Imprecise Sampling and Direct Decision Making

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In decision theory, uncertainty is generally modeled through probabilities or probability intervals. Data however, when collected by sampling, do not provide probabilities (resp. lower/upper probabilities) but frequencies (resp. lower/upper frequencies). Discrepancies between the former and the latter are taken into account by the model presented: axiomatic requirements are shown to imply that the ordering of the decisions must only depend on quadruple (GEU, u, U, N), where GEU is the generalized expected utility evaluation of the decision that would result from the assimilation of frequencies to lower/upper probabilities; u and U are the utility levels of, respectively, the worst and best reachable outcomes; and N is the size of the sample. Additional axioms are given that ensure the existence of an additive utility representing the ordering.

Keywords: direct decision making, imprecise sampling, upper/lower probabilities, belief functions.

1 Introduction

Expected Utility (EU) theory applies to situations of *risk*: events A have probabilities $\Pi(A)$ which are known to the decision Maker (DM), and each decision d generates a probability distribution $P = \Pi \circ d^{-1}$ (i.e., $P(G) = \Pi(d^{-1}(G))$ for all G) on the outcome set, which is the determining factor for preference:

$$d_1 \succsim d_2 \Leftrightarrow E_{P_1}u \geq E_{P_2}u,$$

where $\succsim$ reads "is preferred or indifferent to" and $E_P u$ denotes the expectation of the DM's von Neumann-Morgenstern (vNM) utility u with respect to probability P (see [18]). The compliance of EU theory with rationality requirements as well as its computational tractability have ensured to it a dominant (although not unchallenged) position in decision making under risk.

However, in real life decision problems the relevant events are seldom naturally endowed with probabilities, which makes it necessary, in a decision aiding perspective, either to adapt the data to the model, which leads to Subjective Expected

Utility (SEU) theory, or to adapt the model to the data, which is the solution we will favor here. Our motivation is that we want choices not to depend on some more or less arbitrary parameters and to be jointly determined by: (i) the objective description of the decision set and of the available data; and (ii) the psychological traits of the DM. Pros and contras of this attitude are discussed in section 5.2.

Real situations of uncertainty may differ from risk in many ways and require diverse specific adaptations of EU theory. In this paper the attention is focussed on the situations where data are obtained by sampling and the collecting process involves some imprecision. Such situations depart from the situation of risk in two ways: (i) data provide frequencies, not probabilities; and (ii) these frequencies are partially undetermined. Therefore, a double adaptation of the EU model will be needed.

The paper is organized as follows: in section 2, we define imprecise sampling and analyze decision making in this situation; in section 3, we recall the main features of some decision models adapted to particular situations of uncertainty; in section 4, we present an axiom system for decision making with imprecise sampling and derive representation theorems; finally, section 5 concludes with a discussion. The proofs are presented in an Appendix (section 6).

2 Imprecise sampling and decision making

2.1 Imprecise sampling

Real decision situations are likely to involve both imprecision about the observations and discrepancies between true and observed frequencies, as suggested by the following generic example:

Example 1
A data bank contains a file for each member of a very large population. Each file is supposed to contain certain items of information on the corresponding member; however, some of the files may have been incompletely filled in, so that they are not necessarily fully informative.

Suppose the question arises whether or not some statement is valid for a randomly selected member of the population. If an exhaustive reading of the files is feasible, and, moreover, all the files contain the appropriate information, the question can be answered in a probabilistic form: the required probability is exactly the percentage of files that satisfy the statement.

Clearly, if some files are incomplete and such that the validity of the statement can neither be claimed nor disproved, one can only come up with lower and upper *bounds for the above percentage, hence, with lower and upper probabilities for the truth of the statement.*

Furthermore, if only a limited sample of the files can be examined, then the preceding probabilities or lower/upper probabilities are no longer accurately assessable and have to be estimated on the basis of the observed frequencies.

We thus describe formally an imprecise sampling situation as follows: Ω, the set of conceivable states of nature, is an infinite set and the existing population from which the sample is extracted is a subset of it, which can be either finite or infinite; in both cases, the sample only describes a finite subpopulation of size N. Moreover, the descriptions are imprecise, and a given observation does not allow the DM to identify completely the state ω but only to determine its belonging to some subset (event) B of Ω (for instance, ω is the list of all physical characteristics of some person, which suffice to identify her, and B is the assertion that she is taller than 1.65 m and her weight is less than 55 kg). Since two different observations may bring in the same information B, data consist of a finite collection $(m(B), B \in \mathcal{B})$, $\mathcal{B} \in 2^\Omega$, $|\mathcal{B}| \leq N$, where $m(B) = k(B)/N$ and $k(B) \in \mathbb{N}^*$ is the number of observations which carry exactly the information "B is true".

From these data, one can derive the observed lower frequency and upper frequency mappings $\Phi, \Psi : \mathcal{A} \subset 2^\Omega \mapsto [0,1]$ defined respectively by

$$\Phi(A) = \sum_{B \subset A} m(B) \quad \text{and} \quad \Psi(A) = \sum_{B \cap A \neq \emptyset} m(B). \tag{1}$$

Φ is interpretable: either (i) as the greatest lower bound (g.l.b.) of the percentage of the observations in which event A is true; or (ii) as the percentage of the observations in which the truth of A can be inferred. A similar interpretation holds for $\Psi(A)$ which satisfies $\Psi(A) = 1 - \Phi(A^c)$, with $A^c = \Omega \backslash A$, for all $A \in \mathcal{A}$.

2.2 Decision making with imprecise sampling

Given this new form of the data, the EU criterion is clearly no longer applicable and must be transformed in some ways. The following example illustrates this necessity.

Example 2

The draw of a R(ed), B(lack) or W(hite) ball from an urn and the previous selection of a decision by the DM determine his/her win according to table 1. "$\{\$0, \$100\}$" indicates ignorance about which of \$0 and \$100 shall be the actual pay-off. Table 2 describes four different states of knowledge concerning the composition of the urn.

In situation S, the comparison of d_1, d_2, and d_3 is a case of choice under risk: d_2 stochastically dominates d_1 (greater probability of winning \$100) and any

A	R	B	W
$d_1(A)$	\$100	\$0	\$0
$d_2(A)$	\$0	\$100	\$0
$d_3(A)$	\$50	\$50	\$50
$d_4(A)$	\$0	{\$0, \$100}	{\$0, \$100}

Table 1: Events and decisions.

	A	R	B	W
S	$\Pi(A)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
S'	$\Pi(A)$	$\frac{1}{3}$	$\pi \in [0, \frac{2}{3}]$	$(1-\pi) \in [0, \frac{2}{3}]$
S''	$\Phi(A)$	33.3%	50%	16.7%
S'''	$\Phi(A)$	33.3%	$\phi \in [0\%, 66.7\%]$	$(1-\phi) \in [0\%, 66.7\%]$

Table 2: Events, probabilities and frequencies.

EU maximizer prefers d_2 to d_1; a risk averse DM prefers d_3 to d_2 whereas a risk prone DM prefers d_2 to d_3.

In situation S', the DM can only ascribe lower and upper bounds to the probabilities of several events, including to the probability of winning with d_2, as is shown in table 3.

Remarkably, as shown in table 4, the same imprecision about the probability of winning is generated, in situation S, for d_4, by the existence of indeterminacies about the outcome values (this is in fact a general property: under mild assumptions, indeterminacies on outcomes and imprecisions on probabilities have equivalent effects).

In neither case does EU theory apply, by lack of a probabilistic description of d_2's and d_4's prospects. An evaluation of d_2 in S' and d_4 in S is provided by

A	$\emptyset$	R	B	W	$R \cup B$	$R \cup W$	$B \cup W$	$R \cup B \cup W$
g.l.b. $\Pi^-(A)$	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1
l.u.b. $\Pi^+(A)$	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	1	1	$\frac{2}{3}$	1

Table 3: Lower and upper bounds for $\Pi(\cdot)$ in S'.

	G	$\emptyset$	$\{\$0\}$	$\{\$100\}$	$\{\$0, \$100\}$	
d_2	$f_2(G)$	0	$\frac{1}{3}$	0	1	$f_2(G) = \Pi^-\left(d_2^{-1}(G)\right)$
in S'	$F_2(G)$	0	1	$\frac{2}{3}$	1	$F_2(G) = \Pi^+\left(d_2^{-1}(G)\right)$
d_4	$f_4(G)$	0	$\frac{1}{3}$	0	1	$f_4(G) = \Pi\left(\{\omega : d_4(\omega) \subset G\}\right)$
in S	$F_4(G)$	0	1	$\frac{2}{3}$	1	$F_4(G) = \Pi\left(\{\omega : d_4(\omega) \cap G \neq \emptyset\}\right)$

Prob inf: f_2, f_4; *Prob sup:* F_2, F_4.

Table 4: Comparison of d_2 *in* S' *and* d_4 *in* S.

Generalized Expected Utility (GEU) theory, an extension of EU theory, described below in subsection 3.3.

Let us now turn to situation S''*, where sampling* N *balls with replacement from the urn has resulted in relative frequencies of* R*,* B *and* W*, which are the same as their probabilities in situation* S*. These frequencies are only estimates of the true ratios, and the more likely to differ greatly from them when* N *is smaller. Therefore, it cannot be excluded that even risk prone DMs may prefer* d_3*, which guarantees* $EU = u(50)$*, to* d_2*, which offers an undetermined* EU*, which may be greater, but may also be smaller than* $u(50)$*. An evaluation of* d_4 *in situation* S'' *is provided by the criterion proposed by DDM theory [7] (see subsection 3.4).*

Finally, we can observe that situation S''' *combines the lack of precision of* S' *with the frequency/probability discordance of* S''*.*

In situation S'''*, the percentage of the observations for which* $R \cup B$ *is true is unknown but has g.l.b.* $\Phi(R \cup B) = 33.3\%$ *and l.u.b.* $\Psi(R \cup B) = 100\%$*.*

With 33.3% *of observations of event* R *and* 66.7% *of observations of event* $B \cup W$*,* $\Phi(R \cup B) = 33.3\%$ *is also the percentage of cases in which* $R \cup B$ *can be inferred (since* $R \subset R \cup B$*) and* $\Psi(R \cup B) = 100\%$ *the percentage of cases in which* $(R \cup B)^c$ *cannot be inferred (since Not*$[R \subset (R \cup B)^c]$ *and Not*$[(B \cup W) \subset (R \cup B)^c]$*), and thus* $R \cup B$ *is possibly true.*

The aim of this paper is to provide an axiomatic justification for a decision model which applies to situations such as S''' where data are provided by imprecise sampling. Not surprisingly, this model is related to both GEU theory and DDM theory. In fact, it can be considered either as adapting GEU theory to frequencies in the same way as DDM theory adapts EU theory, or as extending DDM theory to imprecise data in the same way as GEU theory extends EU theory.

In particular, as in DDM theory, the decision criterion will depend on some new characteristics, the sample size N and the decision range $d(\Omega)$. The next example states the case for this dependence.

Example 3
Consider an urn of the same type as in example 2, and suppose that after N draws, only white balls have been observed. Suppose the DM is asked to choose between two decisions, d_1 and d_2, described as follows:

A	R	B	W
$d_1(A)$	$-\$100$	$\$10$	$\$20$
$d_2(A)$	$\$0$	$\$0$	$\$0$

$S_1 = d_1(\Omega) = \{-\$100, \$10, \$20\}$ and $S_2 = d_2(\Omega) = \{\$0\}$. Now, if $N = 1$, the DM has drawn only one ball and it turned out to be white. Should she take decision d_1 or d_2? With only a 1-size sample, it would not be surprising that d_2 be chosen—especially if the DM is very ambiguity averse. However, for large size samples, the DM should have more confidence in Φ, and, for N large enough, she should take decision d_1, thinking that there is a very low probability to draw a red ball.

This example teaches us that the range of each decision (its potentially observable outcomes), which in general differs (being larger) from the set of its actually observed outcomes, must also be taken into account when comparing decisions.

3 Some situations of uncertainty and associated decision models

3.1 Definitions and notations

Ω, an infinite set, is the set of states of nature and $\mathcal{A} \subset 2^\Omega$ the σ-algebra of events. $\mathcal{C}$ is the outcome set and $\mathcal{G} \subset 2^\mathcal{C}$ its σ-algebra; both $\mathcal{A}$ and $\mathcal{G}$ contain singletons. A decision is a measurable mapping $d : \Omega \mapsto \mathcal{C}$, i.e., $d^{-1}(G) \in \mathcal{A}$ for all $G \in \mathcal{G}$. $S = d(\Omega)$ is the range of d. The set of decisions is denoted by $\mathcal{D}$. $\succsim$ denotes the preference or indifference relation on $\mathcal{D}$.

Different assumptions on the DM's information and behavior lead to different decision models. Let us recall the main features of the models which will serve as a basis for our extension.

3.2 Risk and EU theory

Situation: Risk

The probability Π on $(\Omega, \mathcal{A})$ is known; it determines probability $P = \Pi \circ d^{-1}$ (i.e., $P(G) = \Pi(d^{-1}(G)))$ generated on $(\mathcal{C}, \mathcal{G})$ by decision $d \in \mathcal{D}$.

Decision model: EU theory

The DM's attitude with respect to risk is characterized by its vNM utility u since preference relation $\succsim$ in $\mathcal{D}$ is representable by utility function $d \mapsto E_P u$; when Π has a finite support (i.e., $\Pi(\Omega_0) = 1$ for some finite $\Omega_0 \subset \Omega$), so has P, and $E_P u = \sum_{c \in \mathcal{C}} P(\{c\})u(c)$.

3.3 Imprecise risk and GEU theory

Information on probabilities may be vague and only allow the DM to locate them in probability intervals.

Situation: Imprecise risk

The probability Π on $(\Omega, \mathcal{A})$ is only known to belong to a set $\mathcal{M}$ of probability measures defined by $\mathcal{M} = \{\Pi : \Pi(A) \geq g(A) \text{ for all } A \in \mathcal{A}\}$, where $g : \mathcal{A} \mapsto [0,1]$ is an ∞-monotone capacity:

(i) $g(\emptyset) = 0$; $g(\Omega) = 1$; $A \subset B \Rightarrow g(A) \leq g(B)$;

(ii) $g(\cup_{i \in I} A_i) \geq \sum_{J \subset I; J \neq \emptyset} (-1)^{|J|+1} g(\cap_{j \in J} A_j)$, for all I such that $|I| \geq 2$.

Thus $g(A)$ is the lower probability of A, i.e., the g.l.b. of $\Pi(A)$, and $g^*(A) = 1 - g(A^c)$ is its upper probability. The probability of A can only be located in a probability interval $[g(A), g^*(A)]$.

The uncertainty about the outcome of a decision d can then be characterized by the image of g on $\mathcal{C}$ generated by d, $f_d = g \circ d^{-1}$, i.e., $f_d(G) = g(d^{-1}(G))$ for all $G \in \mathcal{G}$; again, the probability of the outcome of d belonging to G remains unknown but is located in interval $[f_d(G), f_d^*(G)]$. Note that f_d inherits g's properties and is itself an ∞-monotone capacity. We need only consider finitely generated ∞-monotone capacities f_d, i.e., the case where f_d is determined by

$$f_d(G) = \sum_{B \subset G} \phi_d(B), \text{ for all } G \in \mathcal{G}, \tag{2}$$

where $\phi_d : \mathcal{G} \mapsto [0,1]$ is null except on a finite set; ϕ_d is the (generalized) Möbius transform of f_d. This is in particular the case when $\mathcal{C}$ is finite and f_d is a belief function (see [17]).

Decision model: GEU theory

Let $\mathcal{F}$ be the set of finitely generated ∞-monotone capacities on $\mathcal{C}$.

Linear utility theory can be extended to this situation (see [11], [12] and [13]) and leads to Generalized Expected Utility (GEU) theory, in which $\succsim$ in $\mathcal{D}$ is representable by the utility function:

$$d \mapsto E_{f_d}\overline{u} = \sum_{G \in \mathcal{G}} \phi_d(G)\overline{u}(m_G, M_G), \tag{3}$$

where ϕ_d is the Möbius transform of $f_d = g \circ d^{-1}$, characterizable by (2), $m_G = \arg\min\{u(c), c \in G\}$, $M_G = \arg\max\{u(c), c \in G\}$, where u defined by $c \mapsto u(c) =$

$\overline{u}(c,c)$ is interpretable as the vNM utility of the DM, $u(m) \leq \overline{u}(m,M) \leq u(M)$, and $\overline{u}(m,M)$ increases with $u(m)$ and $u(M)$. Note that $\overline{u}$ depends on u, i.e., on the DM's attitude with respect to risk, and on additional parameters reflecting his/her degree of ambiguity aversion (pessimism). Extreme cases of pessimism and optimism correspond respectively to

$$E_{f_d}\overline{u} = \sum_{G \in \mathcal{G}} \phi_d(G) u(m_G) = \inf_{\Pi \in \mathcal{M}} E_\Pi u \circ d$$

and

$$E_{f_d}\overline{u} = \sum_{G \in \mathcal{G}} \phi_d(G) u(M_G) = \sup_{\Pi \in \mathcal{M}} E_\Pi u \circ d$$

(see [11]).

Expression (3) can be interpreted as an expectation, with $\phi_d(G)$ the probability of obtaining an outcome in G when taking decision d, and $\overline{u}(m_G, M_G)$ the evaluation of the prospect of receiving an outcome which can be any member of G. In fact, the same criterion is equally applicable in two separate cases (see example 2):

(i) imprecise probabilities and precise decision mappings;

(ii) precise probabilities and imprecise decision mappings.

3.4 Sampling and DDM theory

When feasible, sampling (or surveying) is commonly used, for efficiency and reliability reasons.

Situation: Sample data

The probability Π on $(\Omega, \mathcal{A})$ is unknown to the DM who entirely bases his/her beliefs concerning $(\Omega, \mathcal{A})$ on a (relative) frequency distribution Φ resulting from N observations ($N \in \mathbb{N}^*$). The relevant information concerning each decision d can thus be assumed to consist exactly in:

(i) the range $S = d(\Omega)$, which is the set of potential outcomes of d;

(ii) the measure $P = \Phi \circ d^{-1}$, which expresses the frequency distribution of d's outcomes, as inferred from the N observations (Note that Φ and P have finite supports).

A justification of this assumption is presented in subsection 4.1.

Decision model: DDM theory: (Direct Decision Making [7])

Further axiomatic requirements (similar to those made in section 4 below) lead to the following result: for a given sample size N, the preference ordering $\succsim$ in $\mathcal{D}$ is representable by the utility function H_N:

$$d \mapsto H_N(d) = h_N(E_P u, u_S, U_S),$$

where u_S and U_S are the worst and the best vNM utility levels in the range S of d. Thus, $E_P u$, which would be the value of d (in the EU model), if P was the true probability generated by d, is corrected by MAXMIN and MAXMAX considerations; these corrections can be expected to become less and less important when N increases, which is allowed by the dependence of H_N on N. The denomination of DDM theory is a reference to the direct inference model of [2] and [3].

4 Imprecise sampling and direct decision making

4.1 Imprecise sampling and uncertainty about decision outcomes

Consider a situation where data result from imprecise sampling of size N, as defined in subsection 2.1, characterized by the finite collection $(m(B), B \in \mathcal{B})$, where $m(B)$ is the proportion of the observations which resulted in the information "B is true"; let us denote as before by Φ and Ψ, respectively, the corresponding lower frequency and upper frequency mappings.

$$\Phi(A) = \sum_{B \subset A} m(B) \quad \text{and} \quad \Psi(A) = \sum_{B \cap A \neq \emptyset} m(B). \tag{4}$$

Property (4) makes Φ a finitely generated ∞-monotone capacity (a slight extension, since Ω is not finite, of the concept of belief function ([17])).

For each decision d, the image of Φ by d, $f_d = \Phi \circ d^{-1}$, inherits its properties and is also a finitely generated ∞-monotone capacity; in fact, $\phi_d : \mathcal{G} \mapsto [0, 1]$, defined by

$$\phi_d(G) = \sum_{d(B) = G} m(B), \tag{5}$$

is the (generalized) Möbius transform of f_d, and determines f_d by:

$$f_d(E) = \sum_{G \subset E} \phi_d(G). \tag{6}$$

The interpretation of f_d follows from that of Φ: let $E \in \mathcal{G}$; if the DM had previously taken decision d, then the outcome would have belonged to E in at least $100 f_d(E)\%$ of the observations (interpretation (i)) or it would have been possible to infer that it belonged to E in $100 f_d(E)\%$ of the observations (interpretation (ii)). Let

$$\mathcal{F}_N = \{f_d = \Phi \circ d^{-1} : d \in \mathcal{D}\}; \tag{7}$$

its members satisfy (6) for ϕ_d given by (5).

We set

$$S_{f_d} = \bigcup \{G \in \mathcal{G} : \phi_d(G) > 0\}. \tag{8}$$

S_{f_d} is called the support of f_d. Necessarily $S_{f_d} \subset S$, where $S = d(\Omega)$ is the range of d. Outcomes not in S_{f_d} could not have resulted from decision d in any of the observations made.

The GEU of f_d is

$$E_{f_d}\overline{u} = \sum_{G \in \mathcal{G}} \phi_d(G)\overline{u}(m_G, M_G), \tag{9}$$

and would provide the evaluation of decision d in the model if lower frequencies were assimilated to lower probabilities, which will only be done at the limit when $N \to \infty$.

4.2 Axiom system

Three different orderings appear in the axiom system: the preference ordering among decisions, $\succsim$, the asymptotic case GEU ordering $\succsim_\infty$ and the partial (dominance) ordering $\succsim^*$ on the decision set defined by:

$$d_1 \succsim^* d_2 \Leftrightarrow u(d_1(\omega)) \geq u(d_2(\omega)) \text{ for all } \omega \in \Omega.$$

In our model, preferences among decisions only depend on triples (f, N, S), where f is the outcome frequency generated from the observed state frequency by decision d, N is the size of the sample and S is the range of the decision. This assumption can be justified as follows.

Consider first the no-sample case ($N = 0$; f undefined), which is a complete ignorance situation. Rational behavior under complete ignorance has been studied in [1], [5] and [6]. Arguments of preference invariance with respect to permutations of the states ω of Ω as well as with respect to refinements (generation of a new state space Ω' by subdividing singletons $\{\omega\}$) lead to the conclusion that preference between pairs of decisions d, d' should only depend on: (i) their images $d(\Omega) = S$ and $d(\Omega') = S'$ in the outcome set; and (ii) dominance; moreover, it can be shown that strict dominance ($d \succ^* d'$) can only have a "second order" influence (in the sense that arbitrary small shifts on the decisions outcomes can annihilate it) when $S = S'$. Our model simply neglects this effect, and only requires the respect of weak dominance (axiom 4).

Consider now the influence of the observations on preference: since preferences should still be invariant with respect to permutations of states that occurred (information on each of them is exactly the same: it occurred) additional data concerning decision d can be summarized by the corresponding observed outcome frequencies and the sampling size N.

Finally, in situations of imprecisely probabilized uncertainty, i.e., when the information available to the DM is characterizable by an ∞-monotone lower probability, the DM is assumed to act according to the GEU criterion. Hence with

$\mathcal{F}_N$, S_{f_d} and $E_f\overline{u}$ defined by (7), (8) and (9) respectively, we require the following axiom:

Axiom 1 Ordering
(i) For $N \in \mathbb{N}^*$ fixed, $\succsim$ is a weak order on triples (f, N, S), where $f \in \mathcal{F}_N$, thus $f = f_d$ for some d, and $S \supset S_{f_d}$.
(ii) $\succsim_\infty$ is a weak order on $\mathcal{F}$, and is representable by a GEU functional $f \mapsto E_f\overline{u}$.

Consider now two decisions d and d' to be compared which have the same outcome range S. Their ranking can only be based on the sampling data. Suppose that, on the basis of a first set of data, d is preferred to d'. The next axiom conveys the very simple idea that, if further sampling exactly confirms the first data, the DM has no reason to revise his/her judgment. Thus:

Axiom 2 Size independence
(i) $(f, N, S) \sim (g, N, S) \quad \Rightarrow \quad (f, 2N, S) \sim (g, 2N, S)$
(ii) $(f, N, S) \succ (g, N, S) \quad \Rightarrow \quad (f, 2N, S) \succ (g, 2N, S)$.

For large size samples, the DM should be very confident in the robustness of the observed frequencies. Hence, for the comparison between decisions (f, N, S) and (g, N, S), the relative values of their GEU evaluations with respect to the observed frequencies, $E_f\overline{u}$ and $E_g\overline{u}$, should become determinant eventually, when $N \to \infty$. This idea is conveyed by the following axiom.

Axiom 3 Continuity
Let $f, g \in \mathcal{F}_N$ be such that $S_f \cup S_g \subset S$. Then $f \succ_\infty g \Leftrightarrow$ there exists $k_0 \in \mathbb{N}^*$ such that, for any $k \geq k_0$, $(f, 2^k N, S) \succ (g, 2^k N, S)$.

The following axiom is a standard rationality requirement in any situation of uncertainty.

Axiom 4 Weak dominance
For any $d_1, d_2 \in \mathcal{D}$ such that $f_1 = \Phi \circ d_1^{-1}$, $f_2 = \Phi \circ d_2^{-1}$, $S_{f_1} \subset S_1 \in \mathcal{G}$ and $S_{f_2} \subset S_2 \in \mathcal{G}$, the following property is true:

$$d_1 \succsim^* d_2 \Rightarrow (f_1, N, S_1) \succsim (f_2, N, S_2).$$

Representation theorems derived from these axioms are studied in the next subsection.

4.3 Representation theorems

General representations

Let $\mathcal{T}$ be the preference space, i.e., $\mathcal{T} = \{(f, N, S)$ such $f \in \mathcal{F}_N$, $N \in \mathbb{N}^*$, $S \in \mathcal{G}$ and $S \supset S_f\}$.

Lemma 5
Suppose that axioms 1 through 3 hold. Then, the DM's preferences on $\mathcal{T}$ depend only on the triple $(E_f\overline{u}, N, S)$.

All proofs are given in the Appendix (section 6).

In the sequel, for technical reasons, we restrict the preference space to $\mathcal{T}^0 = \{(f, N, S) \in \mathcal{T}$ such that $S \supset \{m_S, M_S\}\}$, where $m_S, M_S \in \mathcal{C}$ are such that $\inf_{c\in S} u(c) = u(m_S)$ and $\sup_{c\in S} u(c) = u(M_S)$, and u is the vNM utility function. Thus, S must contain a worst outcome m_S and a best outcome M_S.

Theorem 6
Suppose that axioms 1 through 4 hold. Then, the DM's preferences on $\mathcal{T}^0$ depend only on the quadruple $(E_f\overline{u}, N, u(m_S), u(M_S))$.

In other words, the outcome range S influences the preference ordering only through its worst and best elements, and f is taken into account only through the corresponding generalized expected utility $E_f\overline{u}$; moreover the relative importance of these three factors depends on the sample size N.

Additive representations

It has been shown in theorem 6 that the ordering on triples $(f, N, S) \in \mathcal{T}^0$ is equivalent to the ordering on quadruples $(E_f\overline{u}, N, u(m_S), u(M_S))$. In this subsection, additional axioms are given that ensure the existence of real valued functions h_N^i, $i \in \{1, 2, 3\}$, such that

$$(f, N, S) \succsim (f', N, S') \Leftrightarrow$$
$$h_N^1(E_f\overline{u}) + h_N^2(u(m_S)) + h_N^3(u(M_S)) \geq h_N^1(E_{f'}\overline{u}) + h_N^2(u(m_{S'})) + h_N^3(u(M_{S'})).$$

Note that only triples with the same N, the sample size at the time of the evaluation, have to be compared. Therefore, in the sequel, $N \in \mathbb{N}^*$ is fixed, and the preference space is $\mathcal{T}_N^0 = \{(E_f\overline{u}, u(m_S), u(M_S)) : f \in \mathcal{F}_N$ and $u(m_S) \leq E_f\overline{u} \leq u(M_S)\}$. A generic element of $\mathcal{T}_N^0$ is denoted by the triple (v, u, U) and its elements satisfy $u \leq v \leq U$.

Nb: these inequalities are always assumed to hold any time a triple (v, u, U) appears in an assertion below.

The problem of the existence of an additive utility function on $\mathcal{T}_N^0$ presents two difficulties: (i) $\mathcal{T}_N^0$ is not a full Cartesian product, but only a subset; (ii) as we will see, it may happen that u be neither solvable (see the definition below) nor connected, so that neither classical nor non-classical existence theorems of additive conjoint measurement ([16], [20], and [4]) can be applied.

However, as we shall show, additional axioms allow the existence of an additive utility. Since its existence or its nonexistence does not reflect more or less rationality in the DM's behavior, the validation of these axioms has to remain purely empirical. The following axiom is a necessary condition for the existence of functions h_N^i:

Axiom 7 Second order cancellation axiom
Suppose that (v_j, u_j, U_j), (v'_j, u'_j, U'_j), (v''_j, u''_j, U''_j), $j \in \{1,2\}$, are six elements of $\mathcal{T}_N^0$ such that (v_2, v'_2, v''_2), (u_2, u'_2, u''_2) and (U_2, U'_2, U''_2) are permutations of respectively (v_1, v'_1, v''_1), (u_1, u'_1, u''_1) and (U_1, U'_1, U''_1). Then

$$\left.\begin{array}{l} (v_1, u_1, U_1) \succsim (v_2, u_2, U_2) \\ (v'_1, u'_1, U'_1) \succsim (v'_2, u'_2, U'_2) \end{array}\right\} \Rightarrow (v''_1, u''_1, U''_1) \precsim (v''_2, u''_2, U''_2).$$

In particular, this axiom reflects the following preference consistency:

$$\begin{array}{lcl} (v,u,U) \succsim (v,u',U') & \Leftrightarrow & [(v',u,U) \succsim (v',u',U'), \text{ for all } v'], \\ (v,u,U) \succsim (v',u,U') & \Leftrightarrow & [(v,u',U) \succsim (v',u',U'), \text{ for all } u'], \\ (v,u,U) \succsim (v',u',U) & \Leftrightarrow & [(v,u,U') \succsim (v',u',U'), \text{ for all } U']. \end{array}$$

More generally, it means that, when comparing alternatives, the DM does not take into account the components that are the same in both triples. Note that the equivalence relations above imply the existence of orderings on each component of the triples, as defined below:

$$\begin{array}{lclcl} v \precsim_1 v' & \Leftrightarrow & (v,u,U) \precsim (v',u,U) & \text{for all } u, U; \\ u \precsim_2 u' & \Leftrightarrow & (v,u,U) \precsim (v,u',U) & \text{for all } v, U; \\ U \precsim_3 U' & \Leftrightarrow & (v,u,U) \precsim (v,u,U') & \text{for all } v, u. \end{array}$$

As shown by the next lemma, increases in the worst or best possible utility levels or in the GEU level should always be considered as improvements (in the broad sense) by the DM.

Lemma 8
Assume that axioms 1 through 4 hold. then:

$$\begin{array}{lcl} v \le v' & \Leftrightarrow & v \precsim_1 v'; \\ u \le u' & \Rightarrow & u \precsim_2 u'; \\ U \le U' & \Rightarrow & U \precsim_3 U'. \end{array}$$

Note that the last two relations enable the following behavior:

$$\begin{array}{lll} u < u' & \text{and} & u \sim_2 u', \\ U < U' & \text{and} & U \sim_3 U', \end{array}$$

which may happen when the DM does not pay special attention to the best or the worst possible outcomes when comparing decisions.

The (usual) assumption of restricted solvability with respect to all three components of triples (v, u, U) may appear as unrealistic, at least in the case of small samples, as shown by the following example:

Example 4
Consider a DM who is strongly averse to losses, however small, and systematically discards a decision involving the possibility of losses, unless he firmly believes the probability of a loss to be extremely small. For a small sample size N, unobserved events are never guaranteed to be rare events, and, thus, $\succsim$ is likely to be such that:

$$\text{for all } u' \geq u(0) > u'' \text{ and all } v', U', v'', U'', \\ (v', u', U') \succ (v'', u'', U'').$$

Suppose that v, U, U_0 and v_0 are such that:

$$(v, u(0), U) \succ (v_0, u(0), U_0) \succ (v, u'', U);$$

then, $(v_0, u(0), U_0) \succ (v, u, U)$ for all $u < u(0)$, and $(v, u, U) \succ (v_0, u(0), U_0)$ for all $u \geq u(0)$; thus, there exists no u such that $(v, u, U) \sim (v_0, u(0), U_0)$, and restricted solvability w.r.t. u does not hold.

Since certainty or security effects appear experimentally to be much stronger than potential effects, the symmetrical phenomenon should not be expected to prevent restricted solvability w.r.t. U to hold. As for solvability w.r.t. v and U, it relies on the idea that $u(\mathcal{C})$ is sufficiently rich, for instance is an interval of $\mathbb{R}$. It is therefore of interest to consider the case of restricted solvability w.r.t. v and U but not w.r.t. u.

Axiom 9 restricted solvability w.r.t. v and U
If $(v, u, U) \succsim (v_0, u_0, U_0) \succsim (v', u, U)$, then there exists v'' such that $(v_0, u_0, U_0) \sim (v'', u, U)$. If $(v, u, U) \succsim (v_0, u_0, U_0) \succsim (v, u, U')$, then there exists U'' such that $(v_0, u_0, U_0) \sim (v, u, U'')$.

This axiom is illustrated in figure 1.

For a utility function to exist, it is not sufficient that the preference ordering be a weak order; there should not be "more" indifference classes of $\succsim$ than real

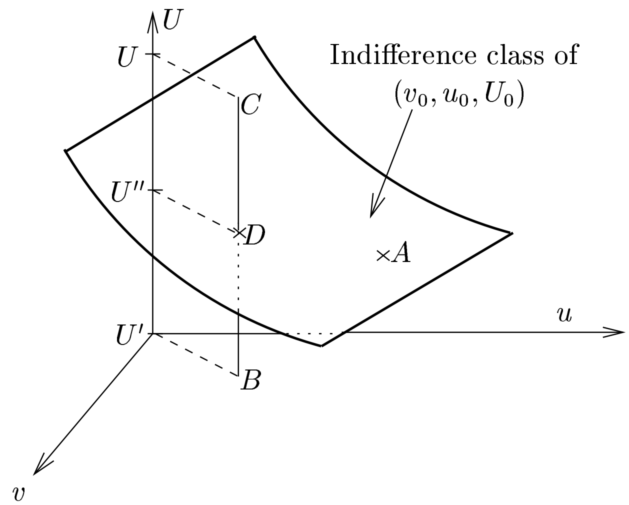

Figure 1: Restricted solvability w.r.t. U: B, C are on the same vertical. If $B \precsim A$ and $A \precsim C$, then there exists D such that $D \sim A$.

numbers. In the additive conjoint measurement framework, this property is ensured by the Archimedean axiom below, which is stated in terms of over-standard sequences (a slight generalization of standard sequences):

Definition 10 over-standard sequence
For any finite or infinite, increasing or decreasing, sequence Z of consecutive relative integers, $(U_z, z \in Z)$ is an over-standard sequence iff: either $(v_0, u_0, U_0) \prec (v_1, u_1, U_0)$ and $(v_0, u_0, U_z) \succsim (v_1, u_1, U_{z+1})$ for all $z, z+1 \in Z$; or $(v_0, u_0, U_0) \succ (v_1, u_1, U_0)$ and $(v_0, u_0, U_z) \precsim (v_1, u_1, U_{z+1})$ for all $z, z+1 \in Z$. Parallel definitions hold when the role of U is exchanged with that of u and v.

Axiom 11 Archimedean axiom
Every bounded over-standard sequence is finite, i.e., if there exist U, U' such that, for all $z \in Z$, $U \precsim_3 U_z \precsim_3 U'$, then Z is finite; and similarly for over-standard sequences w.r.t. u or v.

By definition, the vNM utility is known to be bounded. The last axiom that we require states that those bounds cannot be attained.

Axiom 12
There does not exist $c_m \in \mathcal{C}$ such that $u(c_m) \precsim_2 u(c)$ for all $c \in \mathcal{C}$, nor $c_M \in \mathcal{C}$ such that $u(c) \precsim_3 u(c_M)$ for all $c \in \mathcal{C}$.

Using axioms 1 through 12, the preference ordering on $\mathcal{T}_N^0$ can be shown to be representable by an additive utility function, as stated by the following theorem.

Theorem 13
Suppose that N, the size of the observed sample, is fixed. Assume that axioms 1 through 12 hold. Then, there exist real valued functions, h_N^1, h_N^2, h_N^3, such that, for any $(v, u, U), (v', u', U') \in \mathcal{T}_N^0$,

$$(v, u, U) \succsim (v', u', U') \Leftrightarrow h_N^1(v) + h_N^2(u) + h_N^3(U) \geq h_N^1(v') + h_N^2(u') + h_N^3(U').$$

However, unlike classical representation theorems of additive conjoint measurement, the additive utility representing $\succsim$ on $\mathcal{T}_N^0$ is not an interval scale, i.e., is not unique up to scale and location. In fact, one gets something intermediate between an ordinal and a cardinal representation.

Example 4 (continued)
Suppose that $h_N^1 + h_N^2 + h_N^3$ is an additive utility representing the preferences of the DM of example 4; his/her aversion for losses implies that

$$\inf_{\substack{u \geq u(0) \\ v,U}} \{h_N^1(v) + h_N^2(u) + h_N^3(U)\} \geq \sup_{\substack{u(0) > u \\ v,U}} \{h_N^1(v) + h_N^2(u) + h_N^3(U)\}.$$

Now, for any (v, u, U), let $k_N^1(v) = \alpha h_N^1(v)$, $k_N^3(U) = \alpha h_N^3(U)$ and $k_N^2(u) = \alpha h_N^2(u) + \beta(u)$, where α is an arbitrary positive constant and $\beta : \mathbb{R} \mapsto \mathbb{R}$ is such that

$$\beta(u) = \begin{cases} 1 & \text{if } u \geq u(0) \\ 0 & \text{if } u < u(0). \end{cases}$$

It is clear that $k_N^1 + k_N^2 + k_N^3$ is an additive utility representing the DM's preferences. However, k_N^2 is not an affine transform of h_N^2.

We define below an equivalence relation $\mathcal{O}$ such that the utility function h_N^2 will be cardinal inside each indifference class of $\mathcal{O}$ but not outside. This is in fact a restriction of i-link relation $\mathcal{O}_i$ defined in [8].

Definition 14 I-link relation $\mathcal{O}$
For any u, u', $u \,\mathcal{O}\, u'$ if and only if either $u \sim_2 u'$ or there exist an integer n and a sequence $(u_i)_{i=1}^n$ with $u_0 = u$, $u_n = u'$, such that for any $i \in \{0, \ldots, n-1\}$ there

exist $v_i, U_i, v'_{i+1}, U'_{i+1}$ *such that* $(v'_{i+1}, u_{i+1}, U'_{i+1}) \sim (v_i, u_i, U_i)$*, and such that either* $u_{i+1} \succ_2 u_i$ *for any* $i \in \{0, \ldots, n-1\}$*, or* $u_{i+1} \prec_2 u_i$ *for any* $i \in \{0, \ldots, n-1\}$*.*

The last condition of the above definition may seem restrictive, but in fact is not, because, from any sequence $(u_i)_{i=1}^n$ satisfying all the conditions above but the last one, it is always possible by solvability w.r.t. the other components to extract a sequence satisfying also the last condition.

Under the previous axioms, $\mathcal{O}$ is an equivalence relation.

Theorem 15
Assume that axioms 1 through 12 hold. Then, there exists a set Z *of consecutive relative integers—finite or infinite—and a sequence of elements* $(u_z)_{z \in Z}$*, such that, for any* u*, there exists* $z \in Z$ *such that* $u \,\mathcal{O}\, u_z$*, and, if* $\mathrm{Card}(Z) > 1$*,* $u_{z+1} \succ_2 u_z$ *and Not*$(u_z \,\mathcal{O}\, u_{z+1})$ *for any* $z, z+1$ *in* Z*.*

Assume that $h_N^1 + h_N^2 + h_N^3$ *and* $k_N^1 + k_N^2 + k_N^3$ *are additive utilities representing* $\succsim$ *on* $\mathcal{T}_N^0$*. Then there exist some constants* $\alpha > 0$*,* α_1*,* α_3*, and* β_z*,* $z \in Z$*, such that:*

$$\begin{cases} \text{for any } v, & k_N^1(v) \;=\; \alpha \cdot h_N^1(v) + \alpha_1 \\ \text{for any } U, & k_N^3(U) = \; \alpha \cdot h_N^3(U) + \alpha_3 \\ \text{for any } u \,\mathcal{O}\, u_z, & k_N^2(u) \;=\; \alpha \cdot h_N^2(u) + \beta_z \text{ where, for any } z, z+1 \in Z, \\ \qquad \beta_{z+1} \geq \beta_z & + \,\alpha \cdot \sup\limits_{\substack{u' \,\mathcal{O}\, u_z \\ u' \leq v \leq U}} \{h_N^1(v) + h_N^2(u') + h_N^3(U)\} \\ & - \,\alpha \cdot \inf\limits_{\substack{u' \,\mathcal{O}\, u_{z+1} \\ u' \leq v \leq U}} \{h_N^1(v) + h_N^2(u') + h_N^3(U)\}. \end{cases}$$

5 Discussion and conclusion

5.1 Some questions raised by the model

In the present paper, we have provided a theoretical basis for the dependence of preferences on certain characteristics of the decisions, and for their representation by particular utility functions, when only data from imprecise sampling are known. However, at this stage of our research, no procedure has been yet implemented to elicit those functions. Of course, much work has been done on the construction of utility functions (see e.g. [14]; [15]; [9]; [19]); however, within our framework, some simplifications should be possible because some information about the behavior of the DM is available: in particular, the greater N, the closer his/her criterion is to GEU; moreover, the relative importance of m_S and M_S in the quadruples of subsection 4.3 reflects the degree of pessimism of the DM.

Under uncertainty, a well known pessimism index has been introduced by [10]. The links between the two indices are not obvious, since they address reactions to different kinds of uncertainty: the former is related to his/her trust in the observed sample whereas the latter reflects his/her attitude w.r.t. ambiguity (i.e., situations where only upper and lower probabilities are known). However, it is reasonable to think that the same psychological trait is responsible for overweighting the worst outcomes w.r.t. the best ones in both models, hence establishing a connection between the two pessimism indices.

Another aspect that was not studied in our paper is the effect of new information: how does the attitude of the DM change when the size of the sample increases? Intuitively, the DM should tend to trust more the observed frequency, and to take less into account the worst and best outcomes. On the other hand, the arrival of very vague new data might perhaps decrease the overall trust of the DM and have the opposite effect.

One last aspect that could be investigated is the possibility of justifying particular forms of utility functions. In subsubsection 4.3, additive separable utilities were proved to exist for fixed size samples; in particular, preference orderings representable by functions like:

$$H(f, N, S) = \lambda_N E_f \overline{u} + (1 - \lambda_N)[L(m_S) + K(M_S)],$$

would separate the attitude toward ambiguity (L and K) from the attitude toward imprecision ($E_f \overline{u}$).

5.2 Comparisons with other approaches

Classical parametric statistics assume that the sample distribution is known to belong to a given parameterized family of probability distributions; Bayesian statistics further introduce a prior distribution on the parameter space. There is no objection to these approaches as long as the required information is available, and their well-tried methods are definitely appealing. Moreover, Bayesian statistics are immune to the dynamic inconsistency problems which all other models, including the present one, have much trouble circumventing. On the other hand, when there exists little or no prior information and the likelihood function and the parameter prior are to a large extent arbitrary, so that their choices are guided mostly by technical reasons (normality assumptions; conjugate prior; etc), one can wonder whether these arbitrary elements do not play a decisive role in the selection of the "optimal" decision.

For this reason, decisions models which stick to the data may be worth considering. This preoccupation is of course not new: nonparametric statistics avoid unjustifiable assumptions on distributions; and empirical Bayes methods use priors directly based on the data. Our decision model has been elaborated in the same spirit.

6 Appendix: proofs

Proof of lemma 5

When $(f, N, S) \succ (g, N, S)$, repeated use of axiom 2 shows that $(f, 2^k N, S) \succ (g, 2^k N, S)$ for all $k \in \mathbb{N}^*$, hence, by axiom 3, that $f \succ_\infty g$ and $E_f \overline{u} > E_g \overline{u}$. Thus, $E_f \overline{u} = E_g \overline{u}$ implies that $(f, N, S) \sim (g, N, S)$: preference on $\mathcal{T}$ only depends on f through $E_f \overline{u}$. □

Proof of theorem 6

Theorem 6 states that there exists an ordering on quadruples $(E_f\overline{u}, N, u(m_S), u(M_S))$ that preserves the preference ordering on triples (f, N, S). By lemma 5, the ordering on (f, N, S) can be transformed into an ordering on $(E_f \overline{u}, N, S)$. Let us show that this ordering is representable by an ordering on $(E_f \overline{u}, N, u(m_S), u(M_S))$. In other words, if $\xi = \{(f, N, S) \in \mathcal{T}^0$ such that $E_f \overline{u} = v$, $u(m_S) = u$, $u(M_S) = U\}$, where u, v and U are some arbitrary real constants, then all the elements of ξ belong to the same indifference class of $\succsim$.

Consider a decision d, associated with the triple $(f, N, S) \in \xi$. Since $S \supset \{m_S, M_S\}$, there exist $\omega_m, \omega_M \in \Omega$ such that $d(\omega_m) = m_S$ and $d(\omega_M) = M_S$. Let Ω_N be the set of the observed states of nature. Let $\Omega_N^c = \Omega \backslash \Omega_N$ be the set of unobserved states of nature. Note that Ω_N^c is an infinite set because $\Omega = \Omega_N \cup \Omega_N^c$, $\mathrm{Card}(\Omega_N)$ is finite and $\mathrm{Card}(\Omega)$ is infinite. Let d^+ be the decision defined by:

$$\begin{cases} \text{for any } \omega \in \Omega_N \cap \{\omega_m\}^c, & d^+(\omega) = d(\omega), \\ \text{for any } \omega \in \Omega_N^c \cap \{\omega_m\}^c, & d^+(\omega) = M_S, \\ \text{for } \omega = \omega_m & d^+(\omega) = m_S. \end{cases}$$

By its definition, d^+ is associated with the triple $(f, N, S_f \cup \{m_S, M_S\})$. Moreover, $d^+ \succsim^* d$, hence, by axiom 4, $d^+ \succsim d$, i.e., $(f, N, S_f \cup \{m_S, M_S\}) \succsim (f, N, S)$. Similarly, if d^- is defined by:

$$\begin{cases} \text{for any } \omega \in \Omega_N \cap \{\omega_M\}^c, & d^-(\omega) = d(\omega), \\ \text{for any } \omega \in \Omega_N^c \cap \{\omega_M\}^c, & d^-(\omega) = m_S, \\ \text{for } \omega = \omega_M & d^-(\omega) = M_S, \end{cases}$$

then d^- is associated with $(f, N, S_f \cup \{m_S, M_S\})$, and $d \succsim d^-$. Hence $(f, N, S) \sim (f, N, S_f \cup \{m_S, M_S\})$.

Similarly, for any $(f', N, S') \in \xi$, $(f', N, S') \sim (f', N, S_{f'} \cup \{m_{S'}, M_{S'}\})$. With $u(m_{S'}) = u(m_S)$ and $u(M_{S'}) = u(M_S)$, since S_f and $S_{f'}$ are finite sets, the values of $d^+(\omega)$ and $d^-(\omega)$ can be modified on a subset of $\Omega_N^c \cap \{\omega_m, \omega_{m'}, \omega_M, \omega_{M'}\}^c$, so that the new decisions d_1^+ and d_1^- generate $(f, N, S_f \cup S_{f'} \cup \{m_S, m_{S'}, M_S, M_{S'}\})$; since $d^+ \succsim^* d_1^+ \succsim^* d^-$ and $d^+ \succsim^* d_1^- \succsim^* d^-$, we get, by axiom 4, $d \sim d_1^+ \sim d_1^-$, hence $(f, N, S_f) \sim (f, N, S_f \cup S_{f'} \cup \{m_S, m_{S'}, M_S, M_{S'}\})$.

By the same reasoning, $(f', N, S') \sim (f', N, S_f \cup S_{f'} \cup \{m_S, m_{S'}, M_S, M_{S'}\})$. Now, by axioms 2 and 3, since $E_f\overline{u} = E_{f'}\overline{u} = v$,

$$(f', N, S_f \cup S_{f'} \cup \{m_S, m_{S'}, M_S, M_{S'}\}) \sim (f, N, S_f \cup S_{f'} \cup \{m_S, m_{S'}, M_S, M_{S'}\});$$

hence, $(f, N, S) \sim (f', N, S')$. So, all the elements of ξ belong to the same indifference class. □

Proof of lemma 8

Consider quadruples (v, N, u, U) and (v', N, u, U). Let $S \in \mathcal{G}$ be such that $\inf_{c \in S} u(c) = u$ and $\sup_{c \in S} u(c) = U$. By theorem 6 and axiom 2, (v, N, u, U) and (v', N, u, U) are ordered as (v, N, S) and (v', N, S) and, for every $k \in \mathbb{N}^*$, as $(v, 2^k N, S)$ and $(v', 2^k N, S)$, hence, by axiom 3, as v and v'. So $(v, N, u, U) \precsim (v', N, u, U) \Leftrightarrow v \leq v'$.

Suppose that $u \leq u'$. Let Ω_N be the set of the observed states of nature, and $\Omega_N^c = \Omega \backslash \Omega_N$. Consider a decision d generating quadruple (v, N, u', U) and such that for $\omega_1, \omega_2 \in \Omega_N^c$, $\omega_1 \neq \omega_2$, $d(\omega_1) = d(\omega_2) = U$. Let d' be defined as: $d'(\omega) = d(\omega)$ for $\omega \neq \omega_2$, and $d'(\omega_2) = u$. Obviously, d' generates (v, N, u, U), and since $d' \precsim^* d$, by axiom 4, $(v, N, u, U) \precsim (v, N, u', U)$.

A similar proof holds for the third statement of the lemma. □

For convenience, we will prove theorem 13 and theorem 15 jointly. But before proceeding to the proof, we need define a property that is usually associated with restricted solvability:

Definition 16 Essentialness

The first component of $\mathcal{T}_N^0$ is said to be essential if and only if there exist v, v' such that $v \succ_1 v'$. Parallel definitions hold for the other components of $\mathcal{T}_N^0$.

Proof of theorem 13 and theorem 15

The principle of the proof is to construct an additive utility on each equivalence class of $\mathcal{O}$, and to "fit" together these functions to form a global additive utility on $\mathcal{T}_N^0$. Although, as pointed out in [21, section 2], this principle of proof is bound to fail in general, we will show that it works in our case.

Consider an arbitrary real number u_0 such that there exists a decision d with $\inf_{\omega \in \Omega} u(d(\omega)) = u_0$. Let us show that there exists an additive utility (unique up to scale and location) $h_N^1 + h_N^2 + h_N^3$ representing $\succsim$ on the subset $\{(v, u_0, U)\}$ of $\mathcal{T}_N^0$. Note that this set is not a full Cartesian product since the constraints are $u_0 \leq v \leq U$, and therefore the existence of an additive utility does not follow from the classical existence theorems of additive conjoint measurement (see [16] and [20]).

First step : existence of an additive utility on $\{(v, u_0, U) : u_0 < U\}$

For all $v_0 \succ_1 u_0$, i.e., for all $v_0 > u_0$, the set $\{(v, u_0, U) : v \leq v_0 \leq U\}$ is a full Cartesian product satisfying:

- restricted solvability w.r.t. all the components;
- essentialness w.r.t. components v and U: indeed, since $v_0 \succ_1 u_0$, $\{v : u_0 \leq v \leq v_0\} = \{v : u_0 \precsim_1 v \precsim_1 v_0\}$ is obviously essential. By definition of $\mathcal{T}_N^0$, $v_0 = E_f \overline{u}$, for a frequency f generated on the outcome set from the observed sample of size N; therefore, the support of f is finite and there exists $c \in \mathcal{C}$ such that $v_0 \leq u(c)$; by axiom 12, there exists c' such that $u(c') \succ_3 u(c)$, and, by lemma 2, $u(c') > u(c)$; hence, $\{U : v_0 \leq U\}$ is also essential;
- the second order cancellation axiom;
- an Archimedean axiom that implies the classical one.

Therefore, by a result of [16, theorem 13, page 302], there exists an additive utility $h_N^1 + h_N^2 + h_N^3$, unique up to scale and location, representing $\succsim$ on $\{(v, u_0, U) : v \leq v_0 \leq U\}$.

For $v_1 \succ_1 u_0$, $v_1 \neq v_0$ ($v_1 > v_0$ for instance), a similar reasoning proves that there exist additive utility functions, unique up to scale and location, representing $\succsim$ on the following Cartesian products:

$$\begin{array}{l} \{(v, u_0, U) : v \leq v_1 \leq U\}, \\ \{(v, u_0, U) : v \leq v_0 < v_1 \leq U\}. \end{array}$$

Since the last set is actually the intersection of the two preceding ones, i.e.,

$$\{(v, u_0, U) : v \leq v_0 \leq U\} \cap \{(v, u_0, U) : v \leq v_1 \leq U\},$$

and since all the functions are unique up to scale and location on their respective sets, they can be rescaled so that they coincide on the last set. More precisely, the additive utility on the first set can be extended to a function which is also an additive utility on the second set. Note that, up to this point, we are not yet sure that this function is actually a utility function on the union of the two sets (see [21, section 2]), which remains to be proved. For this purpose, consider two elements:

$$\begin{array}{l} (v', u_0, U') \ \in \{(v, u_0, U) : v \leq v_0 \leq U\}, \\ (v'', u_0, U'') \in \{(v, u_0, U) : v \leq v_1 \leq U\}. \end{array}$$

If $U' \geq v_1$, then, since by construction $h_N^1 + h_N^2 + h_N^3$ is a utility function on $\{(v, u_0, U) : v \leq v_1 \leq U\}$, the following equivalence is true:

$$\left\{ \begin{array}{l} (v', u_0, U') \succsim (v'', u_0, U'') \Leftrightarrow \\ \qquad h_N^1(v') + h_N^2(u_0) + h_N^3(U') \geq h_N^1(v'') + h_N^2(u_0) + h_N^3(U''). \end{array} \right. \tag{10}$$

If, on the contrary, $U' < v_1$, then

- either there exists $(v''', u_0, U''') \in \{(v, u_0, U) : v \leq v_0 < v_1 \leq U\}$ such that $(v''', u_0, U''') \sim (v', u_0, U')$. Then, (v''', u_0, U''') also belongs to $\{(v, u_0, U) : v \leq v_0 \leq U\}$, and, since by construction $h_N^1 + h_N^2 + h_N^3$ is a utility function on $\{(v, u_0, U) : v \leq v_0 \leq U\}$, the following equality holds:

$$h_N^1(v') + h_N^2(u_0) + h_N^3(U') = h_N^1(v''') + h_N^2(u_0) + h_N^3(U''').$$

  But (v''', u_0, U''') also belongs to $\{(v, u_0, U) : v \leq v_1 \leq U\}$, which implies that

$$\begin{cases} (v'', u_0, U'') \succsim (v''', u_0, U''') \Leftrightarrow \\ \qquad h_N^1(v'') + h_N^2(u_0) + h_N^3(U'') \geq h_N^1(v''') + h_N^2(u_0) + h_N^3(U'''). \end{cases}$$

  By transitivity of $\geq$ on $\mathbb{R}$, (10) holds;

- or there exists no element $(v''', u_0, U''') \in \{(v, u_0, U) : v \leq v_0 < v_1 \leq U\}$ such that $(v''', u_0, U''') \sim (v', u_0, U')$. Then $(v', u_0, U') \prec (u_0, u_0, v_1)$; otherwise, by lemma 8, and since $U' < v_1$ and $v' \leq v_0$, the following relation would hold:

$$(u_0, u_0, v_1) \precsim (v', u_0, U') \precsim (v', u_0, v_1) \precsim (v_0, u_0, v_1),$$

  which would imply, by restricted solvability w.r.t. the first component, that there exists $v''' \in [u_0, v_0]$ such that $(v''', u_0, v_1) \sim (v', u_0, U')$, and contradict our hypothesis that no such element exists. Again by lemma 8, $(u_0, u_0, v_1) \precsim (v'', u_0, U'')$. Therefore,

$$(v', u_0, U') \prec (u_0, u_0, v_1) \precsim (v'', u_0, U''),$$

  and since $h_N^1 + h_N^2 + h_N^3$ is a utility function on $\{(v, u_0, U) : v \leq v_0 \leq U\}$ and on $\{(v, u_0, U) : v \leq v_1 \leq U\}$, one gets

$$h_N^1(v') + h_N^2(u_0) + h_N^3(U') < h_N^1(v'') + h_N^2(u_0) + h_N^3(U'').$$

So, to conclude, $h_N^1 + h_N^2 + h_N^3$ represents $\succsim$ on

$$\{(v, u_0, U) : v \leq v_0 \leq U\} \cup \{(v, u_0, U) : v \leq v_1 \leq U\}.$$

Let us extend this representation to $\{(v, u_0, U) : u_0 < U\}$. Note first that, for $v_2 \notin \{v_0, v_1\}$, $v_2 \succ_1 u_0$, the extensions of $h_N^1 + h_N^2 + h_N^3$ to $\{(v, u_0, U) : v \leq v_0 \leq U\} \cup \{(v, u_0, U) : v \leq v_1 \leq U\}$ and $\{(v, u_0, U) : v \leq v_0 \leq U\} \cup \{(v, u_0, U) : v \leq v_2 \leq U\}$ must coincide (by uniqueness up to scale and location) on the intersection of these sets, so that the value of $h_N^1 + h_N^2 + h_N^3$ on any (v, u_0, U) is uniquely determined. Second, $\succsim$ is representable by $h_N^1 + h_N^2 + h_N^3$ on

$$\{(v,u_0,U) : v \le v_0 \le U\} \cup \{(v,u_0,U) : v \le v_1 \le U\} \\ \cup \{(v,u_0,U) : v \le v_2 \le U\}.$$

Since two arbitrary elements of this set necessarily belong to the union of two of them, the preceding result applies. Now, since v_1 and v_2 are arbitrary, although satisfying $v_1, v_2 \succ_1 u_0$, any two elements (v', u_0, v_1) and (v'', u_0, v_2) of $\{(v, u_0, U) : u_0 < U\}$ can be compared through $h_N^1 + h_N^2 + h_N^3$, so that this function represents $\succsim$ on the whole set $\{(v, u_0, U) : u_0 < U\}$. Moreover, by the process of construction, this function is clearly unique up to scale and location on this set.

Second step : existence of an additive utility on $\{(v, u, U) : u \in \{u_0, u_1\},\ u_0\, \mathcal{O}\, u_1,\ u_0 < u_1,\ u_0 < U\}$.

By definition 14, either (i) $u_1 \sim_2 u_0$, in which case $(v, u_1, U) \sim (v, u_0, U)$ for all v, U; therefore, if $h_N^1 + h_N^2 + h_N^3$ is an additive utility function on $\{(v, u_0, U) : u_0 < U\}$, it is also an additive utility on $\{(v, u, U) : u \in \{u_0, u_1\},\ u_0 < U\}$; or (ii) there exist $p \in \mathbb{N}$ and a sequence $(u^i)_{i=0}^p$ such that:

- $u^0 = u_0$, $u^p = u_1$,
- for all $i \in \{0, \ldots, p-1\}$, $u^i \prec_2 u^{i+1}$,
- for all $i \in \{0, \ldots, p-1\}$, there exist $v_i, U_i, v'_{i+1}, U'_{i+1}$ such that $(v_i, u^i, U_i) \sim (v'_{i+1}, u^{i+1}, U'_{i+1})$.

According to the first step, there exists an additive utility $h_N^1 + h_N^2 + h_N^3$, unique up to scale and location, representing $\succsim$ on $\{(v, u^0, U) : u^0 < U\}$. Note also that by axiom 7, for any $i \in \{1, \ldots, p\}$ and for all v, v', U, U',

$$(v, u^i, U) \succsim (v', u^i, U') \Leftrightarrow (v, u^0, U) \succsim (v', u^0, U'),$$

so that, for any value of $h_N^2(u^i)$, the restriction of $h_N^1 + h_N^2 + h_N^3$ to $\{(v, u^i, U) : u^0 < U\}$ also represents $\succsim$.

To summarize, there exists a sequence $(h_N^2(u^i))_{i=0}^p$ such that $h_N^1 + h_N^2 + h_N^3$ represents $\succsim$ on all the sets $\{(v, u^i, U) : u^0 < U\}$. There remains now to define a value for each $h_N^2(u^i)$ such that $h_N^1 + h_N^2 + h_N^3$ is also representing $\succsim$ on $\{(v, u, U) : u \in \{u^i, i = 0, \ldots, p\},\ u^0 < U\}$.

By definition of relation $\mathcal{O}$, for all $i \in \{0, \ldots, p-1\}$, there exist v_i, U_i, v'_{i+1}, U'_{i+1} such that

$$(v_i, u^i, U_i) \sim (v'_{i+1}, u^{i+1}, U'_{i+1}).$$

Thus, the following equality is a necessary condition for $h_N^1 + h_N^2 + h_N^3$ to represent $\succsim$ on $\{(v, u, U) : u \in \{u^i, i = 0, \dots, p\}, u^0 < U\}$:

$$h_N^2(u^{i+1}) = h_N^2(u^i) + \left[h_N^1(v_i) + h_N^3(U_i)\right] - \left[h_N^1(v'_{i+1}) + h_N^3(U'_{i+1})\right]. \qquad (11)$$

The question is whether or not this condition is also sufficient. All the Cartesian products

$$\{(v, u, U) : u \in \{u^i, u^{i+1}\} \text{ and } u^{i+1} \leq v \leq v' \leq U\}, \text{ for all fixed } v' > u^{i+1},$$

satisfy the following properties:

- restricted solvability w.r.t. 2 components, and essentialness w.r.t. all the components,
- the second order cancellation axiom,
- an Archimedean axiom (axiom 11),
- $u^i \,\mathcal{O}\, u^{i+1}$;

therefore, according to [8], there exist additive utilities, unique up to scale and location, representing $\succsim$ on the following Cartesian products:

$$\{(v, u, U) : u \in \{u^i, u^{i+1}\} \text{ and } u^{i+1} \leq v \leq v' \leq U\}, \text{ for all fixed } v' > u^{i+1},$$

and these functions must be equal to $h_N^1 + h_N^2 + h_N^3$. A proof similar to that of the first step shows that $h_N^1 + h_N^2 + h_N^3$ does in fact represent $\succsim$ on

$$\{(v, u, U) : u \in \{u^i, u^{i+1}\} \text{ and } u^{i+1} \leq v \leq U\}.$$

So, to summarize, $h_N^1 + h_N^2 + h_N^3$ represents $\succsim$ on the set above and on $\{(v, u^i, U) : u^0 < U\}$. Now, let us show that it actually represents $\succsim$ on

$$\{(v, u^i, U) : u^0 < U\} \cup \{(v, u^{i+1}, U)\}.$$

(Note that for the second set, it is useless to presume that $U > u^0$ since we already know that $U \geq u^{i+1}$). Consider any couple of elements, say (v', u^i, U') and (v'', u^{i+1}, U''), of the last set. Note that

$$\{(v, u^{i+1}, U)\} \subset \{(v, u, U) : u \in \{u^i, u^{i+1}\}, u^{i+1} \leq v \leq U\}.$$

Two cases must be examined:

- If there exists $(v''', u^i, U''') \in \{(v,u,U) : u \in \{u^i, u^{i+1}\},\ u^{i+1} \le v \le U\}$ indifferent to (v', u^i, U'), then, since $h_N^1 + h_N^2 + h_N^3$ is representing $\succsim$ on $\{(v, u^i, U) : u^0 < U\}$, the following equality holds:

$$h_N^1(v') + h_N^2(u^i) + h_N^3(U') = h_N^1(v''') + h_N^2(u^i) + h_N^3(U'''),$$

  and since (v''', u^i, U''') and (v'', u^{i+1}, U'') belong to $\{(v,u,U) : u \in \{u^i, u^{i+1}\},\ u^{i+1} \le v \le U\}$,

$$\begin{cases} (v''', u^i, U''') \succsim (v'', u^{i+1}, U'') \Leftrightarrow \\ \qquad h_N^1(v''') + h_N^2(u^i) + h_N^3(U''') \ge h_N^1(v'') + h_N^2(u^{i+1}) + h_N^3(U''). \end{cases}$$

  So, by transitivity,

$$\begin{cases} (v', u^i, U') \succsim (v'', u^{i+1}, U'') \Leftrightarrow \\ \qquad h_N^1(v') + h_N^2(u^i) + h_N^3(U') \ge h_N^1(v'') + h_N^2(u^{i+1}) + h_N^3(U''). \end{cases}$$

- If, on the other hand, there exists no $(v''', u^i, U''') \in \{(v,u,U) : u \in \{u^i, u^{i+1}\},\ u^{i+1} \le v \le U\}$ indifferent to (v', u^i, U'), then $(v', u^i, U') \prec (u^{i+1}, u^i, u^{i+1})$; otherwise, either $v' \ge u^{i+1}$, in which case $U' \ge u^{i+1}$ and $(v', u^i, U') \in \{(v,u,U) : u \in \{u^i, u^{i+1}\},\ u^{i+1} \le v \le U\}$, which contradicts our hypothesis; or $v' < u^{i+1}$, which implies, by lemma 8, that $(v', u^i, u^{i+1}) \prec (u^{i+1}, u^i, u^{i+1}) \precsim (v', u^i, U')$; but then, by definition of $\succsim_3$, $u^{i+1} \prec_3 U'$. Again by lemma 8, $(u^{i+1}, u^i, u^{i+1}) \precsim (v', u^i, U') \prec (u^{i+1}, u^i, U')$ and, by restricted solvability w.r.t. the third component, there exists U''' such that $u^{i+1} \precsim_3 U''' \precsim_3 U'$ and such that $(v', u^i, U') \sim (u^{i+1}, u^i, U''')$, which also contradicts our hypothesis.

  So, still by lemma 8, one gets

$$(v', u^i, U') \prec (u^{i+1}, u^i, u^{i+1}) \precsim (v'', u^{i+1}, U'')$$

  and, since $h_N^1 + h_N^2 + h_N^3$ represents $\succsim$ on $\{(v, u^i, U) : u^0 < U\}$ and on $\{(v,u,U) : u \in \{u^i, u^{i+1}\},\ u^{i+1} \le v \le U\}$,

$$\begin{aligned} h_N^1(v') + h_N^2(u^i) + h_N^3(U') &< h_N^1(u^{i+1}) + h_N^2(u^i) + h_N^3(u^{i+1}) \\ &\le h_N^1(v'') + h_N^2(u^{i+1}) + h_N^3(U''). \end{aligned}$$

Thus, for all $i \in \{0, \ldots, p-1\}$, $h_N^1 + h_N^2 + h_N^3$ represents $\succsim$ on

$$\{(v,u,U) : u \in \{u^i, u^{i+1}\} \text{ and } u^0 < U\}.$$

By induction, using the same principle, it is easy to show that $h_N^1 + h_N^2 + h_N^3$ represents $\succsim$ on $\{(v,u,U) : u \in \{u^i, i = 0, \ldots, p\},\ u^0 < U\}$. Moreover, by (11), this utility function is unique up to scale and location.

Third step : existence of an additive utility on $\{(v,u,U) : u \,\mathcal{O}\, u_0$ *and* $U > \inf_{u' \,\mathcal{O}\, u_0} u'\}$.

With $h_N^1 + h_N^2 + h_N^3$ already defined on $\{(v,u^0,U) : U > u^0\}$, let u_1 and u_2 be two real numbers such that $u_0 \,\mathcal{O}\, u_1$, $u_0 \,\mathcal{O}\, u_2$. According to the definition of $\mathcal{O}$, there exists a sequence $(u^i)_{i=0}^p$ i-linking u_0 and u_1. According to the preceding step, $h_N^1 + h_N^2 + h_N^3$ can be extended and represent $\succsim$ on $\{(v,u,U) : u \in \{u^i, i = 0,\dots,p\}$ and $U > \inf\{u_0,u_1\}\}$; moreover, this function is unique up to scale and location. Similarly, there exists a sequence $(w^i)_{i=0}^q$ i-linking u_0 and u_2, and $h_N^1 + h_N^2 + h_N^3$ can be extended in order to represent $\succsim$ on $\{(v,u,U) : u \in \{w^i, i = 0,\dots,q\}$ and $U > \inf\{u_0,u_2\}\}$; and this function is also unique up to scale and location. Let us show that both extensions are compatible and that they define an additive utility, unique up to scale and location, on $\{(v,u,U) : u \in \{u^i, i = 0,\dots,p\} \cup \{w^i,\ i = 0,\dots,q\}$ and $U > \inf\{u_0,u_1,u_2\}\}$. Without loss of generality, we can suppose that $u_2 > u_1$ and Not$(u_2 \sim_2 u_1)$.

Suppose that $u_1 \leq u_0 \leq u_2$. Then $u_1 \precsim_2 u_0 \precsim_2 u_2$ and, consequently, $u^i \precsim_2 u_0$ for all $i \in \{0,\dots,p\}$, and, $w^j \succ_2 u_0$ for all $j \in \{1,\dots,q\}$. Thus h_N^2 is unambiguously defined on $\{u^i, i = 0,\dots,p\} \cup \{w^i, i = 1,\dots,q\}$. h_N^1 and h_N^3 are also well defined since, by lemma 8, preference at fixed u does not depend on u, and additive utilities are unique up to scale and location. But is $h_N^1 + h_N^2 + h_N^3$ a utility function? Sequence $(z^i)_{i=0}^{p+q} = (u^p, u^{p-1},\dots,u^0,w^1,\dots,w^q)$ is such that

- $z^0 = u_1$, $z^{p+q} = u_2$,
- $z^i \prec_2 z^{i+1}$ for all $i \in \{0,\dots,p+q-1\}$,
- for all $i \in \{0,\dots,p+q-1\}$, there exist $v_i, U_i, v'_{i+1}, U'_{i+1}$ such that $(v_i,z^i,U_i) \sim (v'_{i+1}, z^{i+1}, U'_{i+1})$.

Therefore, (z^i) i-links u_1 and u_2, and, according to the preceding step, there exists an additive utility $k_N^1 + k_N^2 + k_N^3$, unique up to scale and location, representing $\succsim$ on $\{(v,u,U) : u \in \{z^i, i = 0,\dots,p+q\},\ z^0 < U\}$. Since the last set contains the union of $\{(v,u,U) : u \in \{u^i, i = 0,\dots,p\}$ and $U > u_1\}$ and $\{(v,u,U) : u \in \{w^i, i = 0,\dots,q\}$ and $U > u_0\}$, $h_N^1 + h_N^2 + h_N^3$ and $k_N^1 + k_N^2 + k_N^3$ must coincide. Therefore, $h_N^1 + h_N^2 + h_N^3$ is an additive utility function, and is, moreover, unique up to scale and location.

Suppose now that $u_0 \leq u_1 \leq u_2$. Let $(z^i)_{i=0}^r$ be the maximal subsequence constituted by the elements of sequences $(u^i)_{i=0}^p$ and $(w^i)_{i=0}^q$, and such that $z^i \prec_2 z^{i+1}$ for all $i \in \{0,\dots,r-1\}$. Then this sequence i-links u_0, u_1 and u_2. Indeed, when two consecutive elements of (z^i) previously belonged to (u^i), $i < p$ or to (w^i), $i < q$, then, by definition of (u^i) and (w^i), there exist $v_i, U_i, v'_{i+1}, U'_{i+1}$ such that

$$(v_i, z^i, U_i) \sim (v'_{i+1}, z^{i+1}, U'_{i+1}).$$

Otherwise, consider the case in which $z^k = u^i$, $z^{k+1} = w^j$ and $z^{k'} = u^{i+1}$, $k' > k+1$. By definition of sequence (u^i), there exist $v_i, U_i, v'_{i+1}, U'_{i+1}$ such that

$$(v_i, u^i, U_i) \sim (v'_{i+1}, u^{i+1}, U'_{i+1}).$$

Therefore, and since, by definition, $z^k \prec_2 z^{k+1} \prec_2 z^{k'}$,

$$\begin{aligned}(v'_{i+1}, z^{k+1}, U'_{i+1}) \prec (v_i, z^k, U_i) &\sim (v'_{i+1}, z^{k'}, U'_{i+1}) \\ &\prec (\max\{v_i, z^{k+1}\}, z^{k+1}, \max\{U_i, z^{k+1}\}).\end{aligned}$$

Since $z^{k'} \succ_2 z^k$,

- either $v_i \succ_1 v'_{i+1}$. Thus $(v'_{i+1}, z^{k+1}, U'_{i+1}) \prec (v_i, z^k, U_i) \prec (v_i, z^{k+1}, U_i)$. But then, either $(v'_{i+1}, z^{k+1}, U_i) \prec (v_i, z^k, U_i) \prec (v_i, z^{k+1}, U_i)$, in which case, by restricted solvability w.r.t. the first component, there exists v''_{i+1} such that $(v''_{i+1}, z^{k+1}, U_i) \sim (v_i, z^k, U_i)$, or $(v'_{i+1}, z^{k+1}, U'_{i+1}) \prec (v_i, z^k, U_i) \precsim (v'_{i+1}, z^{k+1}, U_i)$, in which case, by restricted solvability w.r.t. the third component, there exists U''_{i+1} such that $(v'_{i+1}, z^{k+1}, U''_{i+1}) \sim (v_i, z^k, U_i)$.

- or $U_i \succ_3 U'_{i+1}$. Thus $(v'_{i+1}, z^{k+1}, U'_{i+1}) \prec (v_i, z^k, U_i) \prec (U_i, z^{k+1}, U_i)$. But then, either $(v'_{i+1}, z^{k+1}, U'_{i+1}) \prec (v_i, z^k, U_i) \prec (v'_{i+1}, z^{k+1}, U_i)$, and by restricted solvability w.r.t. the third component, there exists U''_{i+1} such that $(v'_{i+1}, z^{k+1}, U''_{i+1}) \sim (v_i, z^k, U_i)$, or $(v'_{i+1}, z^{k+1}, U_i) \prec (v_i, z^k, U_i) \precsim (U_i, z^{k+1}, U_i)$ and, by restricted solvability w.r.t. the first component there exists v''_i such that $(v''_i, z^{k+1}, U_i) \sim (v_i, z^k, U_i)$.

Therefore, when $z^k = u^i$, $z^{k+1} = w^j$ and $z^{k'} = u^{i+1}$, $k' > k+1$, z^k and z^{k+1} are elements of an i-linking sequence. A similar proof holds when $z^k = w^i$, $z^{k+1} = u^j$ and $z^{k'} = w^{i+1}$, $k' > k+1$. Consequently, (z^k) is a sequence i-linking u_0, u_1 and u_2.

Thus, according to the preceding step, there exists an additive utility $k^1_N + k^2_N + k^3_N$, unique up to scale and location, representing $\succsim$ on $\{(v, u, U) : u \in \{z^i, i = 0, \ldots, r\}$ and $U > z_0\}$. But this utility function also represents $\succsim$ on $\{(v, u, U) : u \in \{u^i, i = 0, \ldots, p\}$ and $U > z_0\}$ and on $\{(v, u, U) : u \in \{w^i, i = 0, \ldots, q\}$ and $U > z_0\}$. Therefore, it must coincide with $h^1_N + h^2_N + h^3_N$. So $h^1_N + h^2_N + h^3_N$ is an additive utility, and is unique up to scale and location.

This is sufficient to conclude that $h^1_N + h^2_N + h^3_N$ is an additive utility on $\{(v, u, U) : u \mathcal{O} u_0$ and $\inf_{u \,\mathcal{O}\, u_0} u < U\}$; it is in fact remarkable that the principle of construction never questions what was previously constructed, but instead extends the domain of definition of the utility function.

Fourth step : existence of an additive utility on $\{(v, u, U) : u \,\mathcal{O}\, u_0$ *or* $u \,\mathcal{O}\, u_1$, *and* $U > \inf_{u' \,\mathcal{O}\, u_0} u'\}$ *(*$u_1 > u_0$ *and Not*$(u_1 \,\mathcal{O}\, u_0)$*).*

Suppose that $\mathcal{O}$ has only one equivalence class. Then, by axiom 12, there exists no $c_m \in \mathcal{C}$ such that $u(c_m) \precsim_2 u(c)$ for all $c \in \mathcal{C}$. So, for any $c_m \in \mathcal{C}$, there exists $c \in \mathcal{C}$ such that $u(c) \prec_2 u(c_m)$. Therefore, by lemma 8, $u(c) < u(c_m)$. Consequently,

$$\{(v,u,U) : \inf u < U\} = \{(v,u,U)\},$$

and theorems 13 and 15 are proved.

If, on the contrary, there exists more than one class, then consider two real numbers u_0 and u_1 such that $\text{Not}(u_1 \,\mathcal{O}\, u_0)$ and $u_1 \succ u_0$. By the previous steps, it is known that there exists an additive utility $h_N^1 + h_N^2 + h_N^3$, unique up to scale and location, on

$$\{(v,u,U) : u \,\mathcal{O}\, u_0 \text{ and } \inf_{u' \,\mathcal{O}\, u_0} u' < U\},$$

and another additive utility $k_N^1 + k_N^2 + k_N^3$, also unique up to scale and location, on

$$\{(v,u,U) : u \,\mathcal{O}\, u_1 \text{ and } \inf_{u' \,\mathcal{O}\, u_1} u' < U\}.$$

Note that, by axiom 7,

$$(v,u_0,U) \succsim (v',u_0,U') \Leftrightarrow (v,u_1,U) \succsim (v',u_1,U'),$$

so that the restrictions of $h_N^1 + h_N^3$ and $k_N^1 + k_N^3$ to $\{(v,U) : u_0, u_1 \leq v \text{ and } u_0, u_1 < U\}$ must coincide. Therefore, there exist some constants $\alpha > 0$ and $\beta_1, \beta_3 \in \mathbb{R}$ such that

$$\begin{aligned} k_N^1 &= \alpha\, h_N^1 + \beta_1 \\ k_N^3 &= \alpha\, h_N^3 + \beta_3. \end{aligned}$$

By subtracting β_i to k_N^i and dividing by α, the ordering represented by the functions is not changed. Consequently, one can suppose that $k_N^1 = h_N^1$ and $k_N^3 = h_N^3$. Let us show that h_N^1, h_N^2 on $\{u \,\mathcal{O}\, u_0\}$, k_N^2 on $\{u \,\mathcal{O}\, u_1\}$, and h_N^3 are bounded.

By axiom 12, there exists v_0 such that $u_1 \prec_1 v_0$. Consequently, $[u_1, v_0] \times [u_0, u_1] \times \{U : U \geq v_0\}$ is a full Cartesian product satisfying:

- restricted solvability w.r.t. 2 components and essentialness w.r.t. all the components,
- the second order cancellation axiom,
- an Archimedean property (axiom 11),
- $\text{Not}(u_0 \,\mathcal{O}\, u_1)$.

So, by [8], $\succsim$ is representable by an additive utility $f_N^1 + f_N^2 + f_N^3$ on $[u_1, v_0] \times [u_0, u_1] \times \{U : U \geq v_0\}$, and the functions corresponding to the first and third components are bounded and are unique up to scale and location. But since $[u_1, v_0] \times \{u_0\} \times \{U : U \geq v_0\} \subset [u_1, v_0] \times [u_0, u_1] \times \{U : U \geq v_0\}$, $f_N^1 + f_N^2 + f_N^3$ and $h_N^1 + h_N^2 + h_N^3$ must coincide (up to scale and location). Therefore, h_N^3 has a l.u.b.

By axiom 12, there exists $u_2 \prec_2 u_0$. If h_N^3 had no g.l.b. on $\{u : u \,\mathcal{O}\, u_1\}$, there would exist an infinite strictly decreasing over-standard sequence w.r.t. the third component, (U^i), of mesh $\{(u_0, u_2), (u_0, u_0)\}$ starting at (u_0, u_0, U^0) and bounded by (u_0, u_0, u_0), which contradicts the Archimedean axiom.

h_N^2 has also a l.u.b. on $\{u : u \,\mathcal{O}\, u_0\}$ else there would exist an infinite increasing over-standard sequence (u^k), of mesh $\{(u_1, u_1), (u_1, v_0)\}$, such that $u^k \,\mathcal{O}\, u_0$, and bounded by (u_1, u_1, u_1), which contradicts the Archimedean axiom. Similarly, h_N^2 has a g.l.b. on $\{u : u \,\mathcal{O}\, u_1\}$.

h_N^1 has a l.u.b. on $\{u : u \,\mathcal{O}\, u_0\}$ else there exists an infinite strictly increasing over-standard sequence w.r.t. the first component, (w^i), of mesh $\{(u_0, u_1), (u_0, v_0)\}$ starting at (u_0, u_0, u_0) and bounded by (u_1, u_1, u_1), which contradicts the Archimedean axiom. Similarly, if h_N^1 had no g.l.b. on $\{u : u \,\mathcal{O}\, u_1\}$, then there would exist an infinite strictly decreasing over-standard sequence w.r.t. the first component, (w^i), of mesh $\{(u_0, v_0), (u_0, u_1)\}$ starting at (u_1, u_1, v_0) and bounded by (u_0, u_0, u_0), which contradicts the Archimedean axiom.

Therefore, since h_N^1, h_N^2 and h_N^3 are bounded, a necessary and sufficient condition for h_N to represent $\succsim$ on $\{(v, u, U) : u \,\mathcal{O}\, u_0 \text{ or } u \,\mathcal{O}\, u_1, \text{ and } \inf_{u' \,\mathcal{O}\, u_0} u' < U\}$ is that

$$\inf_{\substack{u \,\mathcal{O}\, u_1 \\ u \leq v \leq U}} \{h_N^1(v) + h_N^2(u) + h_N^3(U)\} \geq \sup_{\substack{u \,\mathcal{O}\, u_0 \\ u \leq v \leq U}} \{h_N^1(v) + h_N^2(u) + h_N^3(U)\}. \qquad (12)$$

Fifth step : existence of an additive utility on $\mathcal{T}_N^0$.

Equivalence classes of $\mathcal{O}$ are naturally ordered by $>$. Indeed, if $\text{Not}(u_0 \,\mathcal{O}\, u_1)$, then either $u_0 \prec_2 u_1$ or $u_1 \prec_2 u_0$; in both cases, lemma 8 ensures that $u_0 < u_1$ or $u_1 < u_0$. Moreover, as just proved above, given u^0, there is only a finite number of these classes between those containing u^0 and any u. Thus, all classes can be enumerated as a double sequence with representative elements u^z, $z \in Z \subset \mathbb{Z}$, such that the elements of Z are consecutive and $u^z < u^{z+1}$. The additive utility can then be successively extended to

$$\{(v, u, U) : u \,\mathcal{O}\, u^0 \text{ or } u \,\mathcal{O}\, u^1, \text{ and } \inf_{u' \,\mathcal{O}\, u^0} u' < U\},$$
$$\{(v, u, U) : u \,\mathcal{O}\, u^{-1} \text{ or } u \,\mathcal{O}\, u^0 \text{ or } u \,\mathcal{O}\, u^1, \text{ and } \inf_{u' \,\mathcal{O}\, u^{-1}} u' < U\}, \text{etc} \ldots$$

At each step, for instance the extension from $\{(v, u, U) : u \,\mathcal{O}\, u_z, z \in \{z', z' + 1, \ldots, z''\}$ to $\{(v, u, U) : u \,\mathcal{O}\, u_z, z \in \{z' - 1, z', \ldots, z''\}$, the proof is similar to that in in part 4 and relies on the fact that

- h_N^1 is bounded below and above on $\{v : v \geq \inf_{u \,\mathcal{O}\, u^{z'-1}} u\}$,
- h_N^2 is bounded below on $\{u : u \,\mathcal{O}\, u^{z'}\}$, and above on $\{u : u \,\mathcal{O}\, u^{z'-1}\}$,
- h_N^3 is bounded below and above on $\{U : U > \inf_{u \,\mathcal{O}\, u^{z'-1}} u\}$.

And thus the inequality corresponding to (12) can always be satisfied. (The extension on the other side can be justified in a similar way). So, by induction on z, the additive utility function can be extended on each $\{(v, u, U) : u \,\mathcal{O}\, u^z$ and $U > \inf_{u' \,\mathcal{O}\, u^z} u'\}$. This process of construction ensures that the function thus created represents $\succsim$ over $\{(v, u, U) : U > \inf_{u'} u'\}$. Indeed, this is obvious when Z is a finite set; if, on the contrary, Z is infinite, then $h_N^1 + h_N^2 + h_N^3$ does not represent $\succsim$ on $\{(v, u, U) : U > \inf_{u'} u'\}$ only if one the following two cases occur:

- there exists (v, u, U) such that $(v, u, U) \precsim (v', u^z, U')$ for all $z \in Z$, v', U', and
$$\lim_{\substack{z \to -\infty \\ u,v,U \to \inf_{u' \,\mathcal{O}\, u^z} u'}} h_N^1(v) + h_N^2(u^z) + h_N^3(U) = -\infty.$$
- there exists (v, u, U) such that $(v, u, U) \succsim (v', u^z, U')$ for all $z \in Z$, v', U', and
$$\lim_{\substack{z \to +\infty \\ u \to \inf_{u' \,\mathcal{O}\, u^z} u' \\ v,U \to \sup\{u(c) : c \in \mathcal{C}\}}} h_N^1(v) + h_N^2(u^z) + h_N^3(U) = +\infty.$$

But both cases are impossible because u^z is an over-standard sequence, and, by axiom 11, when bounded, must be finite.

Consequently, $h_N^1 + h_N^2 + h_N^3$ represents $\succsim$ on $\{(v, u, U) : U > \inf_{u'} u'\}$. But since, by axiom 12, $\inf_u$ is not reached, $h_N^1 + h_N^2 + h_N^3$ can be extended to represent $\succsim$ on $\{(v, u, U)\} = \mathcal{T}_N^0$. The uniqueness property is derived directly from (12) and the uniqueness up to scale and location inside equivalence classes of $\mathcal{O}$. □

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