Copyless Cost Register Automata
Bounded Ambiguity vs Determinism

Benjamin Monmege
joint work with Théodore Lopez and Jean-Marc Talbot

Laboratoire d'Informatique et Systèmes — Aix-Marseille University

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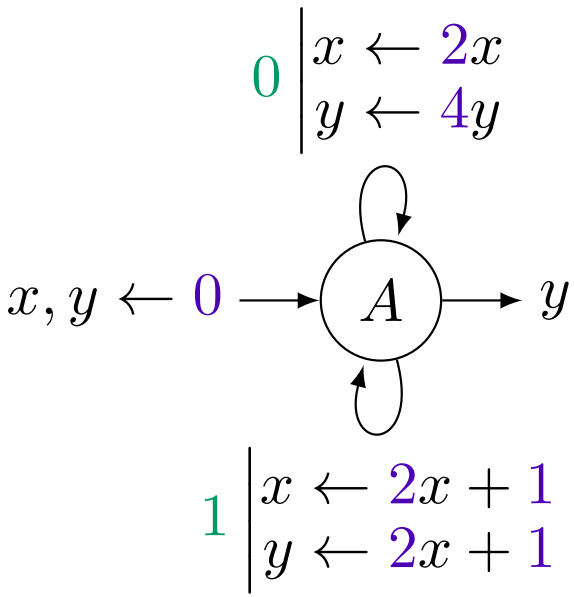
Quantitative models

Classical Framework	Quantitative Framework
Q: "Is it possible ?"	Q: "How much does it cost?"
A: Yes or No	A: Weight
Automaton	Weighted Automaton
MSO logic	Weighted logic
	Cost Register Automaton (CRA)

Weights on a semiring:

Natural	$(\mathbb{N}, +, \times)$	Count ambiguity
Tropical	$(\mathbb{N}, \min, +)$	Shortest path
Boolean	$(\{\top, \bot\}, \vee, \wedge)$	Classical setting
Language	$(\mathcal{P}(\Sigma^*), \cup, \cdot)$	Transducers, SST [Alur & Černý FSTTCS'10]

Cost Register Automaton (CRA) [Alur et al. LICS'13]



- States: A
- Registers: x, y
- Initial valuation
- Final outputs
- Transitions: update registers
- Deterministic

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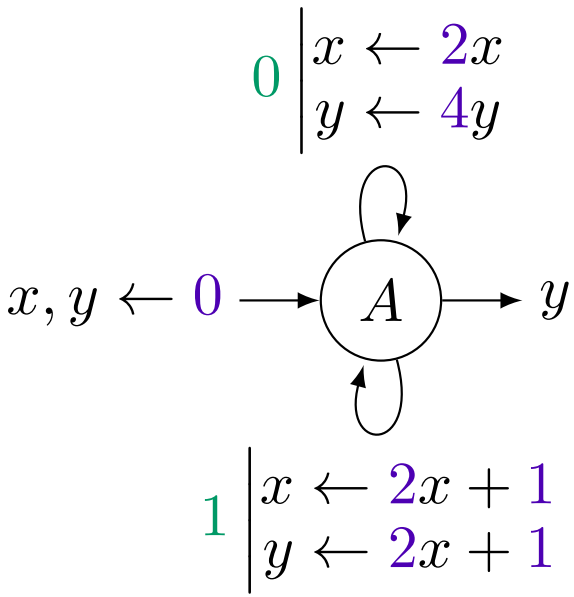

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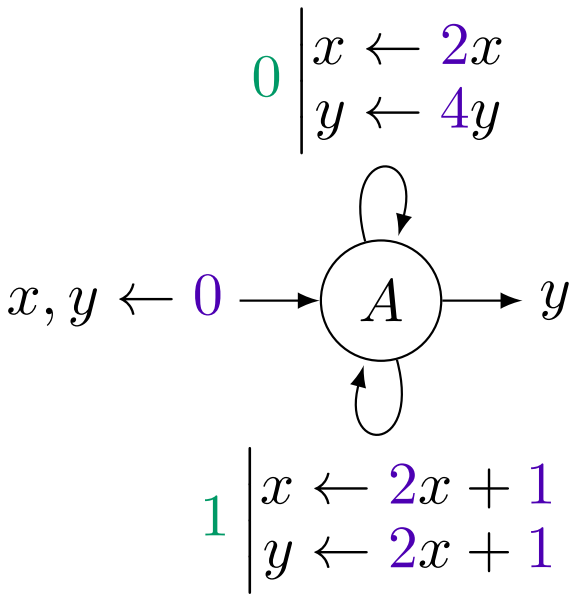
On input 10100:

$$\begin{matrix} x = 0 \\ y = 0 \end{matrix} \xrightarrow{1} \begin{matrix} x = 1 \\ y = 1 \end{matrix} \xrightarrow{0} \begin{matrix} x = 2 \\ y = 4 \end{matrix} \xrightarrow{1} \begin{matrix} x = 5 \\ y = 5 \end{matrix} \xrightarrow{0} \begin{matrix} x = 10 \\ y = 20 \end{matrix} \xrightarrow{0} \begin{matrix} x = 20 \\ y = 80 \end{matrix}$$

Copyless CRA

In updates and final outputs, each register is used at most once.



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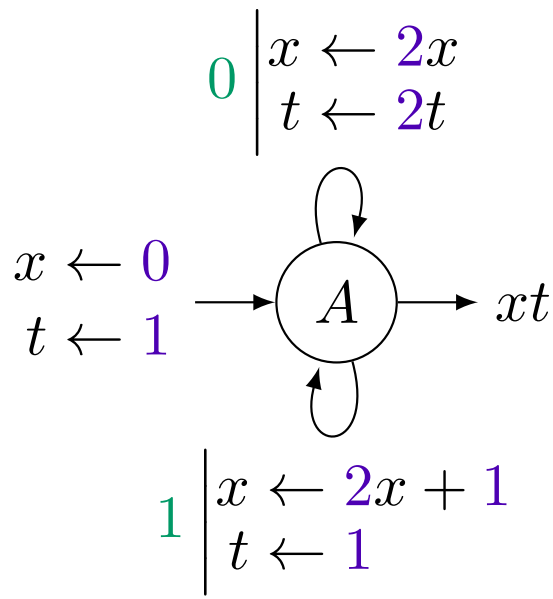

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Known results on (copyless) CRA

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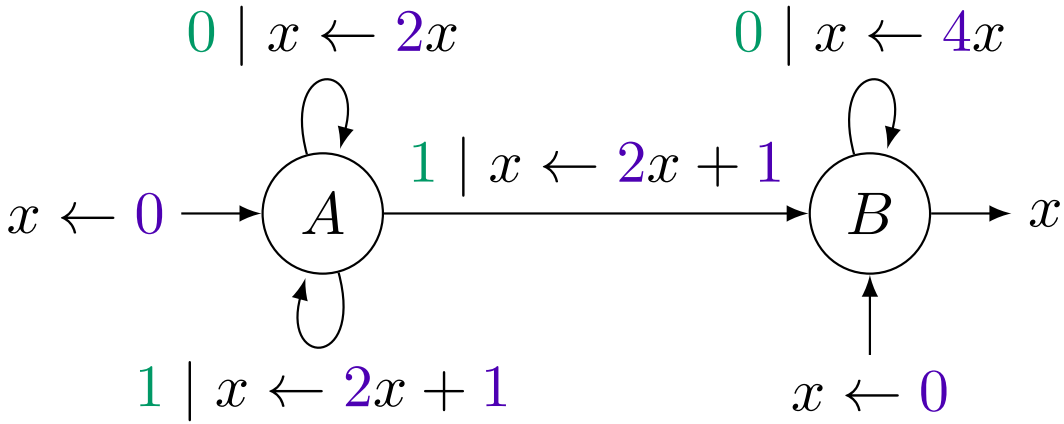
- Copyless CRA $\subsetneq$ WA over commutative semirings [Mazowiecki, Riveros, 2016]

Unambiguous Non-Deterministic CRA

NCRA with at most one run for each input word.



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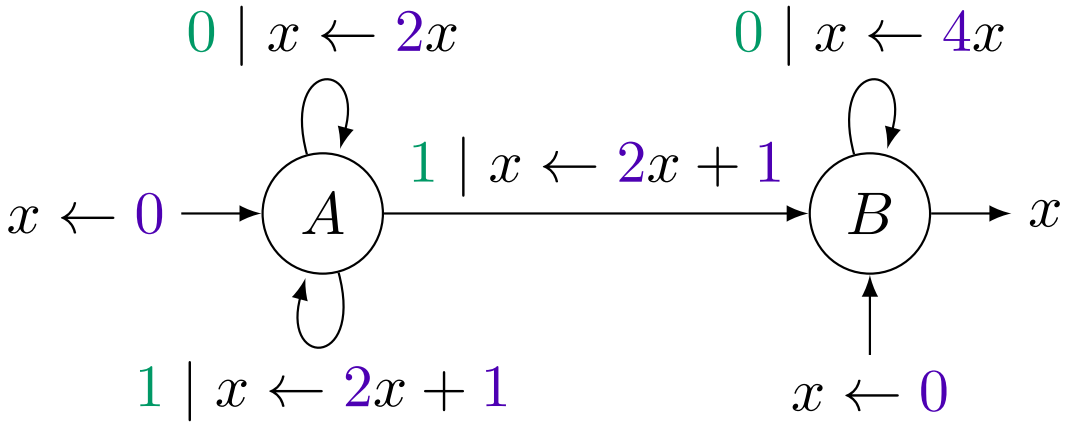

- Modelling via regular look-ahead

Open problem [Mazowiecki, Riveros 2016]

copyless CRA vs unambiguous copyless NCRA ?

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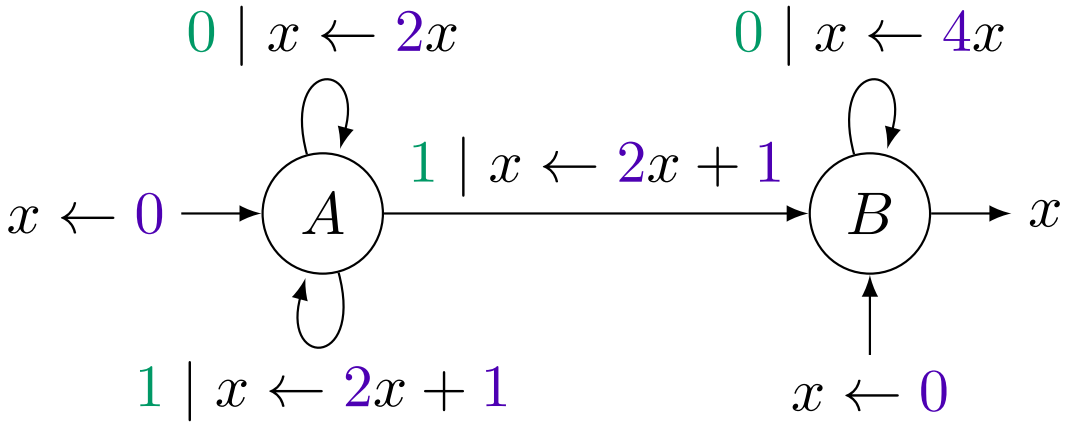

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copyless CRA vs unambiguous copyless NCRA ?

Deterministic WA $\subsetneq$ *Unamb. WA* $\subsetneq$ *Finitely-amb. WA* $\subsetneq$ *WA*

Main Result

Theorem

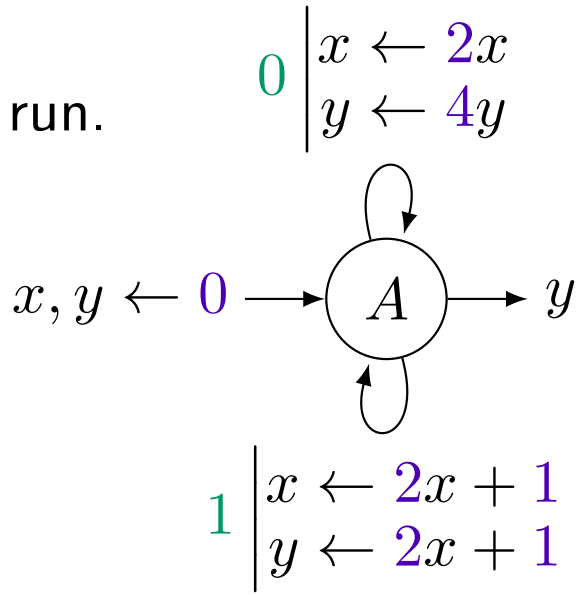
Unambiguous copyless NCRA are as expressive as copyless CRA

Proof scheme

Unambiguous Copyless NCRA $\longrightarrow$ $\diamond$-less CRA $\longrightarrow$ Copyless CRA

Flow Graph

Flow of registers of a CRA along a run.



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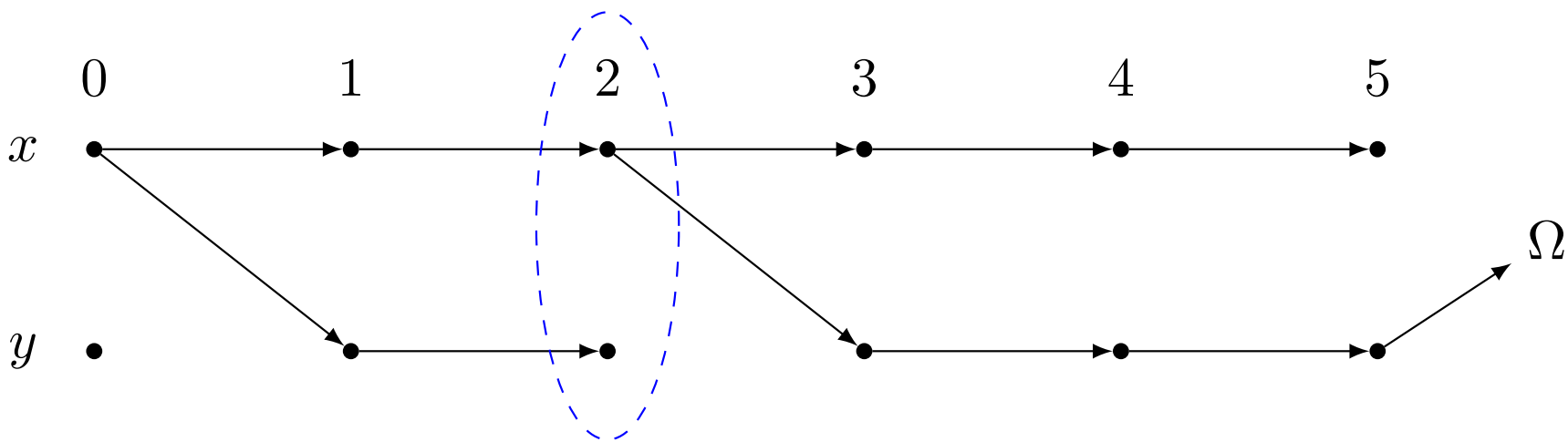

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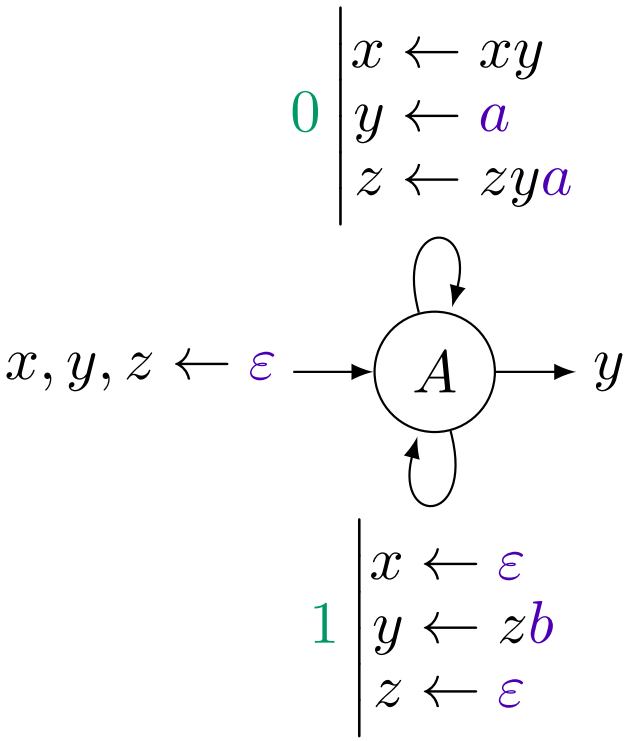

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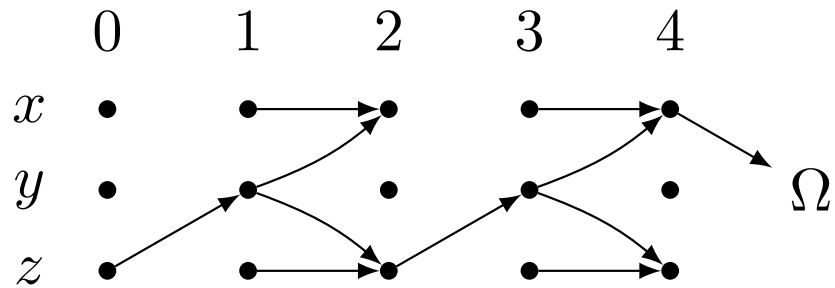
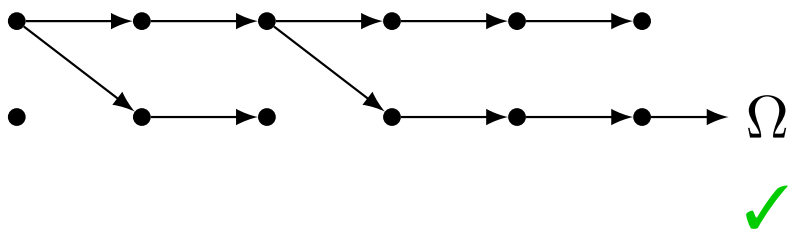


◇-less Flow Graph

Contains no diamonds.



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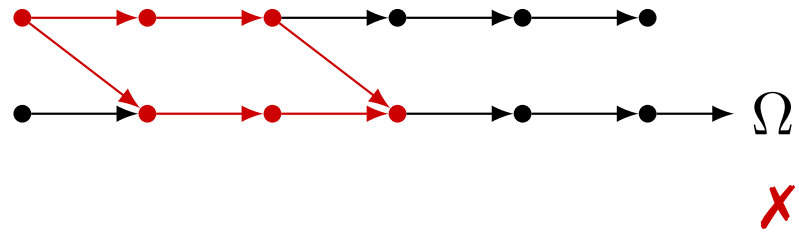

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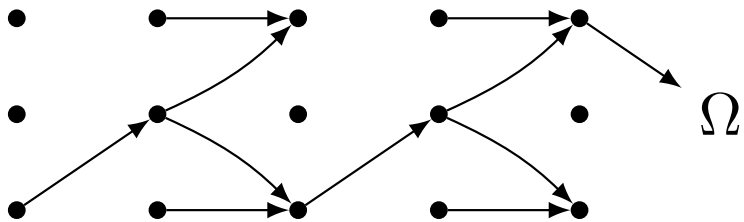

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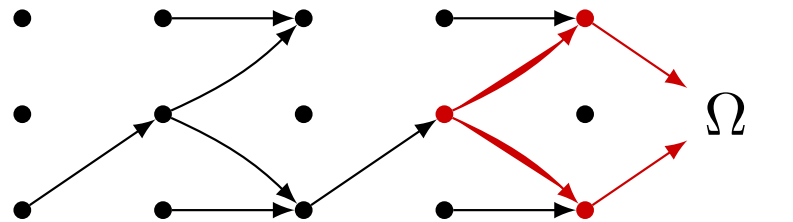

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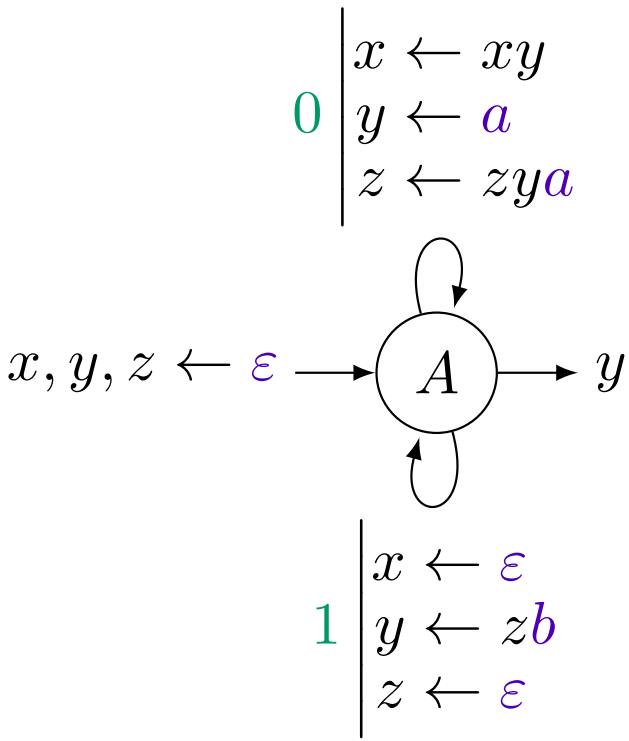
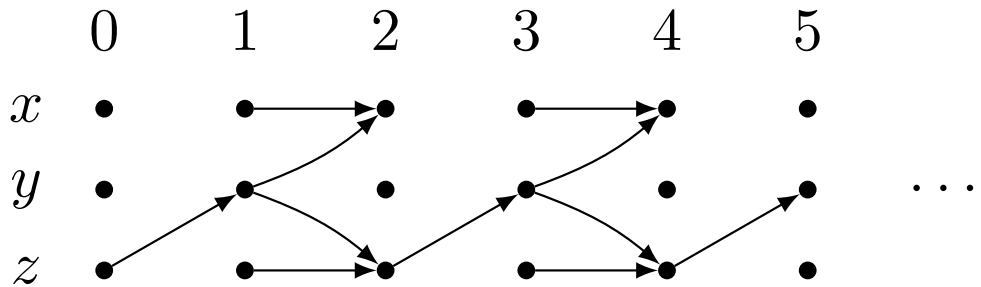
$\diamond$-less CRA

Flow graph of each run is $\diamond$-less.





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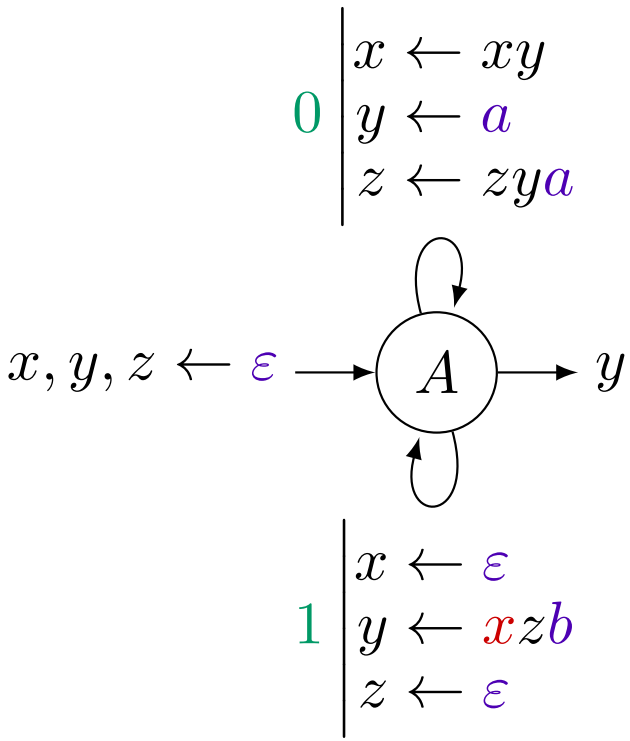

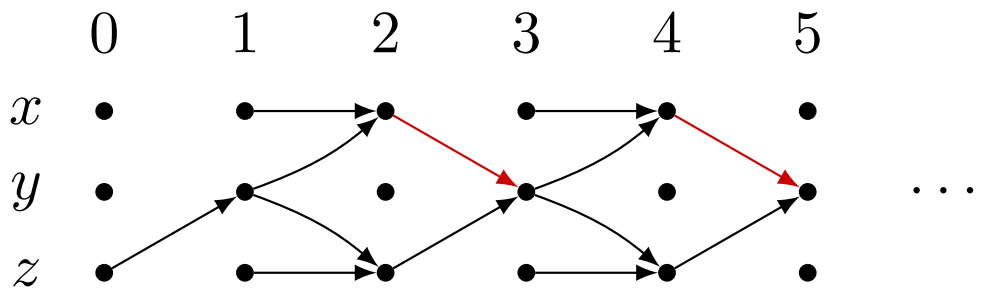

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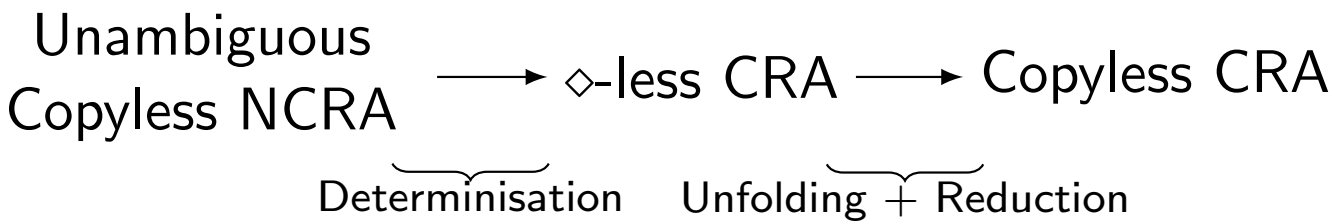
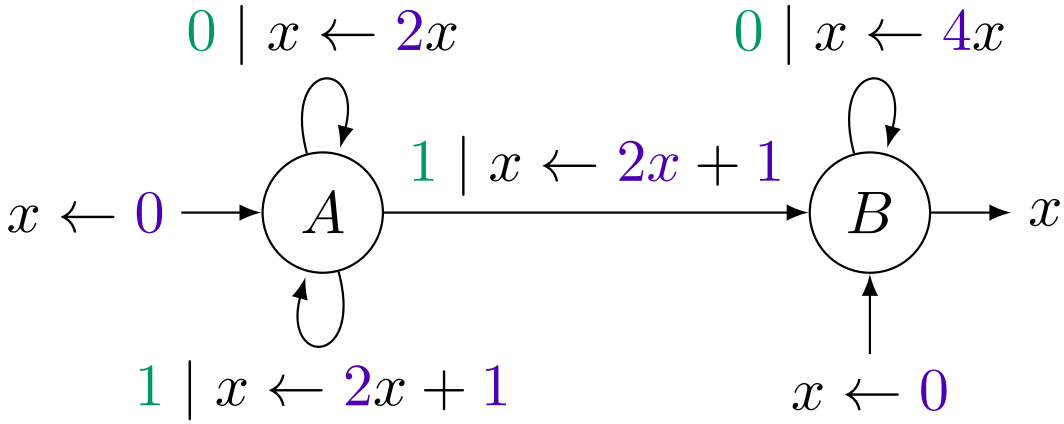


Determinisation



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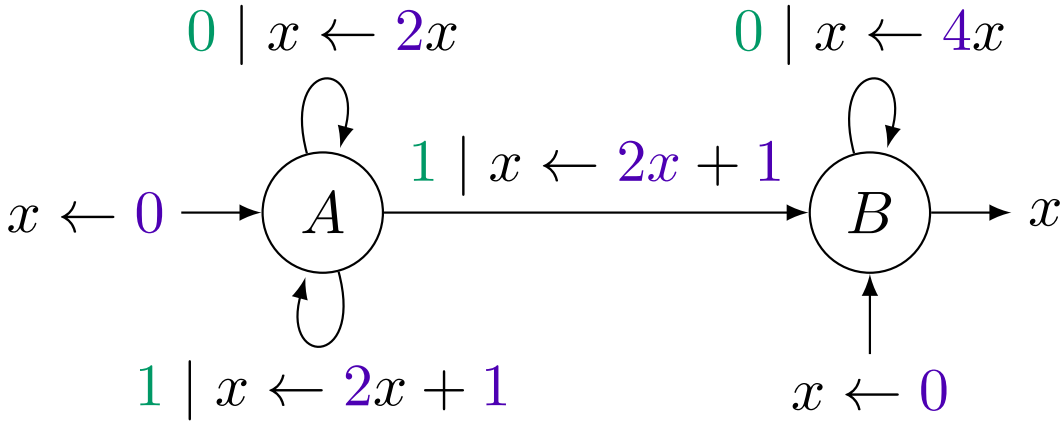

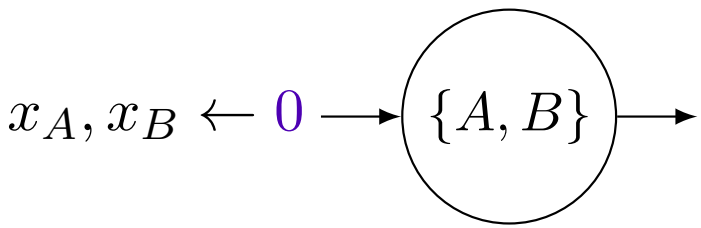

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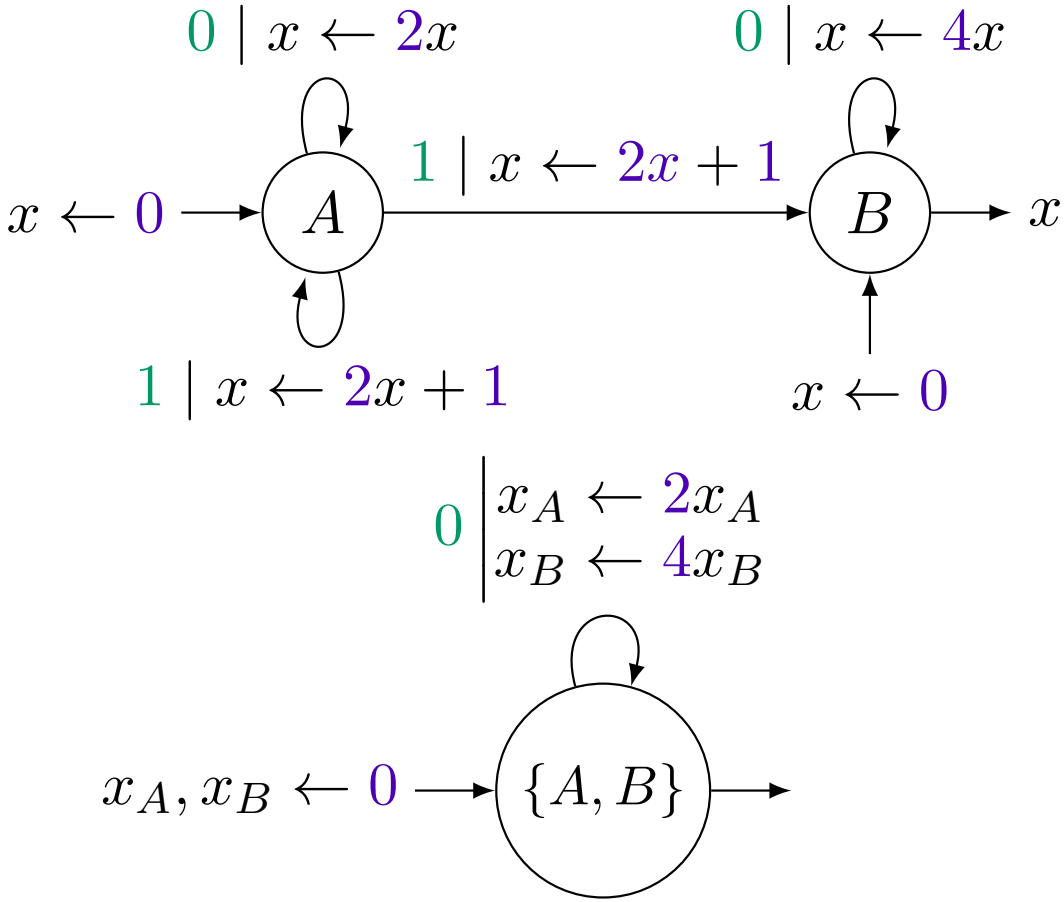

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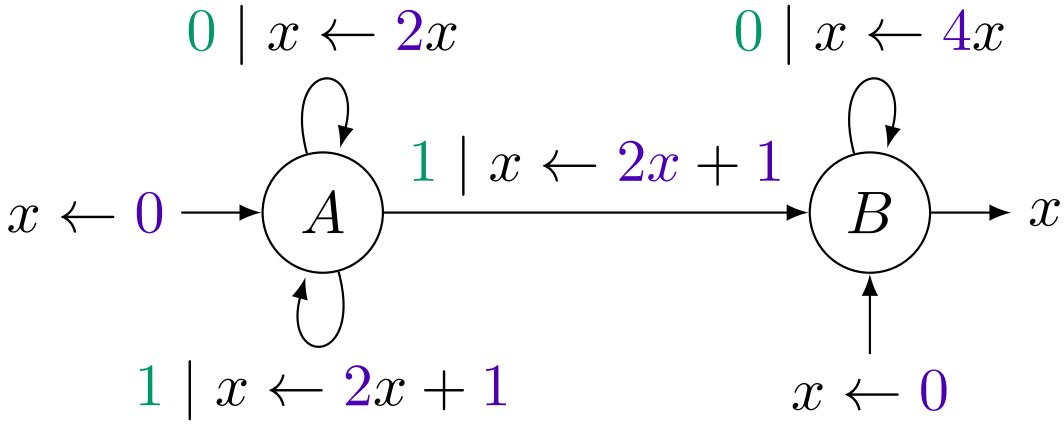

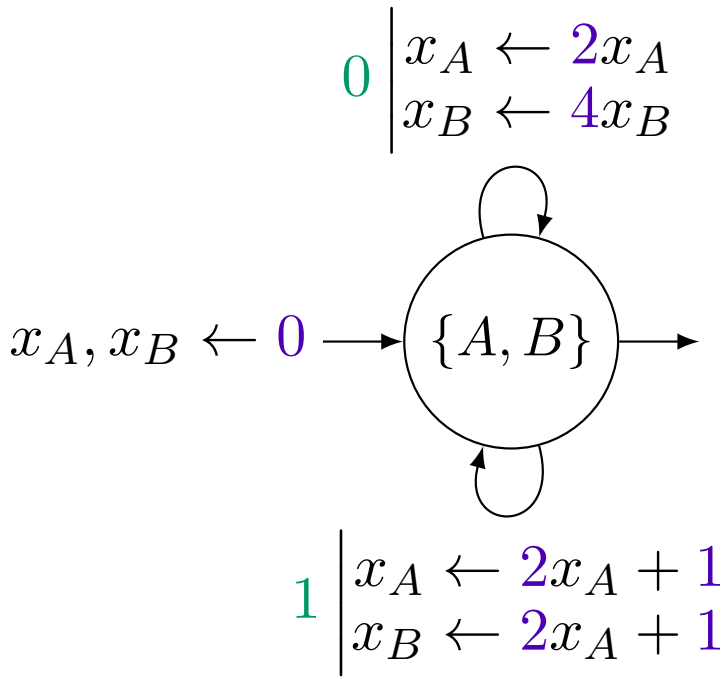

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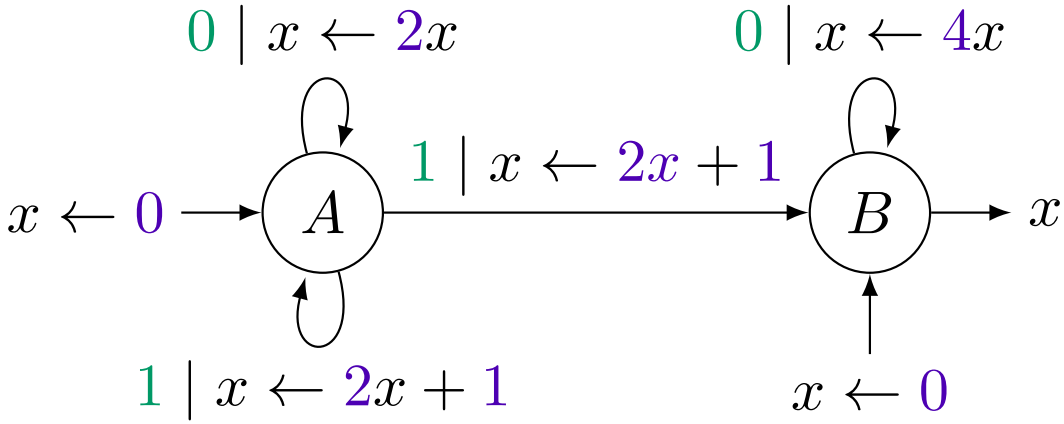

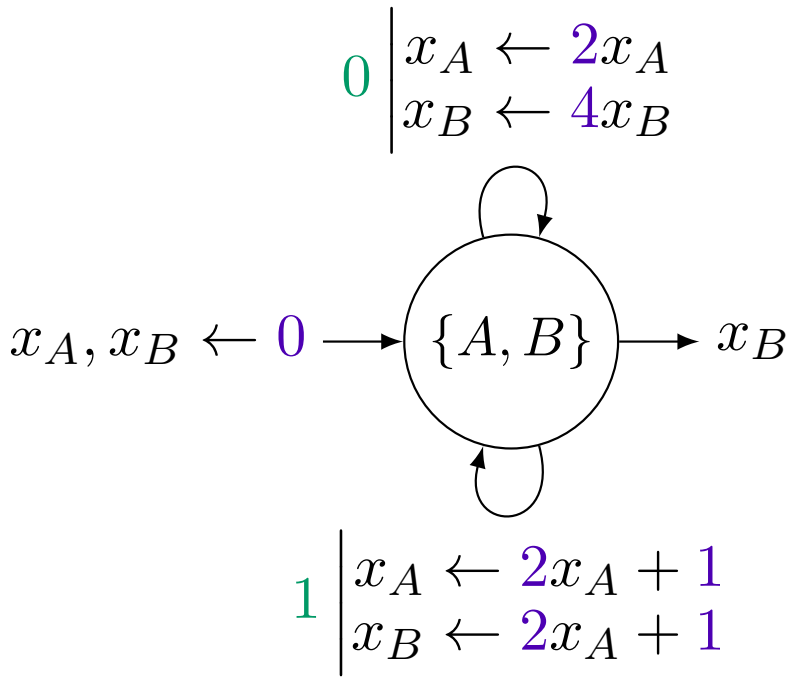

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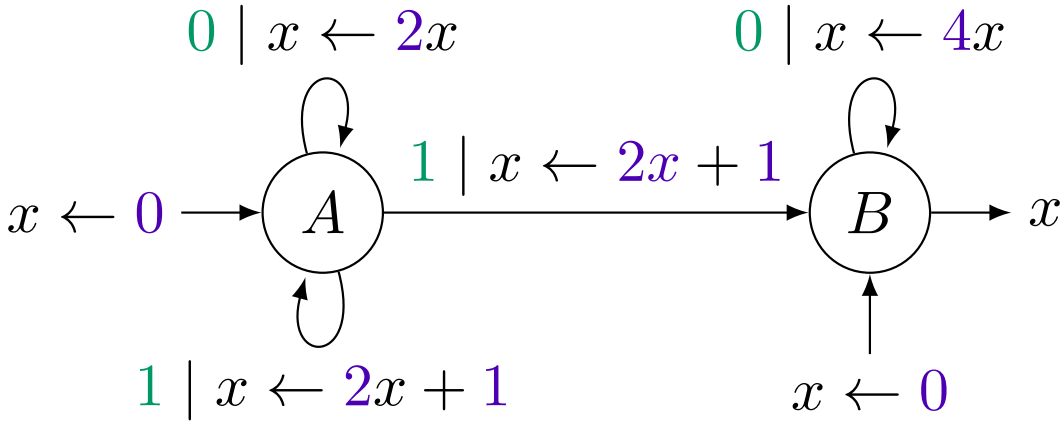
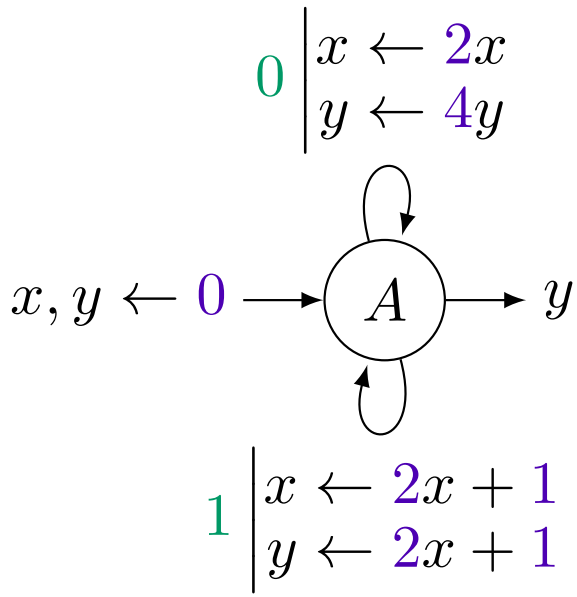
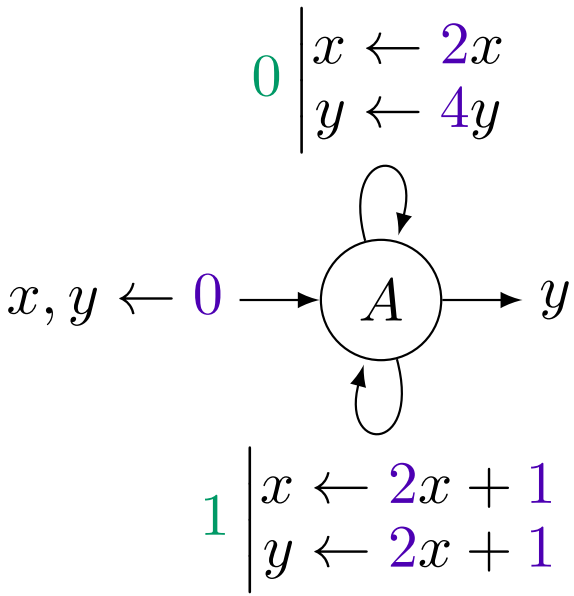




Flow graph with terms



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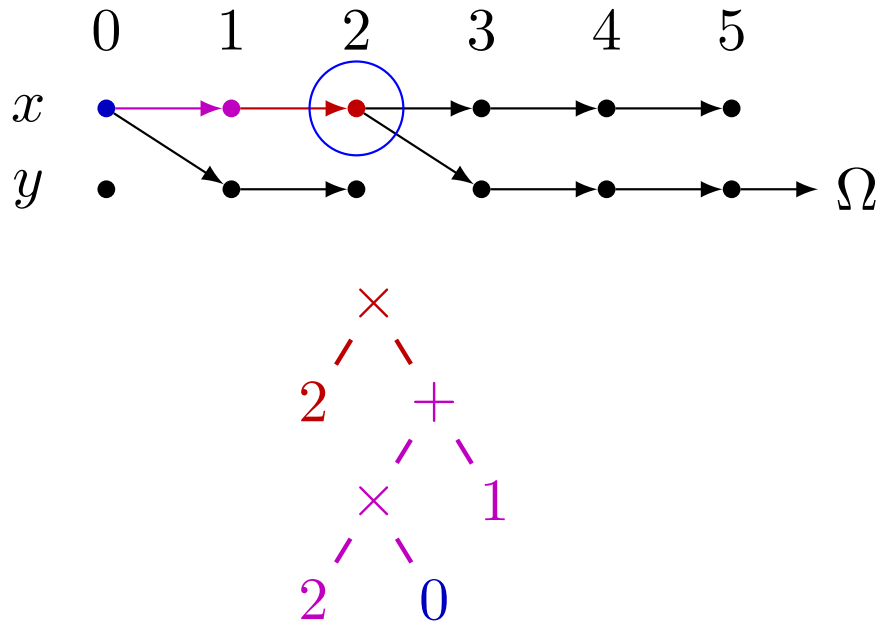

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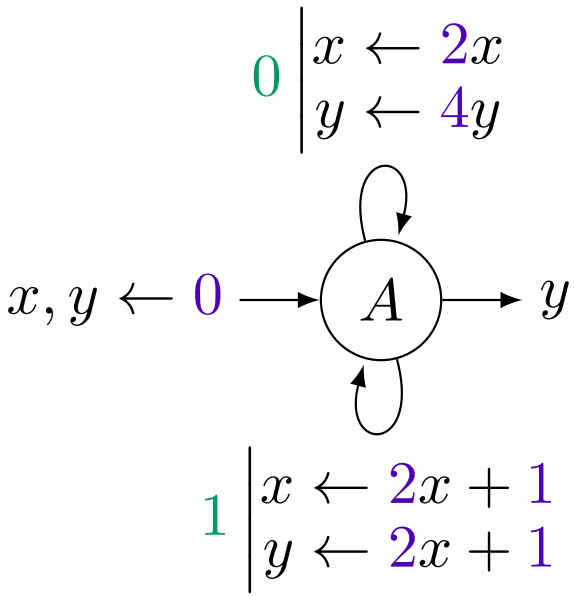

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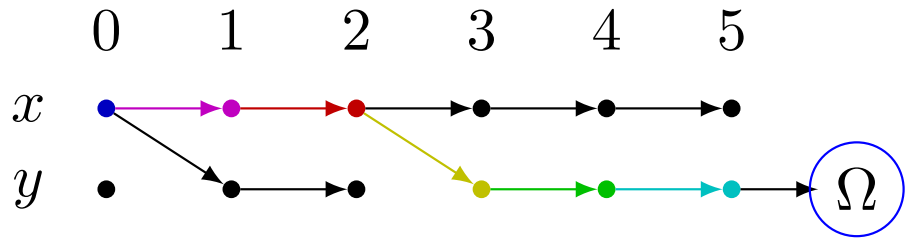
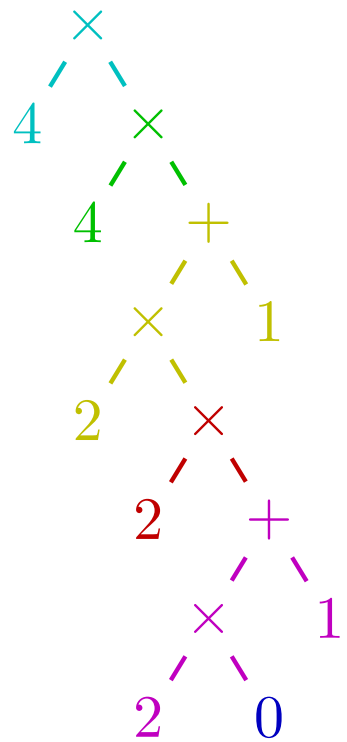
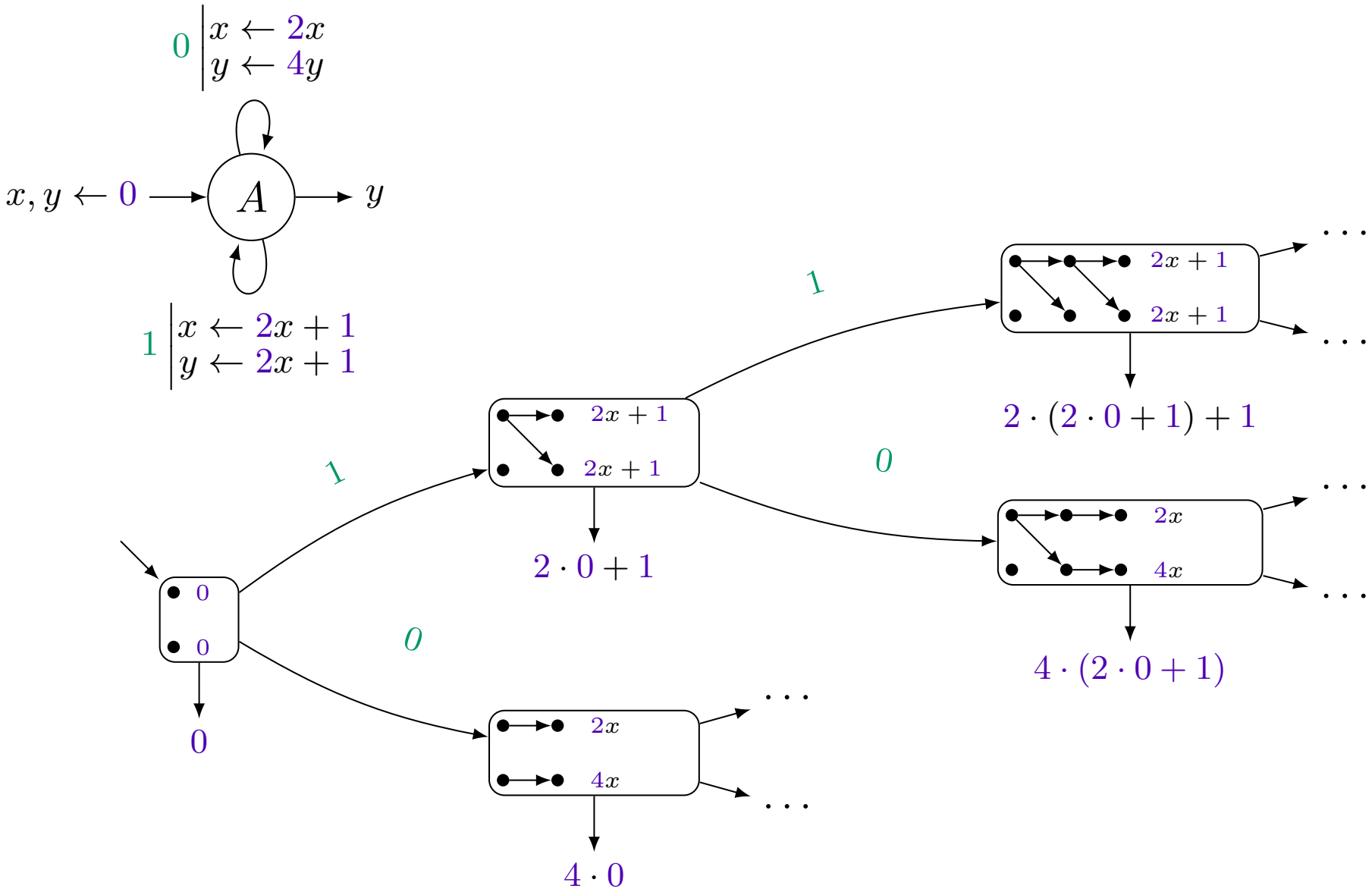
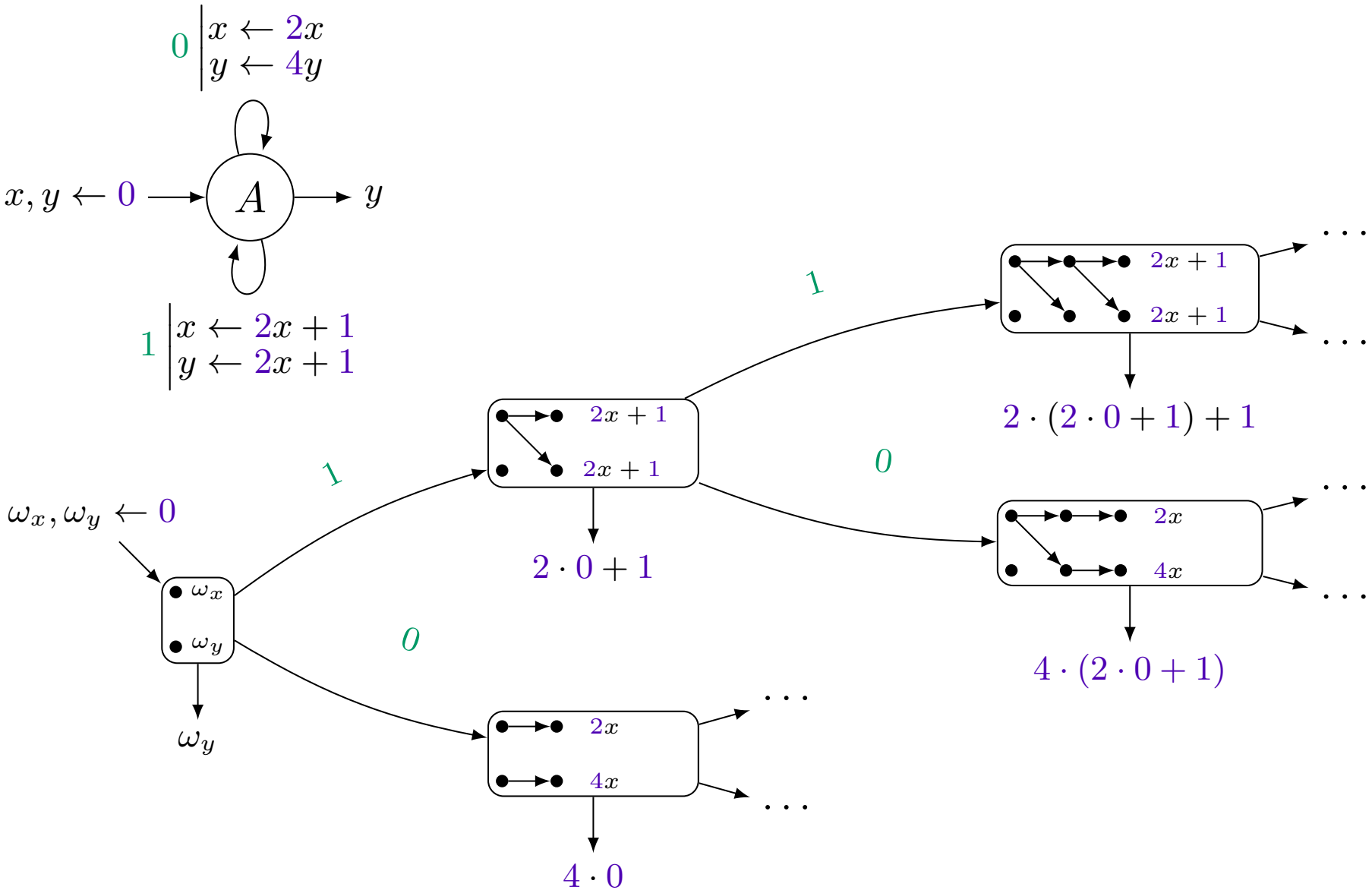




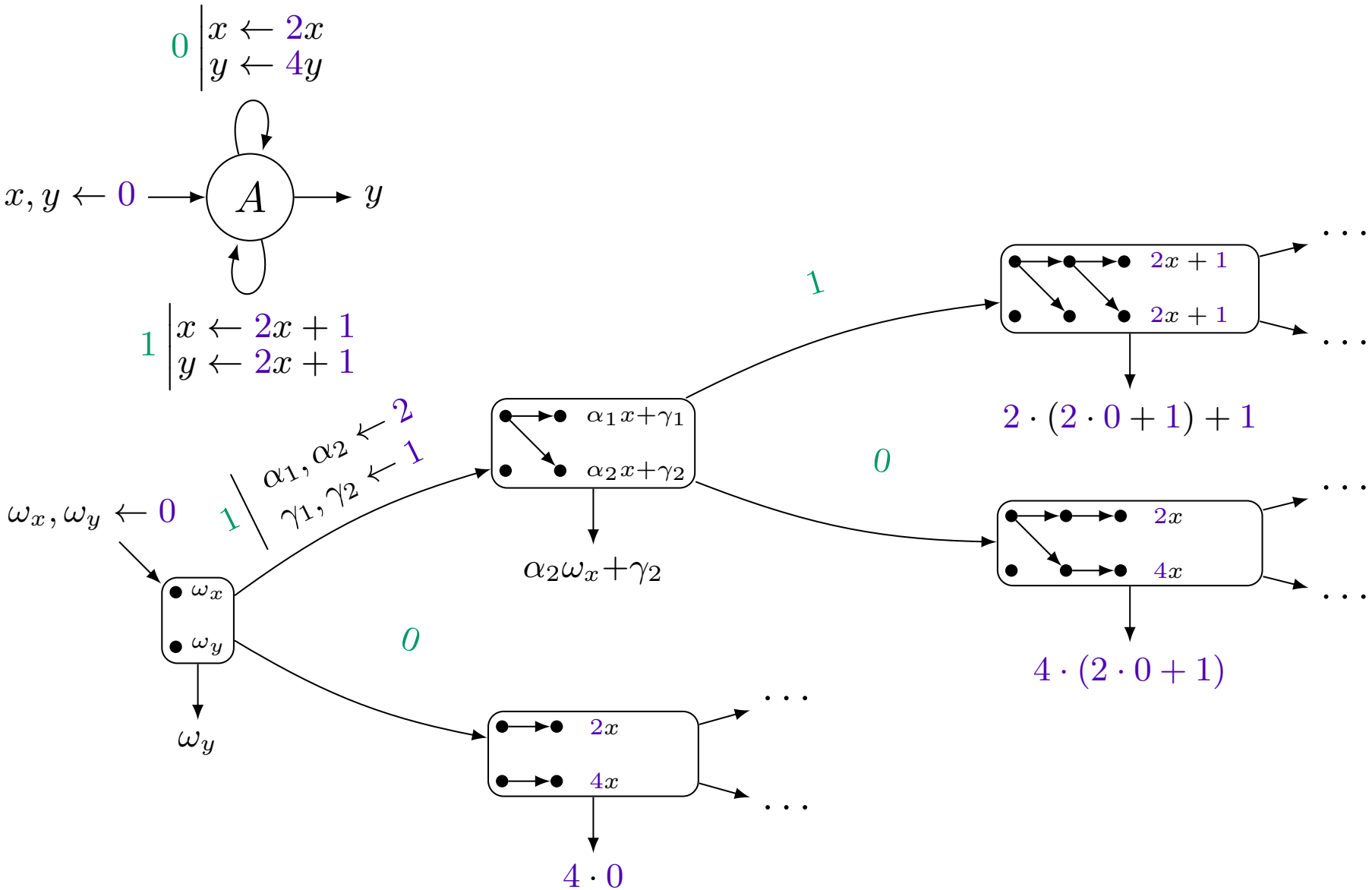
Unfolding


0
$x \leftarrow 2x$
$y \leftarrow 4y$
$x, y \leftarrow 0$
A
y
1
$x \leftarrow 2x + 1$
$y \leftarrow 2x + 1$
1
0
1
0
0
0
$2x + 1$
$2x + 1$
$2 \cdot 0 + 1$
$2x + 1$
$2x + 1$
$2 \cdot (2 \cdot 0 + 1) + 1$
$2x$
$4x$
$4 \cdot (2 \cdot 0 + 1)$
$2x$
$4x$
$4 \cdot 0$

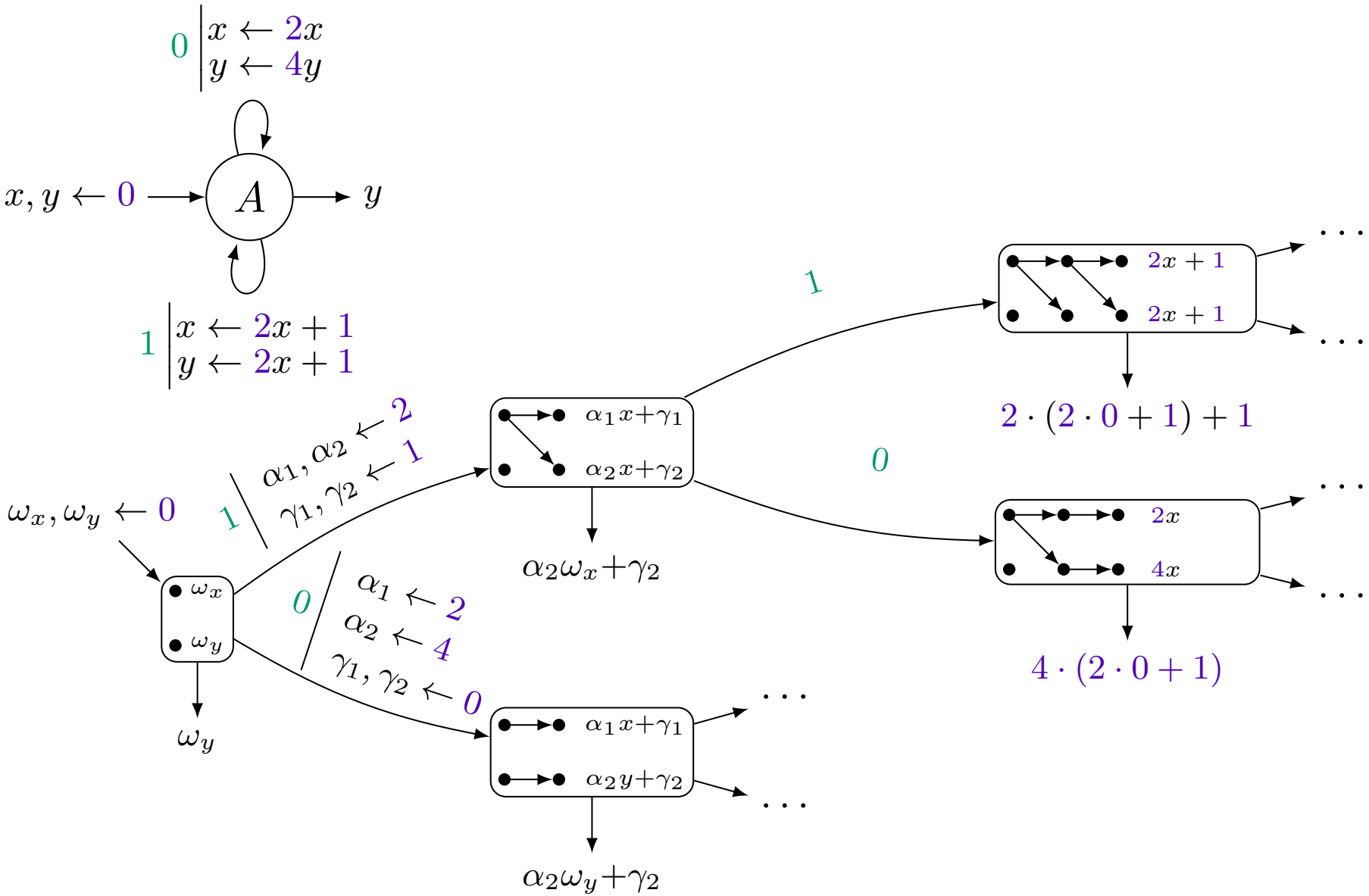
Unfolding


0
$x \leftarrow 2x$
$y \leftarrow 4y$
$x, y \leftarrow 0$
A
y
1
$x \leftarrow 2x + 1$
$y \leftarrow 2x + 1$
$\omega_x, \omega_y \leftarrow 0$
ω_x
ω_y
ω_y
1
0
$2x + 1$
$2x + 1$
$2 \cdot 0 + 1$
1
0
$2x + 1$
$2x + 1$
$2 \cdot (2 \cdot 0 + 1) + 1$
$2x$
$4x$
$4 \cdot (2 \cdot 0 + 1)$
$2x$
$4x$
$4 \cdot 0$

Unfolding


0
$x \leftarrow 2x$
$y \leftarrow 4y$
$x, y \leftarrow 0$
A
y
1
$x \leftarrow 2x+1$
$y \leftarrow 2x+1$
$\omega_x, \omega_y \leftarrow 0$
1
$\alpha_1, \alpha_2 \leftarrow 2$
$\gamma_1, \gamma_2 \leftarrow 1$
ω_x
ω_y
ω_y
$\alpha_1 x+\gamma_1$
$\alpha_2 x+\gamma_2$
$\alpha_2\omega_x+\gamma_2$
1
$2x+1$
$2x+1$
$2 \cdot (2 \cdot 0 + 1) + 1$
0
$2x$
$4x$
$4 \cdot (2 \cdot 0 + 1)$
0
$2x$
$4x$
$4 \cdot 0$

Unfolding


0
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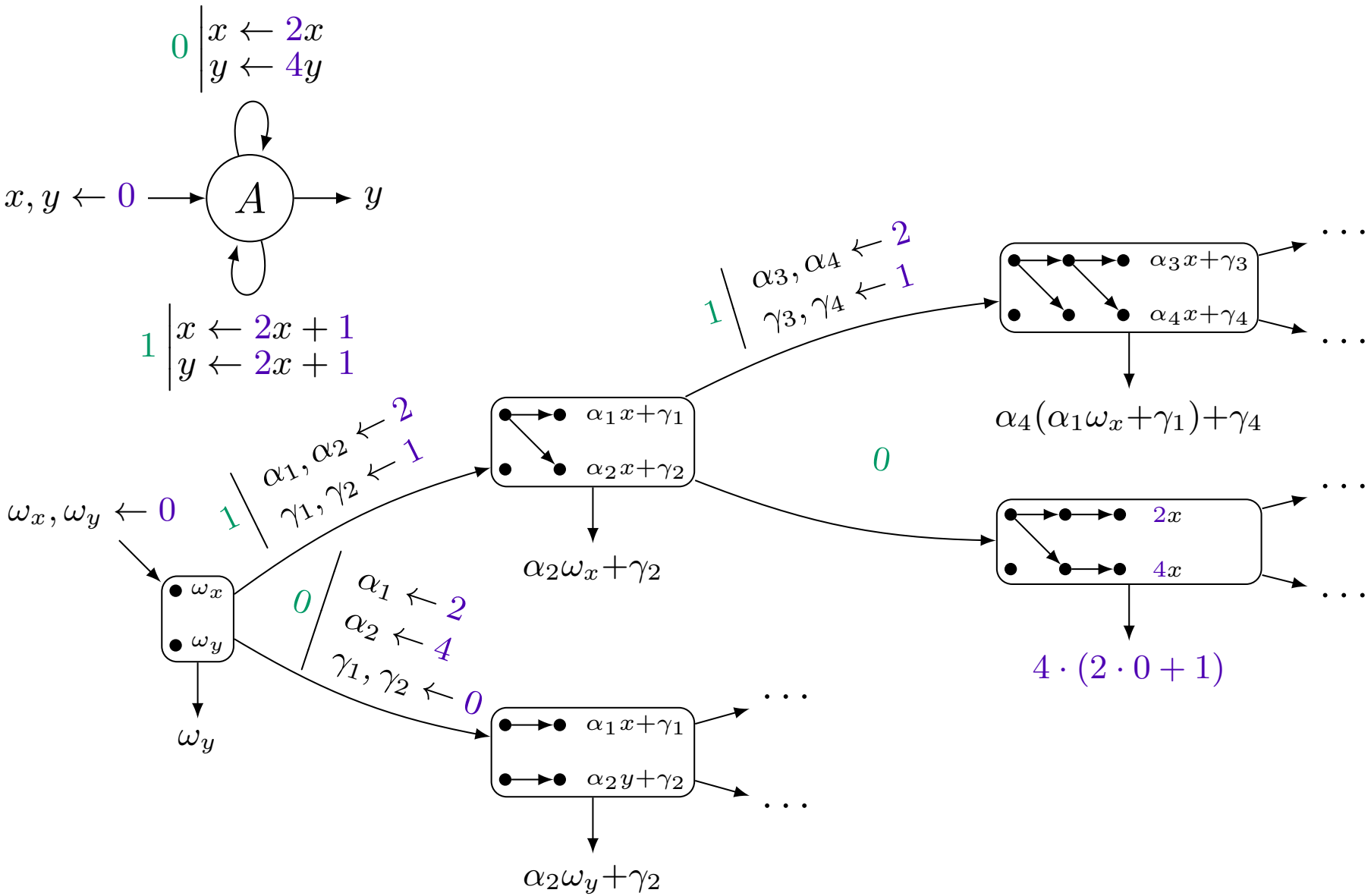
$y \leftarrow 4y$
$x, y \leftarrow 0$
A
y
1
$x \leftarrow 2x+1$
$y \leftarrow 2x+1$
$\omega_x, \omega_y \leftarrow 0$
ω_x
ω_y
ω_y
1
$\alpha_1, \alpha_2 \leftarrow 2$
$\gamma_1, \gamma_2 \leftarrow 1$
0
$\alpha_1 \leftarrow 2$
$\alpha_2 \leftarrow 4$
$\gamma_1, \gamma_2 \leftarrow 0$
$\alpha_1 x+\gamma_1$
$\alpha_2 x+\gamma_2$
$\alpha_2\omega_x+\gamma_2$
$\alpha_1 x+\gamma_1$
$\alpha_2 y+\gamma_2$
$\alpha_2\omega_y+\gamma_2$
1
0
$2x+1$
$2x+1$
$2 \cdot (2 \cdot 0 + 1) + 1$
$2x$
$4x$
$4 \cdot (2 \cdot 0 + 1)$

Unfolding


0
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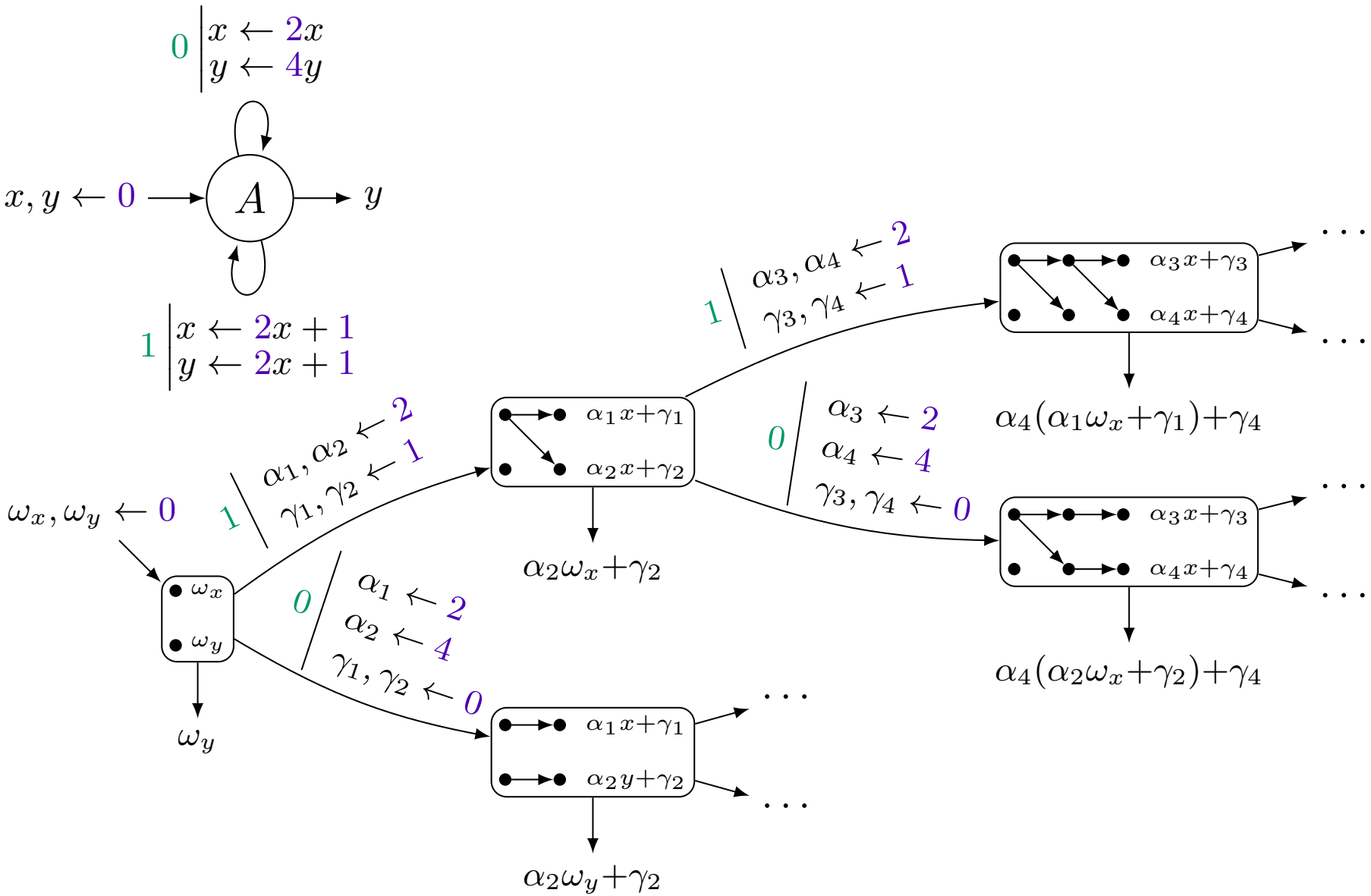
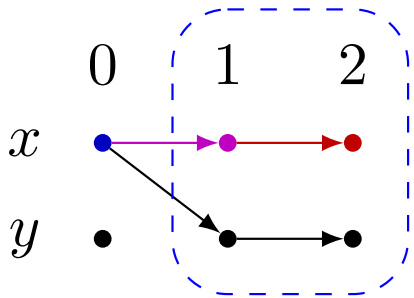
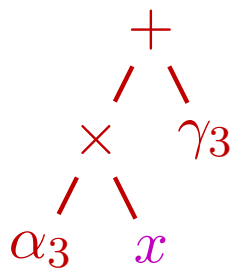
$y \leftarrow 4y$
$x, y \leftarrow 0$
A
y
1
$x \leftarrow 2x+1$
$y \leftarrow 2x+1$
1
$\alpha_3, \alpha_4 \leftarrow 2$
$\gamma_3, \gamma_4 \leftarrow 1$
$\alpha_3 x+\gamma_3$
$\alpha_4 x+\gamma_4$
$\alpha_4(\alpha_1\omega_x+\gamma_1)+\gamma_4$
$\alpha_1 x+\gamma_1$
$\alpha_2 x+\gamma_2$
$\alpha_2\omega_x+\gamma_2$
0
$\omega_x, \omega_y \leftarrow 0$
1
$\alpha_1, \alpha_2 \leftarrow 2$
$\gamma_1, \gamma_2 \leftarrow 1$
$2x$
$4x$
$4 \cdot (2 \cdot 0 + 1)$
ω_x
ω_y
0
$\alpha_1 \leftarrow 2$
$\alpha_2 \leftarrow 4$
$\gamma_1, \gamma_2 \leftarrow 0$
ω_y
$\alpha_1 x+\gamma_1$
$\alpha_2 y+\gamma_2$
$\alpha_2\omega_y+\gamma_2$

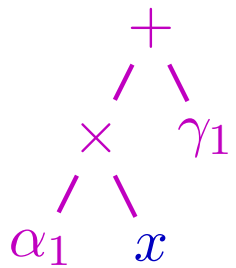
Unfolding


0
$x \leftarrow 2x$
$y \leftarrow 4y$
$x, y \leftarrow 0$
A
y
1
$x \leftarrow 2x+1$
$y \leftarrow 2x+1$
$\omega_x, \omega_y \leftarrow 0$
ω_x
ω_y
1
$\alpha_1, \alpha_2 \leftarrow 2$
$\gamma_1, \gamma_2 \leftarrow 1$
0
$\alpha_1 \leftarrow 2$
$\alpha_2 \leftarrow 4$
$\gamma_1, \gamma_2 \leftarrow 0$
$\alpha_1 x+\gamma_1$
$\alpha_2 x+\gamma_2$
$\alpha_2\omega_x+\gamma_2$
$\alpha_2\omega_y+\gamma_2$
1
$\alpha_3, \alpha_4 \leftarrow 2$
$\gamma_3, \gamma_4 \leftarrow 1$
0
$\alpha_3 \leftarrow 2$
$\alpha_4 \leftarrow 4$
$\gamma_3, \gamma_4 \leftarrow 0$
$\alpha_3 x+\gamma_3$
$\alpha_4 x+\gamma_4$
$\alpha_4(\alpha_1\omega_x+\gamma_1)+\gamma_4$
$\alpha_4(\alpha_2\omega_x+\gamma_2)+\gamma_4$

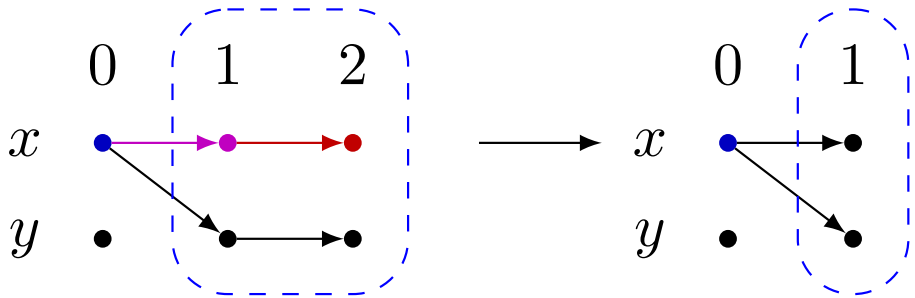
Removal of Copyless Layers

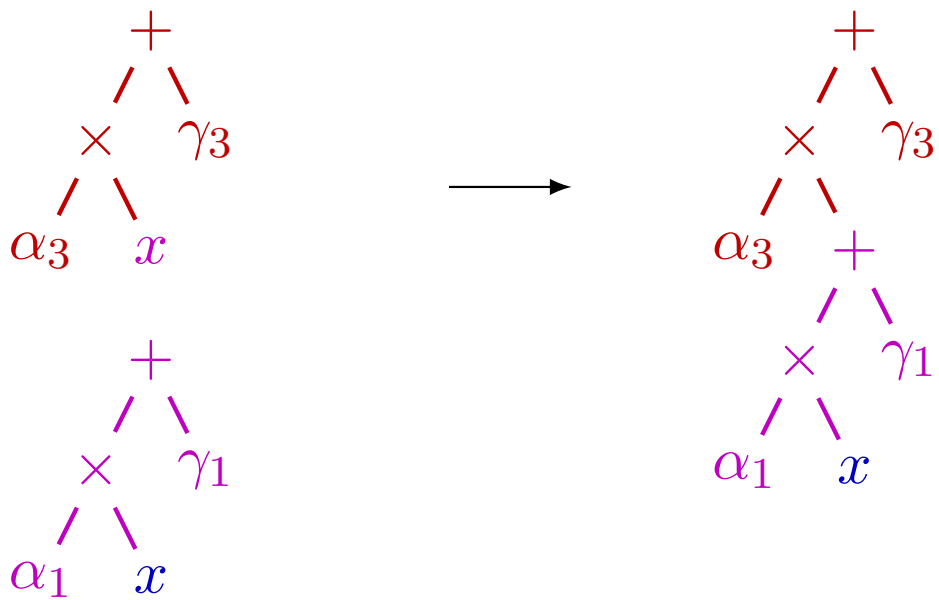




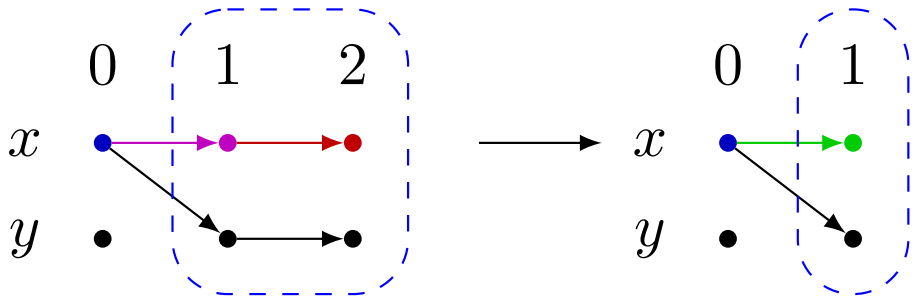


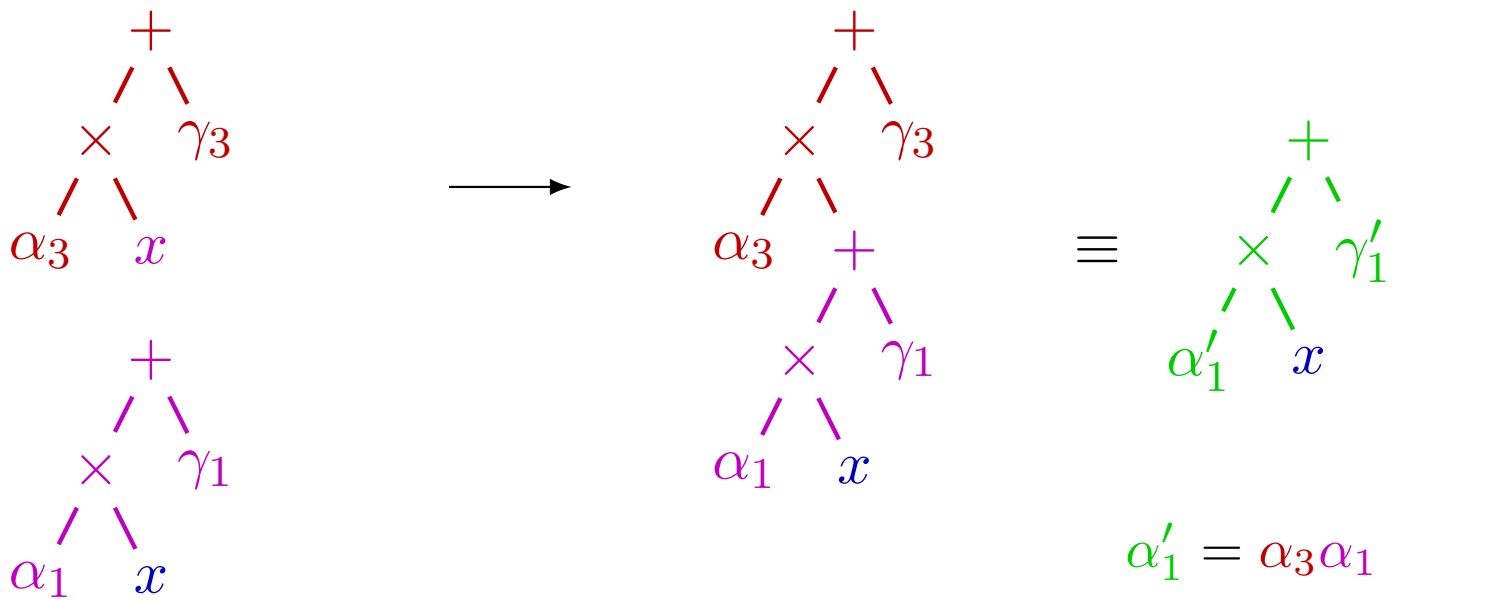
Removal of Copyless Layers





Removal of Copyless Layers

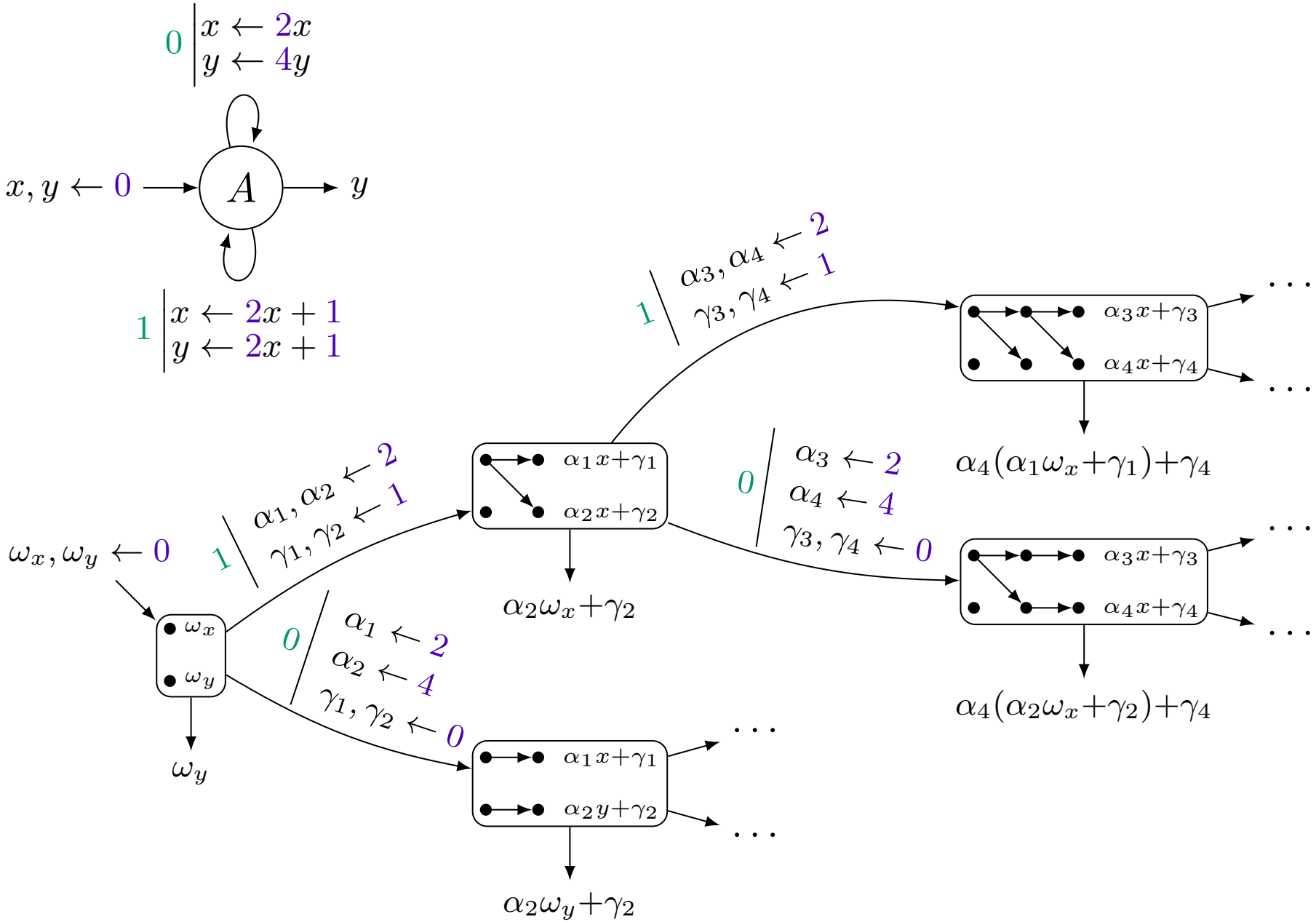




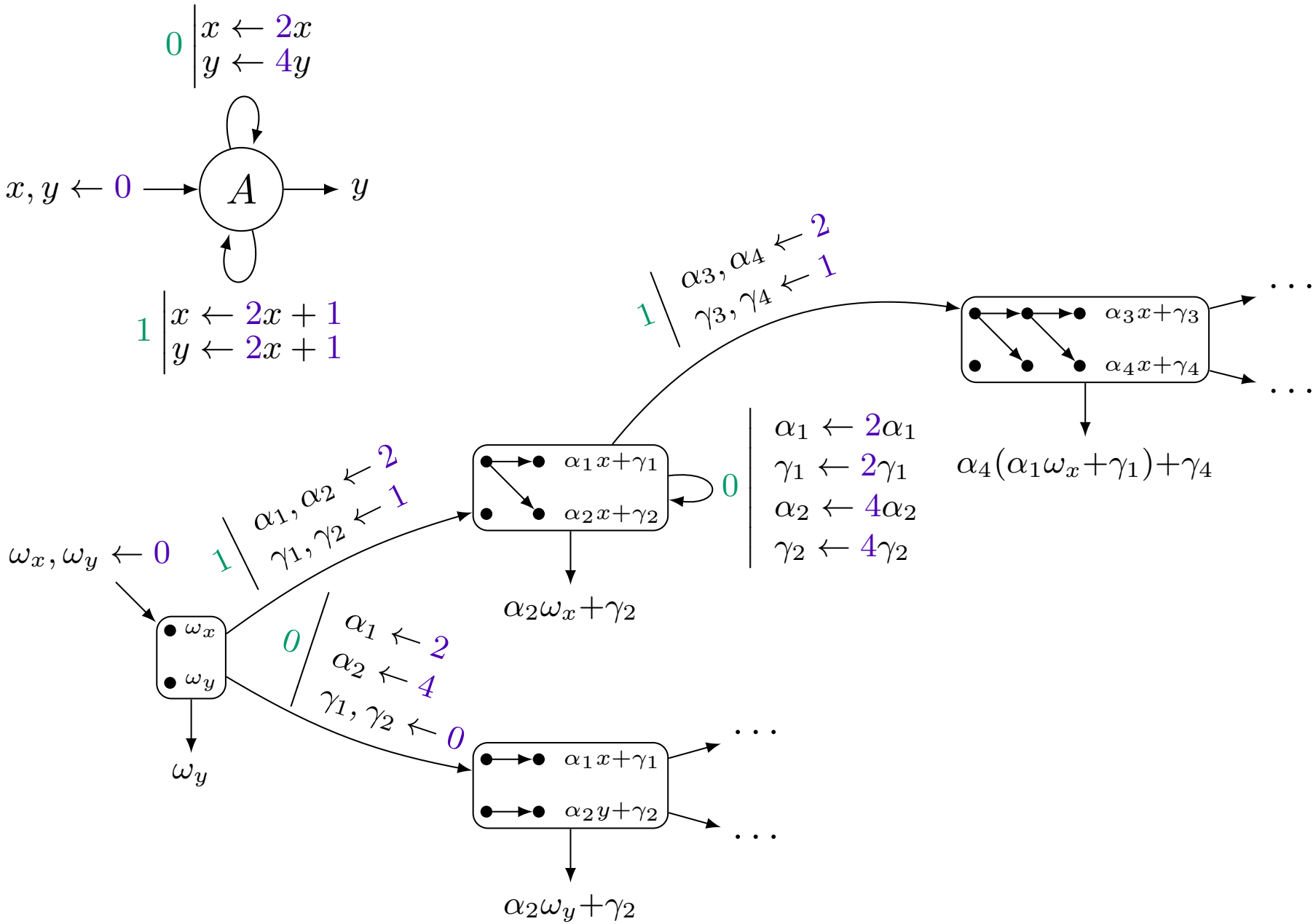
$$\alpha_1' = \alpha_3 \alpha_1$$

$$\gamma_1' = \alpha_3 \gamma_1 + \gamma_3$$

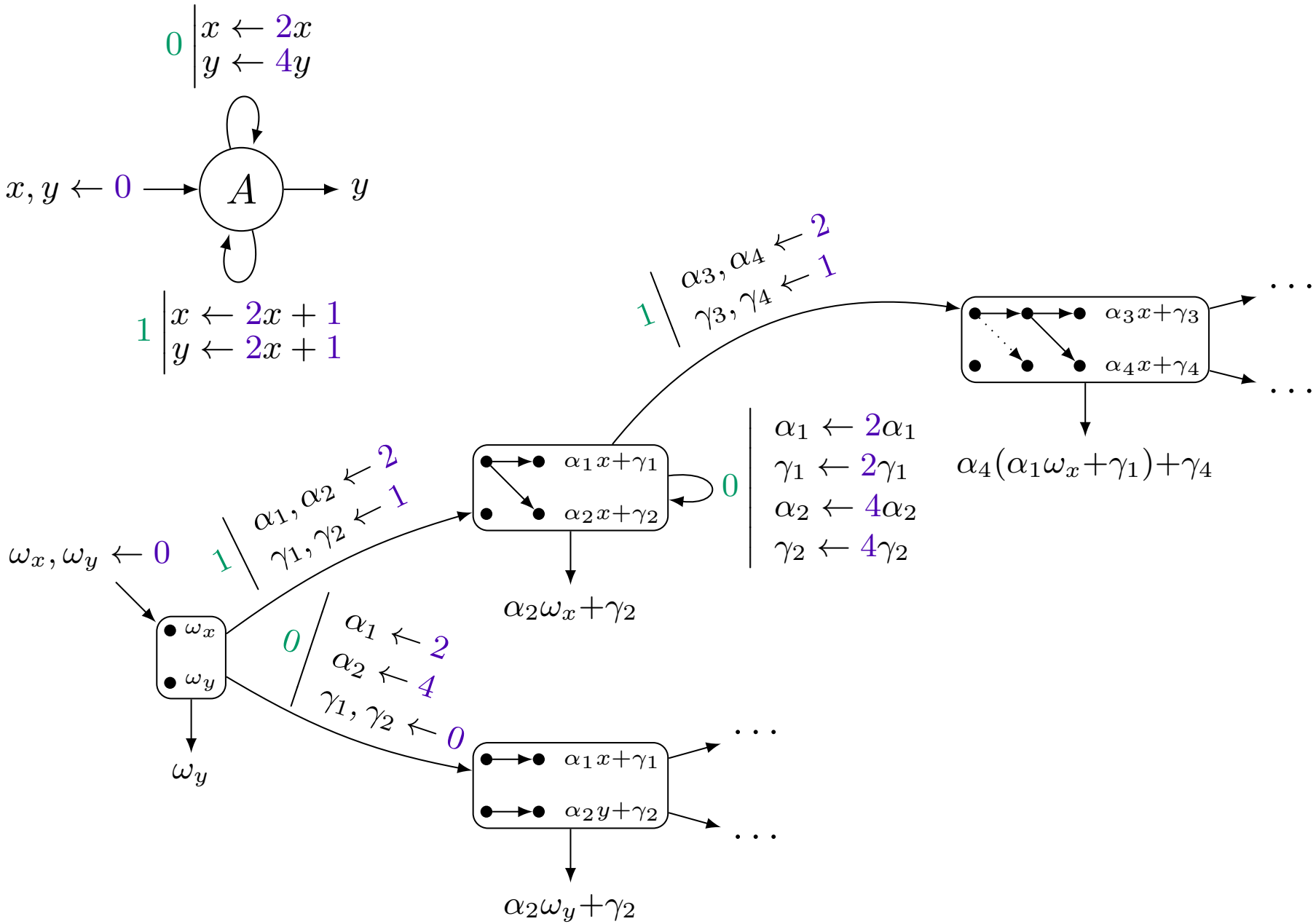
Reduced Unfolding



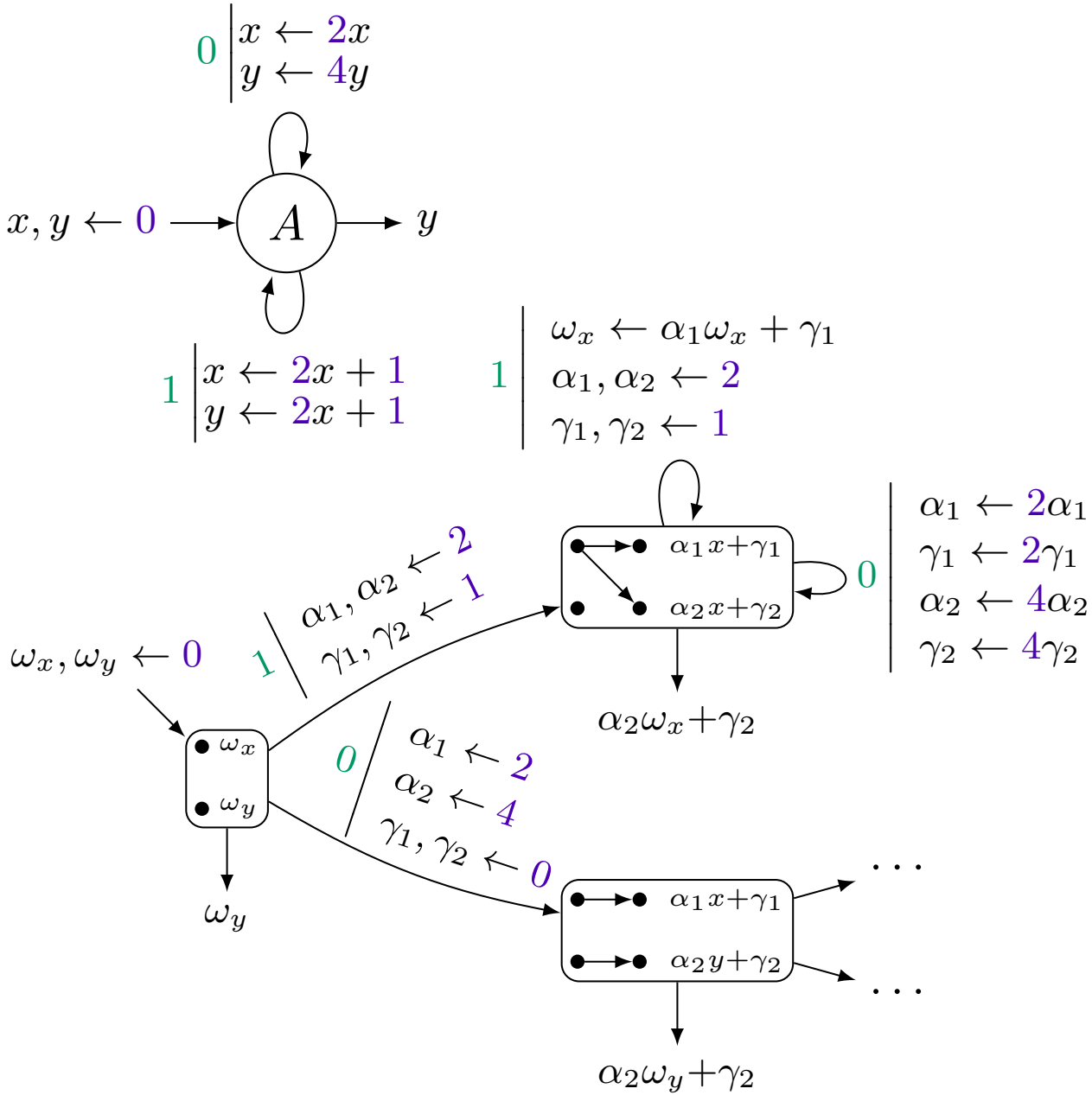
Reduced Unfolding



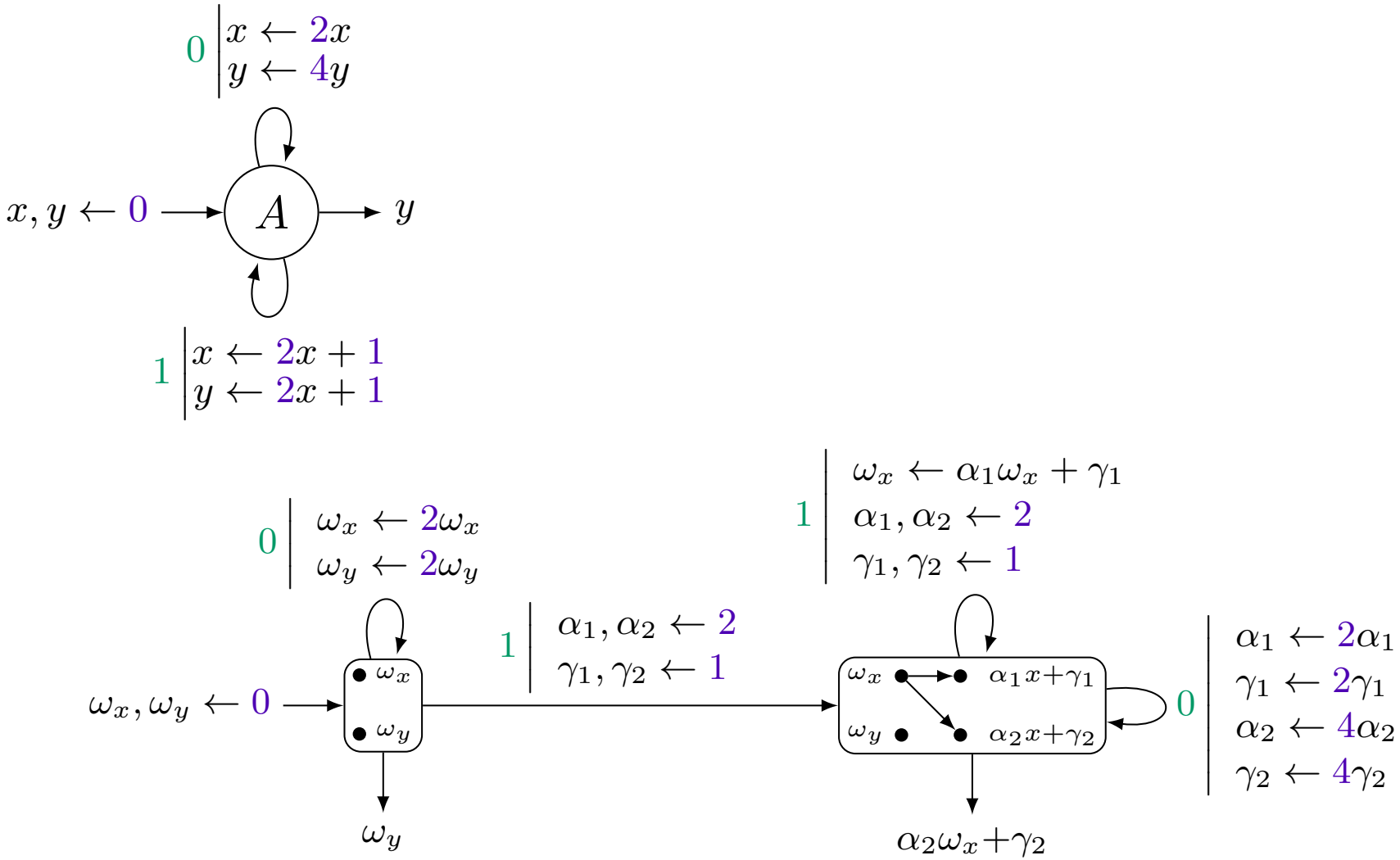
Reduced Unfolding



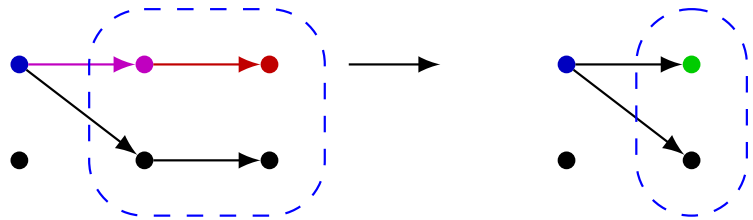
Reduced Unfolding



Reduced Unfolding



Problem 1: Removal of layers make copies



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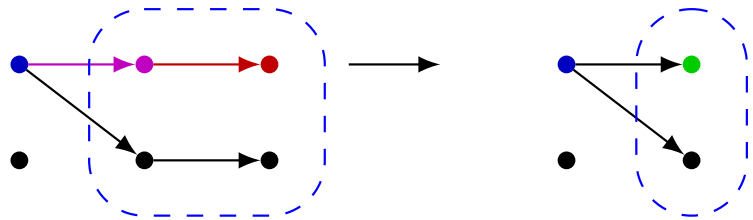

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... but each coefficient is copied only a bounded number of times (proof not too difficult)

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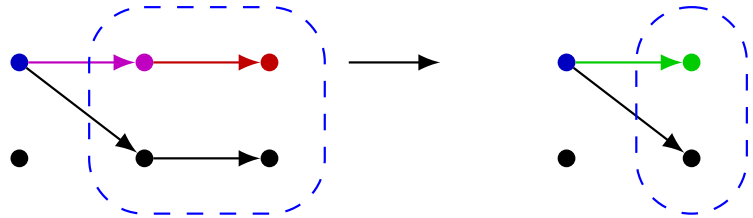

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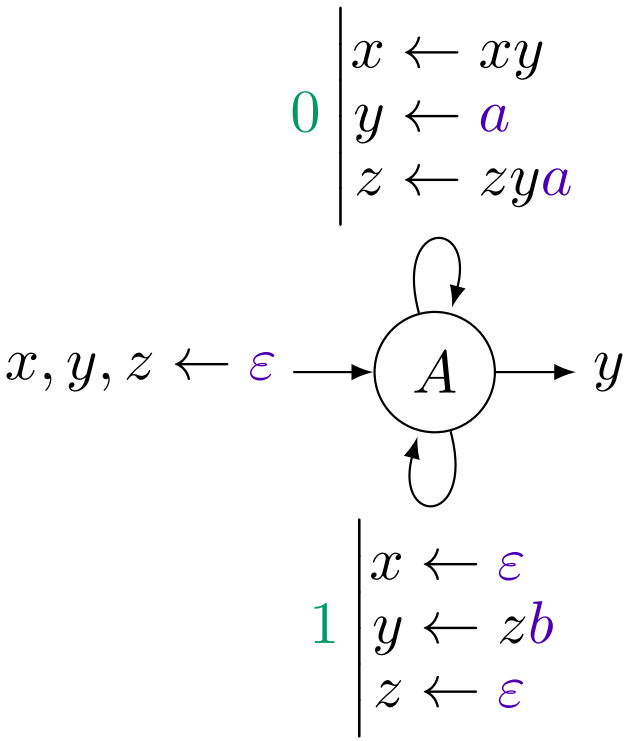
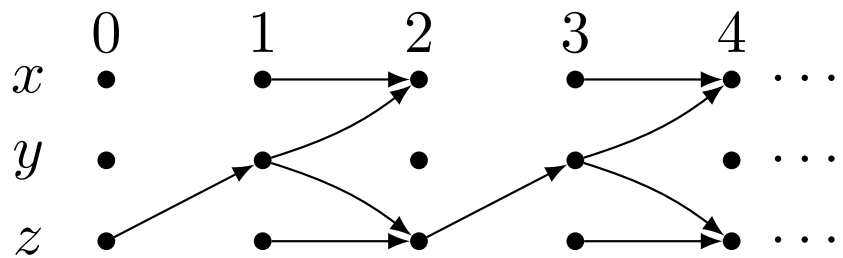
Unambiguous Copyless NCRA $\longrightarrow$ $\diamond$-less CRA $\longrightarrow$ Bounded-copy CRA $\longrightarrow$ Copyless CRA

Determinisation — Unfolding + Reduction — Initiate enough copies

Bounded copyness is not trivial...

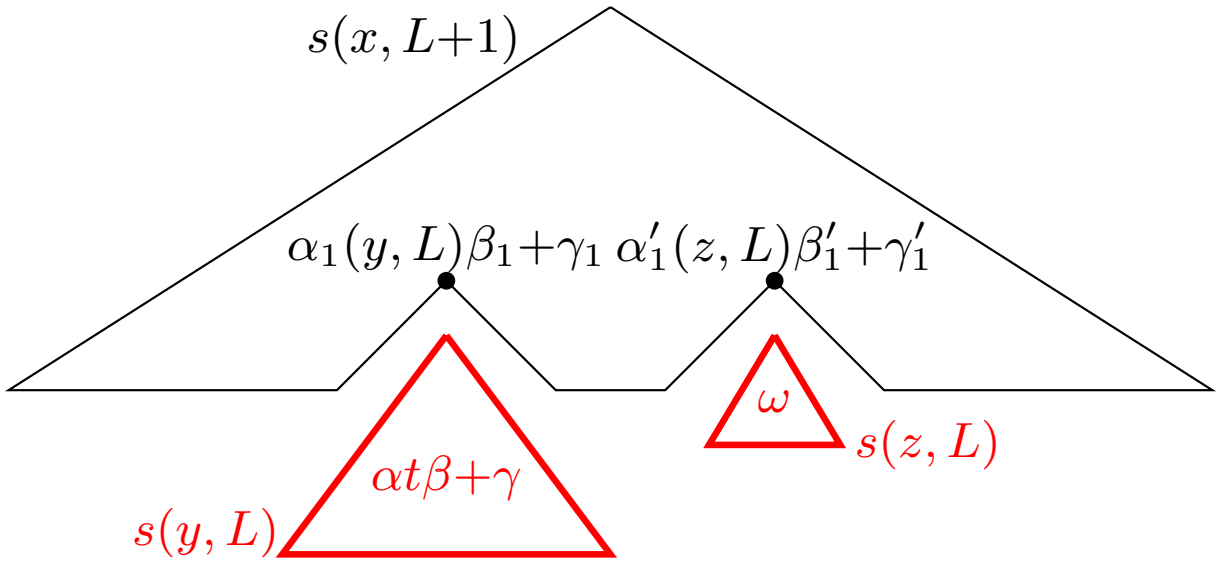


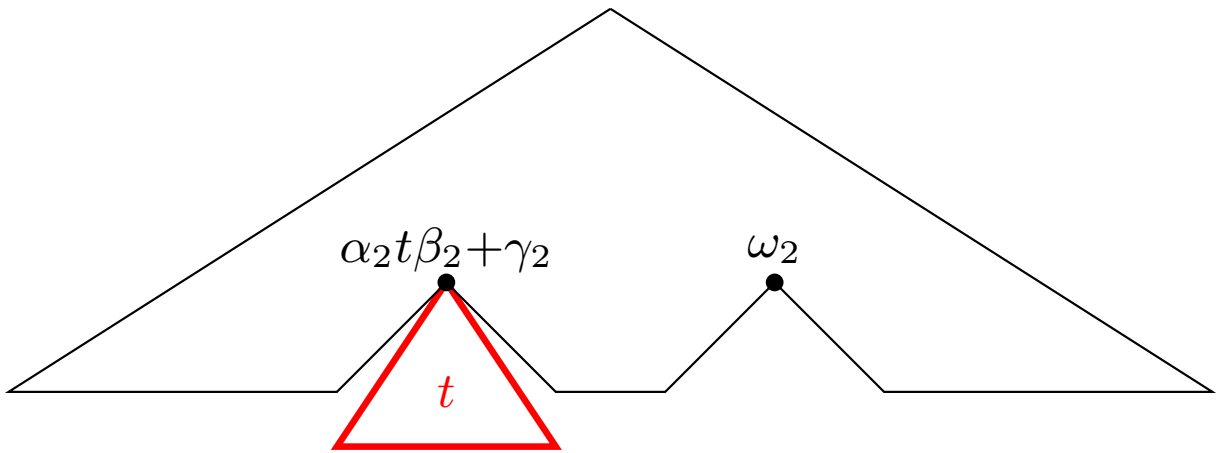
On input word $1010\cdots$:



Problem 2: Abstracted terms with multiple registers

More intricate reshaping into normal forms... more copies





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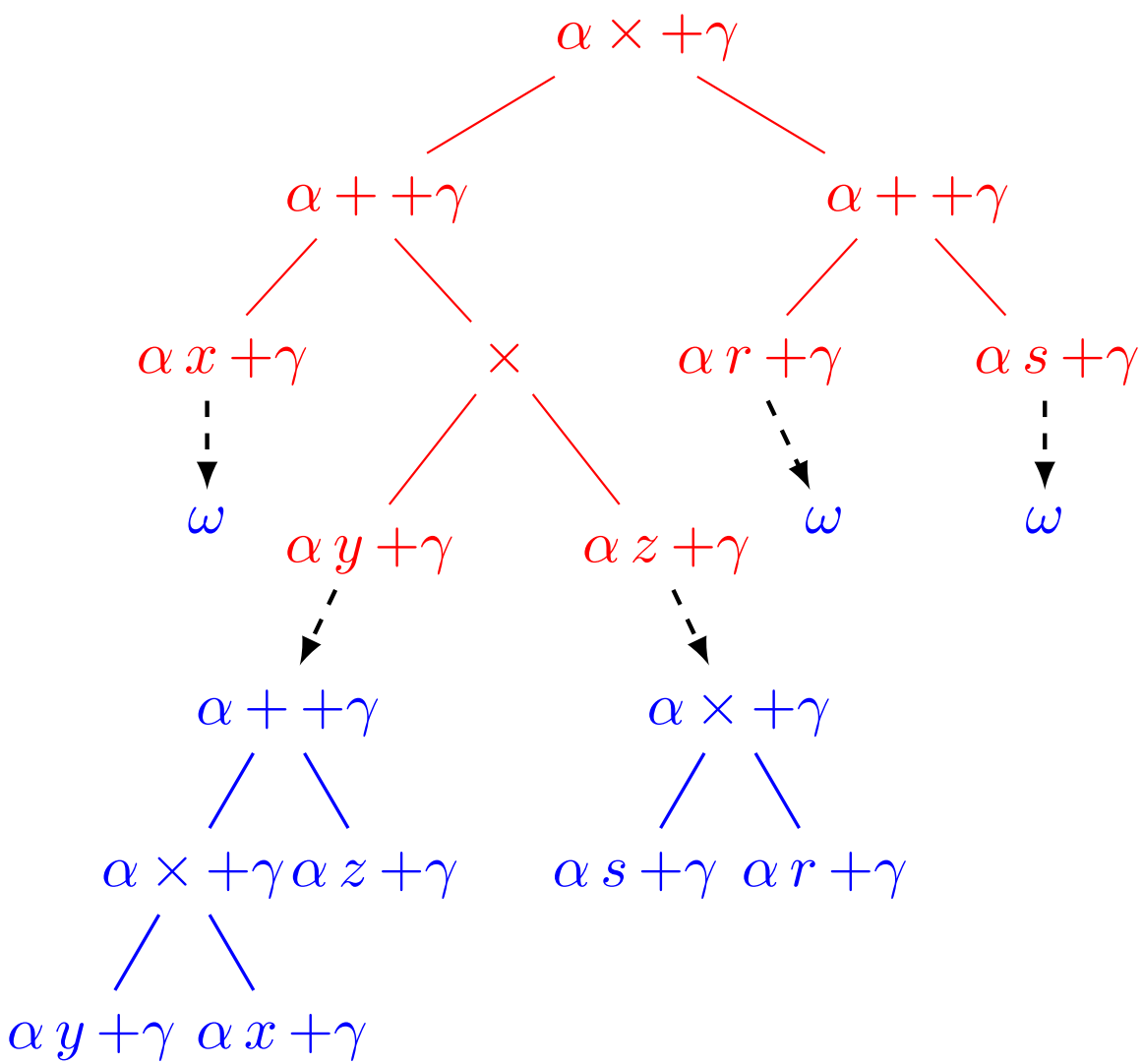
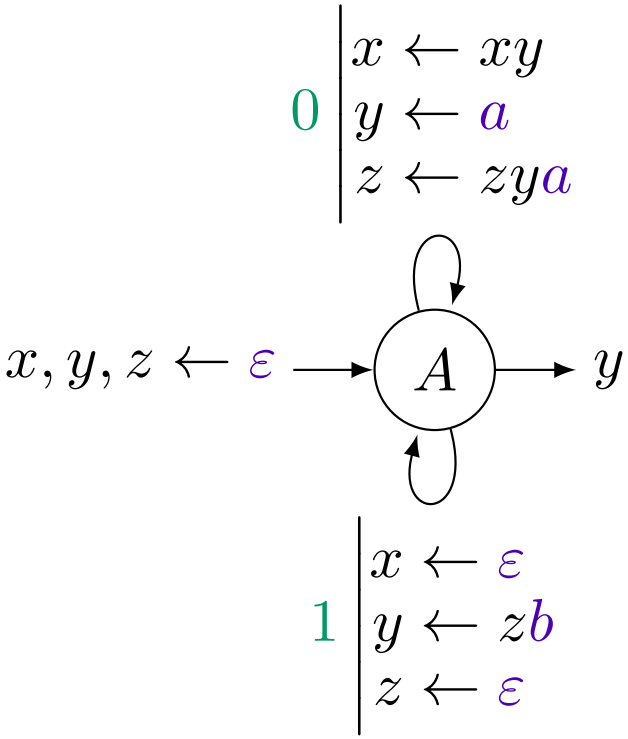
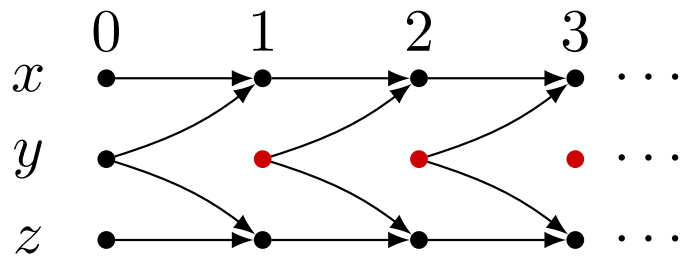


Problem 3: Removing copyless layers is not sufficient



On input word $000\cdots$:



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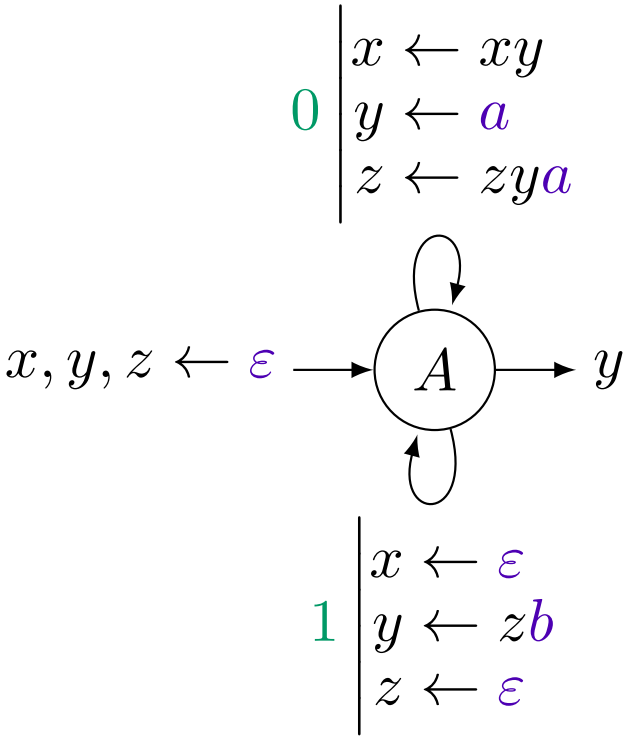

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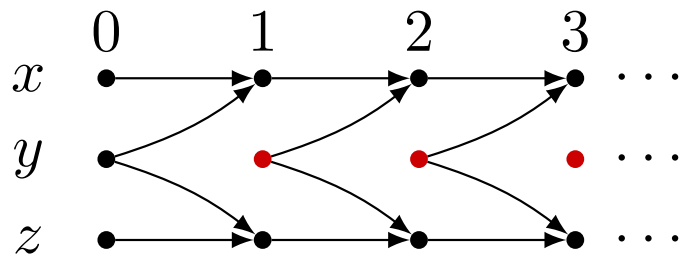

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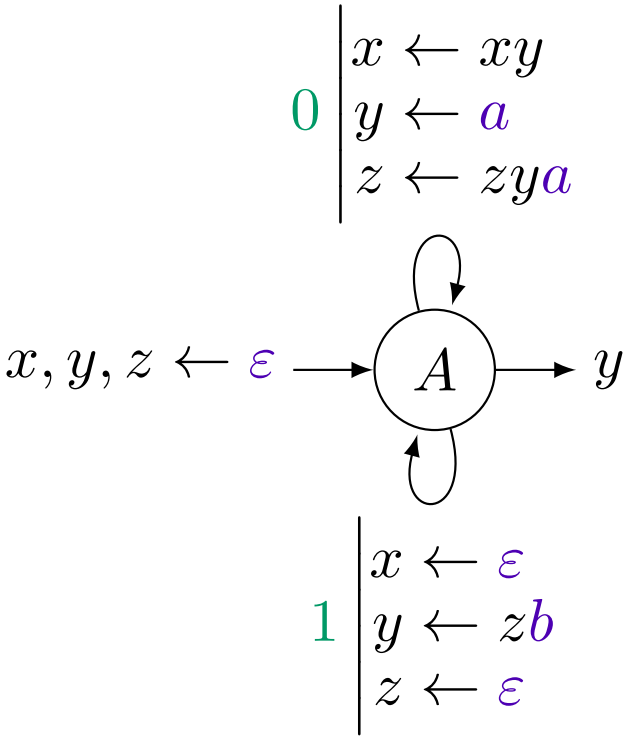

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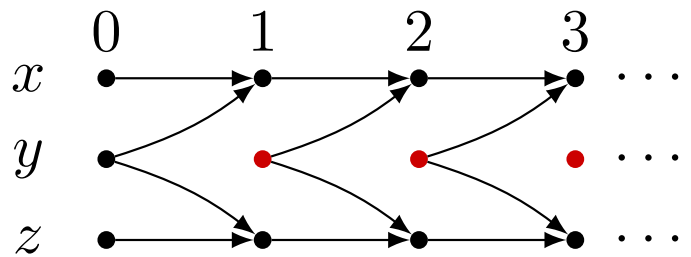

- The new copy of y teaches us that the previous copies of y cannot be copied many times anymore, because of the $\diamond$-less hypothesis...
- We find a notion of *primarily-copyless layers*, that we remove, then proving (much more delicate) that this again creates only a bounded number of copies of coefficients

Summary

Theorem: Results

copyless CRA = Unambiguous copyless NCRA

Summary

Theorem: More results

copyless CRA $=$ Unambiguous copyless NCRA
$=$ finitely-amb. NCRA $\subsetneq$ linearly-amb. NCRA

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Perspectives:

- We build a bounded-copy CRA with an exponential number of copies (and an exponential number of states): this leads to a copyless CRA with a doubly-exponential number of registers... Improvement/optimal?

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Thank you!