utility independence holds, determining an additive utility function for $m > 2$ attributes requires checking a condition similar to that given by indifference (6). Indeed, it can be shown that if there exist some consequences $y \in X$, $x_i, x'_i \in X_i$ and $x_j, x'_j \in X_j$, with $i \neq j$, such that:

$$(x_i x_j y, \frac{1}{2}; x'_i x'_j y, \frac{1}{2}) \sim (x_i x'_j y, \frac{1}{2}; x'_i x_j y, \frac{1}{2}),$$

then utility function u must be additively decomposable [KR93].

Without mutual utility independence, the additive decomposability of function u requires a generalization of the additive independence condition introduced for the two-attribute case. Attributes $X_1, ..., X_n$ are said to be additively independent if, for any consequences $x, x', y, y' \in X$ and any $J \subset N$,

$$(x_J y, \frac{1}{2}; x'_J y', \frac{1}{2}) \sim (x_J y', \frac{1}{2}; x'_J y, \frac{1}{2}).$$

Pollak [Pol67] proposes a slightly different condition which is both necessary and sufficient for the additive decomposability.

As in the case of two-attribute decision problems, choosing between the multiplicative and the additive models for more than two attributes requires checking whether the corresponding conditions are approximately satisfied by the decision maker's preferences. This task is however slightly more complicated as it requests from the decision maker a deeper cognitive effort. Finally, note that, in [KR93, p. 292], Keeney and Raiffa provide another set of conditions that enable checking utility independence while being more economical than those resulting directly from the definition given in this subsection.

4 Elicitation of utility functions

The aim of elicitation of multiattribute utility functions is to assign scores or utilities to the possible actions that can be chosen by the decision maker. These scores can then be used to rank the actions from the least desirable to the most desirable, and conversely. However, the very fact that such scores can be constructed from single-attribute utility functions requires some specific independence conditions to hold. In this section, we will only address the problem of eliciting utility functions in the two-attribute case. Similar methods can be used in situations where there are more than two attributes.

4.1 Elicitation under certainty

Assume that the decision maker faces a decision problem involving two attributes, and that her preferences can be represented by the additive model given by:

$$\text{for all } x, y \in X_1 \times X_2,\ x \succsim y \Longleftrightarrow u_1(x_1) + u_2(x_2) \geq u_1(y_1) + u_2(y_2).$$

It is now well known that if there exist some additional functions v_1 and v_2 satisfying the above equivalence in place of u_1 and u_2 respectively, then there exist $\alpha > 0$ and $\beta_1, \beta_2 \in \mathbb{R}$ such that $v_i(.) = \alpha u_i(.) + \beta_i$ for $i = 1, 2$. As a consequence, the origins of u_1 and u_2 —which can be distinct— can be set as we wish, as well as a common unit for the scales of both u_1 and u_2. Assume that x_i^0 denote the smallest consequence of set X_i, for $i = 1, 2$.

The first step in u_1's and u_2's elicitations consists of setting the origins of their utility scales as follows:

$$u(x_1^0, x_2^0) = u_1(x_1^0) = u_2(x_2^0) = 0. \tag{11}$$

Eliciting single-attribute utility function u_1 now requires choosing a new consequence R_2 such that $R_2 \succ x_2^0$ and determining consequence x_1^1 such that:

$$(x_1^1, x_2^0) \sim (x_1^0, R_2). \tag{12}$$

Intuitively, the closer to x_2^0 (in terms of preferences) the consequence R_2, the closer to x_1^0 the consequence x_1^1. The next step in u_1's elicitation consist of determining a new consequence x_1^2 such that:

$$(x_1^2, x_2^0) \sim (x_1^1, R_2). \tag{13}$$

Translating indifferences (12) and (13) into the additive utilities model and subtracting the resulting equations leads to the following equality:

$$u_1(x_1^1) - u_1(x_1^0) = u_1(x_1^2) - u_1(x_1^1). \tag{14}$$

To summarize, u_1's elicitation amounts to construct a standard sequence of consequences $x_1^0, x_1^1, ..., x_1^{s_1}$ which "covers" X_1 using indifferences:

$$(x_1^i, x_2^0) \sim (x_1^{i-1}, R_2),\ i = 1, ..., s^1.$$

Finally, setting $u_1(x_1^1) = 1$, we get $u_1(x_1^i) = i$, $i = 2, ..., s^1$. Figure 7 illustrates the elicitation process thus described.

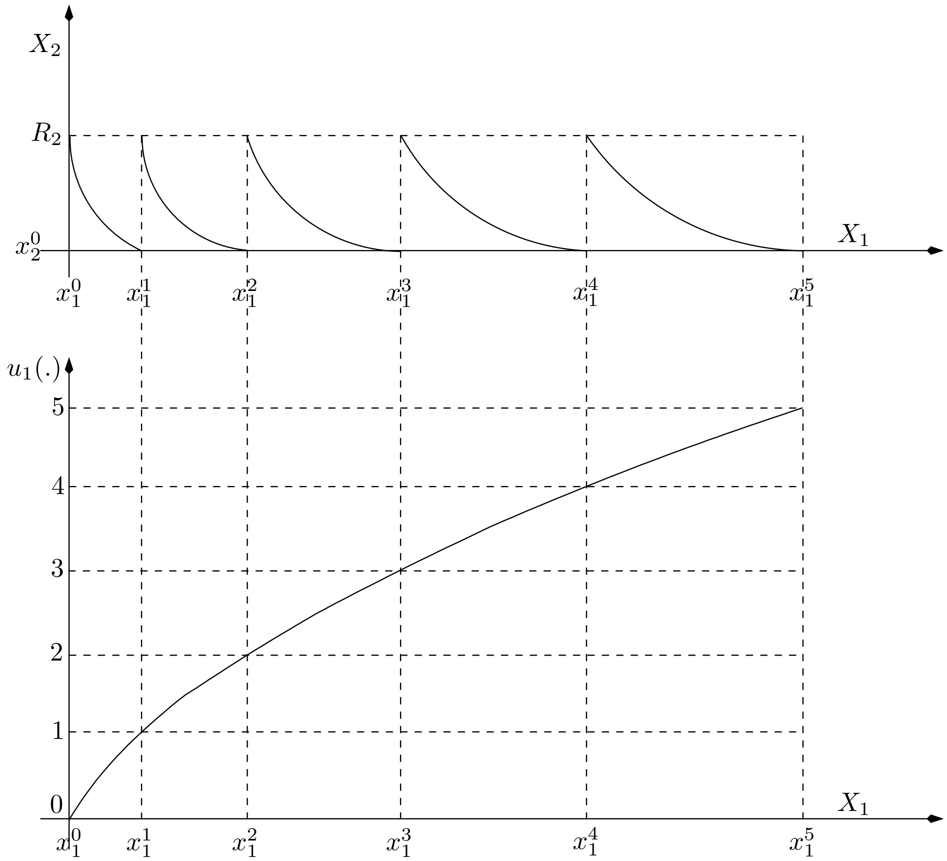

Figure 7: Elicitation of function $u_1(.)$

Similarly, eliciting function u_2 starts by choosing a consequence R_1 such that $R_1 \succ x_1^0$ and determining consequence x_2^1 such that:

$$(x_1^0, x_2^1) \sim (R_1, x_2^0). \tag{15}$$

After the construction of the initial indifference (15), the elicitation process goes on with the construction of a standard sequence of consequences $x_2^0, x_2^1, ..., x_2^{s_2}$ "covering" X_2 and determined using the following indifferences:

$$(x_1^0, x_2^i) \sim (R_1, x_2^{i-1}),\ i = 1, ..., s^2.$$

Figure 8 illustrates graphically the process.

By indifferences (12) and (15), choosing $R_1 = x_1^1$ leads necessarily to $R_2 = x_2^1$. This choice thus results in $u_2(x_2^i) = i$, $i = 1, ..., s^2$.

The value chosen for R_1 can also be different from x_1^1. This results inevitably in $x_2^1 \neq R_2$. In this case, the additive model below is to be used:

$$\text{for all } x \in X_1 \times X_2,\ u(x_1, x_2) = k_1 u_1(x_1) + k_2 u_2(x_2) \tag{16}$$

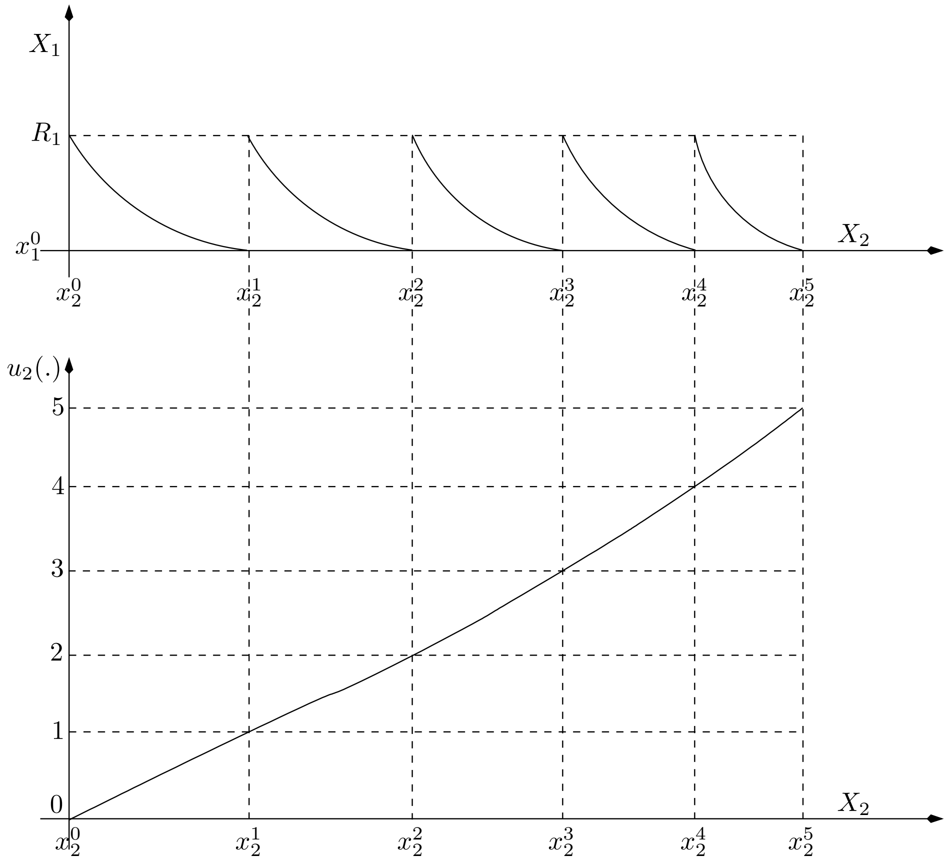

Figure 8: Elicitation of function $u_2(.)$

where $k_1 > 0$ and $k_2 > 0$ are *scaling constants* such that $k_1 + k_2 = 1$. These constant introduce an additional degree of freedom that allows us to assign to u_2 a utility unit independent from that resulting from $u_1(x_1^1) = 1$ and, thus, to set $u_2(x_2^1) = 1$. Determining the scaling constants requires using (or constructing) an additional indifference. Thus, translating indifference (15) in terms of the model described in Equation (16) results in the following equality:

$$\frac{k_2}{k_1} = \frac{u_1(R_1) - u_1(x_1^0)}{u_2(x_2^1) - u_2(x_2^0)} = u_1(R_1).$$

Knowning $u_1(R_1)$ and $k_1 + k_2 = 1$, scaling constants can thus be determined. These allow to link appropriately the utility scales of both u_1 and u_2.

4.2 Elicitation under uncertainty

The essential hypothesis underlying the expected utility-based decision model is that the decision maker's preferences are sufficiently stable that they can be observed through very simple risky choices. These preferences are revealed through her utility function by the analyst. The latter can then uses them to infer the decision maker's preferences over the set of all the possibles actions. Being able to perform this inference is essential: if we are unable to elicit the "appropriate" utility function, it may happen that we propose to the decision maker some ranking of the possible actions that is utterly unrelated to her own preferences.

In the rest of this subsection, we assume that $X_i = [x_i^0, x_i^*]$ for $i = 1, ..., n$. In addition, all the utility functions are considered to be normalized as follows: $u_i(x_i^0) = 0$ and $u_i(x_i^*) = 1$, $i = 1, ..., n$. Of course, these normalizations require some scaling constants, as in the certain case.

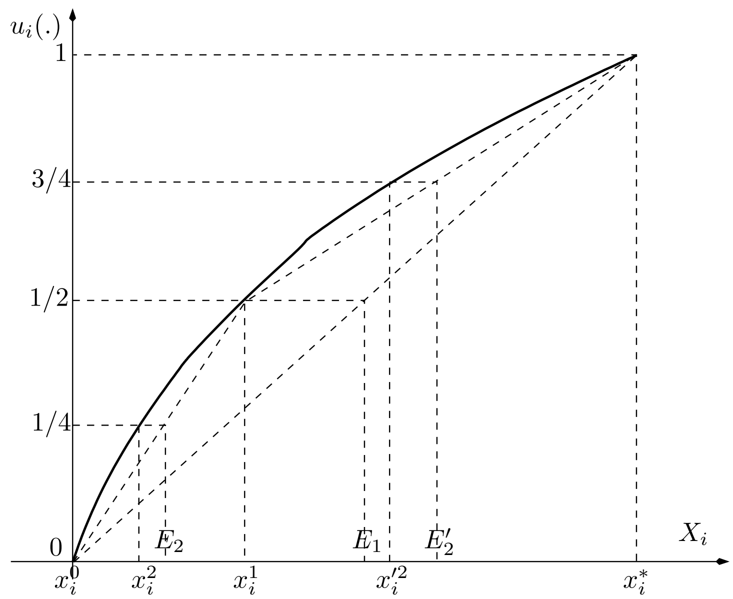

Figure 9: Elicitation of $u_i(.)$ using the fractile method

The most popular method for eliciting single-attribute utility functions is called the *fractile method*. The key idea is to choose a probability p, called a reference probability, and to ask the decision maker to express for which consequence x_i^1 in the interval $[x_i^0, x_i^*]$ she is indifferent between x_i^1 with certainty (hence a degenerated lottery) and lottery $(x_i^*, p; x_i^0, 1-p)$ denoted from now on by $(x_i^*, p; x_i^0)$.

Using the expected utility criterion, we get immediately $u_i(x_i^1) = p$. Applying a similar process to intervals $\left[x_i^0, x_i^1\right]$ and $\left[x_i^1, x_i^*\right]$, two other points of the utility function can be obtained: indifference $x_i^2 \sim (x_i^1, p; x_i^0)$ implies that

$u_i(x_i^2) = p^2$ and indifference $x_i'^2 \sim (x_i^*, p; x_i^1)$ implies that $u_i(x_i'^2) = 2p - p^2$. Iterating this process, we get as many points $(x_i^j, u_i(x_i^j))$ as needed for determining utility function u_i over interval $[x_i^0, x_i^*]$. Figure 9 represents one such iterative utility construction process with reference probability $p = 1/2$ ($\mathrm{E}_1 = px_i^* + (1-p)x_i^0$, $\mathrm{E}_2' = px_i^* + (1-p)x_i^1$, $\mathrm{E}_2 = px_i^1 + (1-p)x_i^0$).

The increasing number of experimental results against expected utility has attracted the attention of many researchers interested in applications of this theory in decision aid. Already at the beginning of the 80's, MacCord and de Neufville [MdN83] showed that there was a direct connection between violations of expected utility and the systematic inconsistencies observed during the elicitation process of the single-attribute utility functions. Among these inconsistencies, it was observed that there exists a systematic dependence between the utility functions and the reference probabilities used for their elicitation. The higher this probability, the more concave the utility function elicited.

Numerous experimental results, dating back to the end of the 40's [PB48], show a systematic trend from the decision makers facing simple risky choices to subjectively transform probabilities. Nowadays, this phenomenon is taken into account in many models of decision making under risk using a probability transformation function (*weighting*) in addition to the utility function (which actually can be thought of as a consequence transformation function). Thus, in both rank dependent utility models [Qui82, TK92] and in Gul's model [Gul91], lottery $P = (x, p; y)$, with $x \succ y$, is evaluated by the utility defined as follows:

$$V(P) = w(p)u(x) + (1 - w(p))u(y) \tag{17}$$

where probability weighting function w is an increasing function from $[0, 1]$ into $[0, 1]$, with $w(0) = 0$ and $w(1) = 1$. When $w(p) = p$ for every $p \in [0, 1]$, we get back $V(P) = E(u, P)$. As compared with the expected utility model, in this new model, probabilities p and $1 - p$ are substituted by *decision weights* $w(p)$ and $(1 - w(p))$ respectively. Knowing that $x \succ y$, it can be easily seen that the weight assigned to a given consequence actually depends on its rank.

Note however that rank dependent utility model (17) cannot be used to elicit function u using the fractile method or a similar method without prior knowledge of transformation function w. Only the tradeoff (TO) method, initially proposed by Wakker and Deneffe [WD96], can avoid this problem.

Eliciting a utility function by the tradeoff method TO essentially consists of constructing a standard sequence of consequences. A standard sequence of positive monetary consequences (gains) is usually constructed as follows. The process starts by the determination of consequence x_1 for which the decision maker is indifferent between lotteries $(x_0, p; R)$ and $(x_1, p; r)$, with $0 \leq r < R < x_0 < x_1$ and $p \in]0, 1[$, r, R, x_0 being set to a fixed value. As shown in Figure 10, the gain induced by substituting x_0 by x_1 on the "p axis" outweights the loss induced by substituting consequence R by r on the "$(1 - p)$ axis".

Next, consequence x_i^2 is determined such that the decision maker is indifferent between $(x_i^1, p; R)$ and $(x_i^2, p; r)$. Using general model (17), both indiffer-

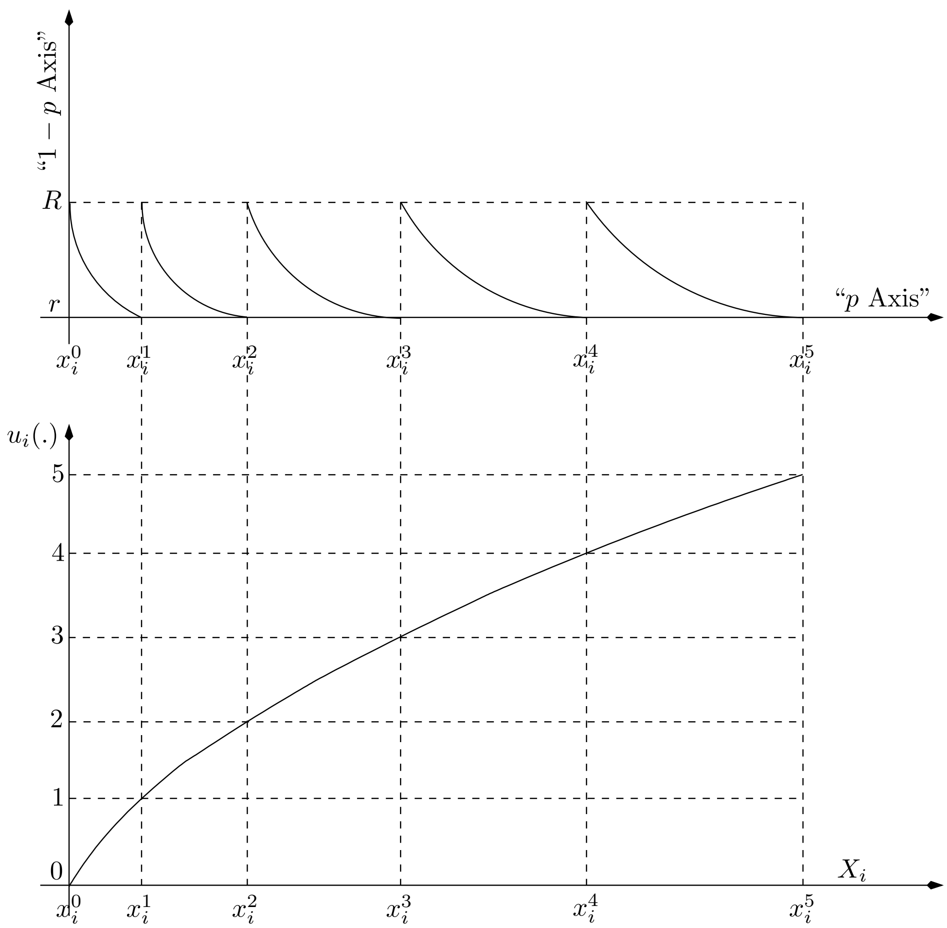

Figure 10: Elicitation of function $u_i(.)$

ences thus constructed induce the following equations:

$$w(p)u_i(x_i^0) + (1 - w(p))u_i(R) = w(p)u_i(x_i^1) + (1 - w(p))u_i(r) \tag{18}$$

$$w(p)u_i(x_i^1) + (1 - w(p))u_i(R) = w(p)u_i(x_i^2) + (1 - w(p))u_i(r) \tag{19}$$

Combining these equations and canceling out terms appearing on both sides of the equalities, we get the following equality:

$$u_i(x_i^1) - u_i(x_i^0) = u_i(x_i^2) - u_i(x_i^1). \tag{20}$$

It results from this equality that consequence x_i^1 is exactly halfway in terms of utilities between consequences x_i^0 and x_i^2. Consequences x_i^0, x_i^1, x_i^2 thus build up a standard sequence. This conclusion clearly also holds under the expected utility hypothesis. Thus, constructing standard sequence of consequences $x_i^0, ..., x_i^q$

requires the construction of q indifferences $(x_i^{j-1}, p; R) \sim (x_i^j, p; r)$, $j = 1, ..., q$. Setting $u_i(x_i^0) = 0$ and $u_i(x_i^q) = 1$, we get $u_i(x_i^j) = j/q$, $j = 1, ..., q$.

In [MW96], Miyamoto and Wakker show that the propositions that enable the decomposition of the von Neumann-Morgenstern utilities still hold even when probabilities are subjectively transformed. This justifies the combination of the new TO utility elicitation method with some classical techniques used for eliciting scaling constants.

Determining scaling constants can be performed in two different ways, often used in combination by the analysts. These two methods can be easily illustrated in the two-dimensional multiattribute case ($n = 2$). Assume that mutual utility independence holds. According to the preceding discussion, we then get:

$$U(x_1, x_2) = k_1 u_1(x_1) + k_2 u_2(x_2) + k k_1 k_2 u_1(x_1) u_2(x_2)$$

with $X_i = [x_i^0, x_i^*]$, $u_i(x_i^0) = 0$, $u_i(x_i^*) = 1$ for $i = 1, 2$ and $k_1 + k_2 + kk_1k_2 = 1$. Constant k can be interpreted as an interaction factor among attributes X_1 and X_2.

Indeed, three scaling constants require three equations to be unambiguously determined. As we already know that $k_1 + k_2 + kk_1k_2 = 1$, we just need two additional independent equations and therefore two additional indifferences under certainty and/or uncertainty.

Assume that $(x_1^0, x_2^*) \succ (x_1^*, x_2^0)$, i.e., that $k_2 > k_1$. By monotonicity, $(x_1^0, x_2^0) \prec (x_1^*, x_2^0)$. It is therefore possible to find a consequence $x_2^{\downarrow}$ ($< x_2^*$) such that $(x_1^0, x_2^{\downarrow}) \sim (x_1^*, x_2^0)$. Translating into the above multilinear form, the following equation obtains:

$$k_2 u_2(x_2^{\downarrow}) = k_1. \tag{21}$$

A second equation, independent from the first one, can be obtained by substituting x_2^0 (in $(x_1^0, x_2^*) \succ (x_1^*, x_2^0)$) by $x_2^{\uparrow}$ ($> x_2^0$) such that $(x_1^0, x_2^*) \sim (x_1^*, x_2^{\uparrow})$. In general, this results in the following equation:

$$k_2 = k_1 + k_2 u_2(x_2^{\uparrow}) + kk_1k_2 u_2(x_2^{\uparrow}). \tag{22}$$

Combined with equality $k_1 + k_2 + kk_1k_2 = 1$, the last two equations enable the determination of the scaling constants.

In the uncertain case, k_1 and k_2 can also be determined by finding probabilities p_1 and p_2 such that:

$$(x_1^*, x_2^0) \sim ((x_1^*, x_2^*), p_1; (x_1^0, x_2^0), 1 - p_1),$$
$$(x_1^0, x_2^*) \sim ((x_1^*, x_2^*), p_2; (x_1^0, x_2^0), 1 - p_2).$$

Translating these indifferences in terms of expected utilities, we get:

$$k_i = p_i, \ i = 1, 2. \tag{23}$$

When probabilities are subjectively transformed, we get $k_i = w(p_i)$, which requires the additional elicitation of function w [Abd00].

When there are more than two attributes in the decision problem, the necessity of having independent and compatible equations for evaluating the scaling constants makes their determination all the more complicated. In [KR93, p. 301-307], Keeney and Raiffa describe for the additive and multiplicative models several procedures avoiding both redundancy and incompatibilities (of these equations).

5 Conclusion

The overview of multiattribute utility theory presented in this chapter is an introduction to a literature with a profusion of results covering a wide domain. We tried to present it in the most homogeneous possible way. We suggest that readers interested in applications of the various techniques described in the chapter read chapters 7 and 8 of [KR93], as well as chapters 15 and 16 of [Cle96]. Chapter 12 of [vWE93] contains also some valuable material.

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