Logics for Weighted Automata

Benjamin Monmege

Aix-Marseille Université, Laboratoire d'Informatique et Systèmes

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Software Verification

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Critical Software

- communication systems
- e-commerce
- health databases
- energy production

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Property to be verified

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TO BE VERIFIED

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Property to be verified

Is the property verified or not by the software?

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- May an *error* state be reached?
- Is there a book written by X, rented by Y?
- Does this leader election protocol permit to elect the leader?

erified

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- May an *error* state be reached?
- Is there a book written by X, rented by Y?
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From **Boolean** to **Quantitative** Verification

- What is the probability for an *error* state to be reached?
- How many books, written by X, have been rented by Y?
- What is the maximal delay ensuring that this leader election protocol permits the election?

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TO BE VERIFIED

Formal Verification

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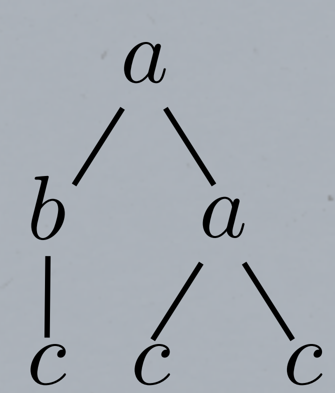
Formal Verification

Property to be verified

Is the property verified or not by the model?

Formal Model

$ababcaabb$



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TO BE VERIFIED

Formal Verification

Property to be verified
Formal Specification

$$(a+b)^\star c(ac)^+$$

$$\forall x \forall y \,(x < y \Rightarrow \exists z \,(x < z < y))$$

$$\mathsf{F}\,\mathsf{G}\,(p\,\mathsf{U}\,q)$$

Is the property verified or not by the model?

Formal Model

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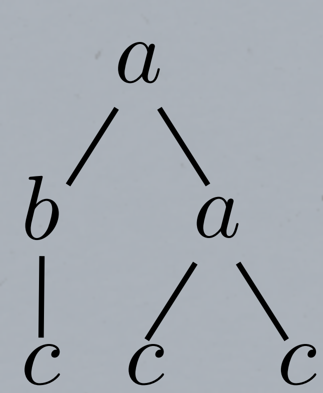

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TO BE VERIFIED

Qualitative/Quantitative

- Qualitative, Boolean: [Büchi 60, Elgot 61, Trakhtenbrot 61]

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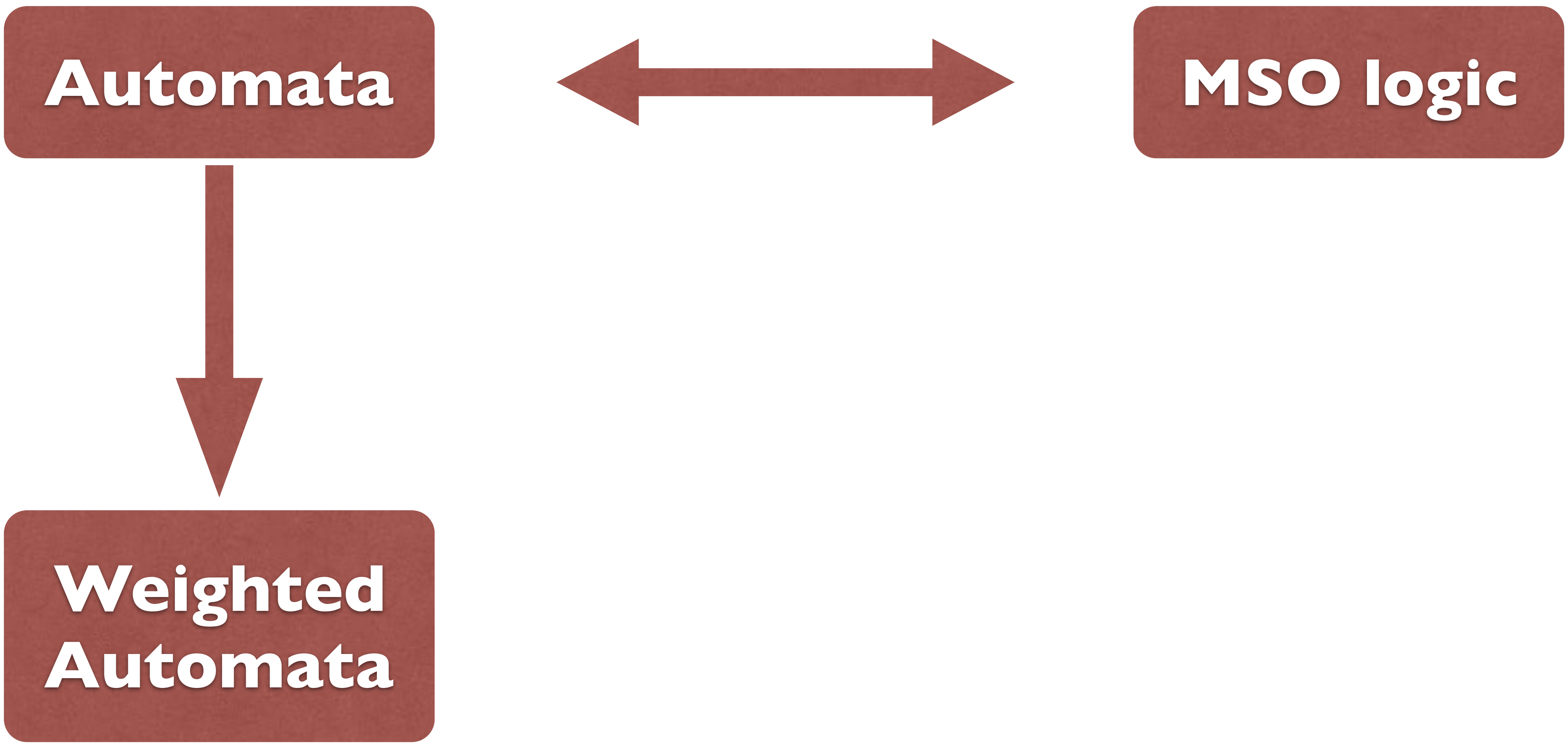

- Quantitative, weights

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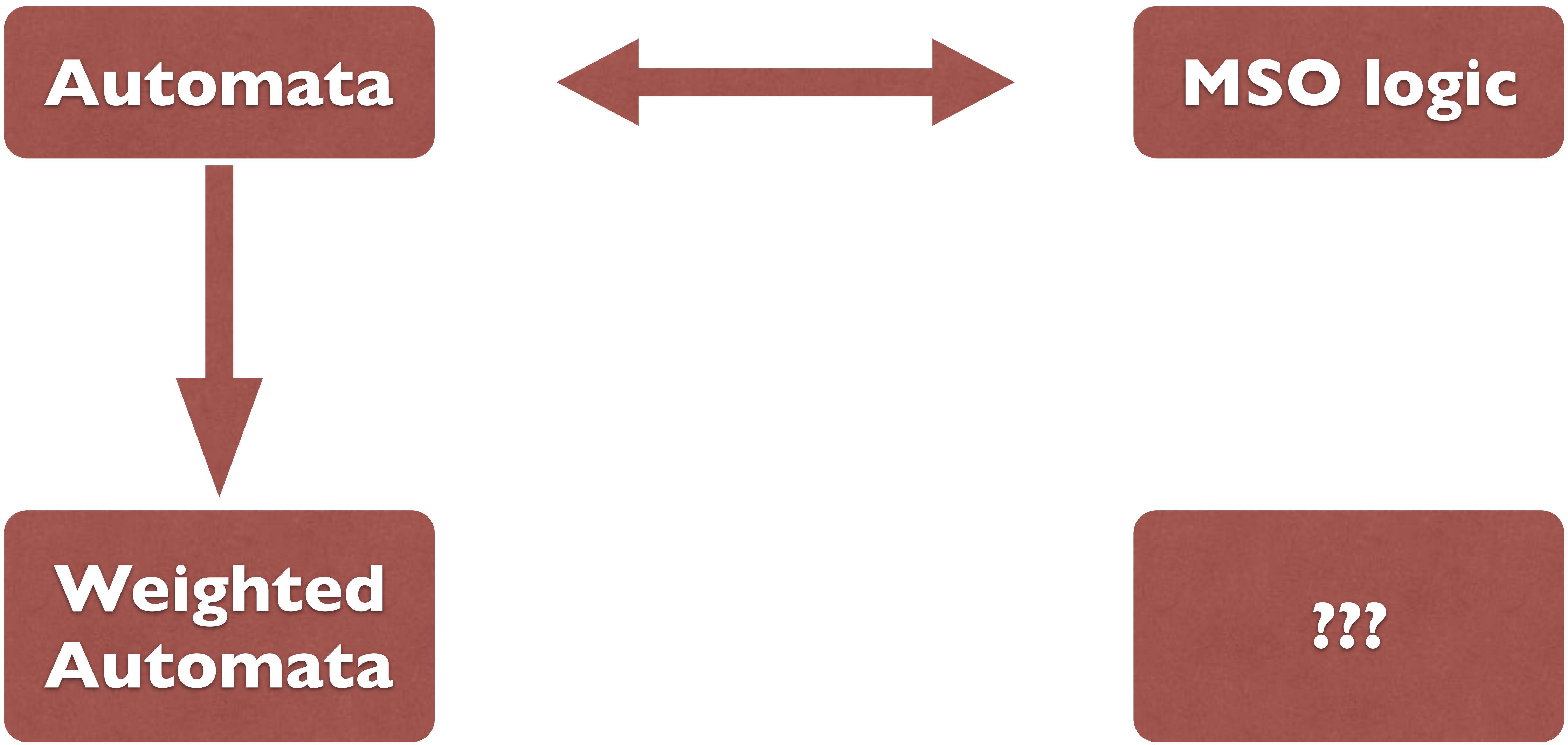

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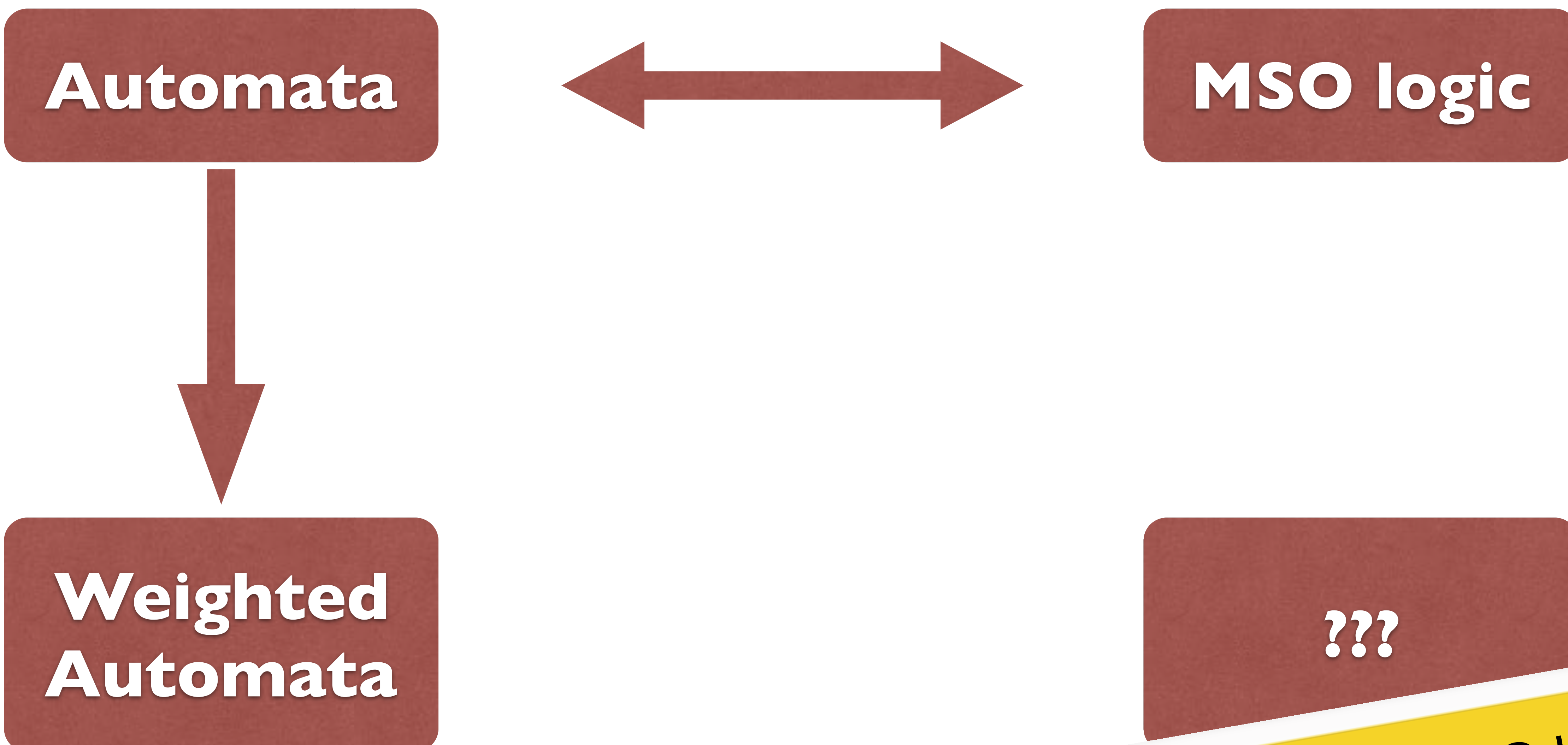

- Quantitative, weights

Find suitable weighted MSO logic

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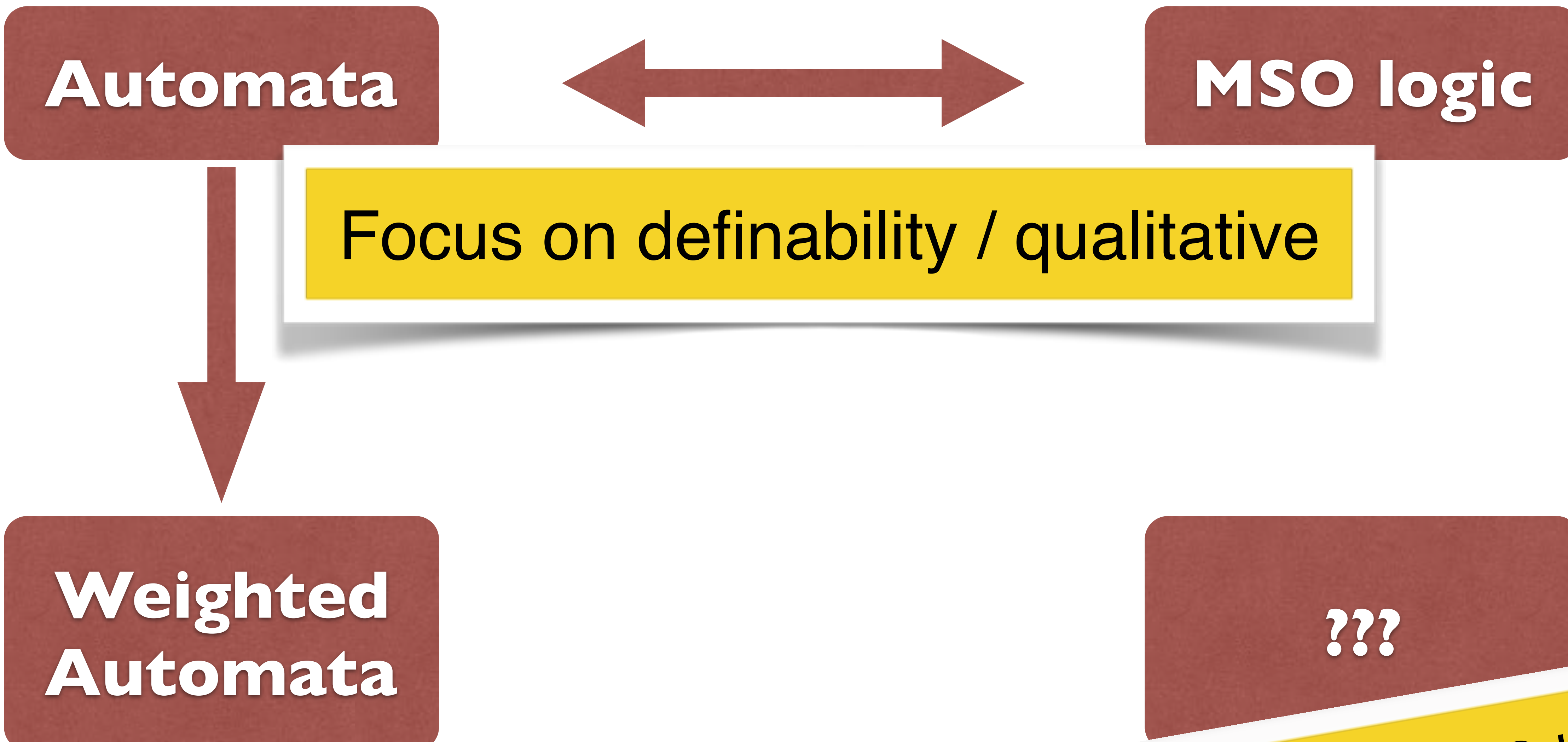

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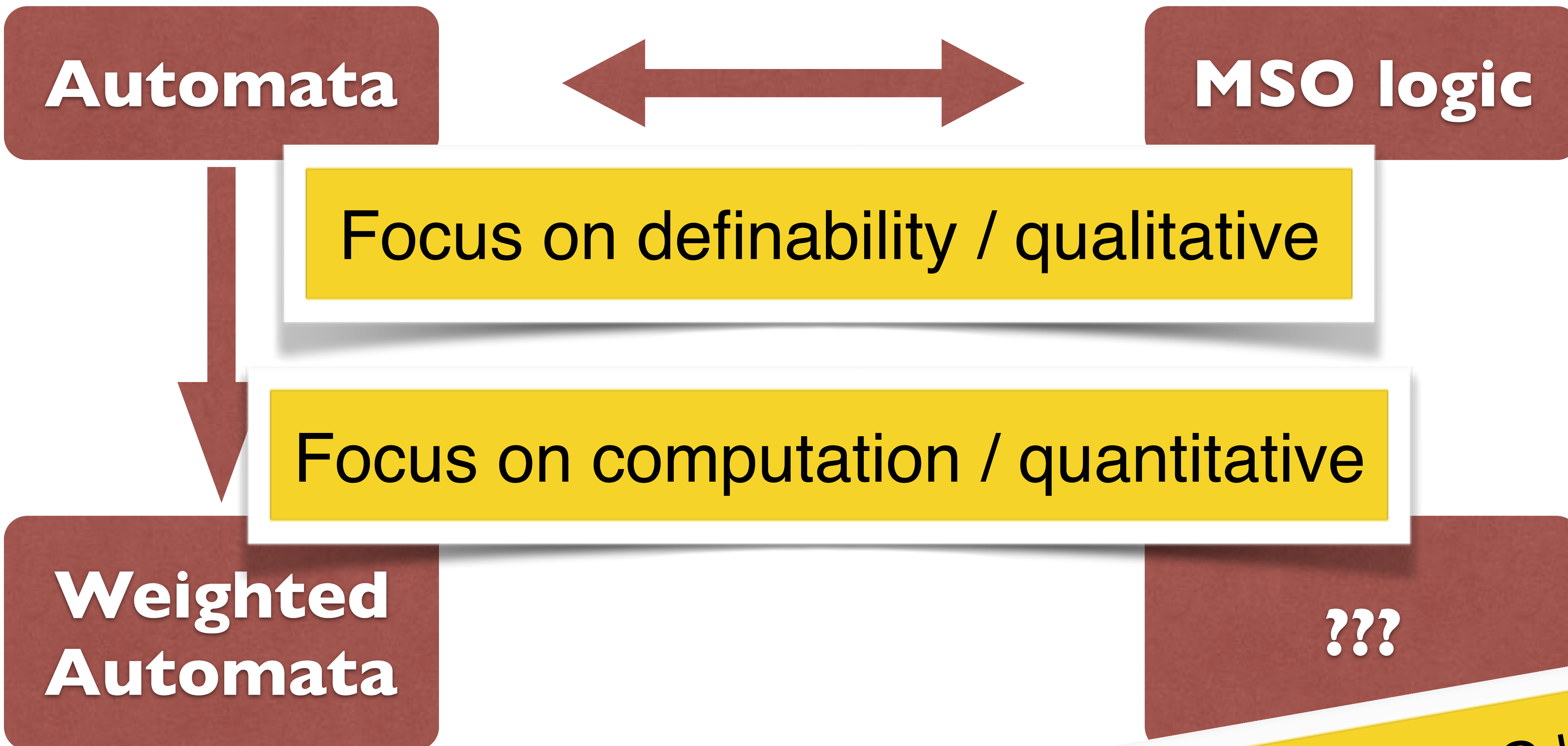

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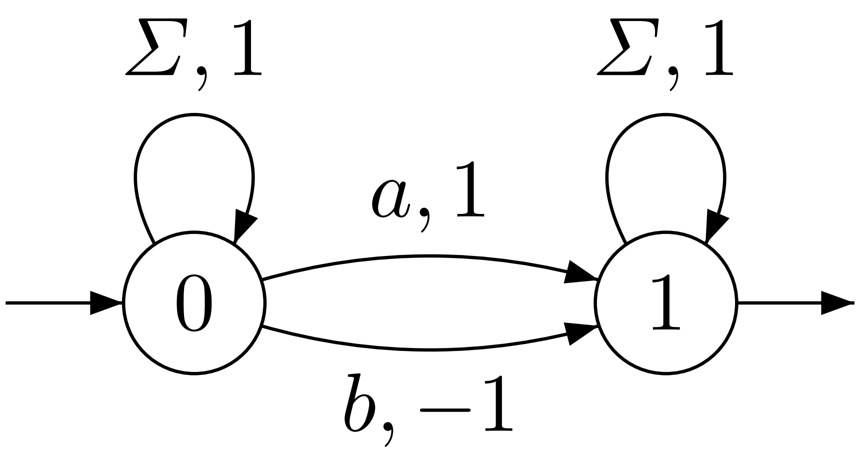
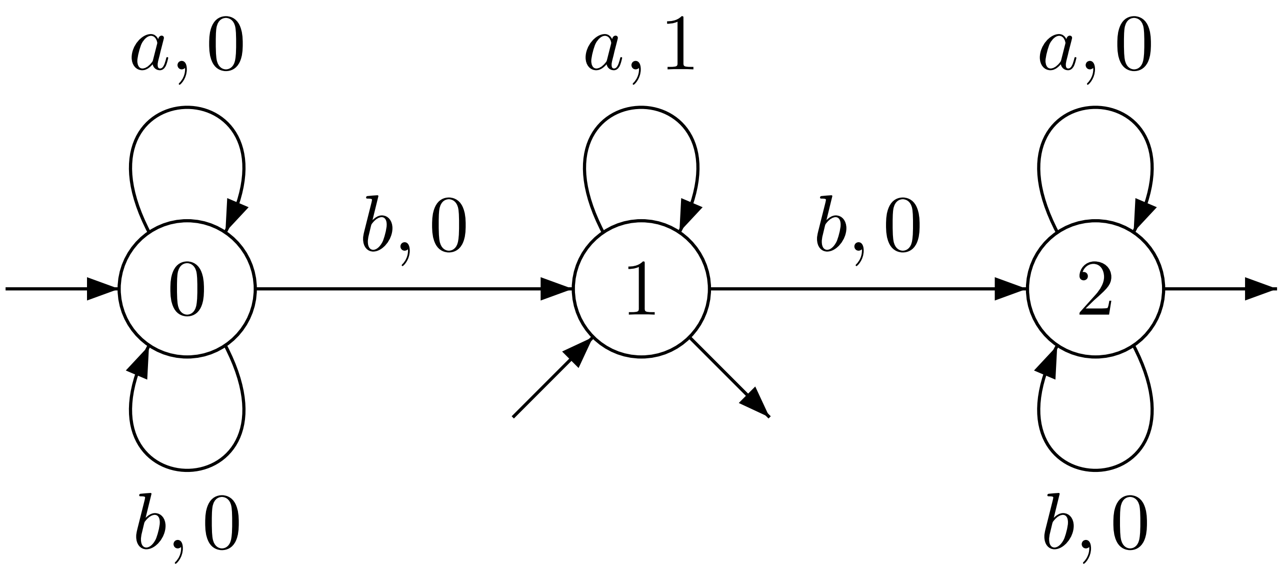
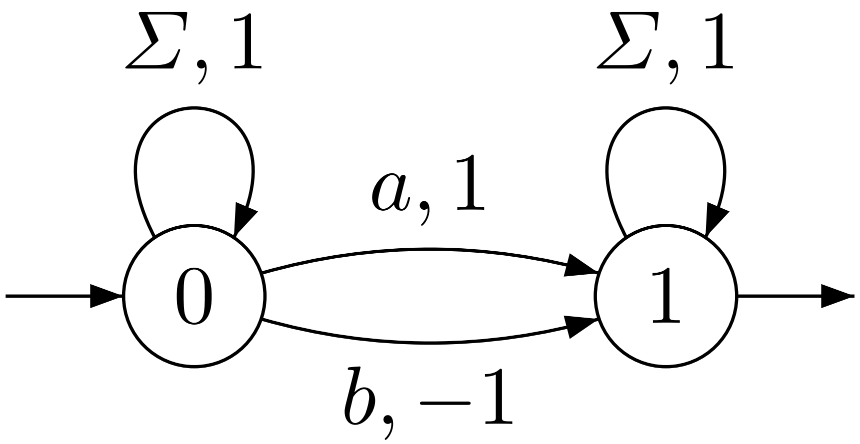
Weighted Automata

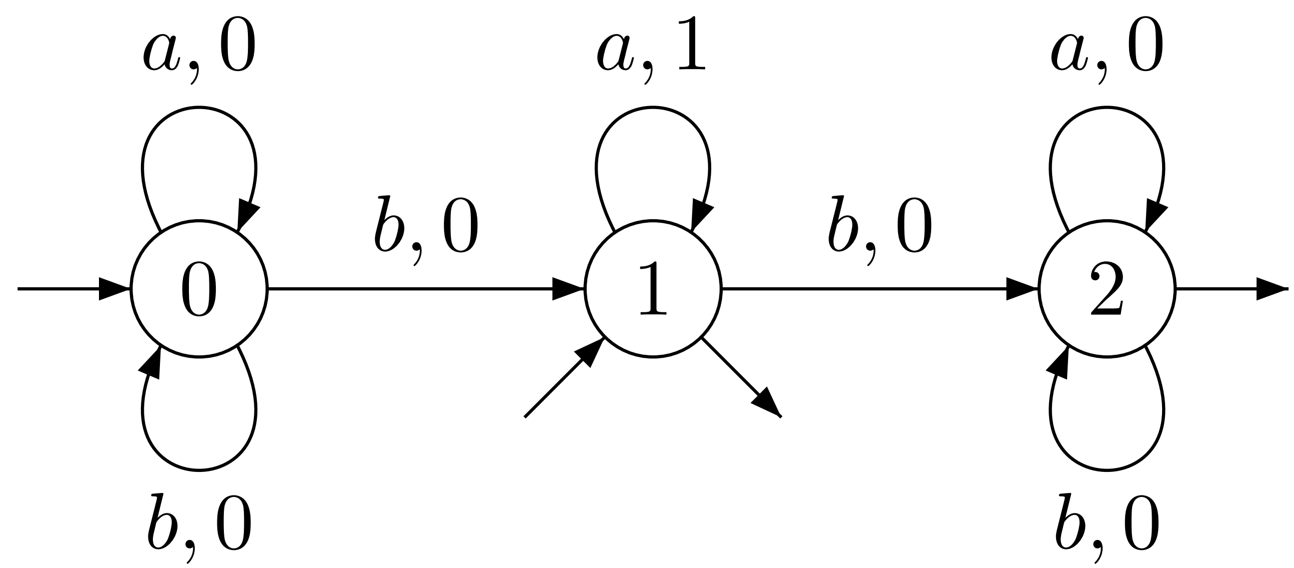




[Schützenberger 61]

Weighted Automata

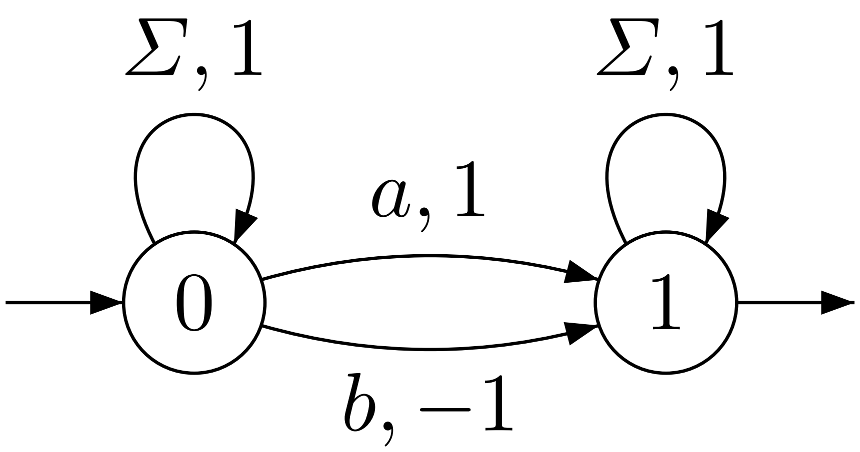


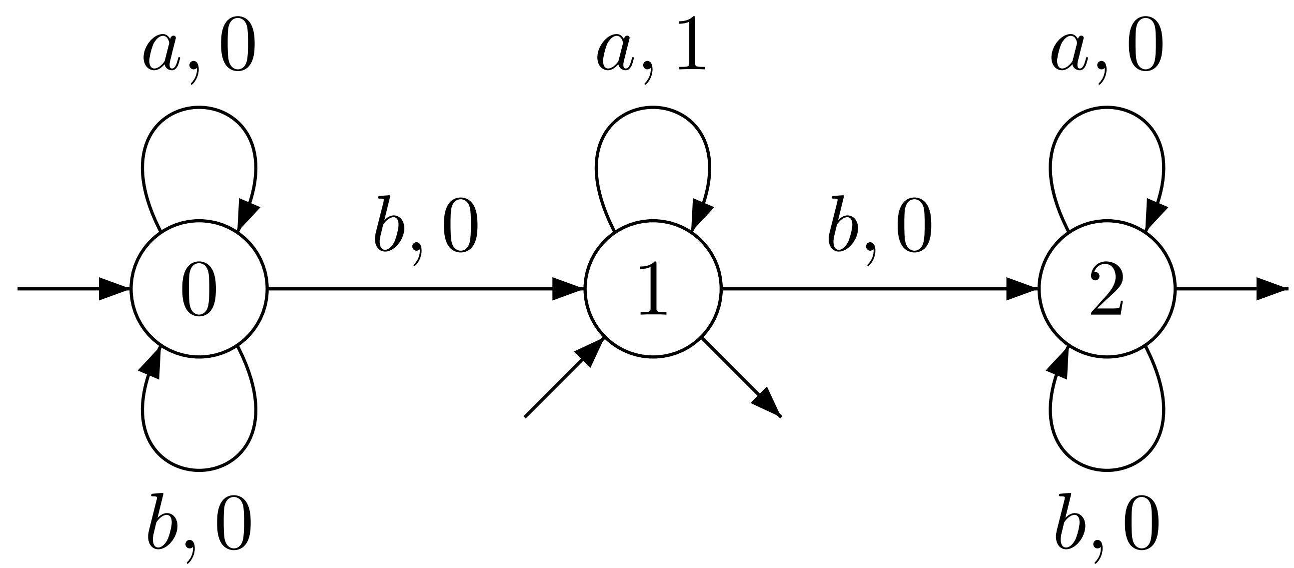


$$(\mathbf{Z}, +, \times, 0, 1)$$

[Schützenberger 61]

Weighted Automata





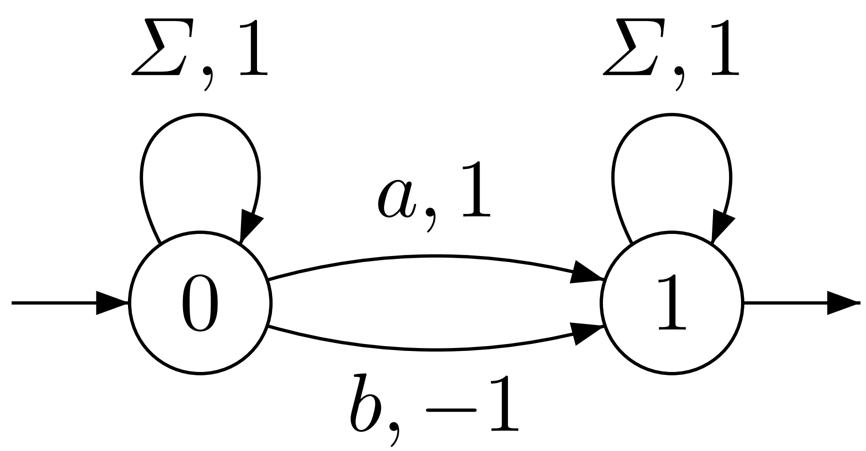
$$(\mathbf{Z}, +, \times, 0, 1)$$

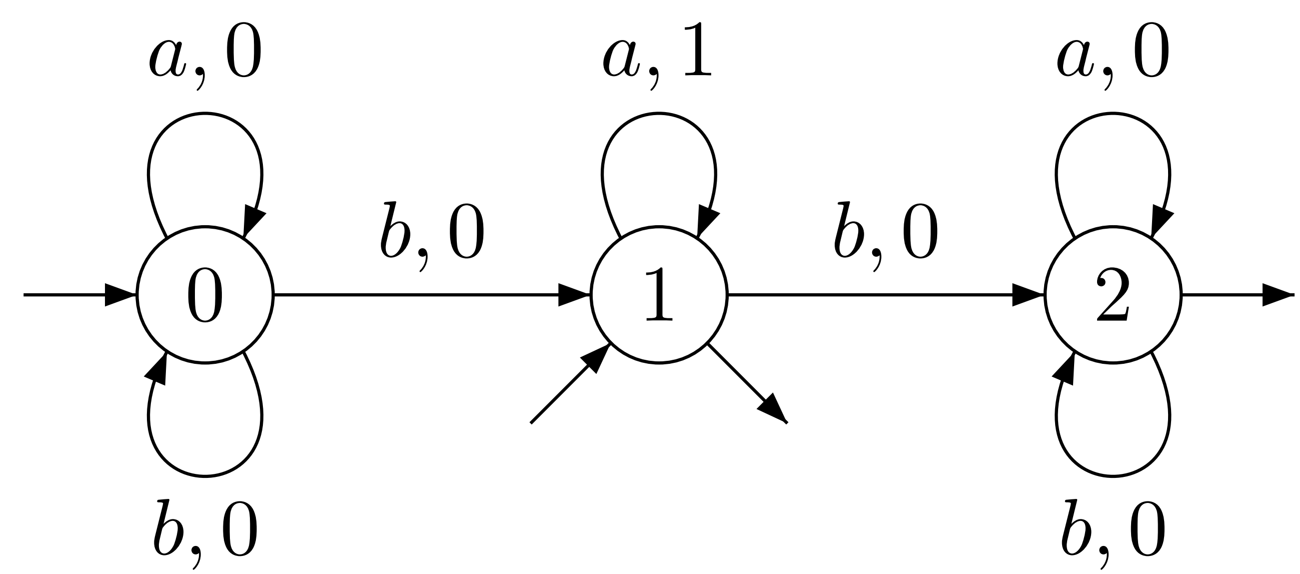
$$0 \xrightarrow{a,1} 1 \xrightarrow{b,1} 1 \xrightarrow{a,1} 1$$

1

[Schützenberger 61]

Weighted Automata





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$0 \xrightarrow{a,1} 1 \xrightarrow{b,1} 1 \xrightarrow{a,1} 1$ — 1

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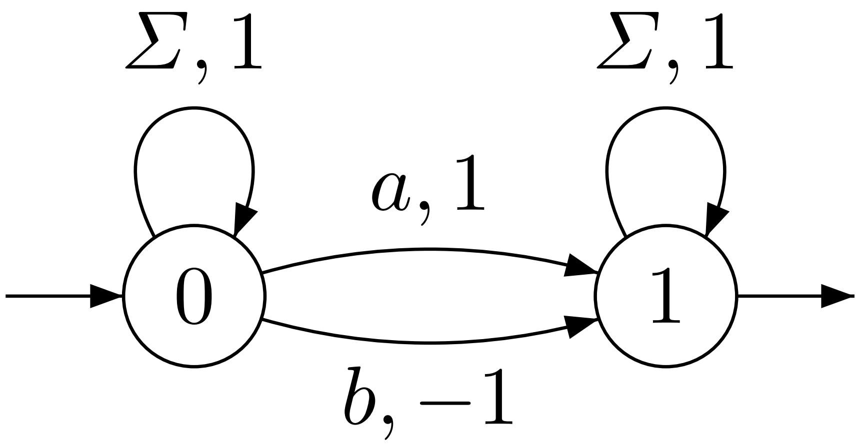

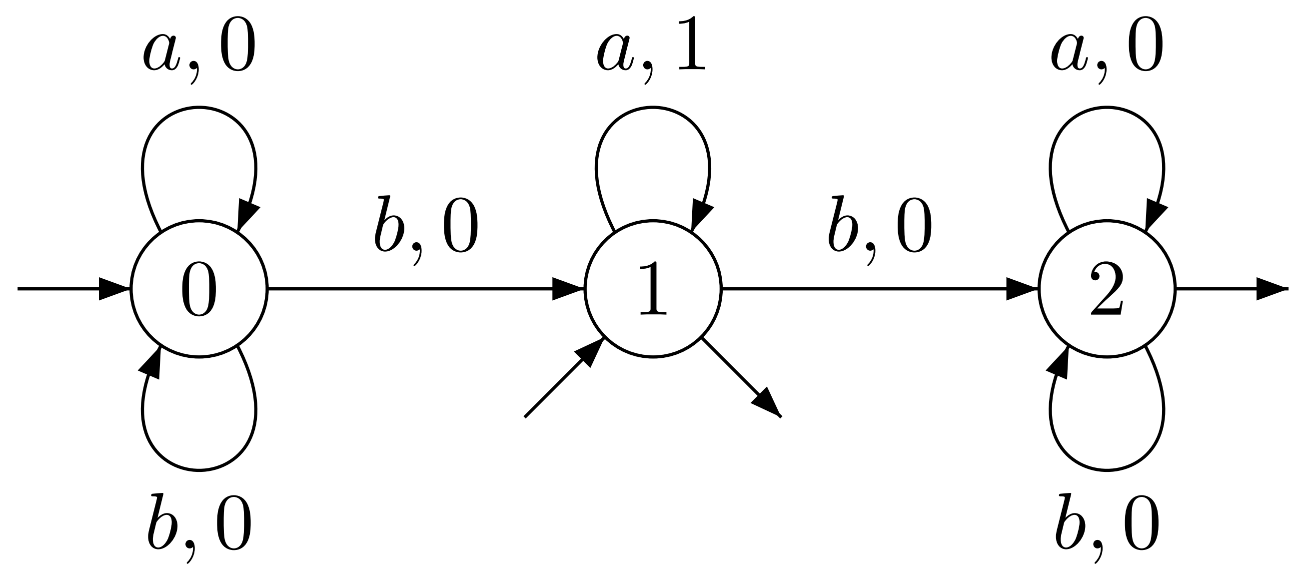

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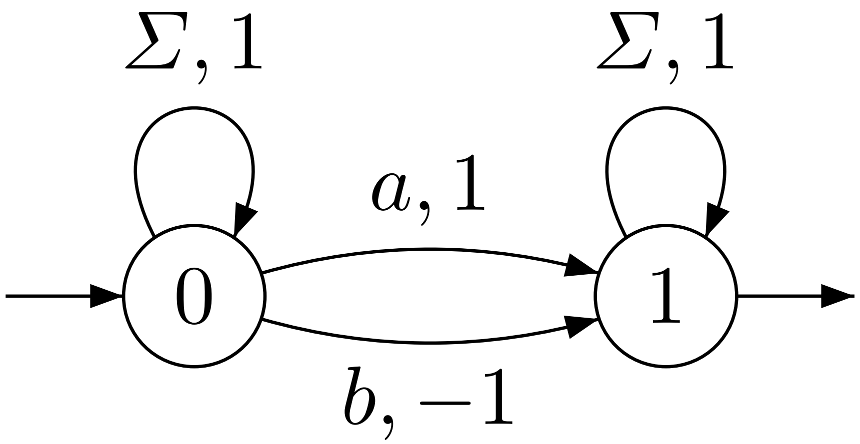

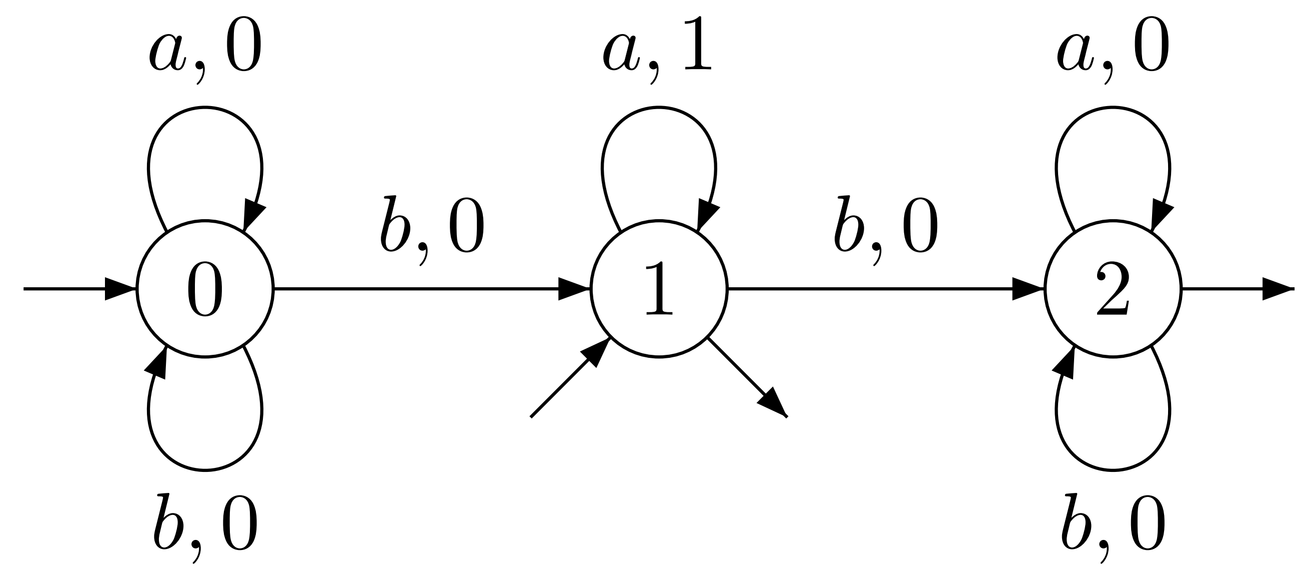

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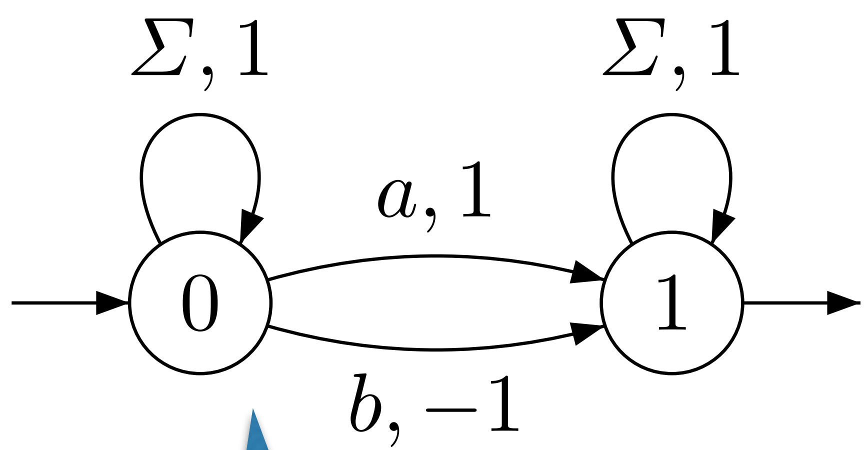
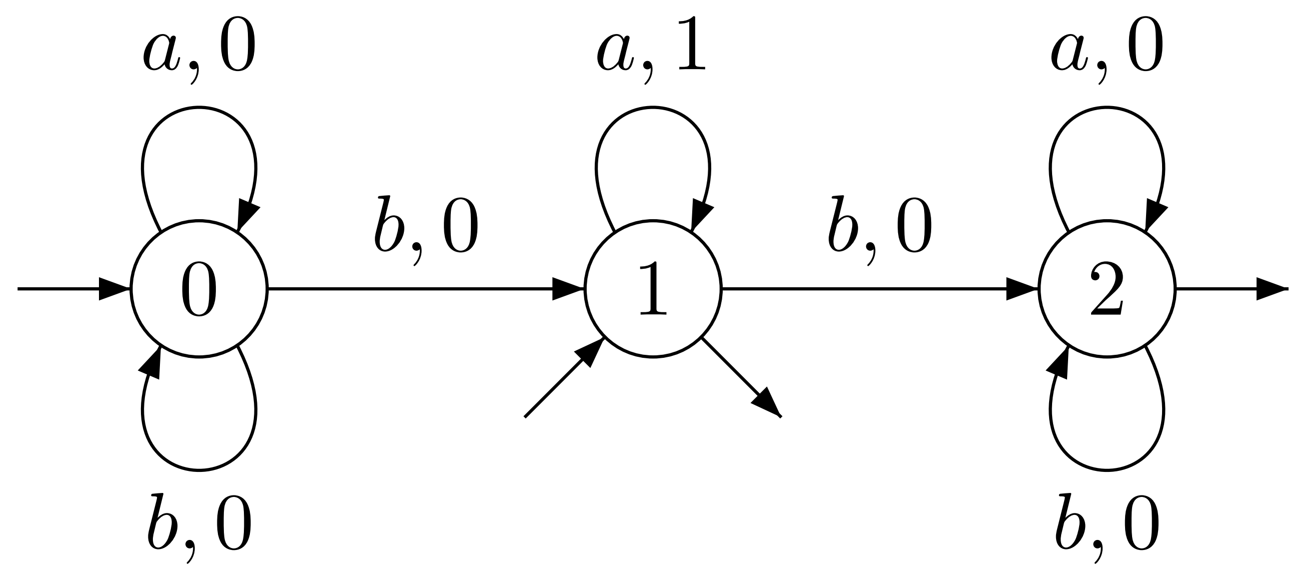
Semantics of *aba*: 1+(-1)+1 = 1

[Schützenberger 61]

Weighted Automata





$\#_a(w) - \#_b(w)$

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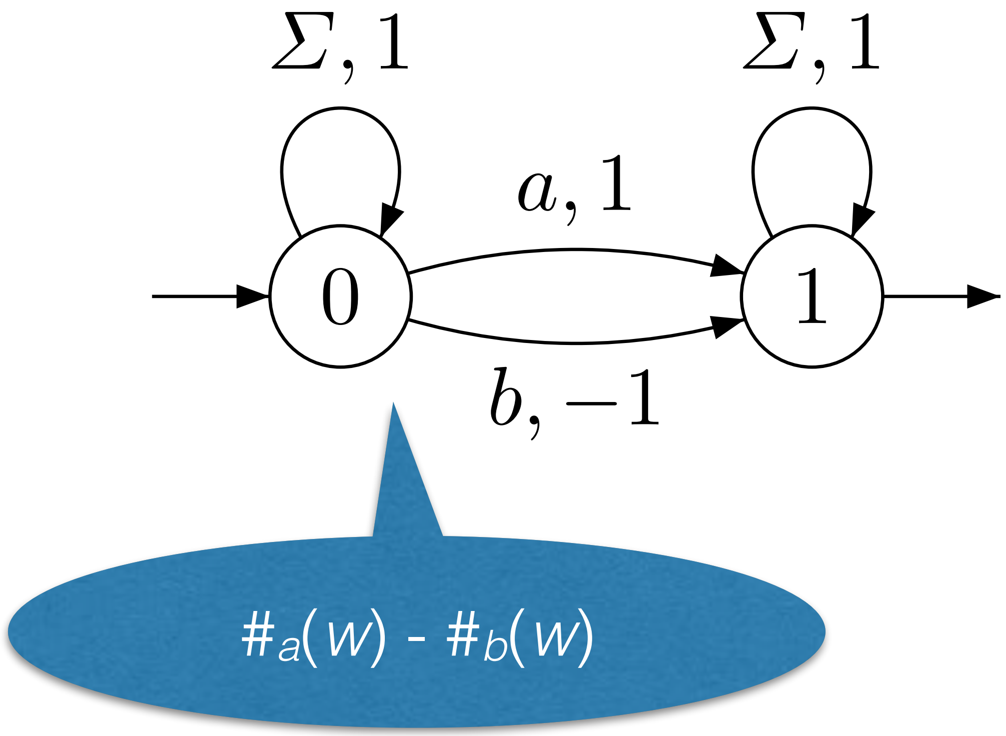

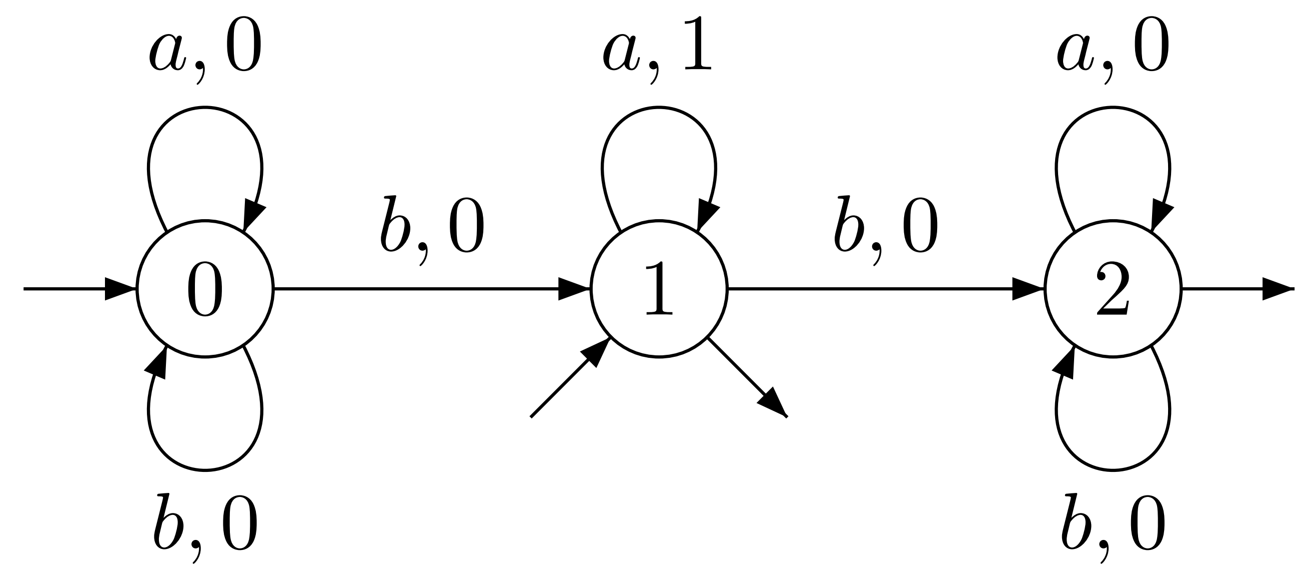

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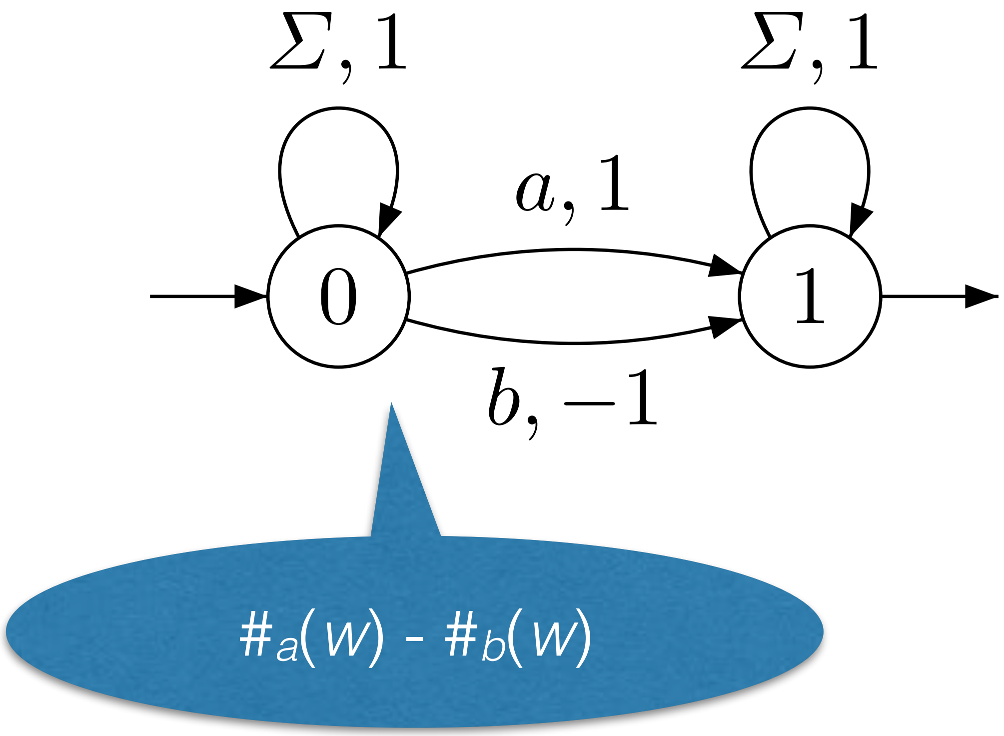

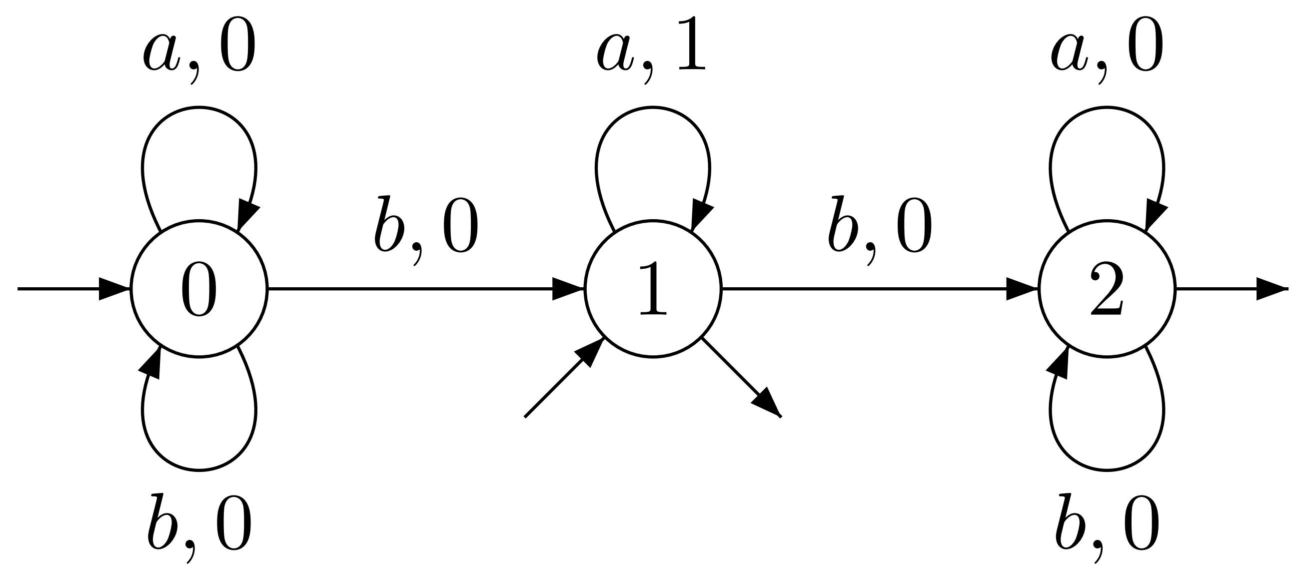

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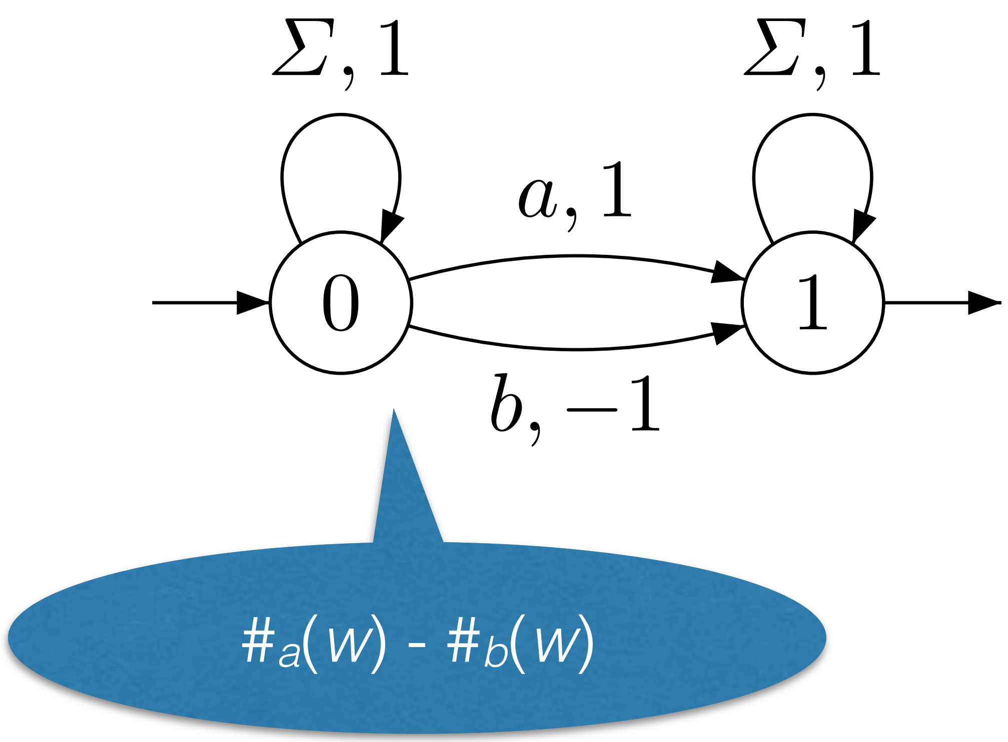

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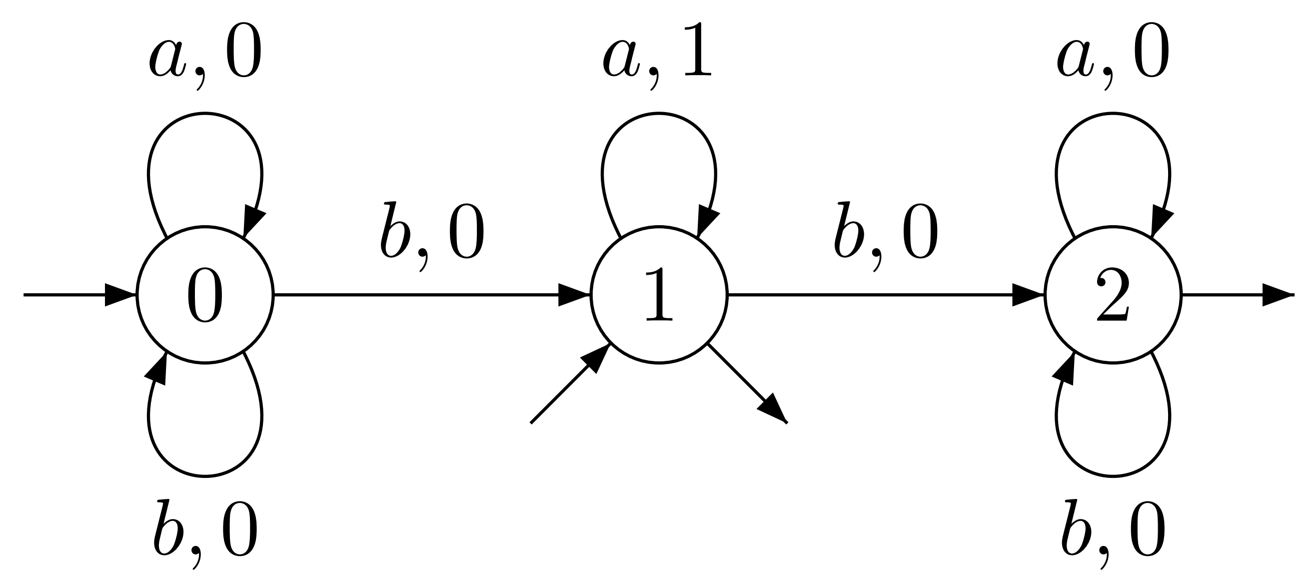

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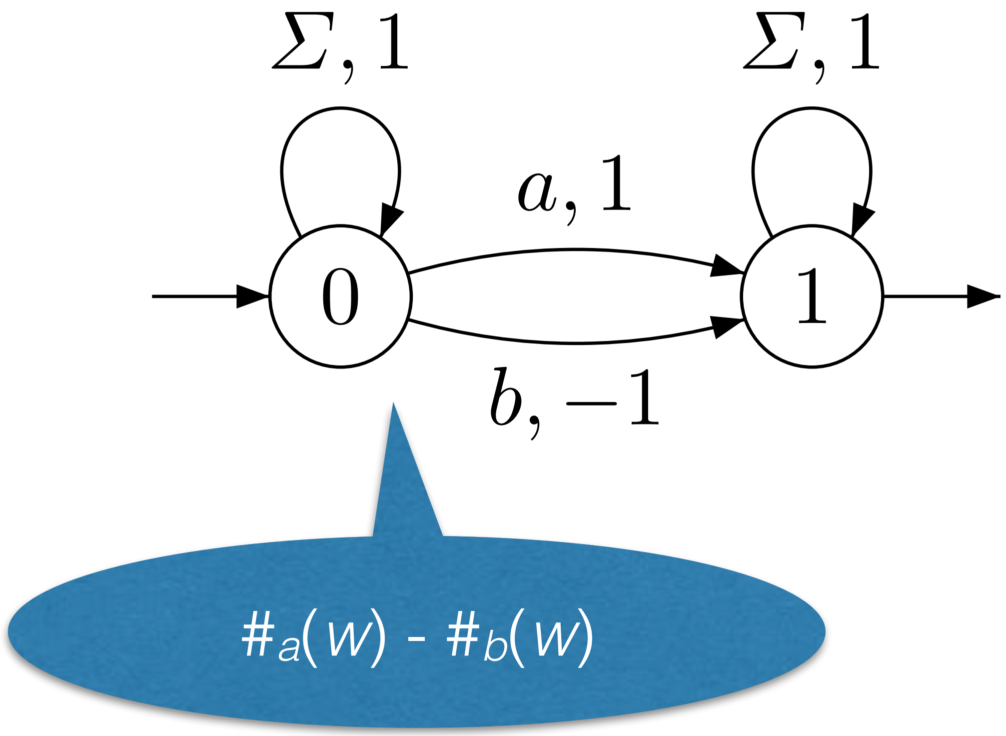

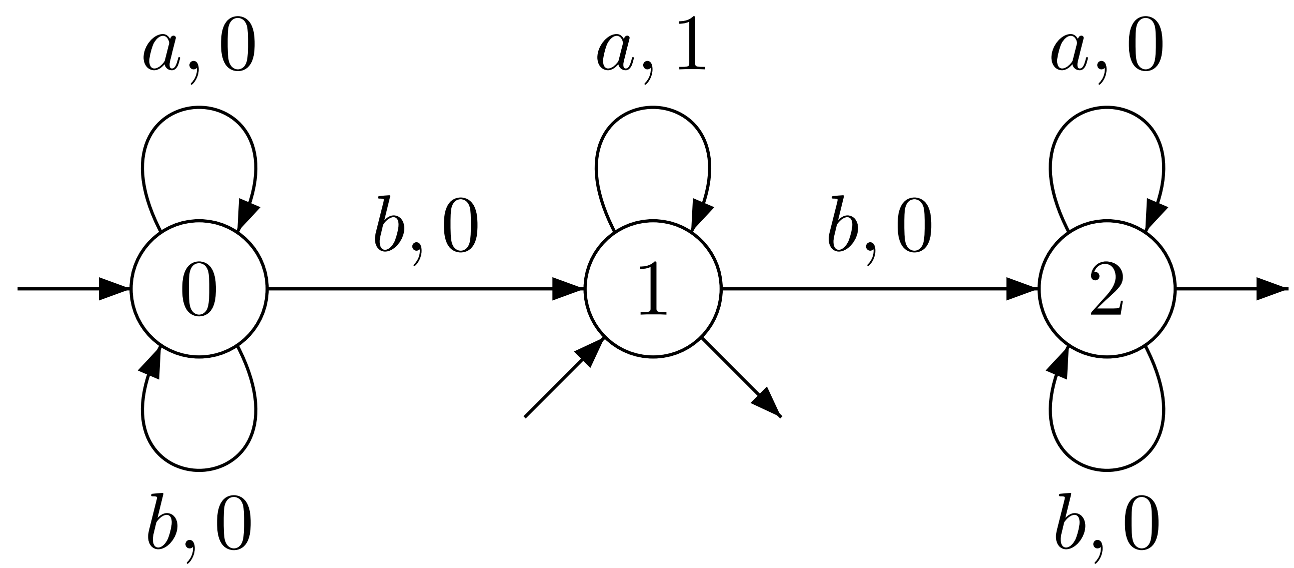

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Semantics of *aba*: 1+(-1)+1 = 1

Semantics of *aaba*: max(2,1) = 2

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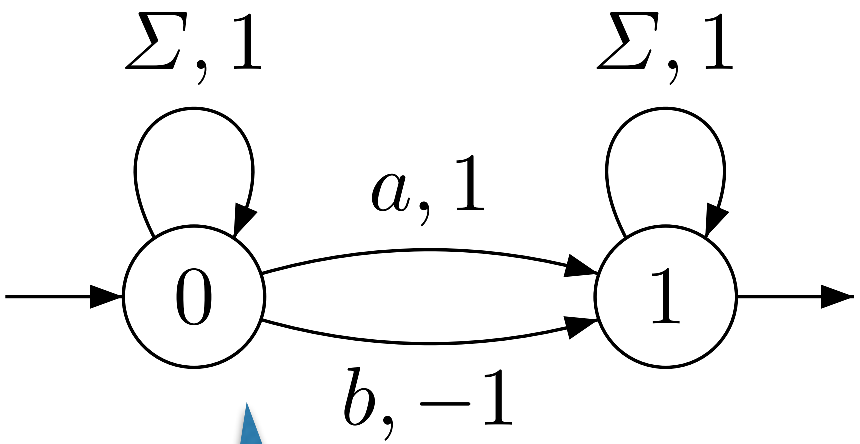
Weighted Automata



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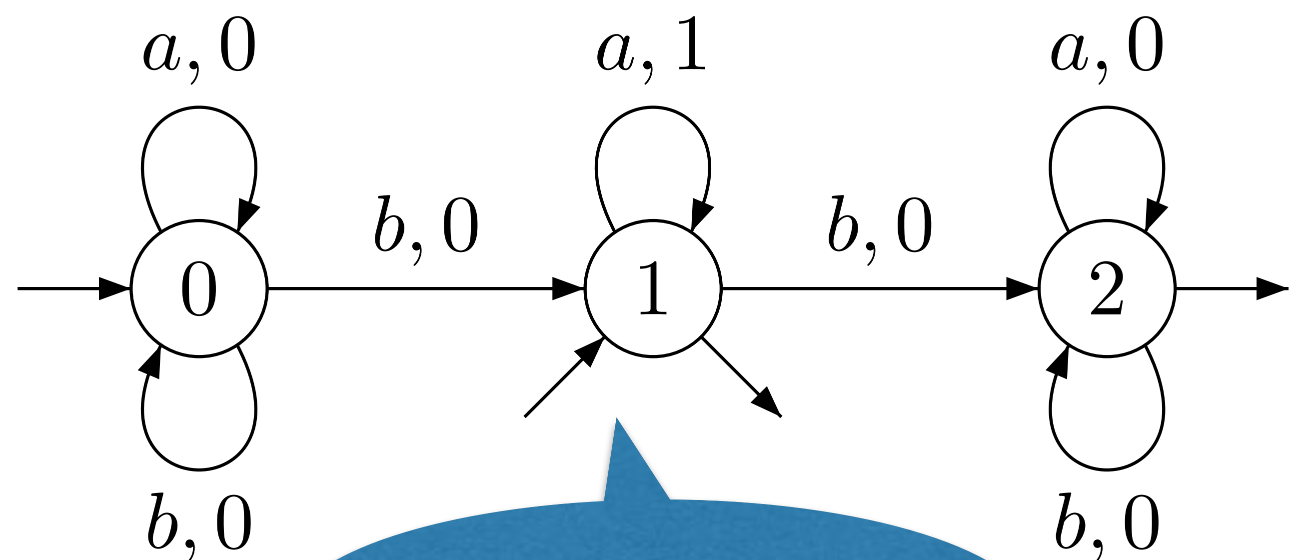

max size of *a*'s blocks

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Weighted Monadic Second Order Logic [Droste&Gastin 05] generalized to trees [Droste&Vogler 06], infinite words [Droste&Rahonis 07], nested words [Mathissen 10] or pictures [Fichtner 11]

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Weighted Temporal Logics:
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Weighted Temporal Logics:
PCTL [Hansson&Jonsson 94], WLTL [Mandrali 12]

- Core weighted logic for weighted automata
- Enhancing the logic to handle more properties: FO vs pebbles
- Deciding weighted FO logic
- A special case: the transducers

Weighted MSO

$$\varphi ::= s \mid P_a(x) \mid x \leqslant y \mid x \in X \mid \neg P_a(x) \mid \neg(x \leqslant y) \mid \neg(x \in X)$$
$$\mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x\, \varphi \mid \forall x\, \varphi \mid \exists X\, \varphi \mid \forall X\, \varphi$$

[Droste&Gastin 2005]

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Negation restricted to atomic formulae

[Droste&Gastin 2005]

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Arbitrary constants from a semiring

Negation restricted to atomic formulae

[Droste&Gastin 2005]

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- Semantics in a semiring $\mathbb{S} = (S, +, \times, 0, 1)$
  - Atomic formulae: 0, 1
  - disjunction, existential quantifications: sum
  - conjunction, universal quantifications: product
- Inspired from the boolean semiring $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$

[Droste&Gastin 2005]

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- Examples

$$\varphi_1 = \exists x\, P_a(x)$$

$$[\![\varphi_1]\!](w) = |w|_a$$

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$$\varphi_1 = \exists x\, P_a(x) \qquad \varphi_2 = \forall x\, \exists y\, (y \leqslant x \wedge P_a(y))$$

$$[\![\varphi_1]\!](w) = |w|_a \qquad [\![\varphi_2]\!](abaab) = 1 \times 1 \times 2 \times 3 \times 3$$

$$[\![\varphi_2]\!](a^n) = n!$$

[Droste&Gastin 2005]

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Too big to be computed by a weighted automaton

[Droste&Gastin 2005]

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Need to restrict weighted MSO

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Theorem: weighted automata = restricted wMSO

[Droste&Gastin 2005]

Weighted MSO

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$$\mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x\, \varphi \mid \forall x\, \varphi \mid \exists X\, \varphi \mid \forall X\, \varphi$$

Theorem: weighted automata = restricted wMSO

[Droste&Gastin 2005]

Weighted MSO

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φ almost boolean

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commutativity

φ almost boolean

Theorem: weighted automata = restricted wMSO

[Droste&Gastin 2005]

Core weighted MSO logic

- Boolean fragment

$$\varphi ::= \top \mid P_a(x) \mid x \leqslant y \mid x \in X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x\, \varphi \mid \forall X\, \varphi$$

[Gastin&Monmege 2015]

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- Step formulae $\quad \Psi ::= s \mid \varphi\,?\,\Psi : \Psi$

[Gastin&Monmege 2015]

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if … then … else …

[Gastin&Monmege 2015]

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$$P_a(x)\,?\,1 : 0$$

[Gastin&Monmege 2015]

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$$P_a(x) \,?\, 1 : (P_b(x) \,?\, -1 : 0)$$

[Gastin&Monmege 2015]

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$$P_a(x)\,?\,1 : (P_b(x)\,?\,-1 : 0)$$

$$x \in X_1\,?\,s_1 : (x \in X_2\,?\,s_2 : \cdots (x \in X_{n-1}\,?\,s_{n-1} : s_n) \cdots)$$

[Gastin&Monmege 2015]

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$$[\![\Psi]\!](w, \sigma) = s$$

some value occurring in Ψ

[Gastin&Monmege 2015]

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$P_a(x)\,?\,1 : 0$

...

$P_a(x)$

A step formula takes finitely many values

For each value, the pre-image is MSO-definable

$x \in X$ ... $X_2\,?\,s_2 : \cdots (x \in X_{n-1}\,?\,s_{n-1} : s_n)\cdots)$

$$[\![\Psi]\!](w, \sigma) = s$$

some value occurring in Ψ

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- core wMSO $\quad \Phi ::= \mathbf{0} \mid \varphi \,?\, \Phi : \Phi \mid \Phi + \Phi \mid \sum_x \Phi \mid \sum_X \Phi \mid \prod_x \Psi$

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no constants

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if … then … else …

Assigns a value from Ψ to each position

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no constants

if … then … else …

Assigns a value from Ψ to each position

$$\{\!|\prod_x \Psi|\!\}(w,\sigma) = \{\!\{(\llbracket\Psi\rrbracket(w,\sigma[x \mapsto i]))_i\}\!\} \in \mathbb{N}\langle R^\star\rangle$$

singleton multiset

Core weighted MSO logic

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- core wMSO $\quad \Phi ::= \mathbf{0} \mid \varphi \,?\, \Phi : \Phi \mid \Phi + \Phi \mid \sum_x \Phi \mid \sum_X \Phi \mid \prod_x \Psi$

- Semantics
  - $\{\!|\, \mathbf{0} \,|\!\}(w,\sigma) = \varnothing$
  - sums over multisets

$$\{\!|\, \prod_x \Psi \,|\!\}(w,\sigma) = \{\!\{ (\llbracket \Psi \rrbracket (w, \sigma[x \mapsto i]))_i \}\!\} \in \mathbb{N}\langle R^\star \rangle$$

Multisets of weight structures

- A run generates a sequence of weights $\mathrm{wgt}(\rho) = s_1 s_2 \cdots s_n$
- Abstract semantics $\{\!|\mathcal{A}|\!\}(w) = \{\{\mathrm{wgt}(\rho) \mid \rho \text{ run on } w\}\}$

multiset

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$$\{\!|\mathcal{A}|\!\} \colon \Sigma^\star \to \mathbb{N}\langle R^\star \rangle$$

multiset

weights of A

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weights of A

- Aggregation $\mathsf{aggr} \colon \mathbb{N}\langle R^\star \rangle \to S$

Multisets of weight structures

Semiring: sum-product

$$\mathsf{aggr}_{\mathrm{sp}}(A) = \sum \prod A = \sum_{r_1 \cdots r_n \in A} r_1 \times \cdots \times r_n$$

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[Droste&Perevoshchikov 2014]

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$$\mathsf{aggr}_{\mathtt{sv}}(A) = \sum \mathsf{Val}(A) = \sum_{r_1 \cdots r_n \in A} \mathsf{Val}(r_1 \cdots r_n)$$

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weights of A

Average value
Discounted value…

- Aggregation $\rightarrow S$

[Droste&Perevoshchikov 2014]

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- Aggregation $\mathsf{aggr} \colon \mathbb{N}\langle R^\star \rangle \to S$
- Concrete semantics $[\![\mathcal{A}]\!] = \mathsf{aggr} \circ \{\!|\mathcal{A}|\!\} \colon \Sigma^\star \to S$

Core weighted MSO logic

$$\varphi ::= \top \mid P_a(x) \mid x \leqslant y \mid x \in X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x\, \varphi \mid \forall X\, \varphi$$

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Theorem: weighted automata = core wMSO

[Gastin&Monmege 2015]

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Theorem: weighted automata = core wMSO

- Abstract semantics ...
- Concrete sema... $\Sigma^\star \to S$

Easy constructive proofs
preservation of the constants
no restriction on core wMSO
no hypotheses on weights

[Gastin&Monmege 2015]

Extensions

More general models than words: trees, nested words, labelled graphs, infinite words…

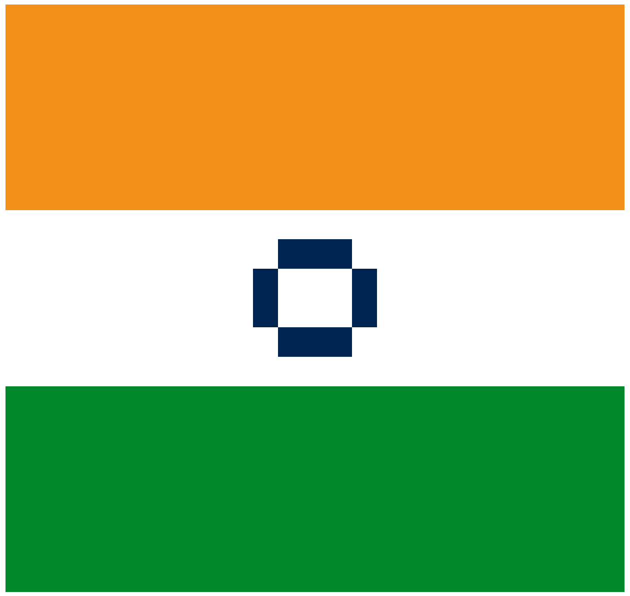
Other logics: other manageable fragment of wMSO formula than core wMSO

More powerful automata: finding equivalent fragments of wMSO

Logical Specifications: Query Examples

Logical Specifications: Query Examples

Is there a line of green pixels?



Logical Specifications: Query Examples

Is there a line of green pixels?

How many lines of green pixels are there?

Logical Specifications: Query Examples

Is there a line of green pixels?

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What is the size of the picture?

Logical Specifications: Query Examples

Is there a line of green pixels?

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What is the size of the picture?

What is the average lightness?

Logical Specifications: Query Examples

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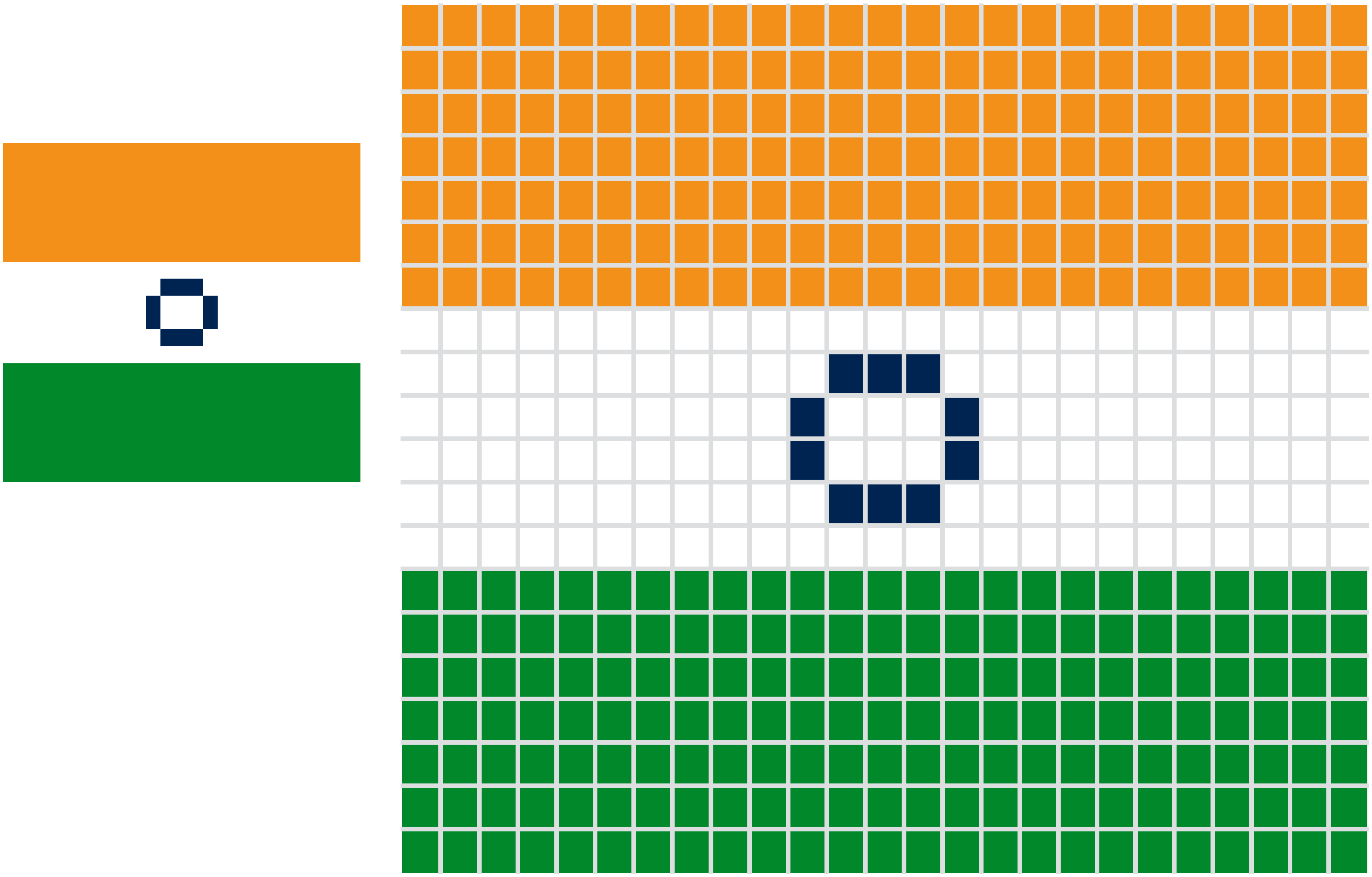
What is the size of the picture?

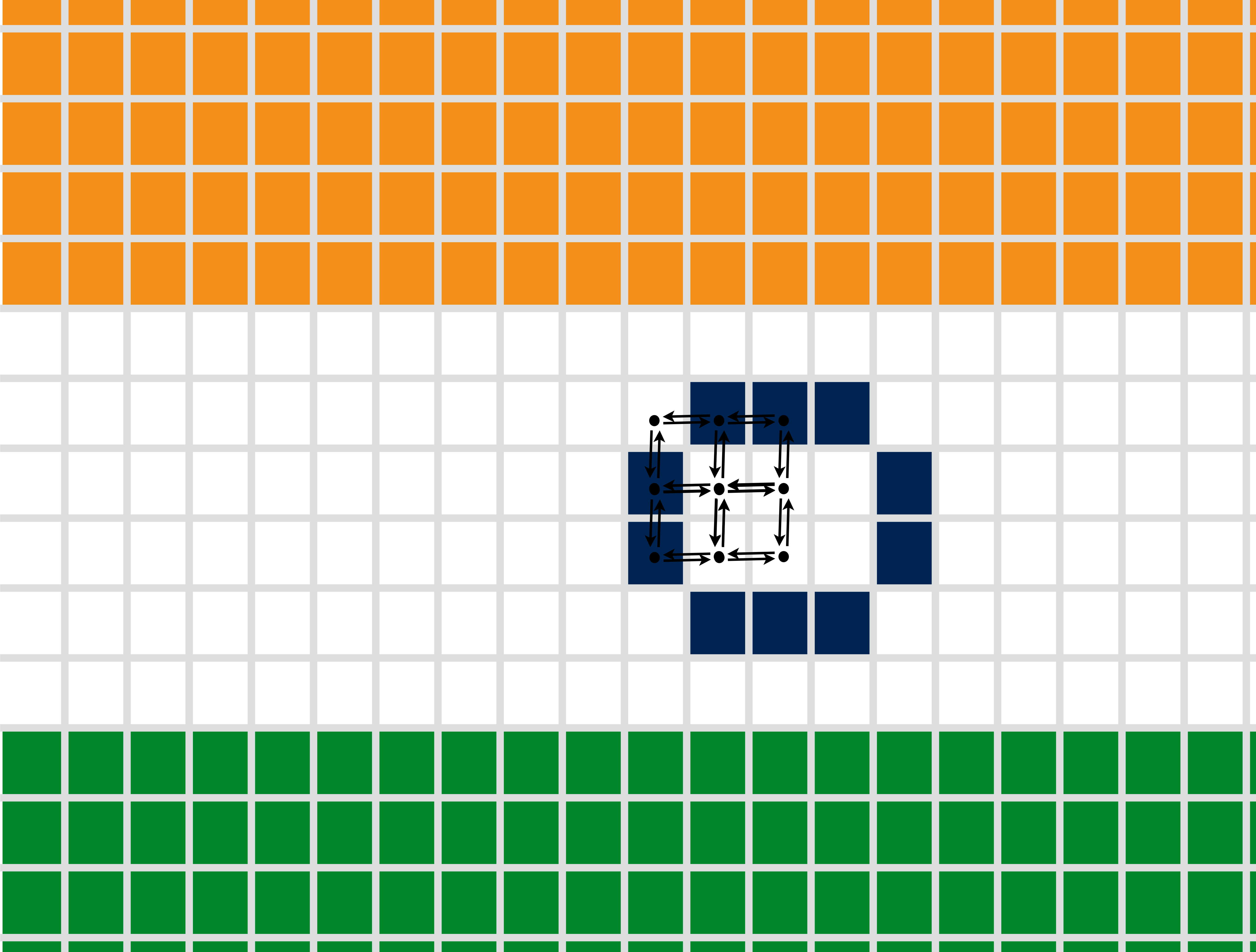
What is the average lightness?

What is the size of the biggest monochromatic rectangle?

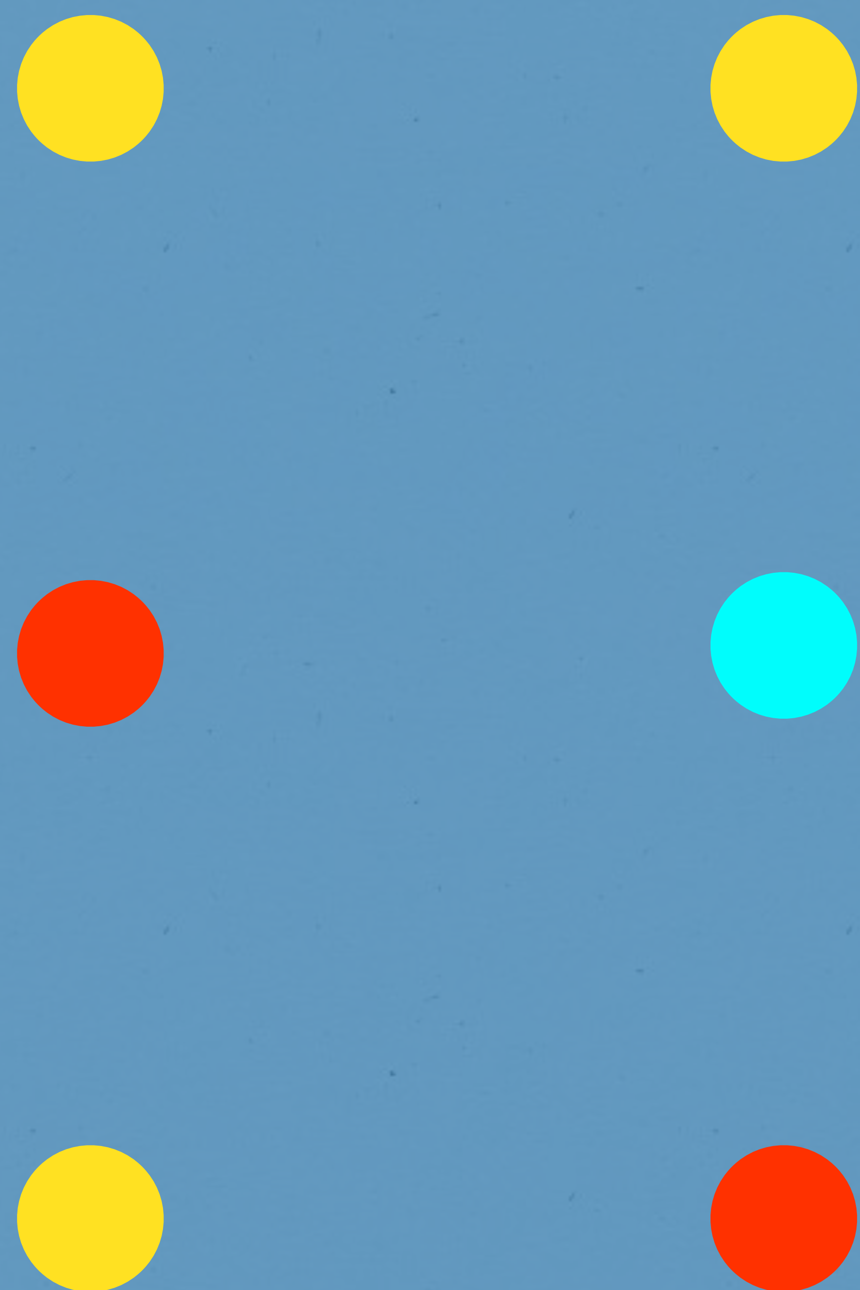
Modelling a picture as a graph

Modelling a picture as a graph



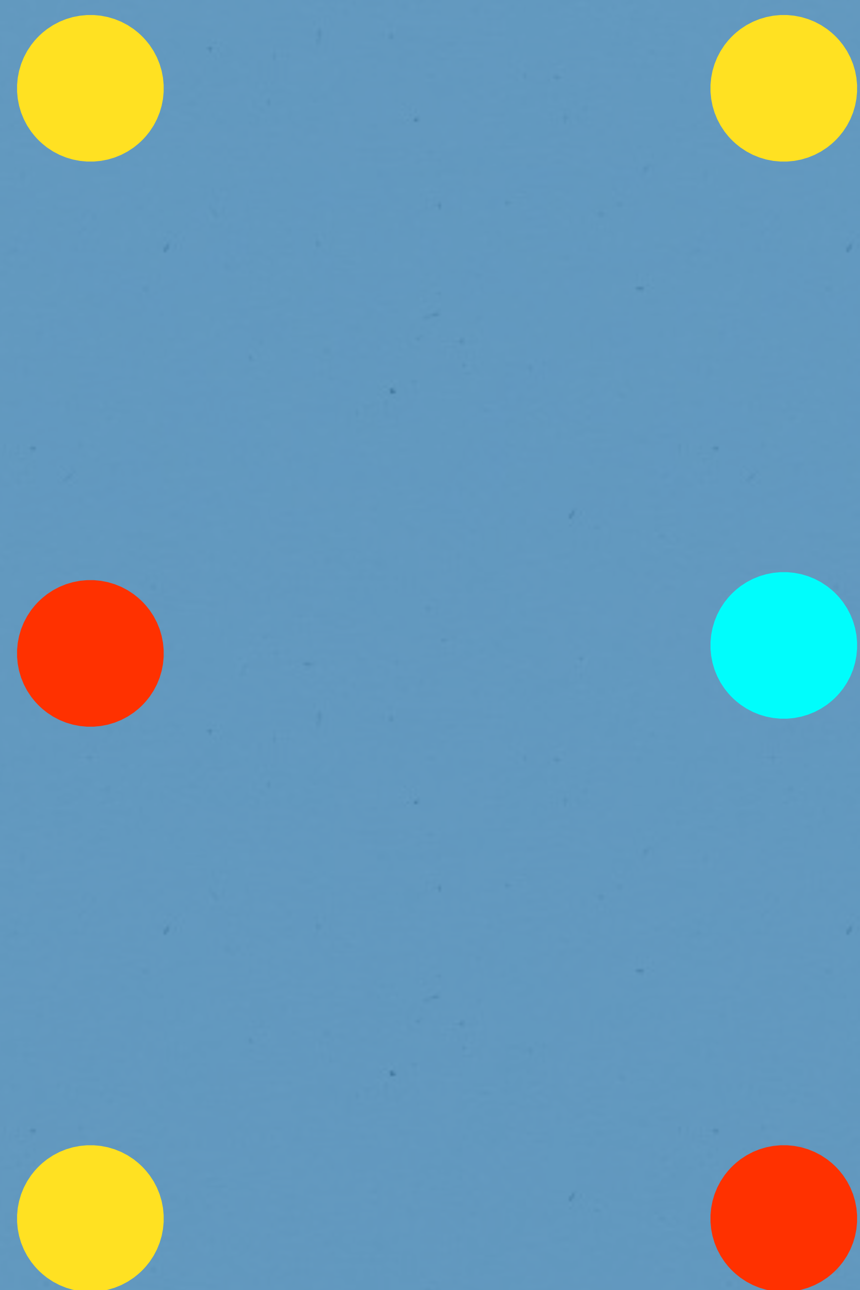


$$G = (V, (E_d)_{d \in D}, \lambda)$$



$$G = (V, (E_d)_{d \in D}, \lambda)$$

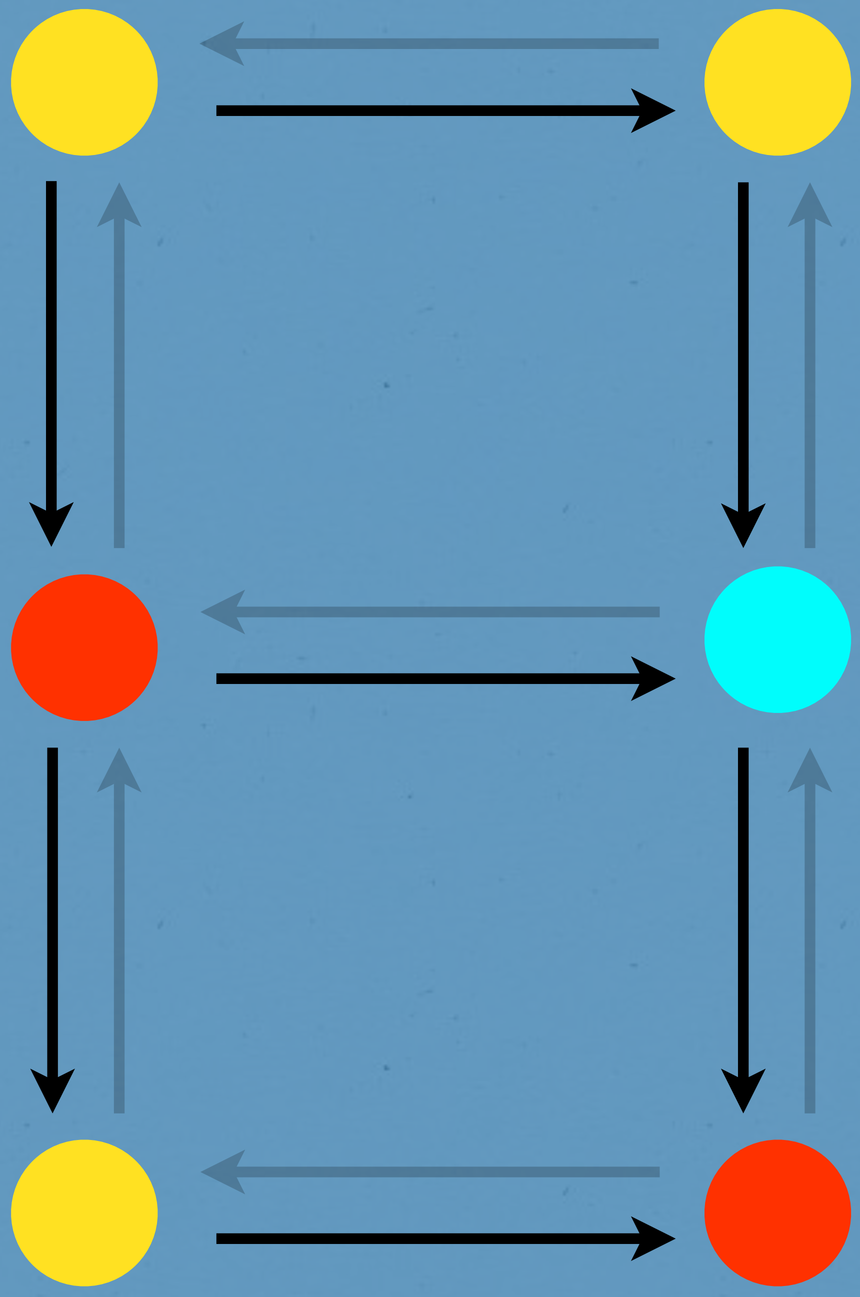
V	set of vertices
λ	labels of vertices



$$G = (V, (E_d)_{d \in D}, \lambda)$$

V	set of vertices
λ	labels of vertices
D	set of directions

$$D = \{\rightarrow, \downarrow\} \cup \{\leftarrow, \uparrow\}$$



$$G = (V, (E_d)_{d \in D}, \lambda)$$

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E_d	set of *d*-edges

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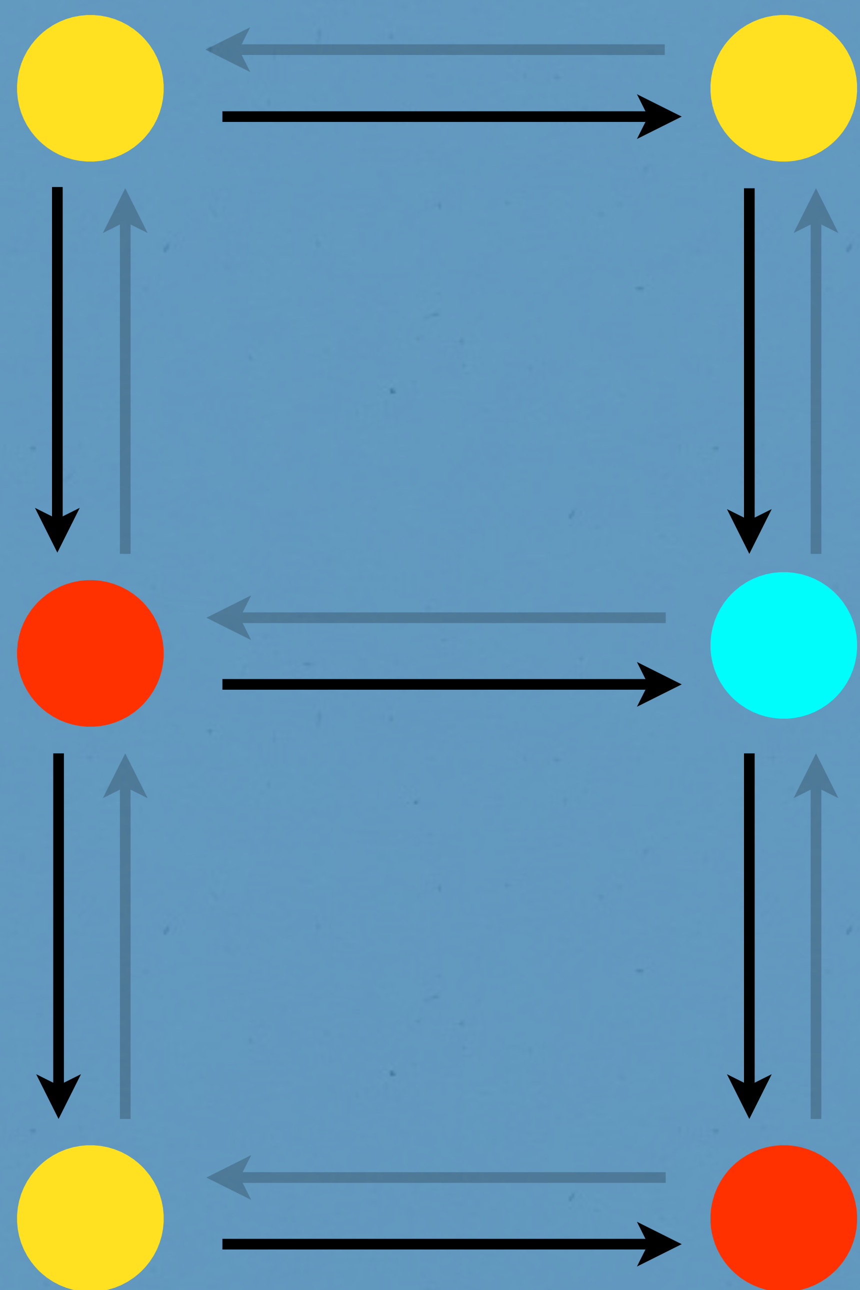

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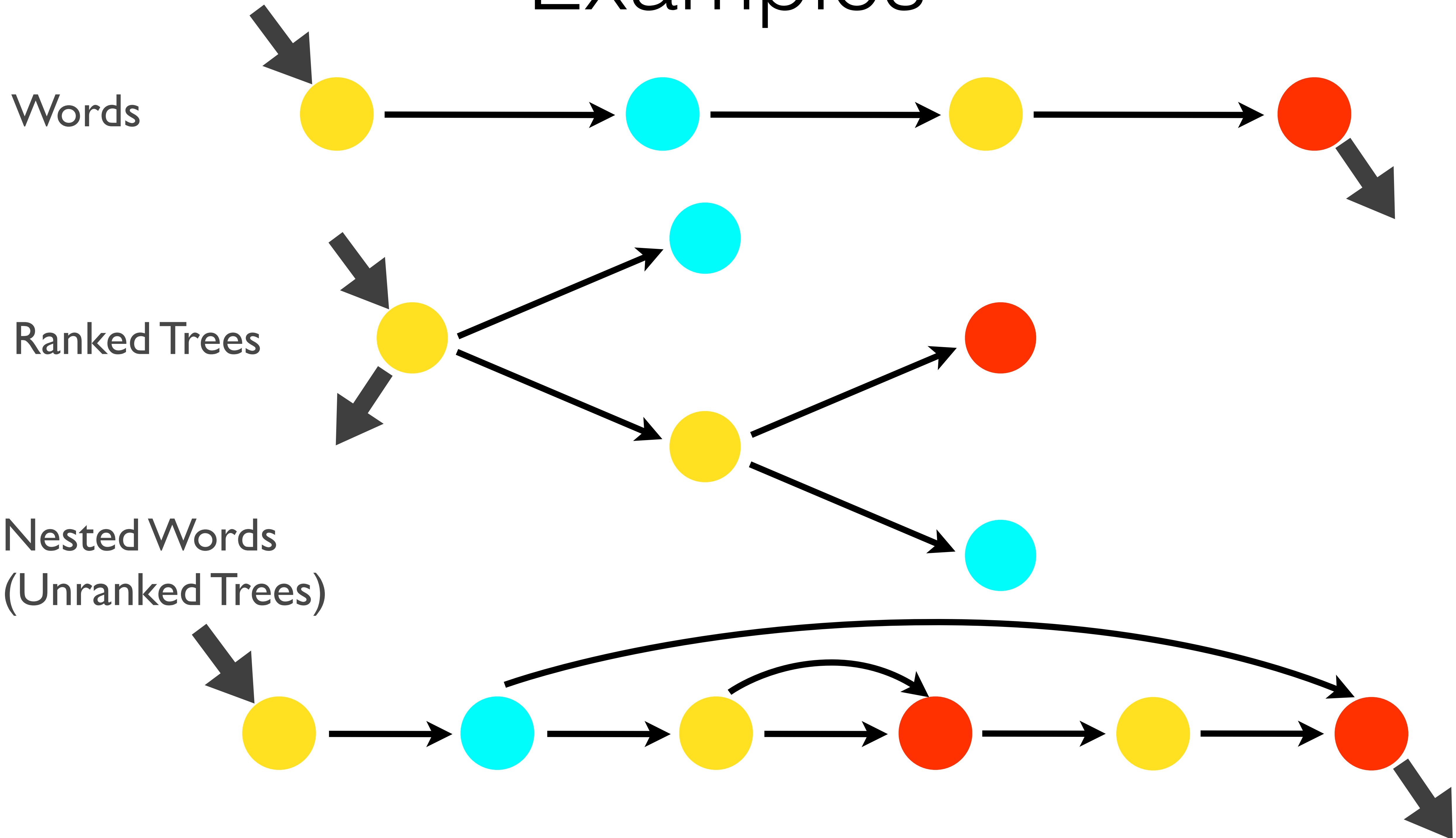
V	set of vertices
λ	labels of vertices
D	set of directions
E_d	set of *d*-edges

deterministic
(hence bounded degree)

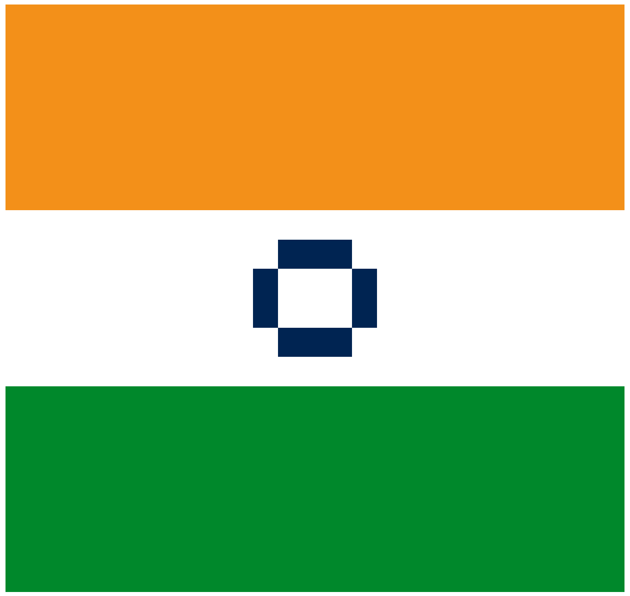
Examples



Mazurkiewicz Traces   Pictures   Graphs of bounded degree

Labyrinths   ...

Logical Specifications: Query Examples



Is there a line of green pixels?

How many lines of green pixels are there?

What is the size of the picture?

What is the average lightness?

What is the size of the biggest monochromatic rectangle?

Logical Specifications: Query Examples

Is there a line of green pixels?

$$\exists x \forall y [(R^*_{\rightarrow}(x,y) \vee R^*_{\rightarrow}(y,x)) \Rightarrow P_{\blacksquare}(y)] ?1 : 0$$

How many lines of green pixels are there?

What is the size of the picture?

Boolean fragment: first-order logic

$$\varphi ::= \top \mid (x = y) \mid \mathsf{init}(x) \mid \mathsf{final}(x) \mid P_a(x) \mid R_d(x,y) \mid R_d^\star(x,y) \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x\, \varphi \mid \forall x\, \varphi$$

$$G, \sigma \models P_a(x) \quad \text{iff.} \quad \lambda(\sigma(x)) = a$$

$$G, \sigma \models R_d(x,y) \quad \text{iff.} \quad (\sigma(x), \sigma(y)) \in E_d$$

$$G, \sigma \models R_d^\star(x,y) \quad \text{iff. there is a } d\text{-path from } \sigma(x) \text{ to } \sigma(y)$$

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How many lines of green pixels are there?

$$\sum_{x} \neg \exists z \, R_{\rightarrow}(z, x) \wedge \forall y \, R^*_{\rightarrow}(x, y) \Rightarrow P_{\blacksquare}(y) ? 1 : 0$$

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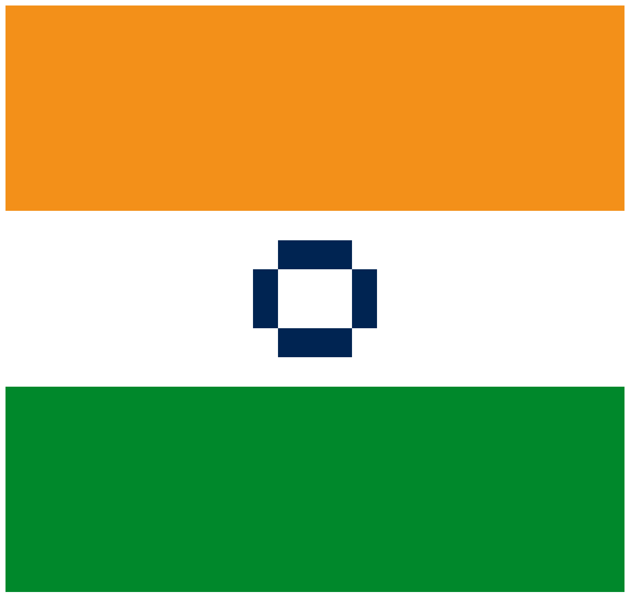

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How many lines of green pixels are there?

$$\sum_x \neg \exists z\, R_\rightarrow(z,x) \wedge \forall y\, R^*_\rightarrow(x,y) \Rightarrow P_{\blacksquare}(y)\,?1 : 0$$

What is the size of the picture?

$$\left(\sum_x \neg \exists y R_\rightarrow(y,x)\,?1 : 0 \right) \times \left(\sum_x \neg \exists y R_\downarrow(y,x)\,?1 : 0 \right)$$

What is the average lightness?

What is the size of the biggest monochromatic rectangle?

Logical Specifications: Query Examples

$(\mathbb{N}, +, \times, 0, 1)$

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$$\exists x \forall y [(R^*_{\rightarrow}(x,y) \vee R^*_{\rightarrow}(y,x)) \Rightarrow P_{\blacksquare}(y)] ? 1 : 0$$

How many lines of green pixels are there?

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Logical Specifications: Query Examples

$(\mathbb{N}, +, \times, 0, 1)$

Is there a line of green pixels?

$$\exists x \forall y [(R^*_{\rightarrow}(x,y) \vee R^*_{\rightarrow}(y,x)) \Rightarrow P_{\color{green}\blacksquare}(y)] ?1 : 0$$

How many lines of green pixels are there?

$$\sum_x \neg \exists z\, R_{\rightarrow}(z,x) \wedge \forall y\, R^*_{\rightarrow}(x,y) \Rightarrow P_{\color{green}\blacksquare}(y) ?1 : 0$$

What is the size of the picture?

$$\left(\sum_x \neg \exists y R_{\rightarrow}(y,x) ?1 : 0 \right) \times \left(\sum_x \neg \exists y R_{\downarrow}(y,x) ?1 : 0 \right)$$

$(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$

What is the average lightness?

$$\sum_x P_{\color{orange}\blacksquare}(x) ?100 : 0 + \sum_x P_{\color{green}\blacksquare}(x) ?150 : 0 + \sum_x P_{\color{navy}\blacksquare}(x) ?220 : 0$$

What is the size of the biggest monochromatic rectangle?

Logical Specifications: Query Examples

$(\mathbb{N}, +, \times, 0, 1)$

Is there a line of green pixels?

$$\exists x \forall y [(R^*_{\rightarrow}(x,y) \vee R^*_{\rightarrow}(y,x)) \Rightarrow P_{\color{green}\blacksquare}(y)] ? 1 : 0$$

How many lines of green pixels are there?

$$\color{red}{\sum_x} \color{black} \neg \exists z\, R_{\rightarrow}(z,x) \wedge \forall y\, R^*_{\rightarrow}(x,y) \Rightarrow P_{\color{green}\blacksquare}(y) ? 1 : 0$$

What is the size of the picture?

$$\left(\sum_x \neg \exists y R_{\rightarrow}(y,x) ? 1 : 0 \right) \color{red}{\times} \color{black} \left(\sum_x \neg \exists y R_{\downarrow}(y,x) ? 1 : 0 \right)$$

$(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$

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$$\sum_x P_{\color{orange}\blacksquare}(x) ? 100 : 0 + \sum_x P_{\color{green}\blacksquare}(x) ? 150 : 0 + \sum_x P_{\color{navy}\blacksquare}(x) ? 220 : 0$$

What is the size of the biggest monochromatic rectangle?

$$\max_{x,y} \left[\varphi_{mono}(x,y) ? 1 : 0 + \left(\sum_z \varphi_{rect}(x,y,z) ? 1 : 0 \right) \right]$$

Logical Specifications: Query Examples

$(\mathbb{N}, +, \times, 0, 1)$

Is there a line of green pixels?

$$\exists x \forall y [(R^*_{\rightarrow}(x,y) \vee R^*_{\rightarrow}(y,x)) \Rightarrow P_{\blacksquare}(y)]\,?1:0$$

How many lines of green pixels are there?

$$\sum_x \neg \exists z\, R_{\rightarrow}(z,x) \wedge \forall y\, R^*_{\rightarrow}(x,y) \Rightarrow P_{\blacksquare}(y)\,?1:0$$

What is the size of the picture?

$$\left(\sum_x \neg \exists y R_{\rightarrow}(y,x)\,?1:0\right) \times \left(\sum_x \neg \exists y R_{\downarrow}(y,x)\,?1:0\right)$$

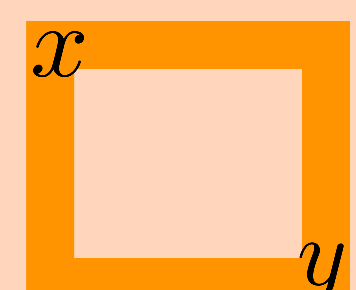
$(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$

What is the average lightness?

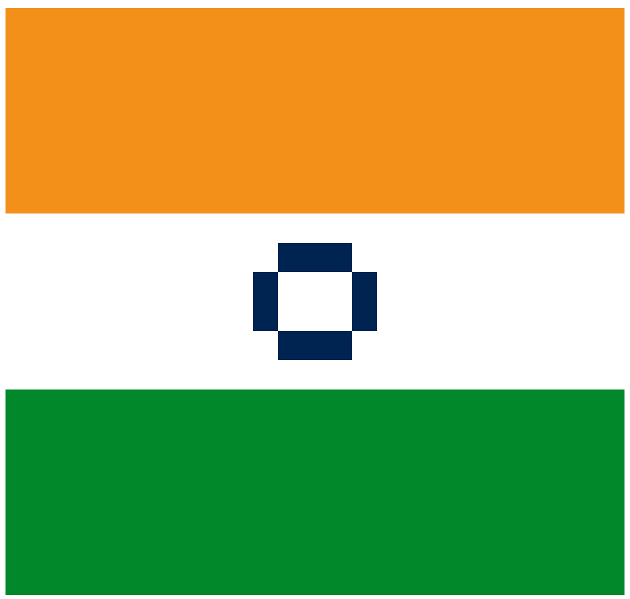
$$\sum_x P_{\blacksquare}(x)\,?100:0 + \sum_x P_{\blacksquare}(x)\,?150:0 + \sum_x P_{\blacksquare}(x)\,?220:0$$

What is the size of the biggest monochromatic rectangle?

$$\max_{x,y}\left[\varphi_{mono}(x,y)\,?1:0 + \left(\sum_z \varphi_{rect}(x,y,z)\,?1:0\right)\right]$$



Logical Specifications: Query Examples



$(\mathbb{N}, +, \times, 0, 1)$

Is there a line of green pixels?

$$\exists x \forall y [(R^*_\rightarrow(x,y) \lor R^*_\rightarrow(y,x)) \Rightarrow P_{\blacksquare}(y)] ?1 : 0$$

How many lines of green pixels are there?

$$\sum_x \neg \exists z\, R_\rightarrow(z,x) \land \forall y\, R^*_\rightarrow(x,y) \Rightarrow P_{\blacksquare}(y) ?1 : 0$$

What is the size of the picture?

$$\left(\sum_x \neg \exists y R_\rightarrow(y,x) ?1 : 0\right) \times \left(\sum_x \neg \exists y R_\downarrow(y,x) ?1 : 0\right)$$

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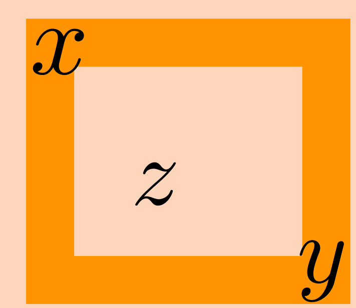

Weighted FO logic

$$\varphi ::= \top \mid P_a(x) \mid x \leqslant y \mid x \in X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x\, \varphi \mid \forall X\, \varphi$$

$$\Psi ::= s \mid \varphi\, ?\, \Psi : \Psi$$

$$\Phi ::= \mathbf{0} \mid \varphi\, ?\, \Phi : \Phi \mid \Phi + \Phi \mid \textstyle\sum_x \Phi \mid \textstyle\sum_X \Phi \mid \textstyle\prod_x \Psi$$

Weighted FO logic

We can keep Boolean MSO or restrict to FO…

$$\varphi ::= \top \mid P_a(x) \mid x \leqslant y \mid x \in X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x\, \varphi \mid \forall X\, \varphi$$

$$\Phi ::= s \,\big|\, \varphi ? \Phi : \Phi \,\big|\, \Phi + \Phi \,\big|\, \Phi \times \Phi \,\big|\, \sum_x \Phi \,\big|\, \prod_x \Phi$$

Reintroduction of the product

Unconditional product quantification

Weighted FO logic

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$$\Phi ::= s \mid \varphi ? \Phi : \Phi \mid \Phi + \Phi \mid \Phi \times \Phi \mid \sum_x \Phi \mid \prod_x \Phi$$

Reintroduction of the product

Unconditional product quantification

$$\varphi_2 = \forall x\, \exists y\, (y \leqslant x \wedge P_a(y)) \qquad [\![\varphi_2]\!](a^n) = n!$$

Weighted FO logic

We can keep Boolean MSO or restrict to FO…

$$\varphi ::= \top \mid P_a(x) \mid x \leqslant y \mid x \in X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x\, \varphi \mid \forall X\, \varphi$$

$$\Phi ::= s \mid \varphi ? \Phi : \Phi \mid \Phi + \Phi \mid \Phi \times \Phi \mid \sum\nolimits_x \Phi \mid \prod\nolimits_x \Phi$$

Reintroduction of the product

Unconditional product quantification

$$[\![\prod\nolimits_x \prod\nolimits_y 2]\!](w) = 2^{|w|^2}$$

Pebble weighted automata

$I \in \mathbb{S}^Q$ $T \in \mathbb{S}^Q$

$$\mathcal{A} = (Q, A, I, \delta, T)$$

$$\delta : Q \times \mathrm{Test} \times \mathrm{Move} \times Q \to \mathbb{S}$$

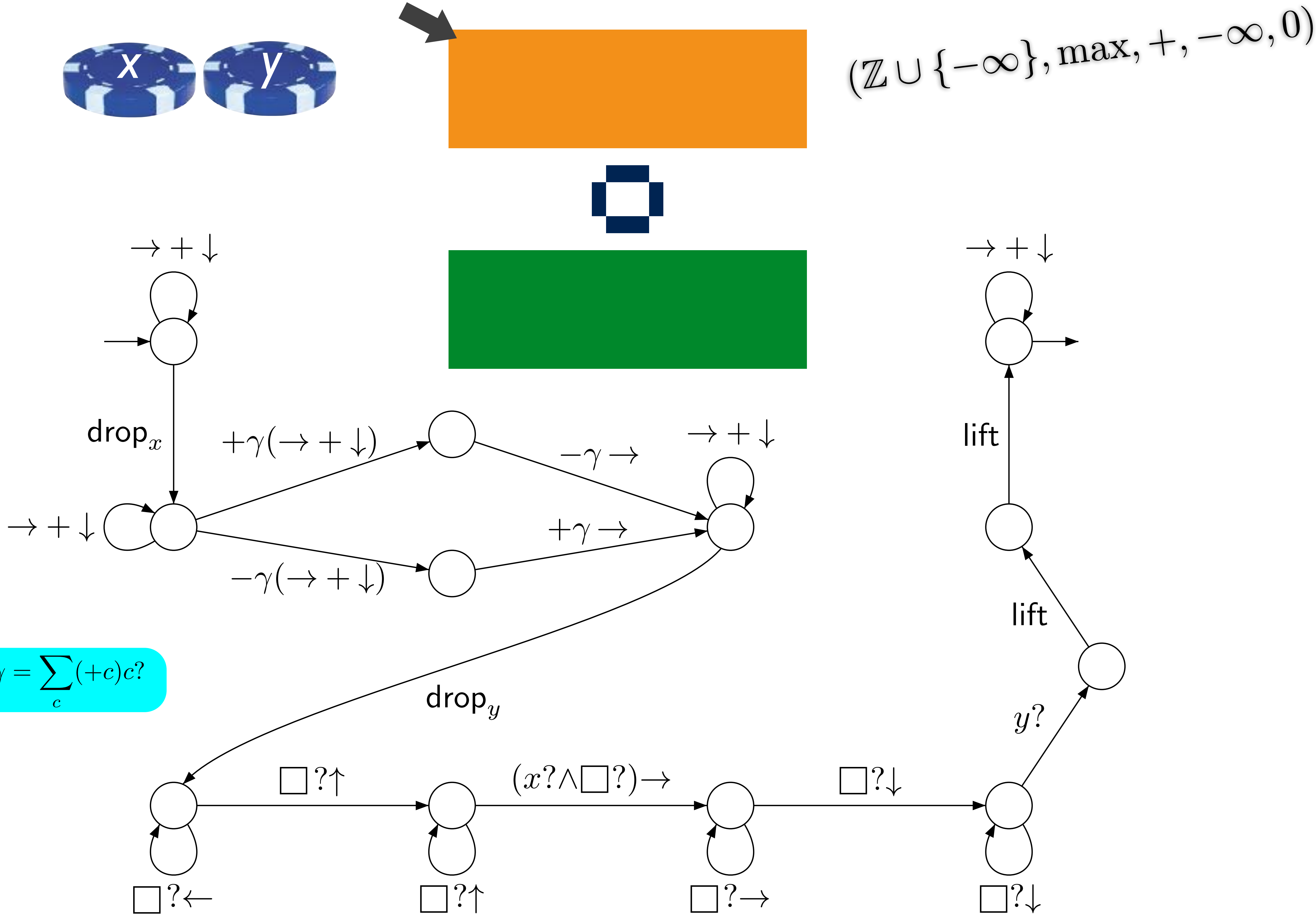
$$\mathrm{Move} = D \cup \{\mathrm{drop}_x, \mathrm{lift} \mid x \in \mathrm{Peb}\}$$

Run as a finite sequence of configurations (W, σ, q, i, π)

with free pebbles $\sigma : \mathrm{Peb} \to \mathrm{pos}(W)$

and a stack of currently dropped pebbles $\pi \in (\mathrm{Peb} \times \mathrm{pos}(W))^*$

Pebble Weighted Automata: An Example

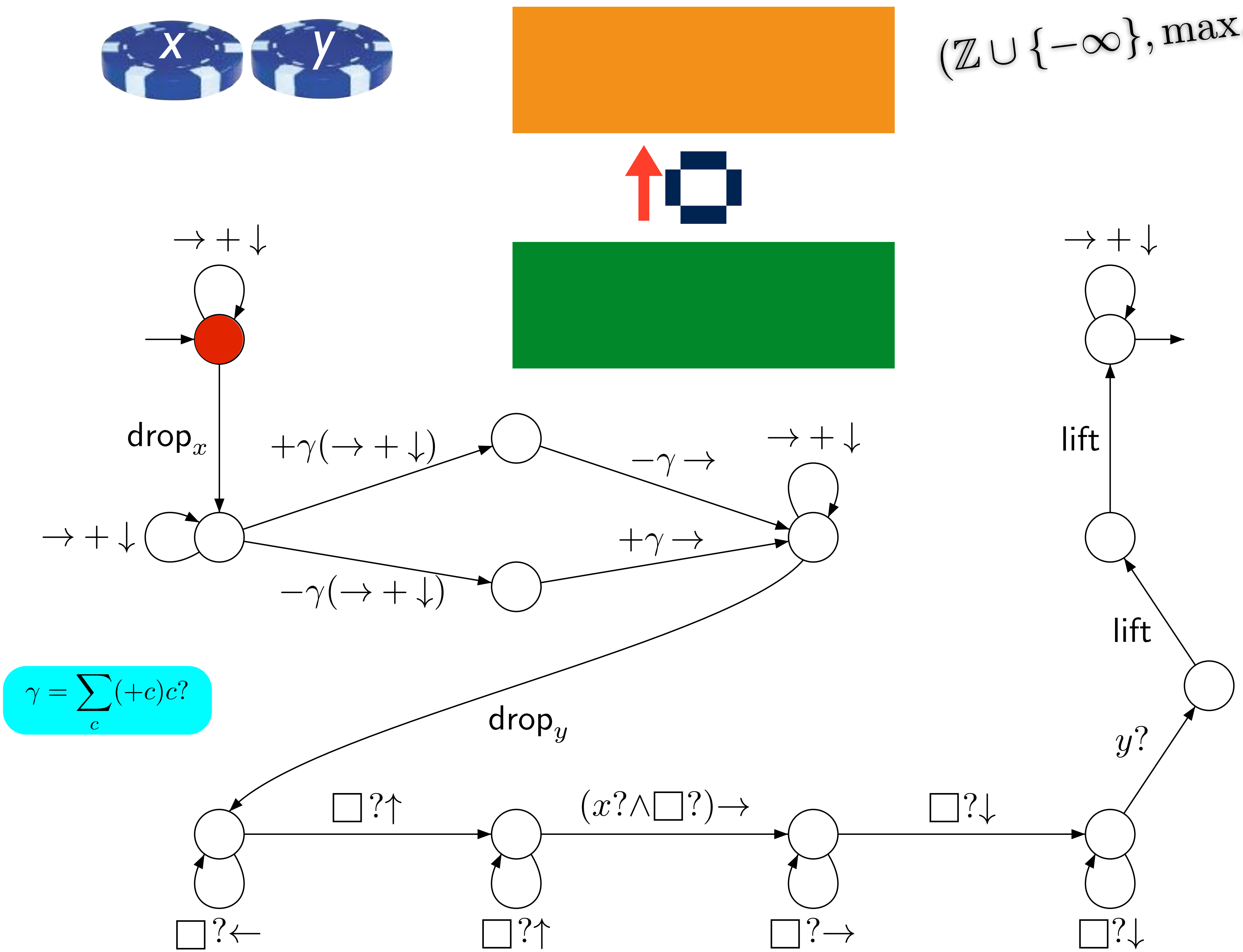


$\gamma = \sum_c (+c)c?$

Pebble automata over words and trees: [Globerman&Harel 96], [Engelfriet&Hoogeboom 99]

Pebble Weighted Automata: An Example

$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$



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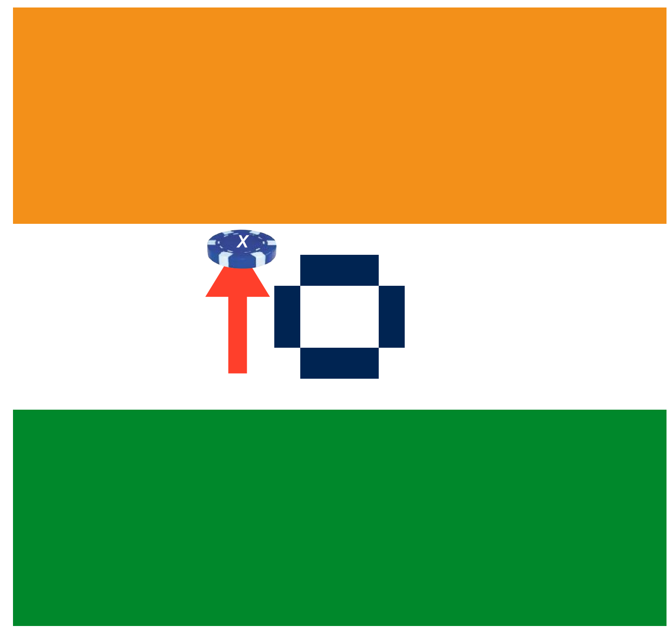

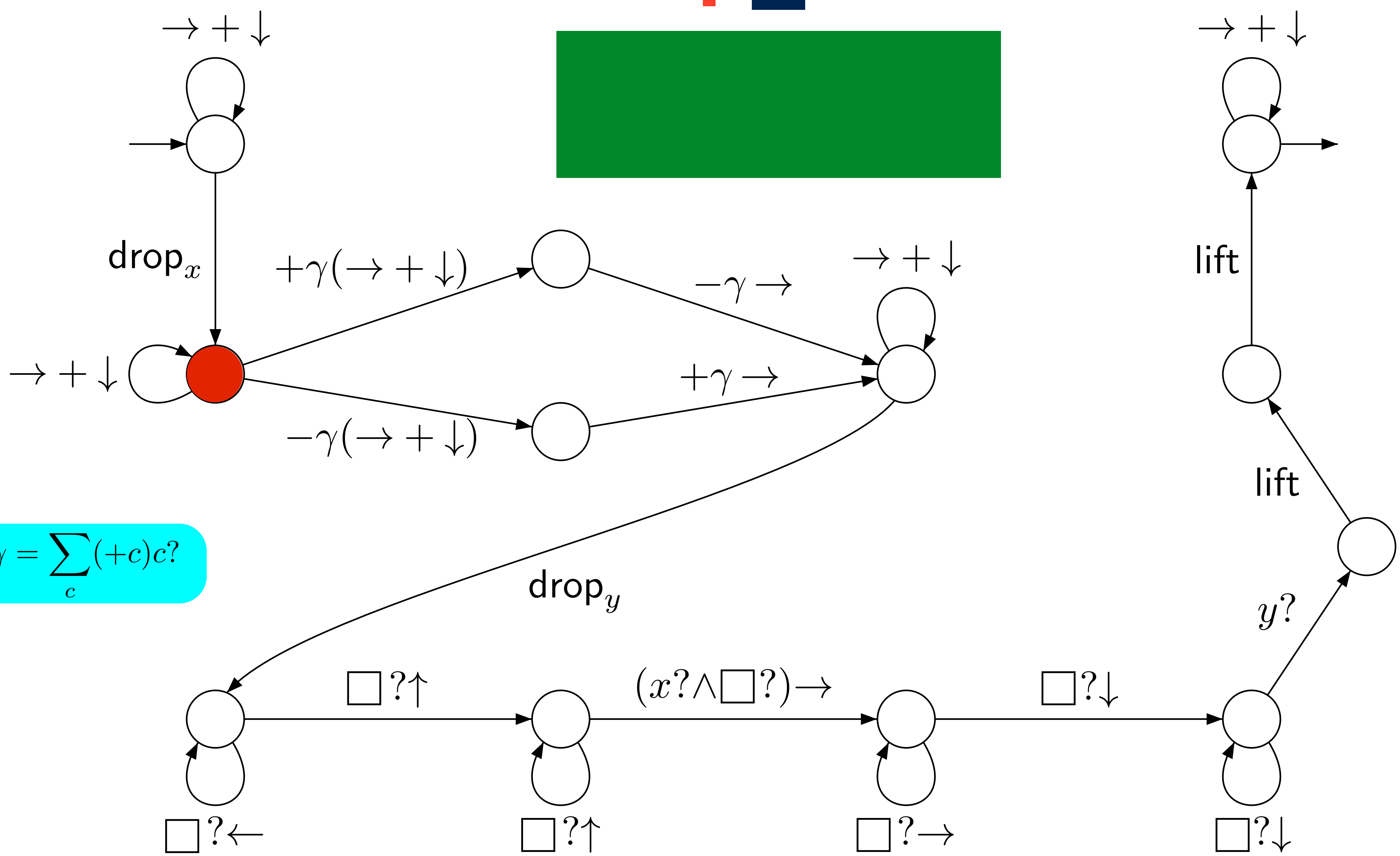

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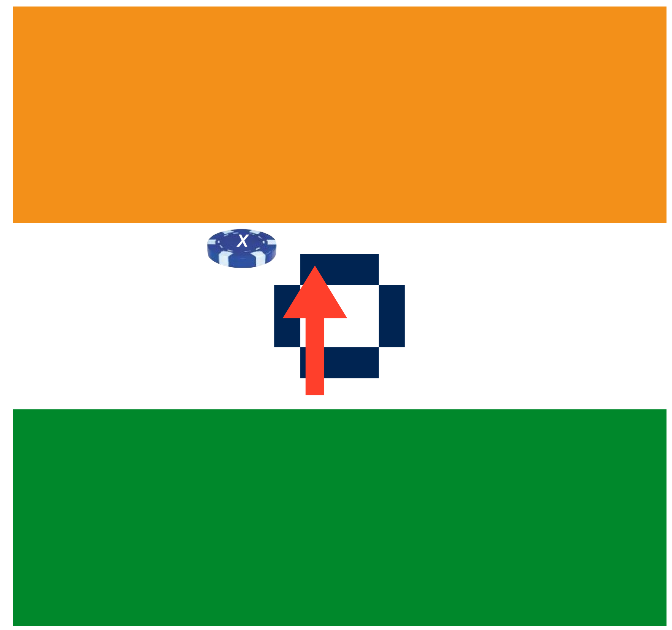

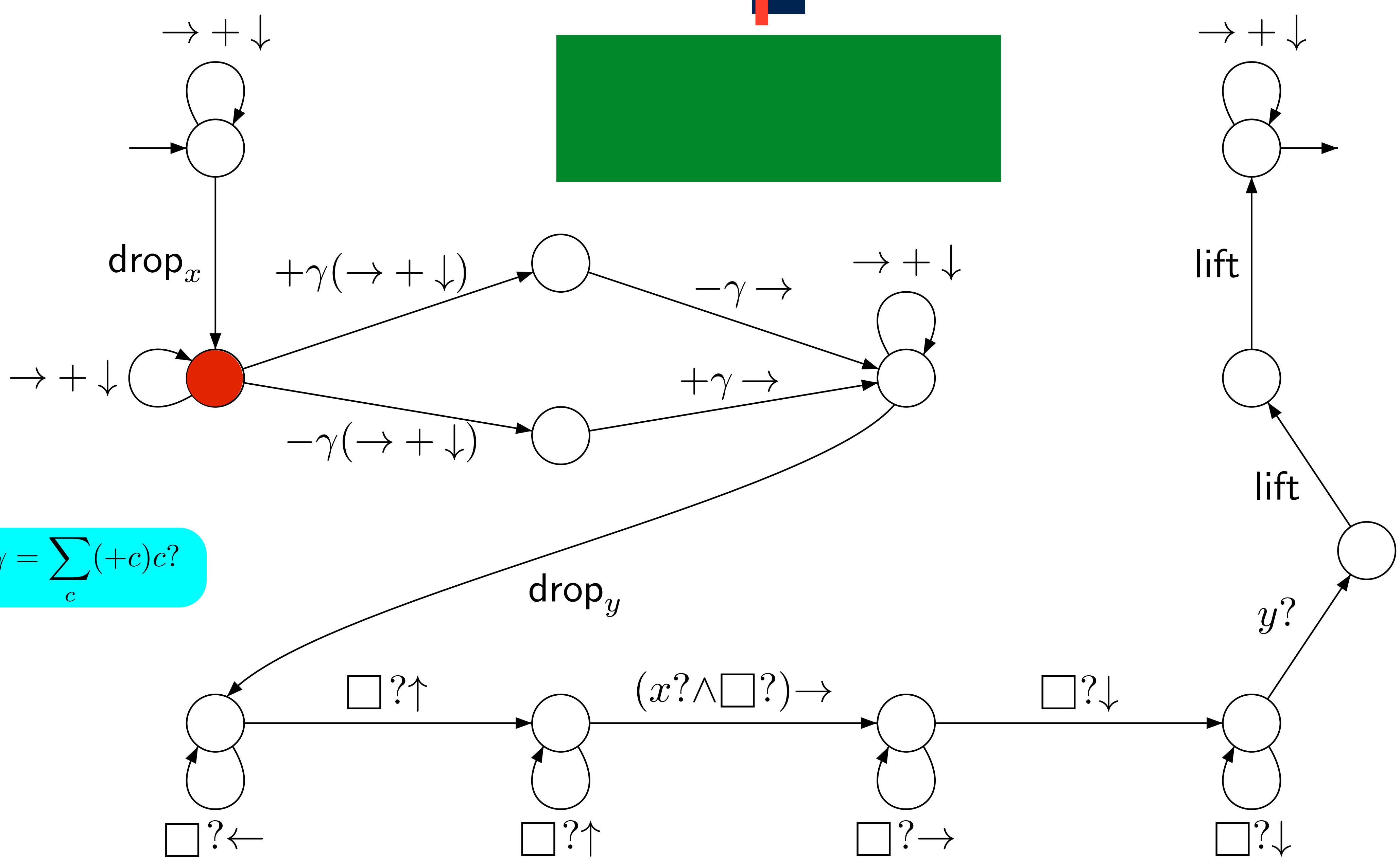

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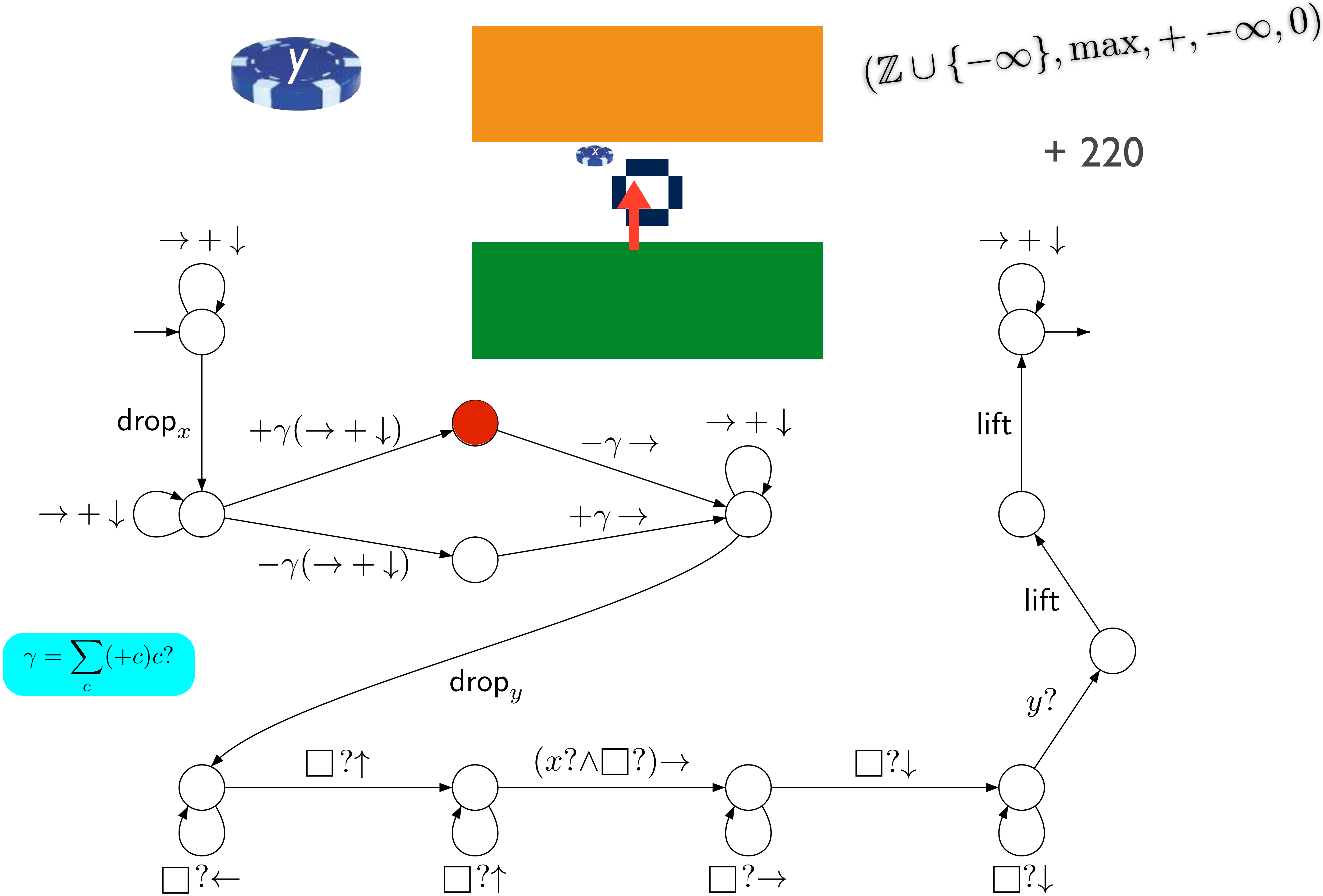

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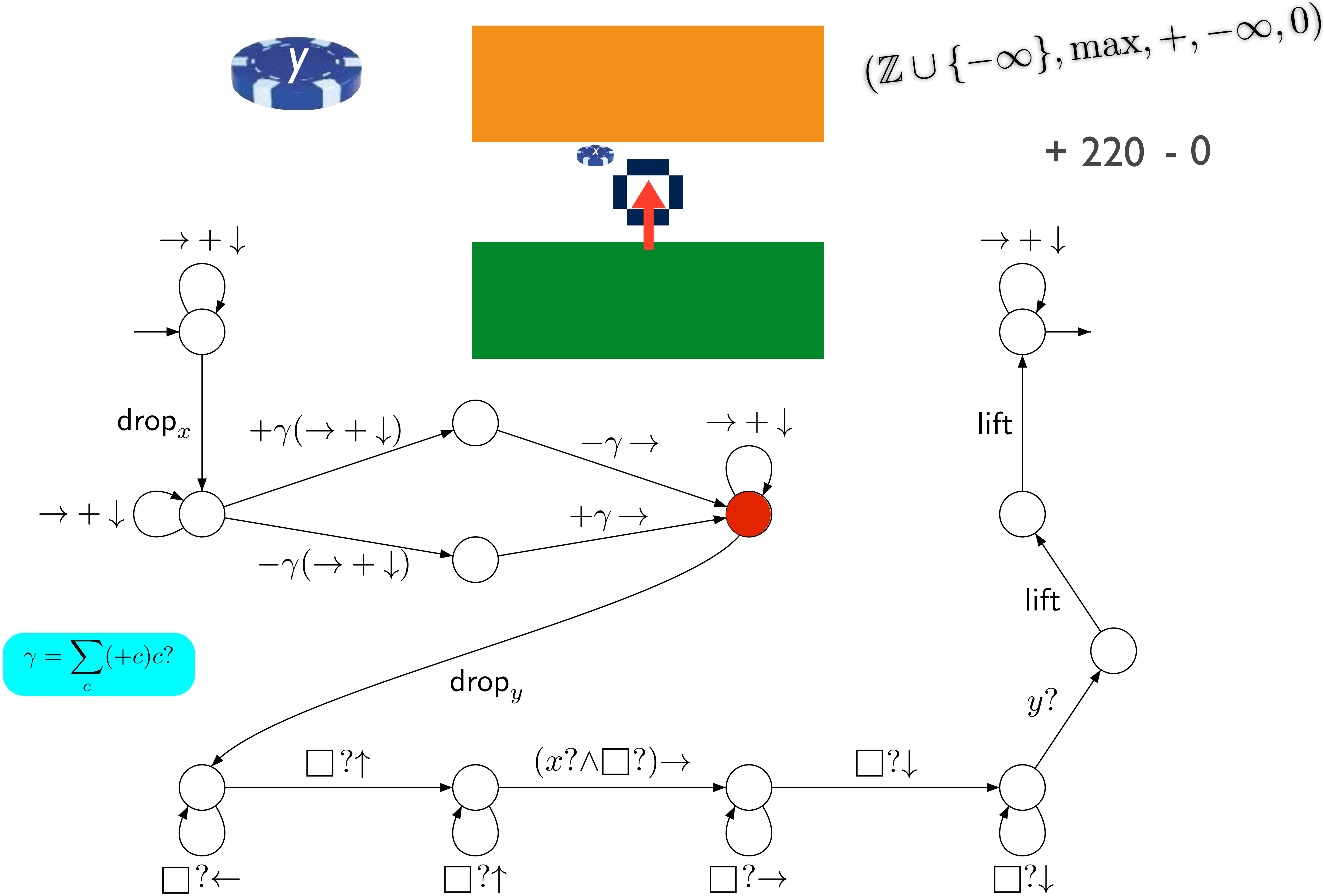

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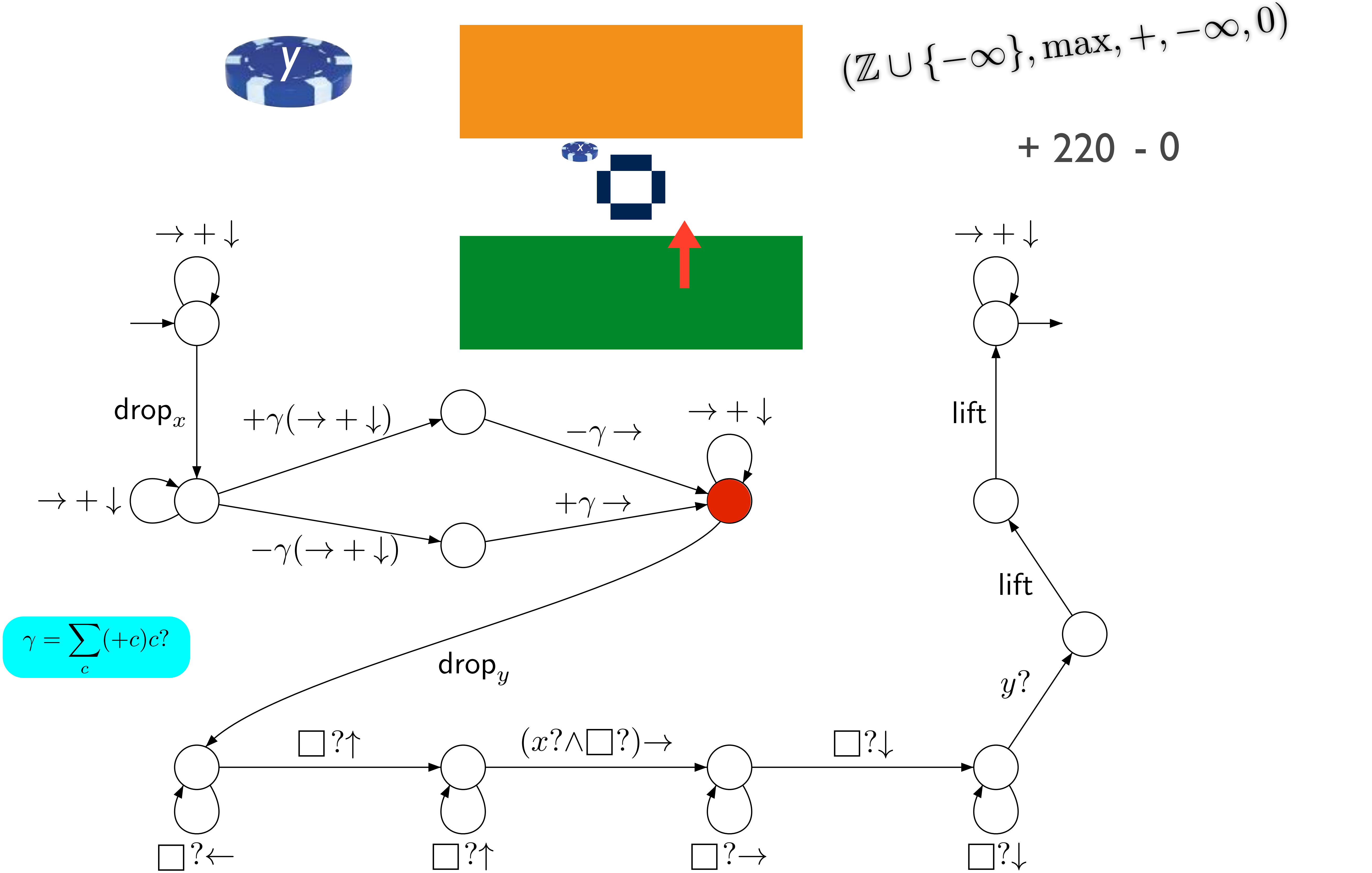

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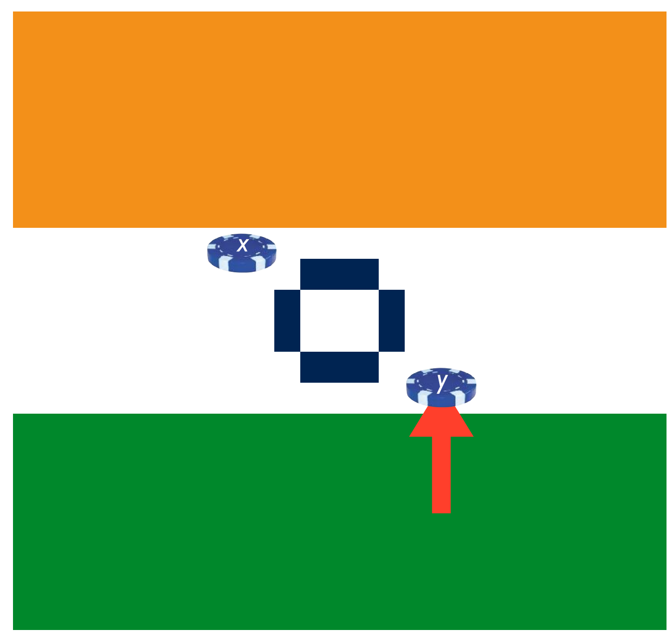
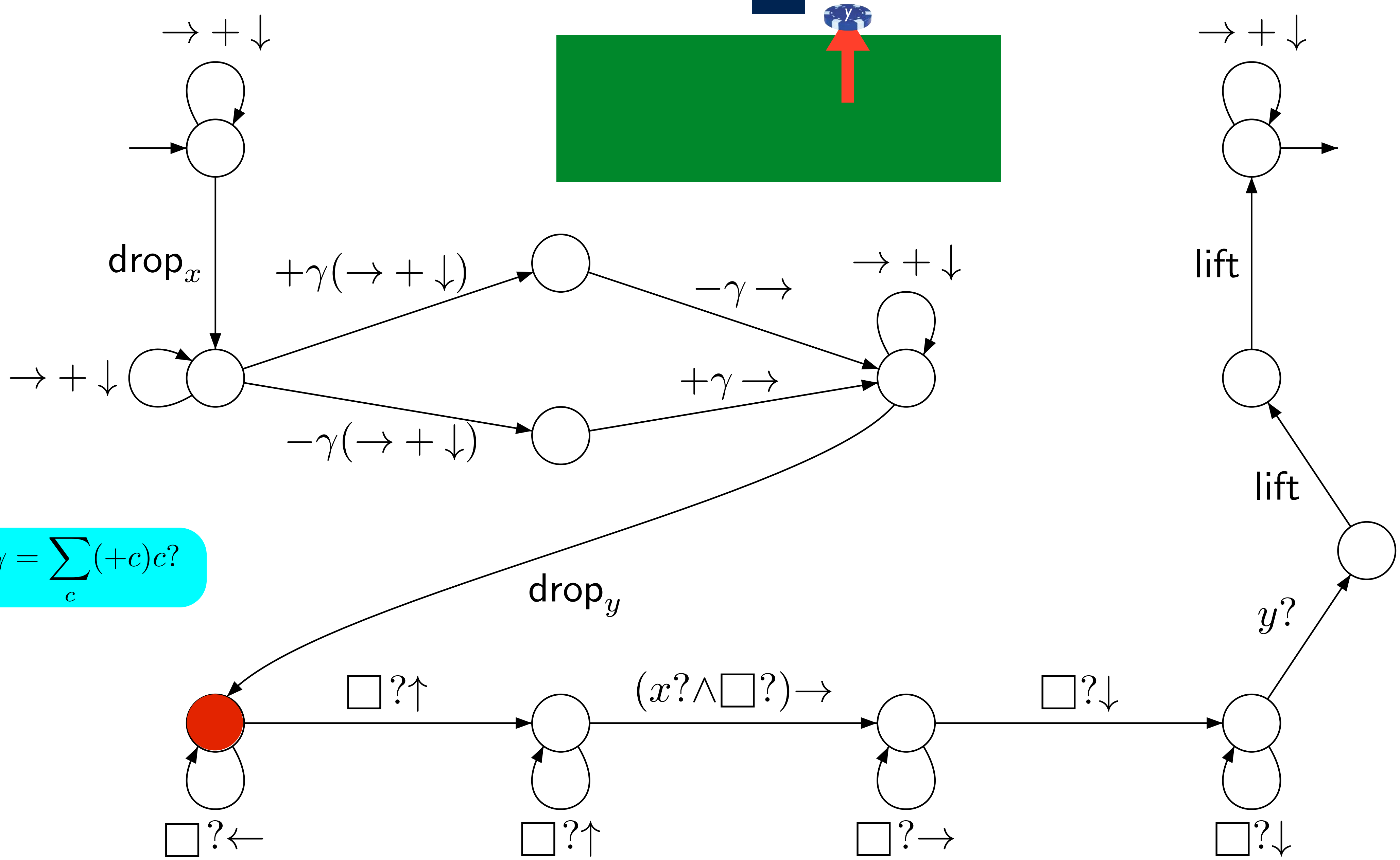


+ 220 - 0



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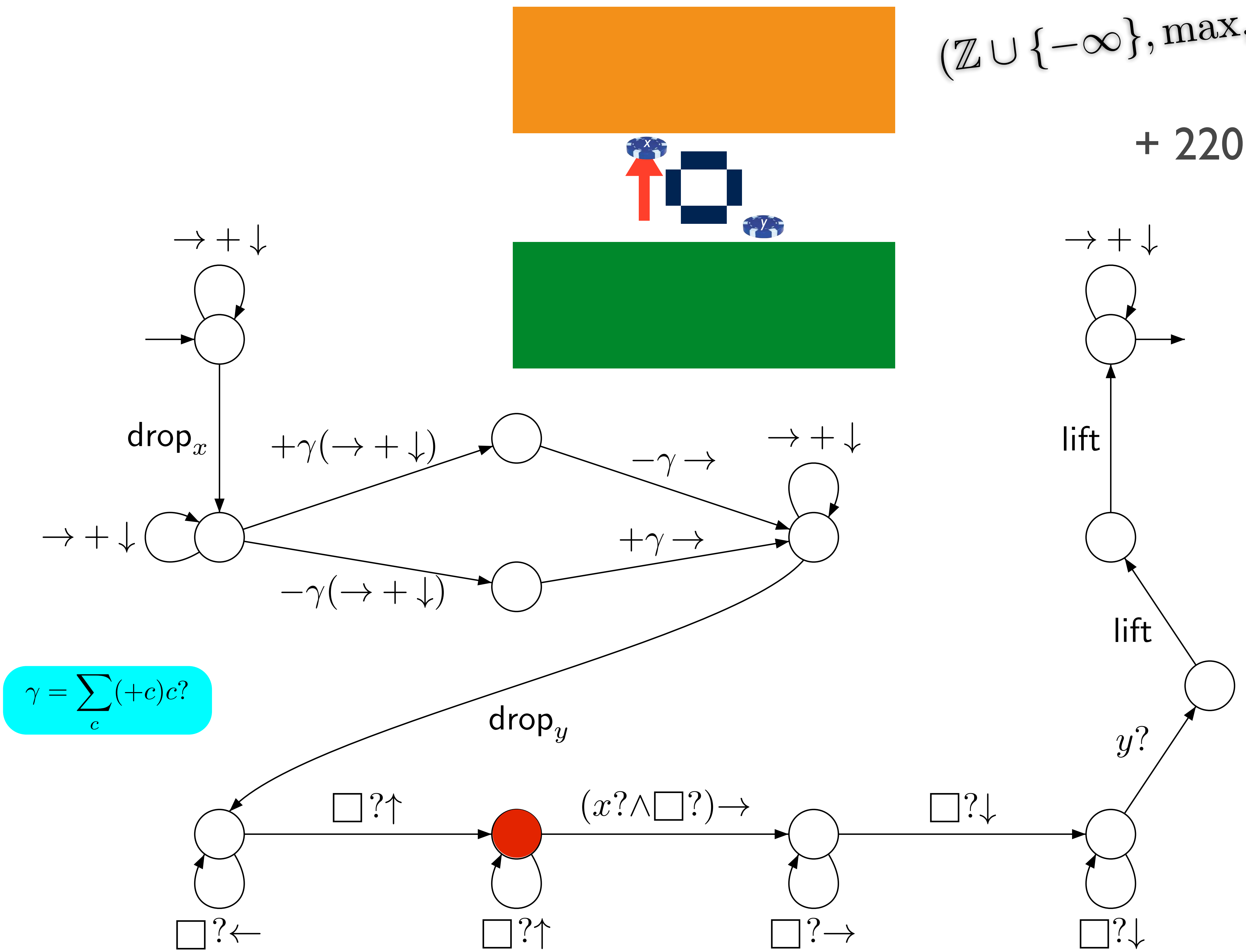

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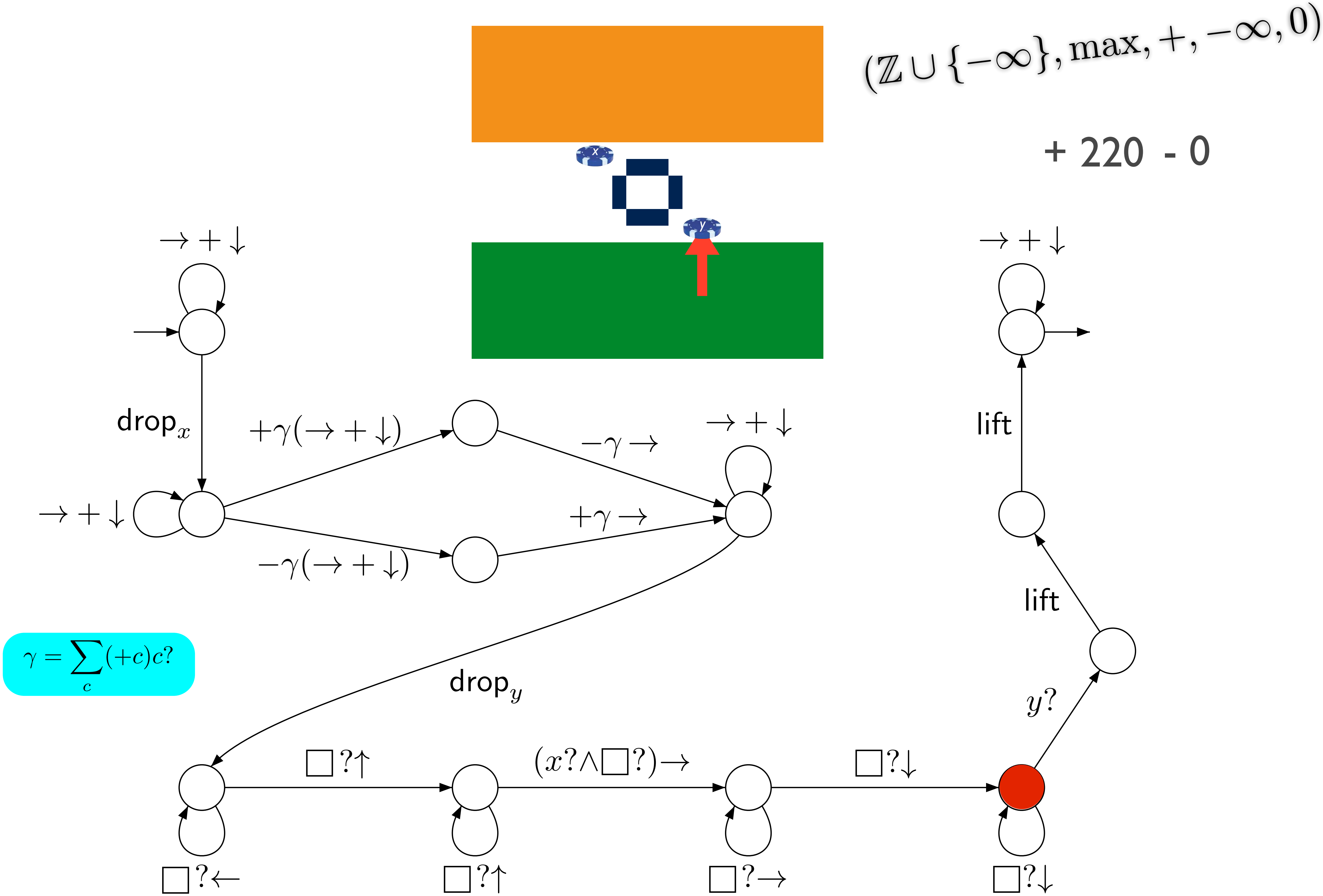

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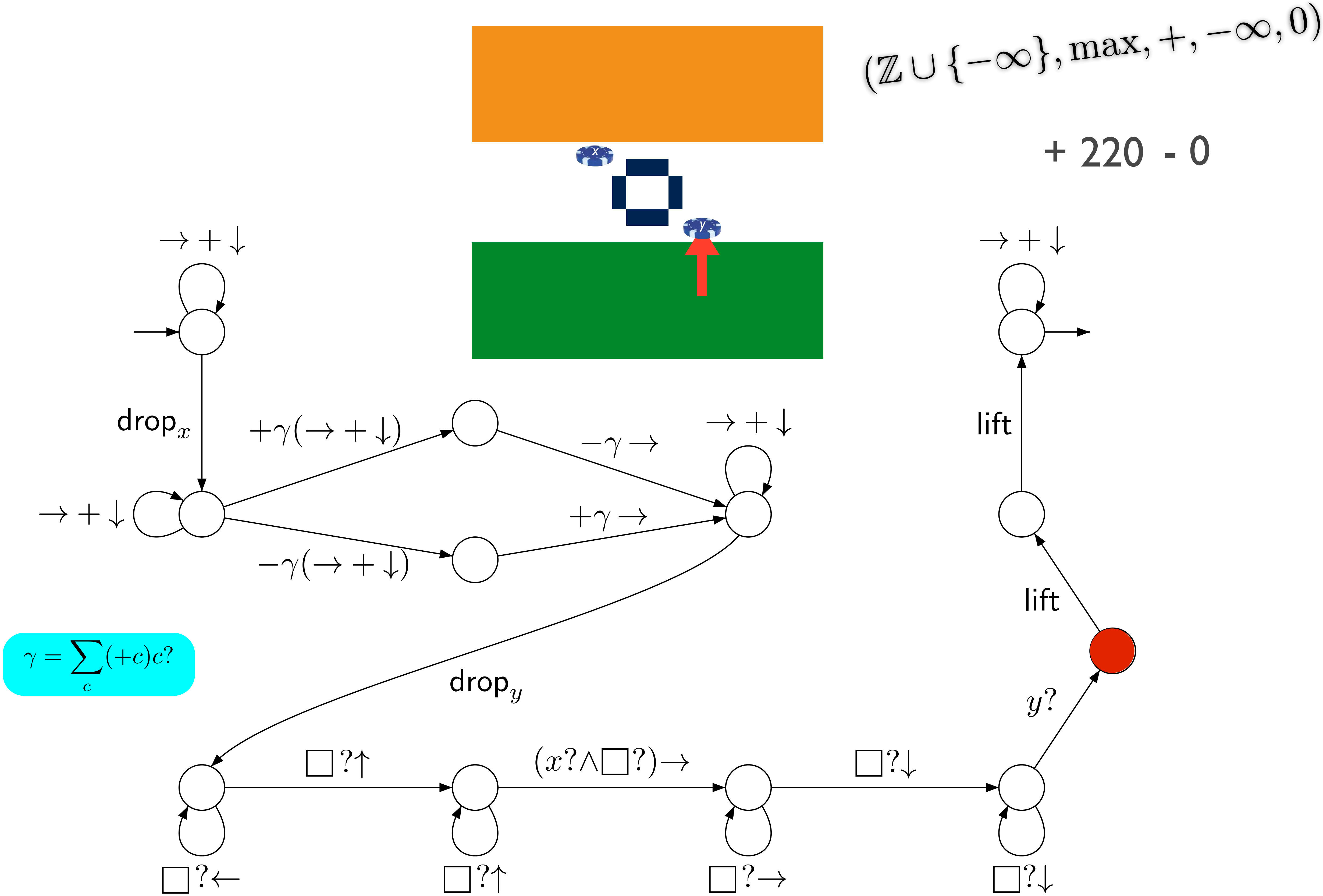

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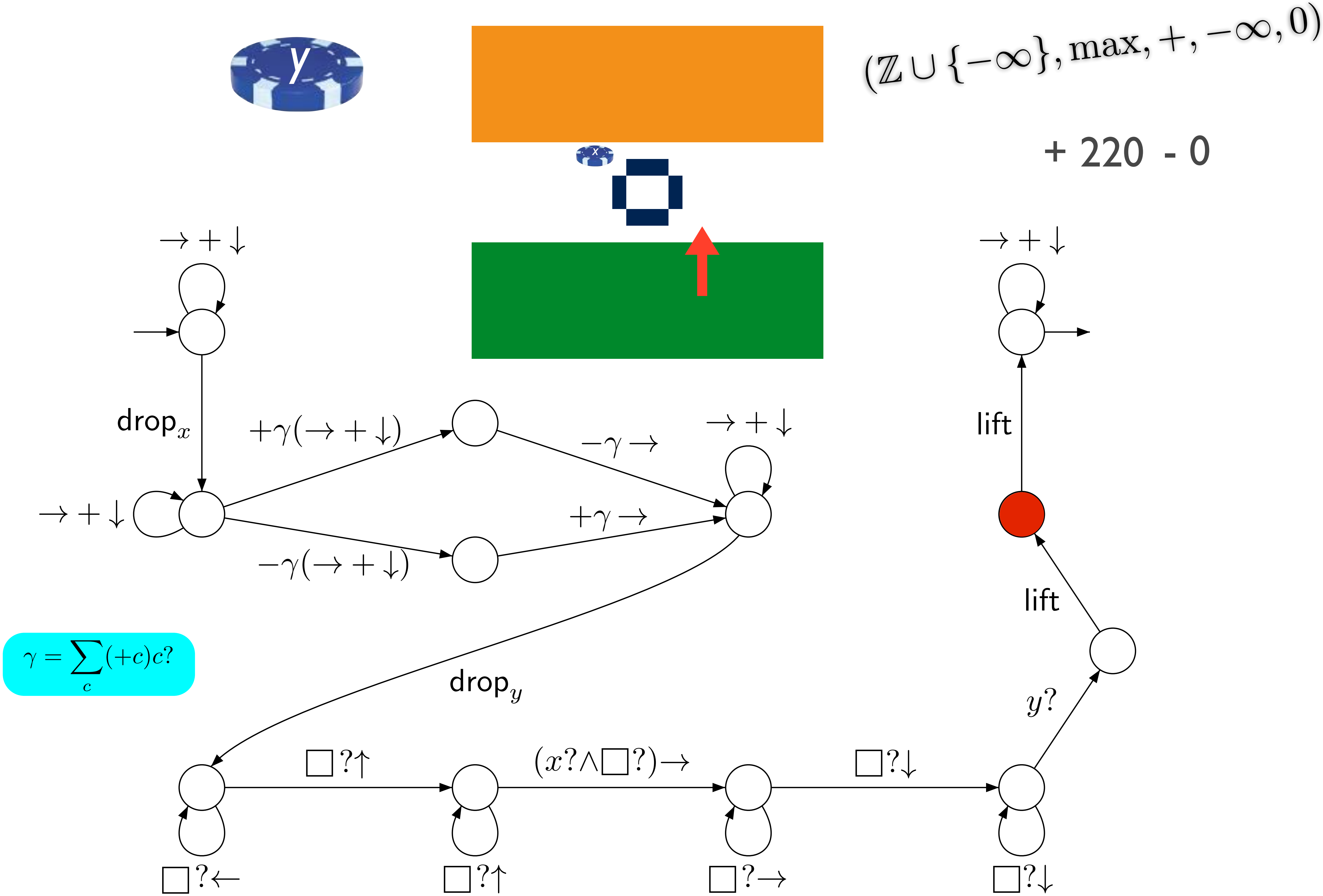

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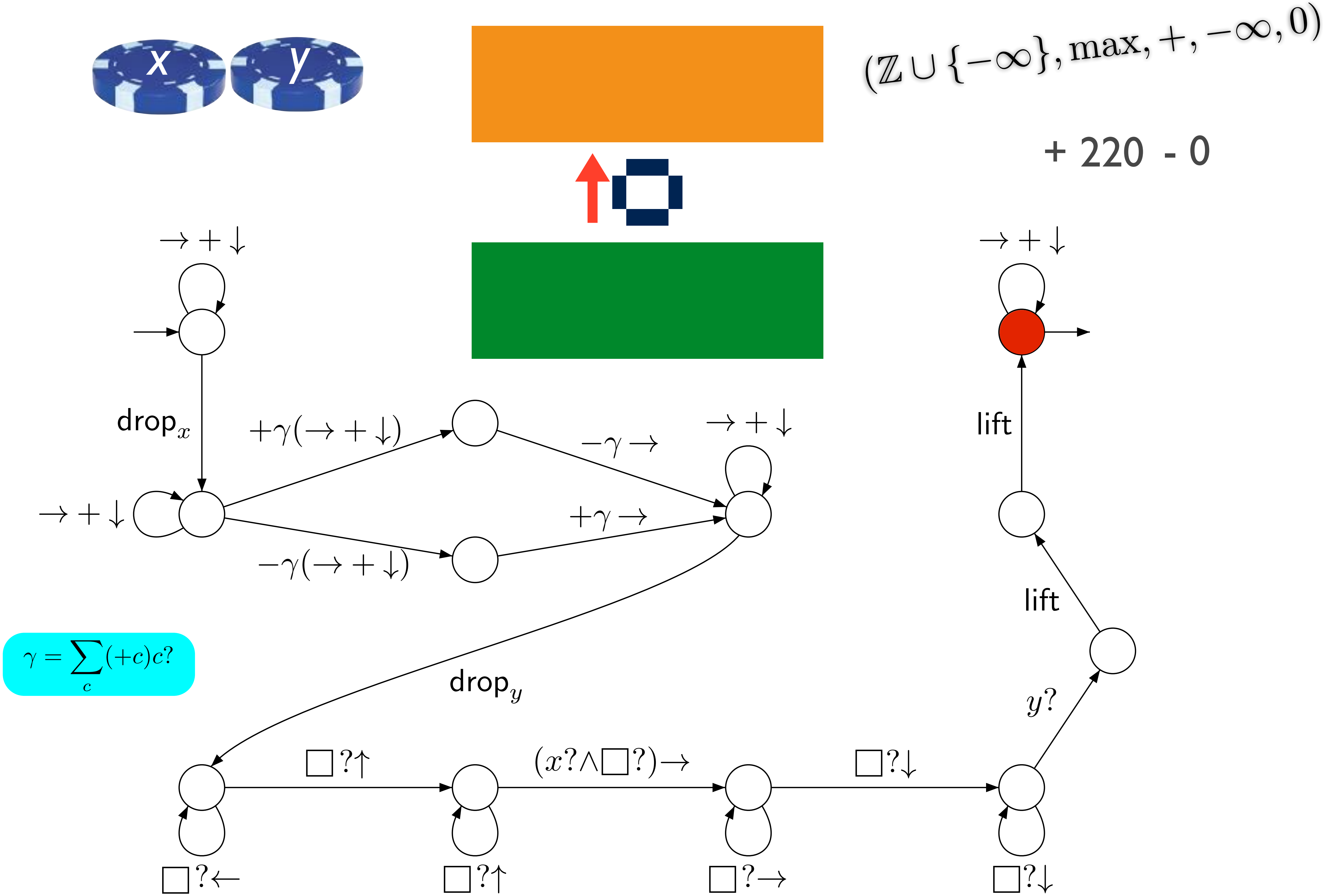

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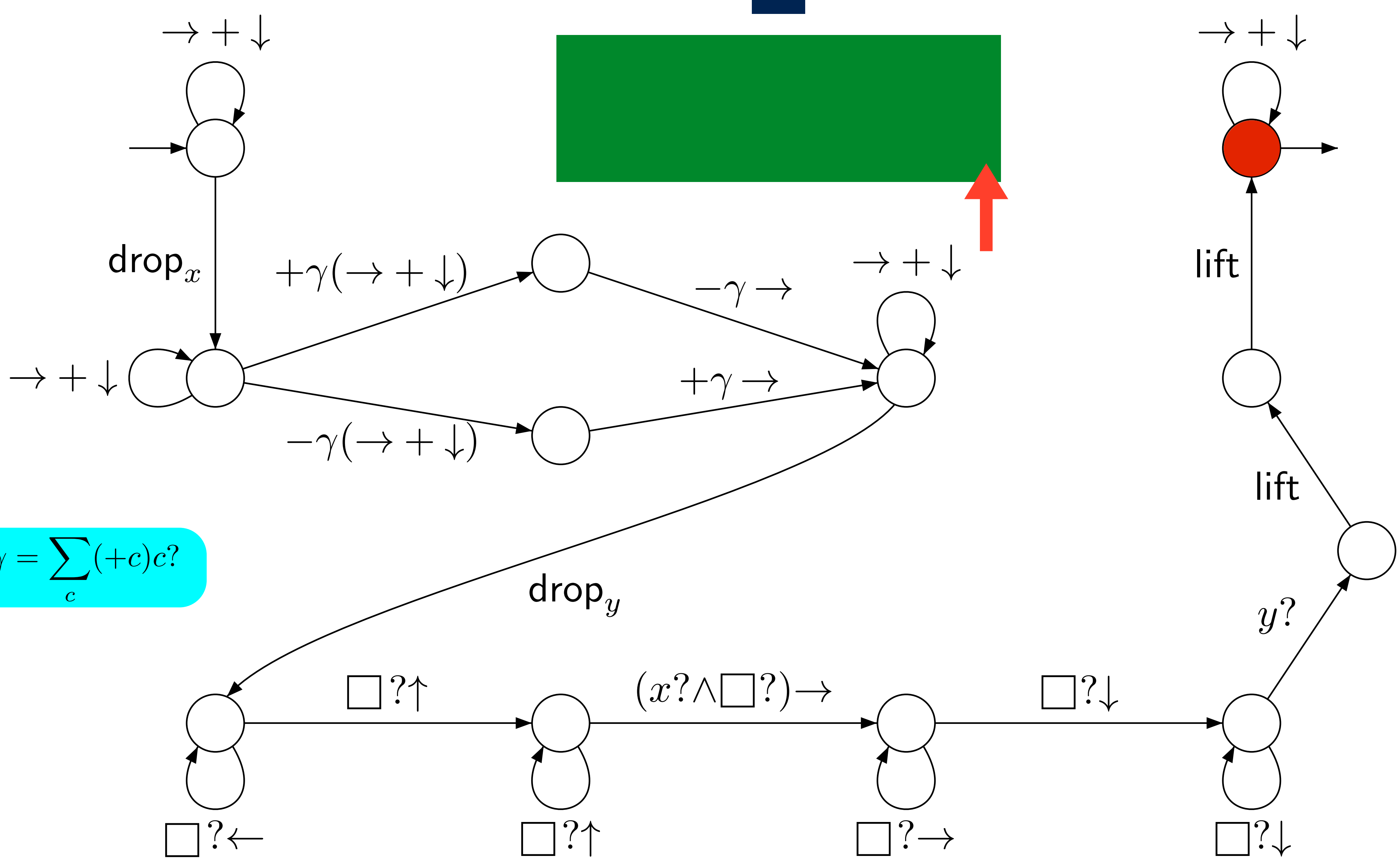

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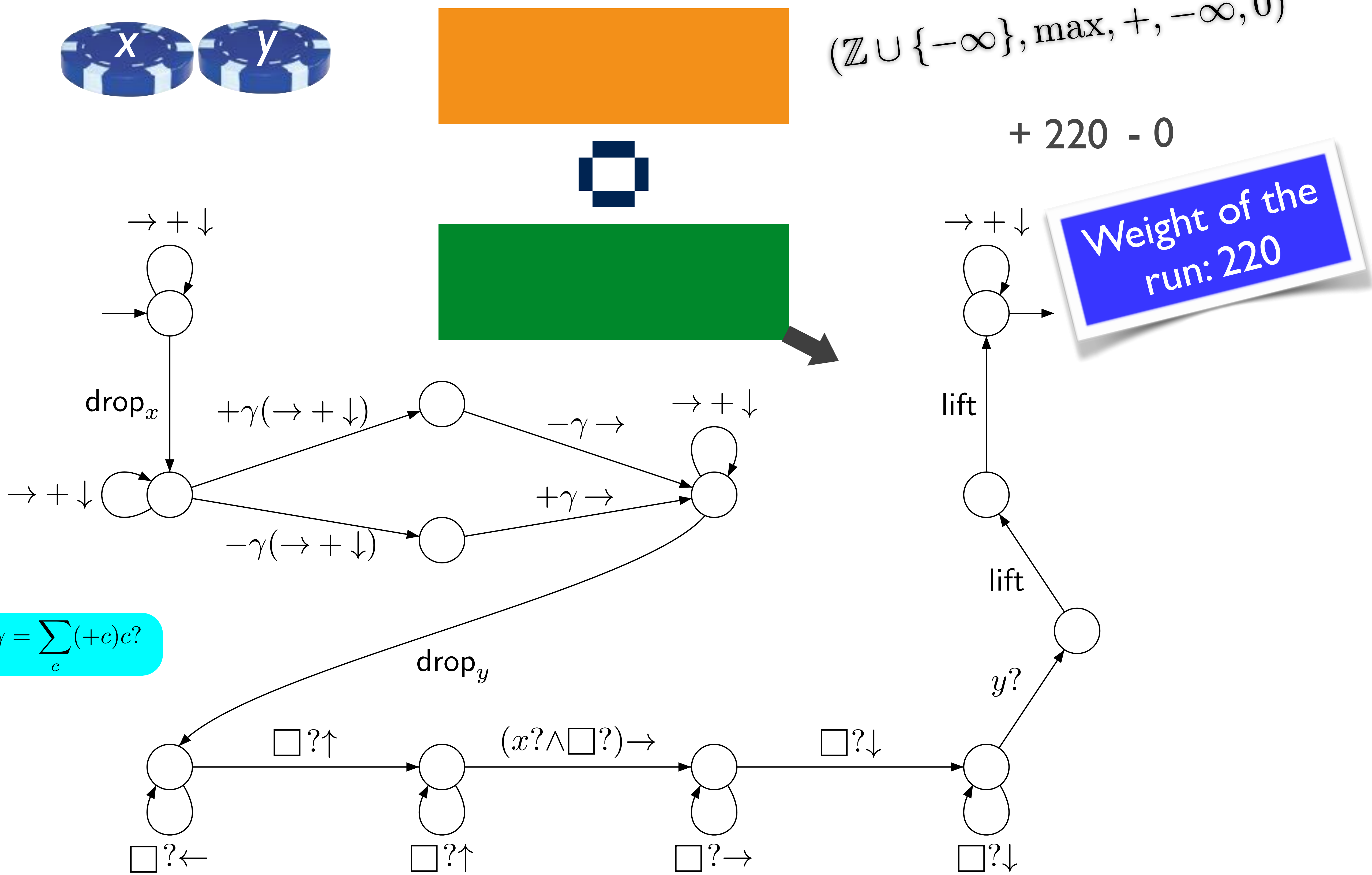

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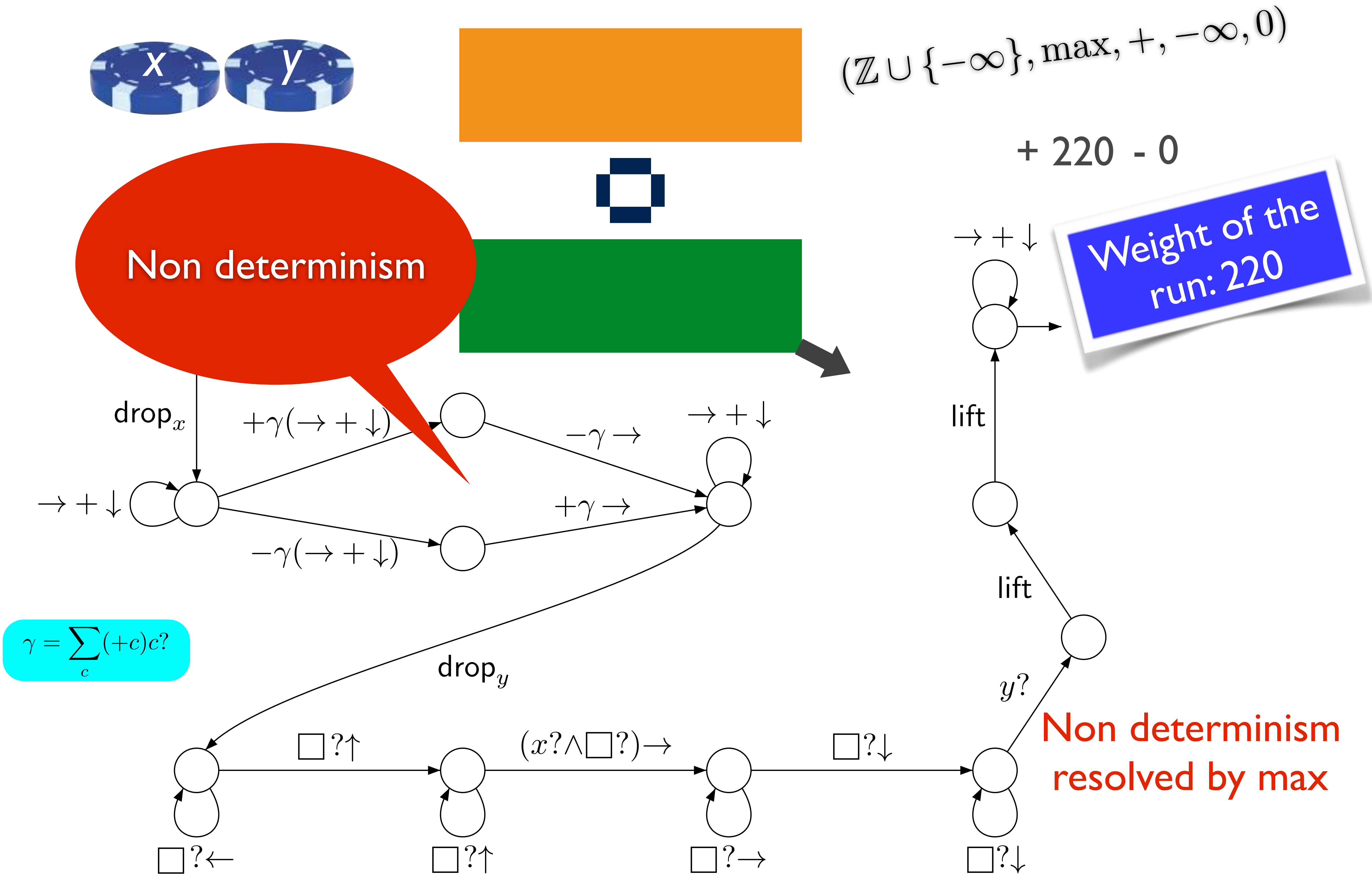

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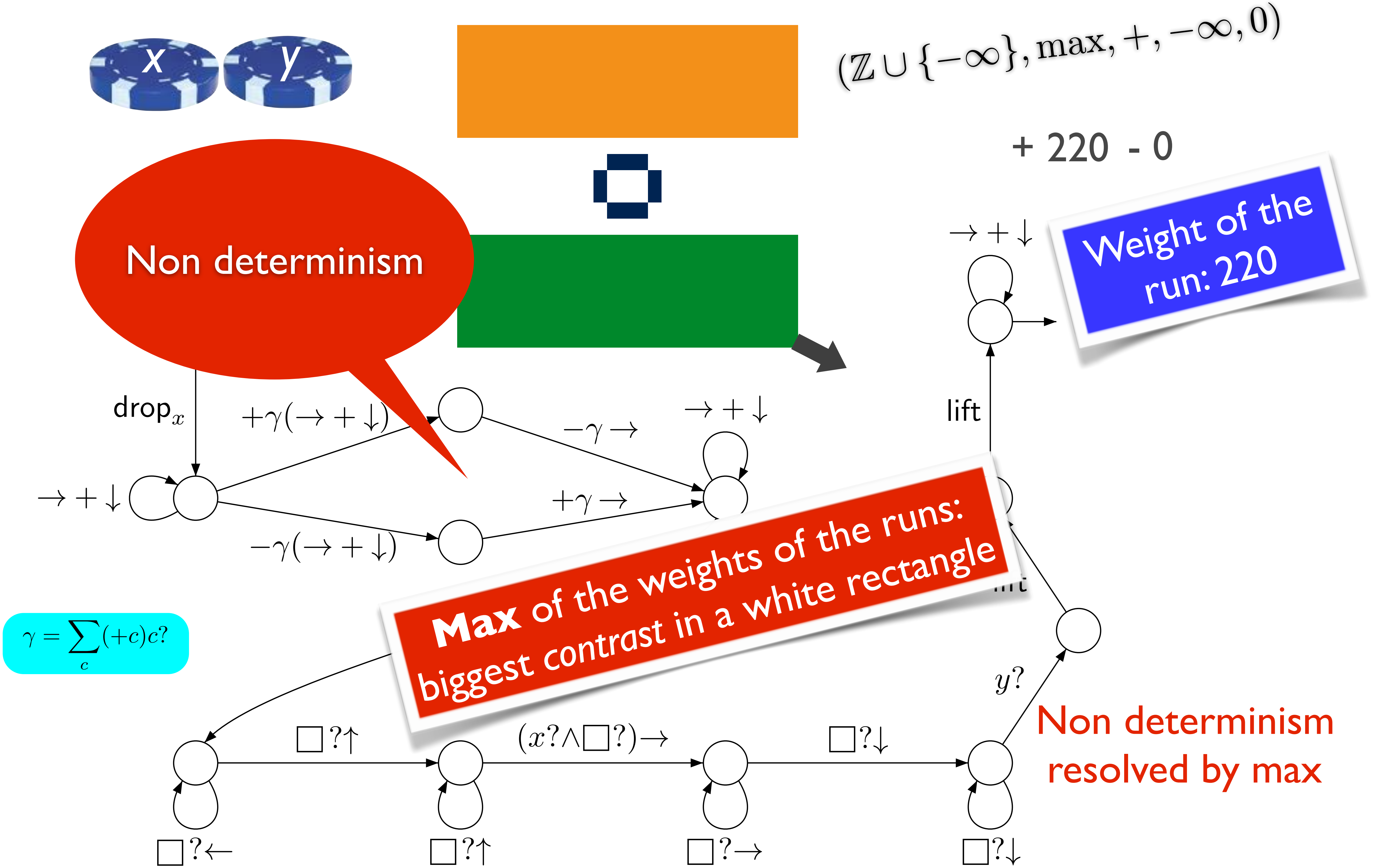

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Translation from Logics to Automata

Theorem: Consider a *searchable* class of graph. Every wFO formula can then be translated into a Pebble Weighted Automaton equivalent over this class of graphs.

WFO ⟶ PWA

Over words: [Bollig&Gastin&Monmege&Zeitoun 2010]
Over nested words: [Bollig&Gastin&Monmege&Zeitoun 2013]

Translation from Logics to Automata

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Which complexity?


Over words: [Bollig&Gastin&Monmege&Zeitoun 2010]
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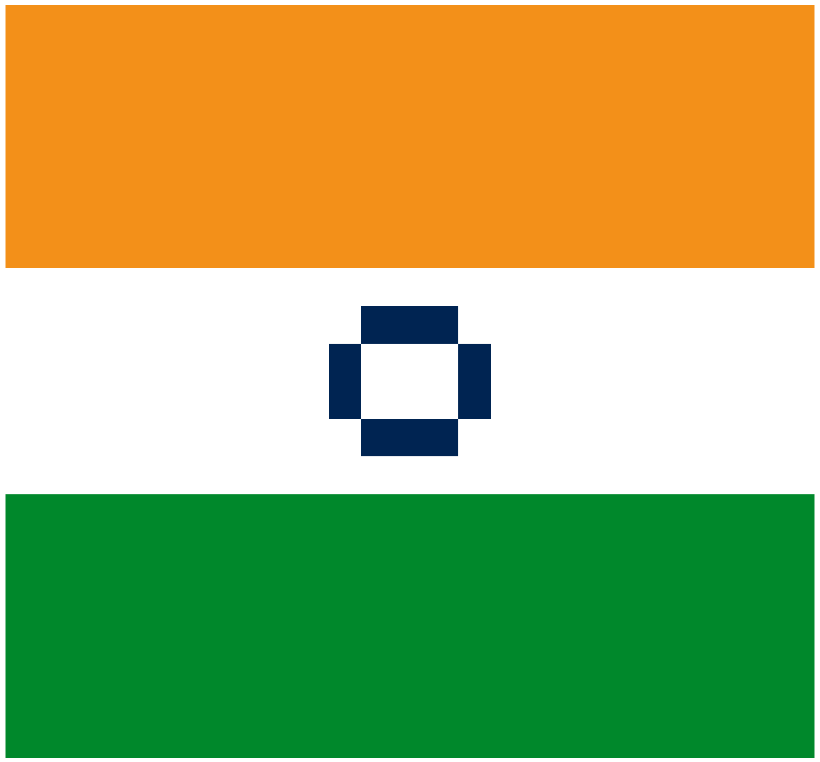
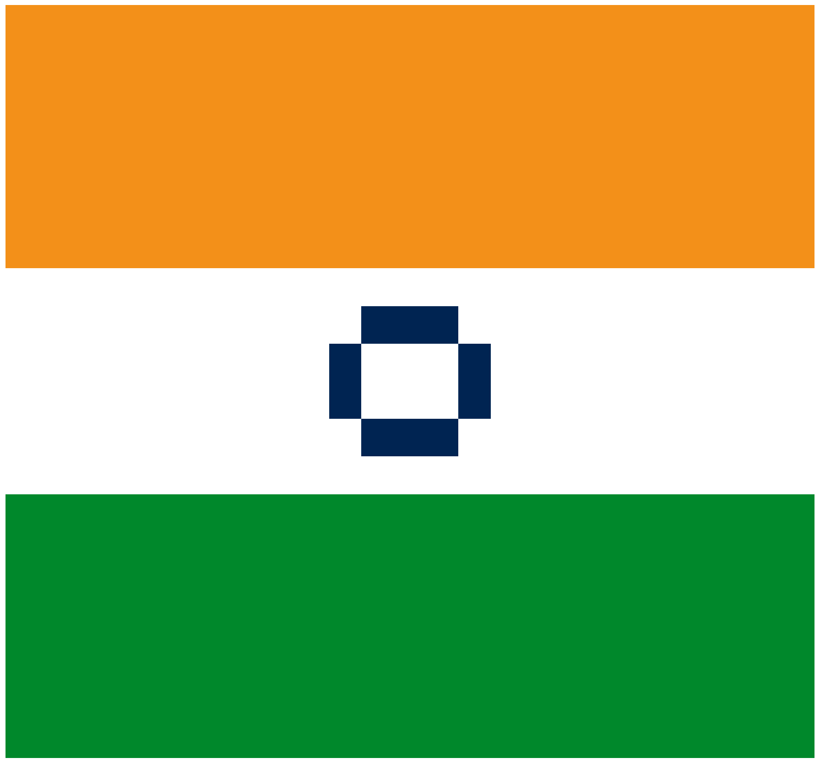
Translation from Logics to Automata



$$\sum_{x} P_{\blacksquare}(x)$$

$(\mathbb{N} \cup \{+\infty\}, +, \times, 0, 1)$

Translation from Logics to Automata



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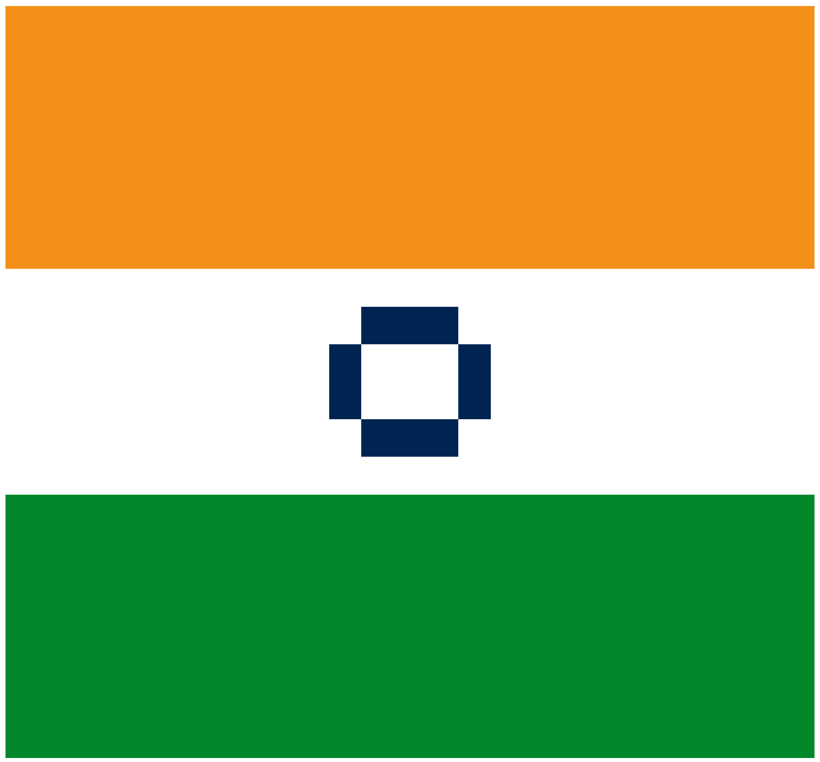
$$\sum_x P_{\blacksquare}(x)$$

use *non-determinism* to count

- a run per position
- each run has the value of the subformula

Translation from Logics to Automata



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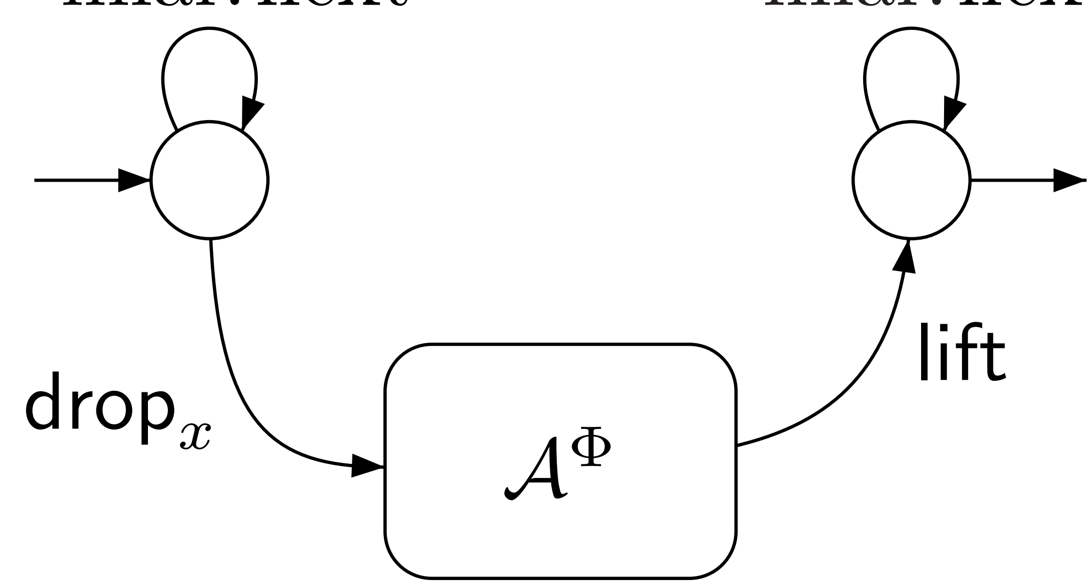

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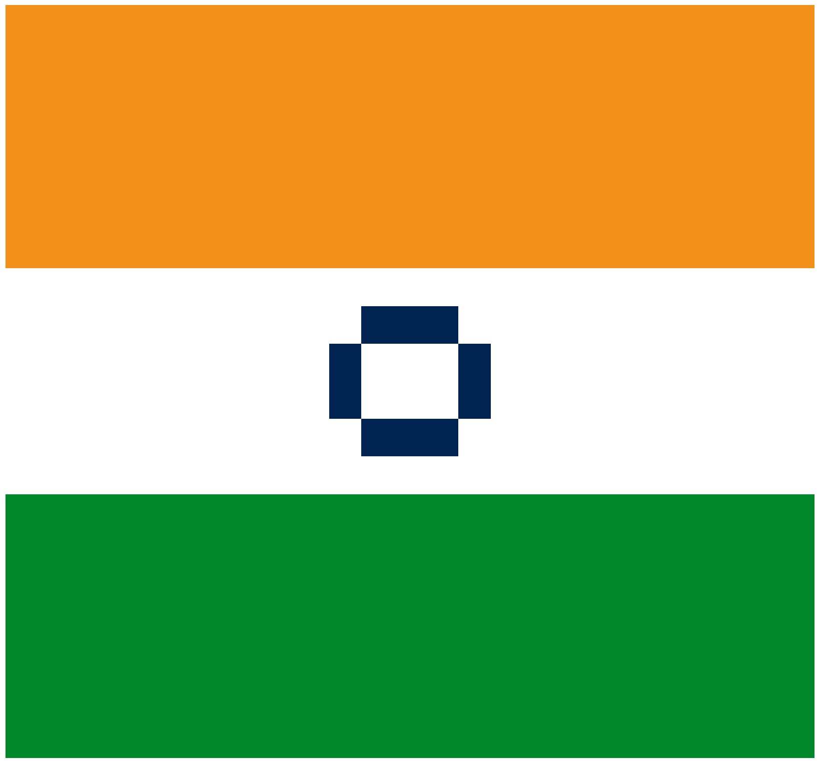

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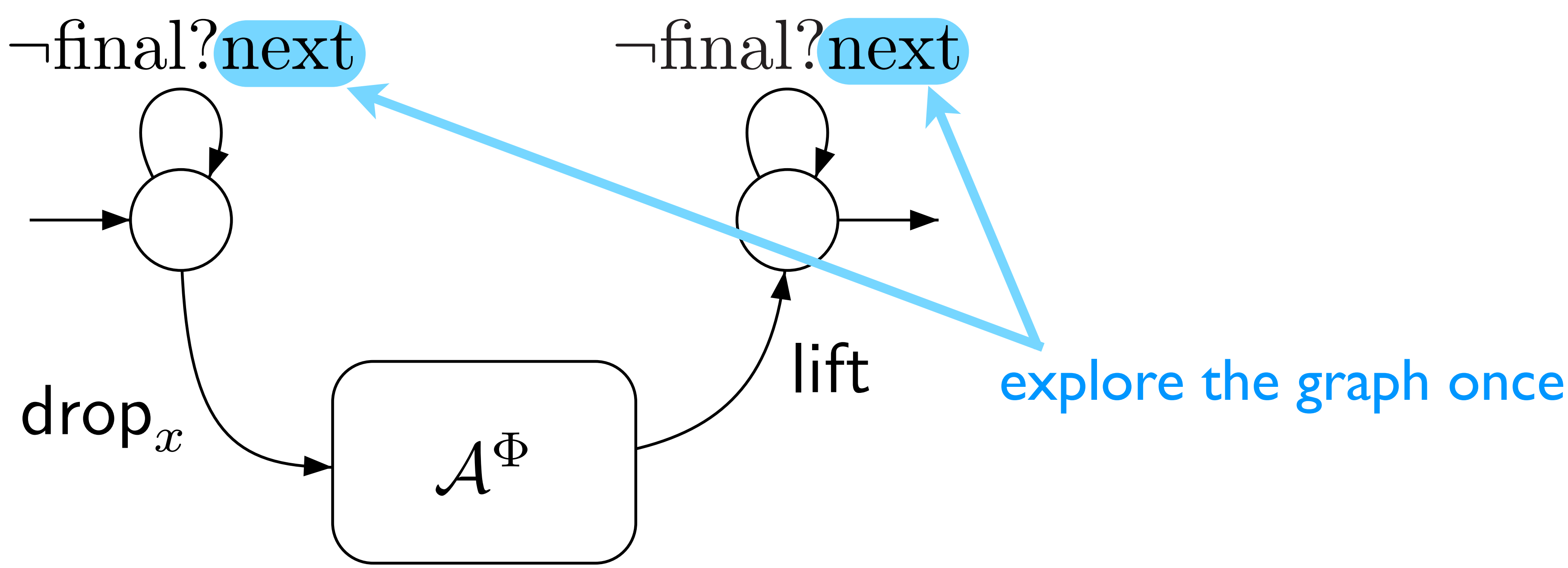

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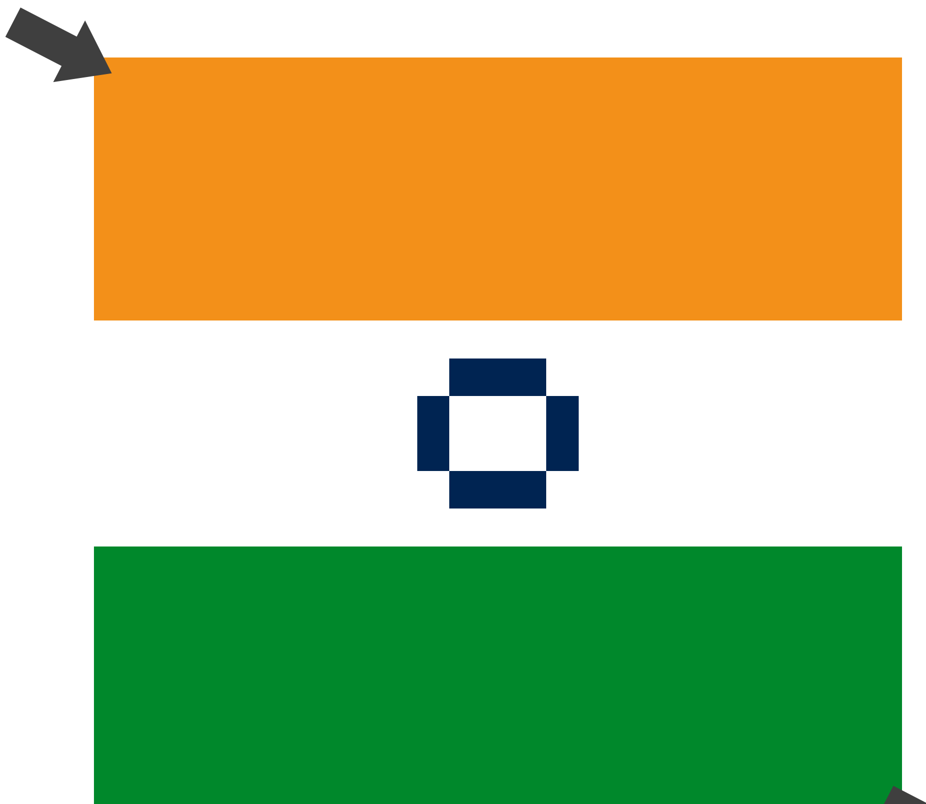


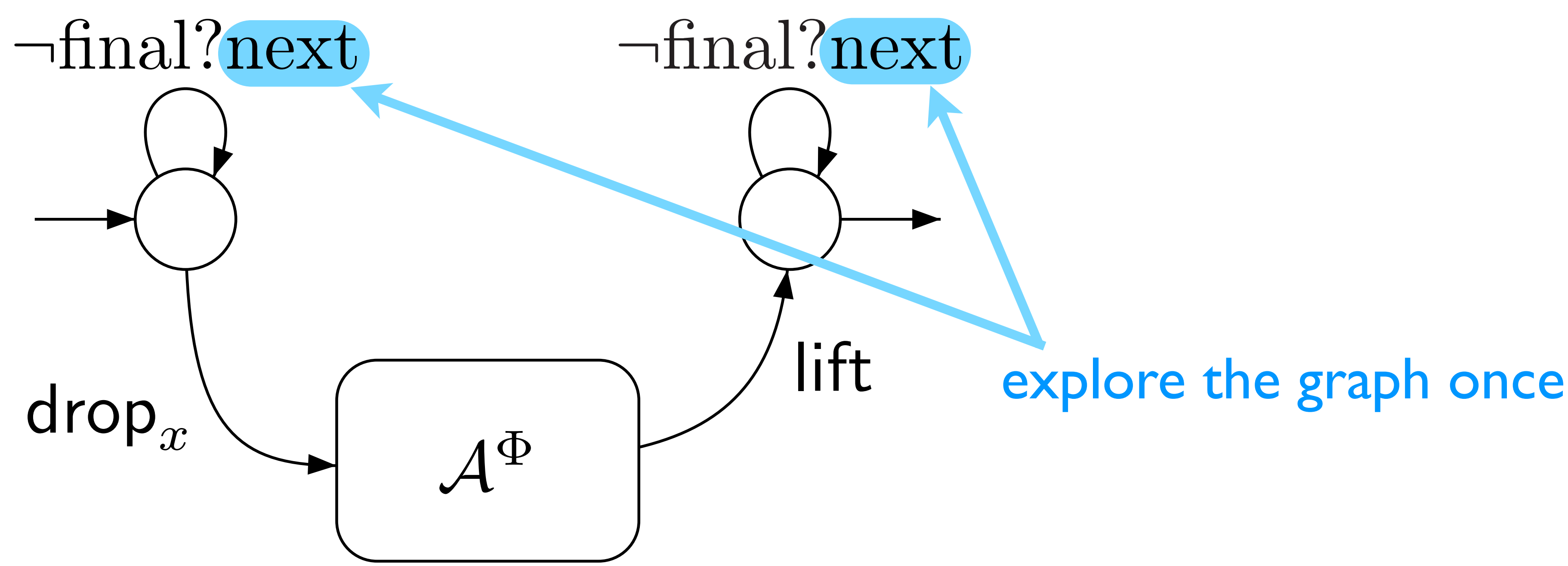
Searchable Graphs

$$G = (V, (E_d)_{d \in D}, \lambda, v^{(i)}, v^{(f)}, \leq)$$

$v^{(i)}$ initial vertex

$v^{(f)}$ final vertex

$\leq$ total order over vertices, computable with navigating automata



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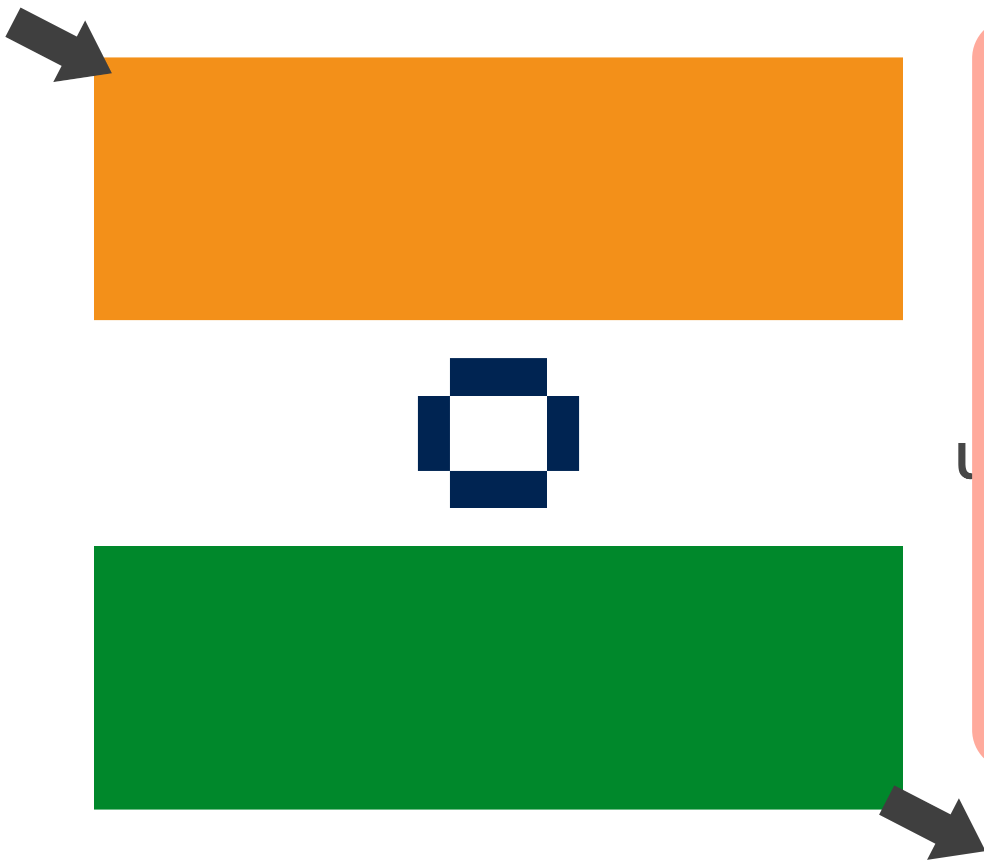

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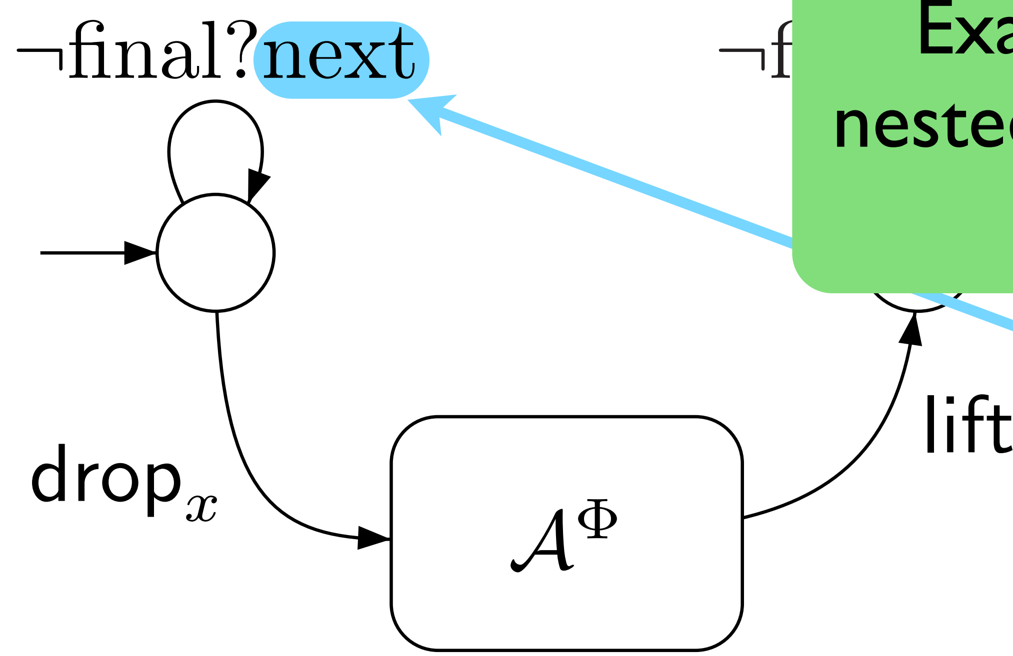
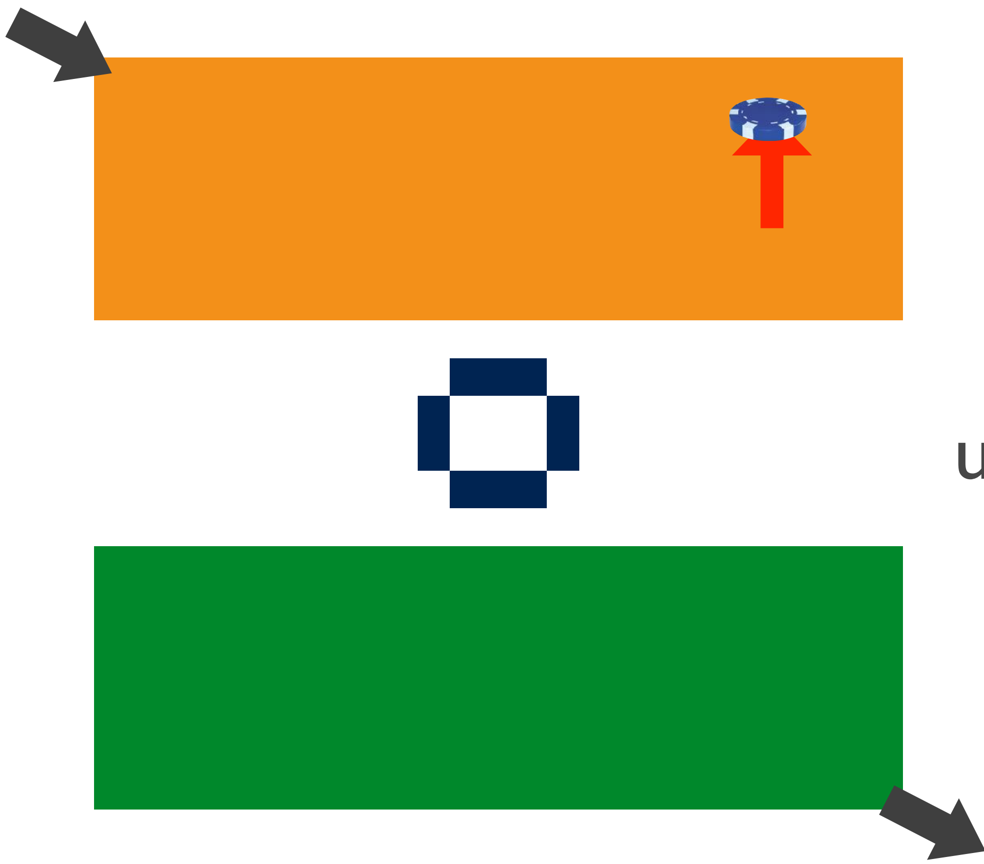


Examples: words, trees, nested words, Mazurkiewicz traces, pictures...

explore the graph once

Translation from Logics to Automata



$(\mathbb{N} \cup \{+\infty\}, +, \times, 0, 1)$

$$\sum_x \Phi(x)$$

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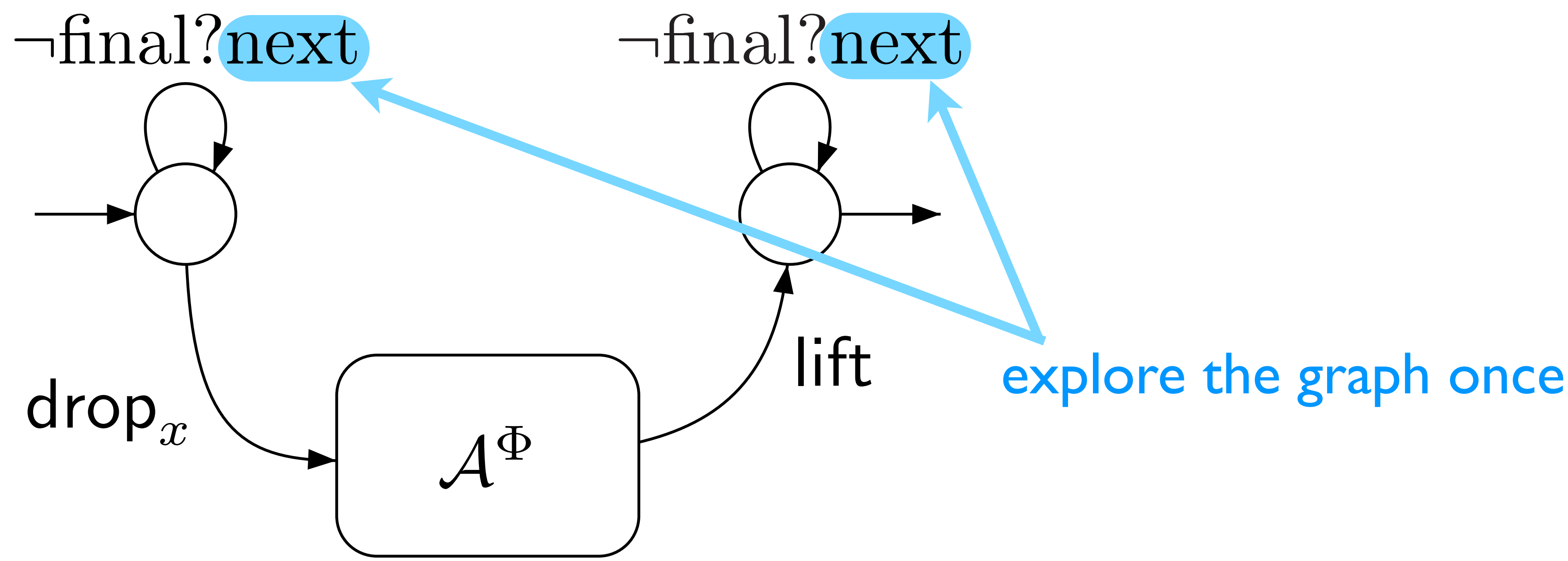

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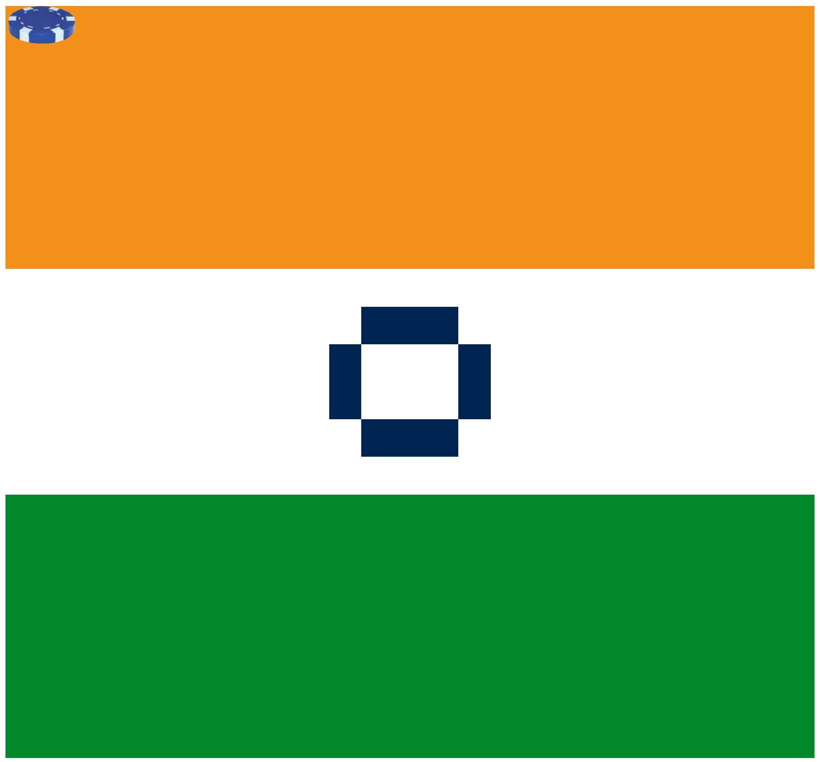
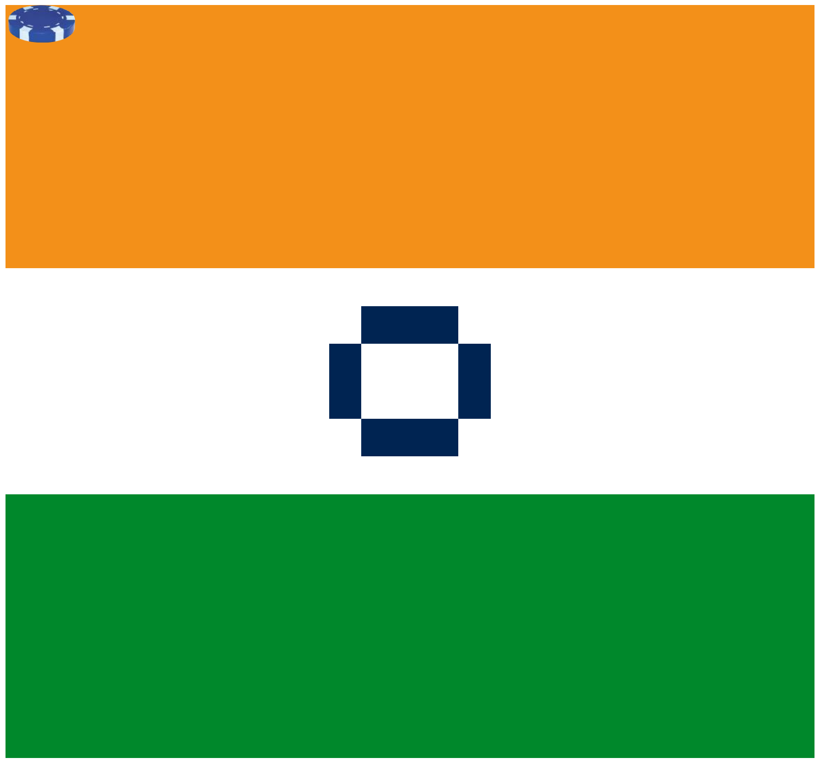


$$\prod_{x} \Phi(x)$$

Translation from Logics to Automata

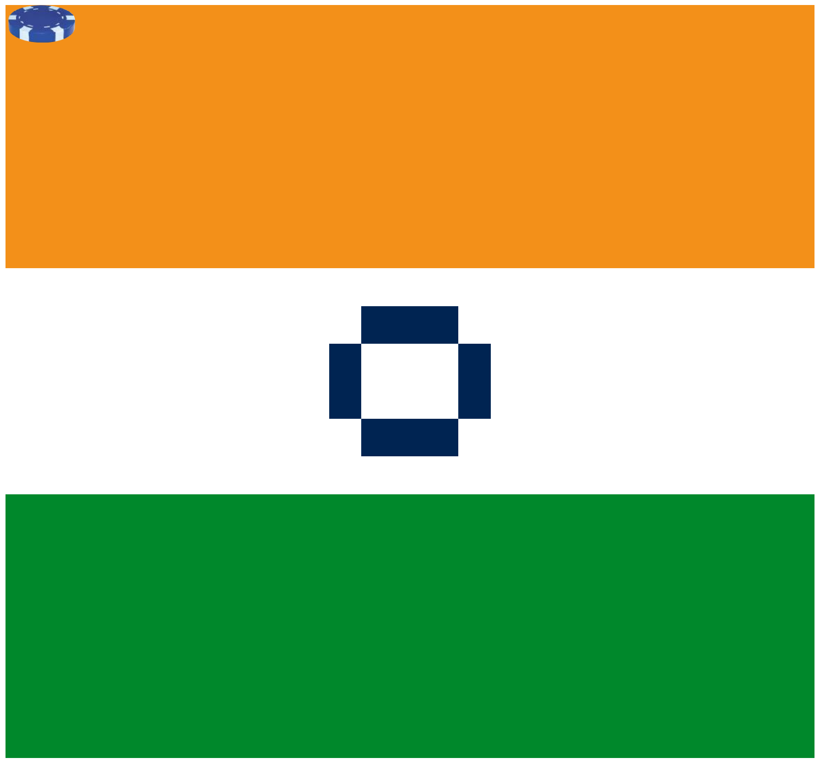


$$\prod_x \Phi(x)$$

use *sequentialization* to multiply

- a single accepting run
- multiply the values of subformula along this run

Translation from Logics to Automata



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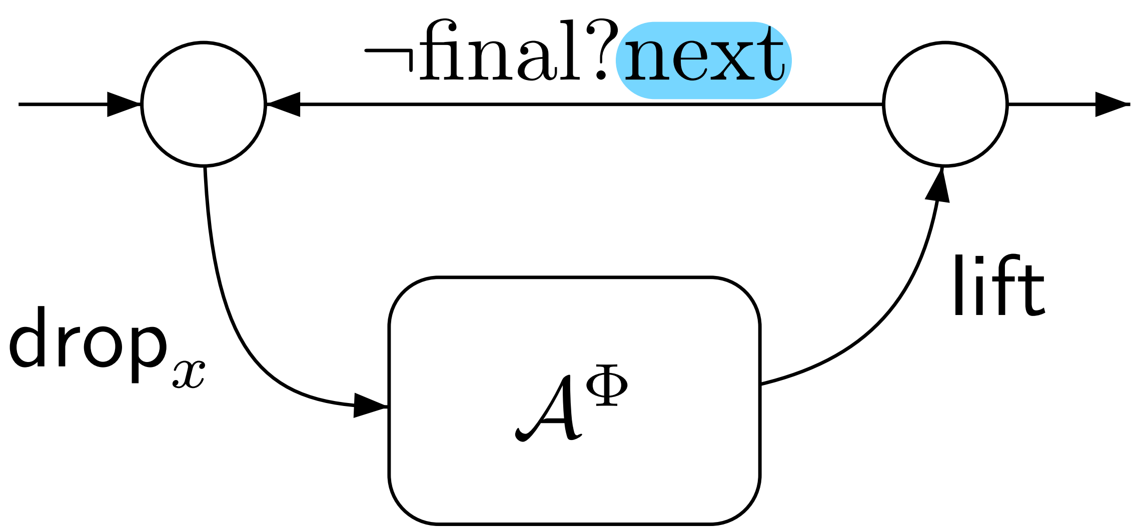
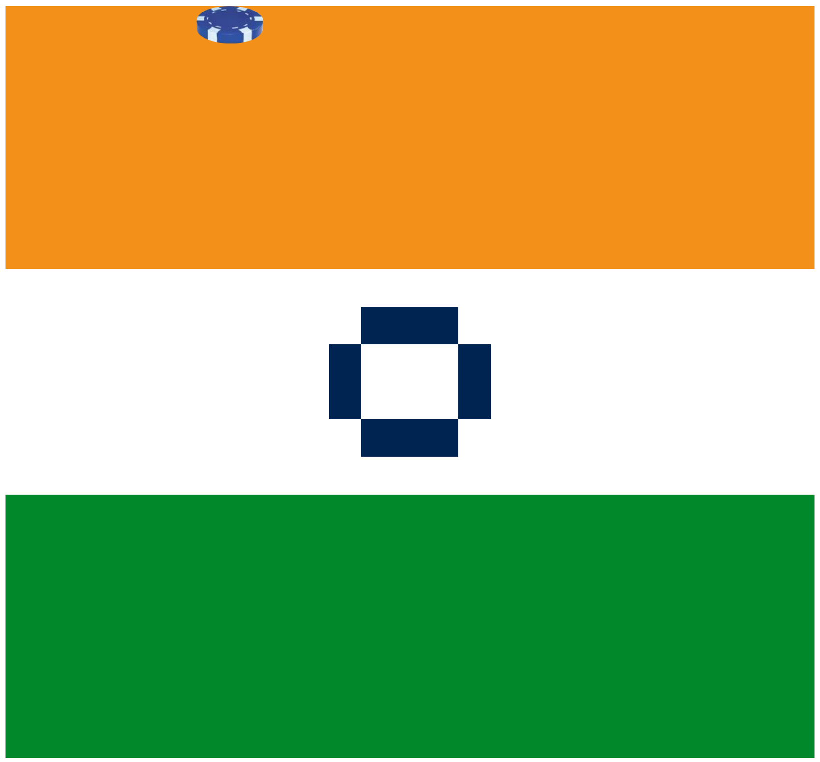


Translation from Logics to Automata



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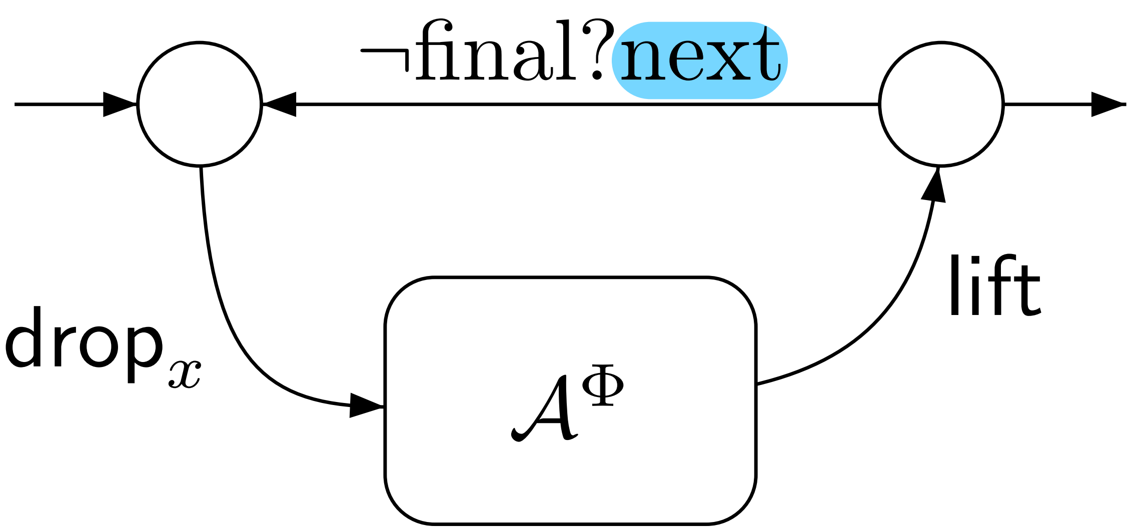


Translation from Logics to Automata

Challenging for the *Boolean* part:
we need unambiguous automata

Translation from Logics to Automata

Challenging for the *Boolean* part:
we need unambiguous automata

Use deterministic automata
of size non-elementary...

Translation from Logics to Automata

Challenging for the *Boolean* part:
we need unambiguous automata

Use deterministic automata
of size non-elementary...

Take advantage of the
navigation and the pebbles to
build linear sized automata

Translation from Logics to Automata

$$(\exists x\, \varphi(x))?3 : 5$$

$$(\mathbb{N} \cup \{+\infty\}, +, \times, 0, 1)$$

Translation from Logics to Automata

$$(\exists x\, \varphi(x))?3 : 5$$

$(\mathbb{N} \cup \{+\infty\}, +, \times, 0, 1)$

use *unambiguous* *non-determinism* to check

- a single accepting run
- run has value 3 or 5 depending on the truth value of the Boolean subformula

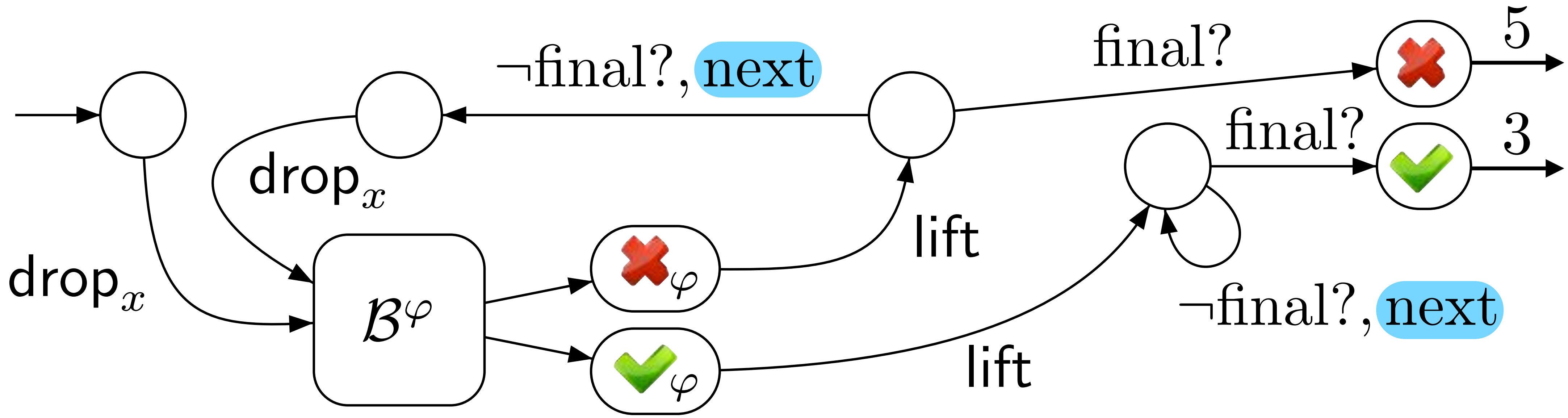
Translation from Logics to Automata

$(\mathbb{N} \cup \{+\infty\}, +, \times, 0, 1)$

$$(\exists x\, \varphi(x))?3 : 5$$

use *unambiguous* *non-determinism* to check

- a single accepting run
- run has value 3 or 5 depending on the truth value of the Boolean subformula



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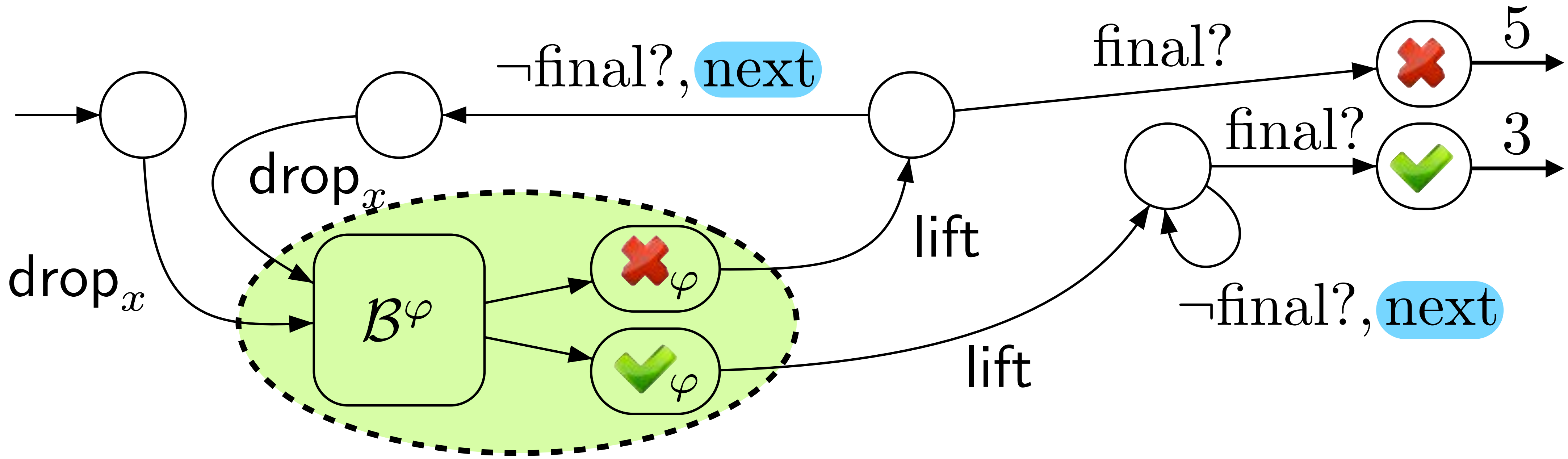

unambiguous automaton for formula $\varphi(x)$

Translation from Logics to Automata

Theorem: Consider a searchable class of graph. Every wFO formula can be translated into a Pebble Weighted Automaton equivalent over this class of graphs.

$$\text{WFO} \xrightarrow{\text{linear time}} \text{PWA}$$

Obtained automata are of linear size with respect to the size of the formula

Logic equivalent to PWA?

- Weighted FO misses a counting capability…
- Solution: weighted transitive closure operation

Over words: [Bollig&Gastin&Monmege&Zeitoun 2010]

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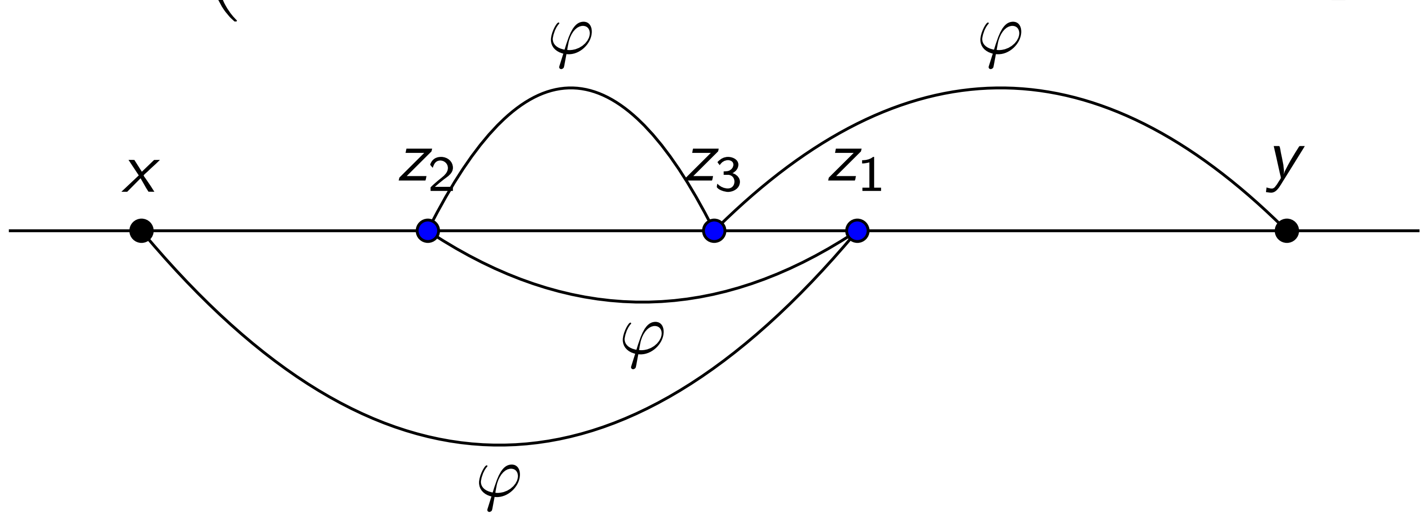
$$\varphi^1(x,y) = \varphi(x,y)$$

$$\varphi^n(x,y) = \exists z_0 \cdots \exists z_n \Big(x = z_0 \wedge z_n = y \wedge \mathsf{diff}(z_0,\dots,z_n) \wedge \big[\textstyle\bigwedge_{1\leq \ell \leq n} \varphi(z_{\ell-1}, z_\ell)\big]\Big)$$



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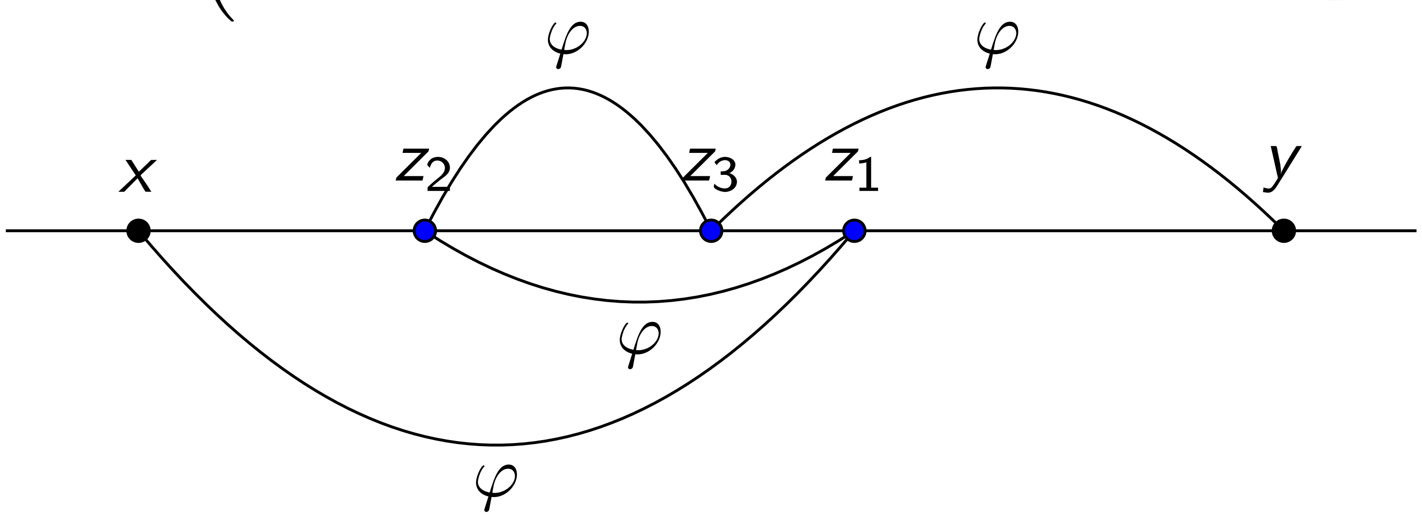
$$\mathrm{TC}_{xy}\varphi = \textstyle\bigvee_{n \geq 1} \varphi^n$$



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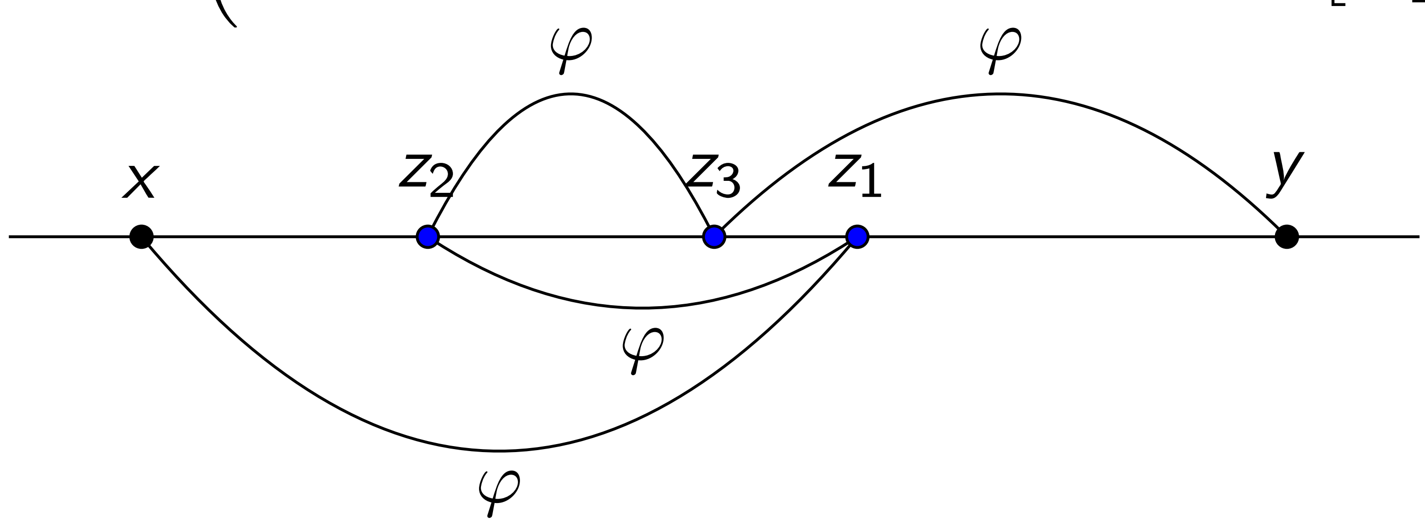
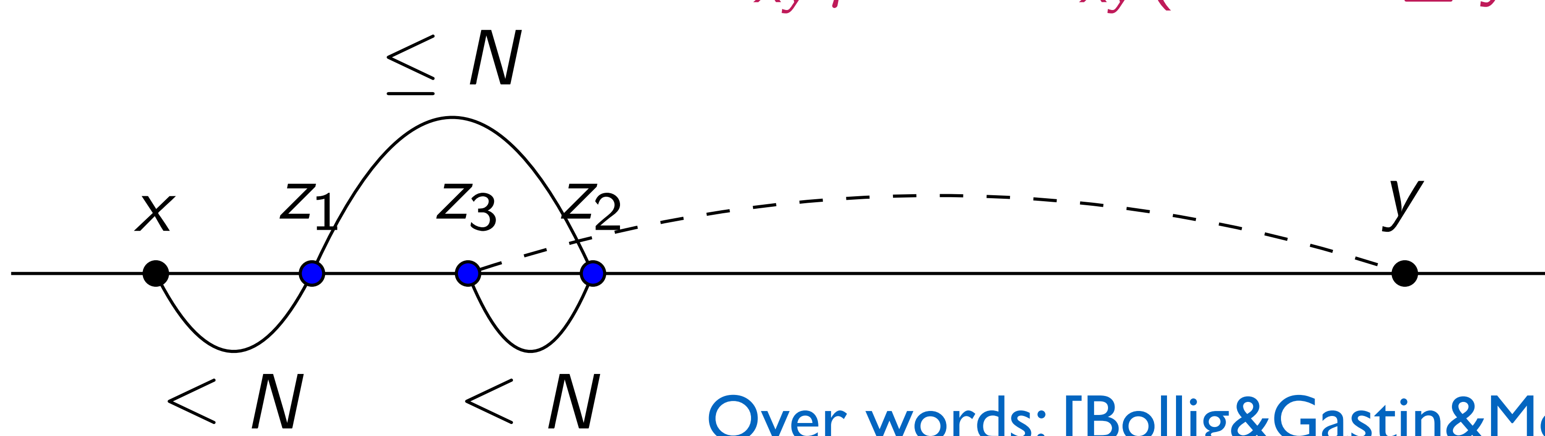


Bounded transitive closure : $N\text{-}\mathrm{TC}_{xy}\varphi = \mathrm{TC}_{xy}(x - N \leq y \leq x + N \wedge \varphi)$



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Theorem: Weighted First Order logic with weighted transitive closure and Pebble Weighted Automata are equivalent for *zonable* and searchable classes of graphs.

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Examples: words, trees, nested words, Mazurkiewicz traces, pictures...

Over words: [Bollig&Gastin&Monmege&Zeitoun 2010]

Deciding wFO?

Input: A pebble weighted automata / A formula of wFO+BTC
Question: Does there exist an equivalent formula in wFO?

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- L is LTL definable
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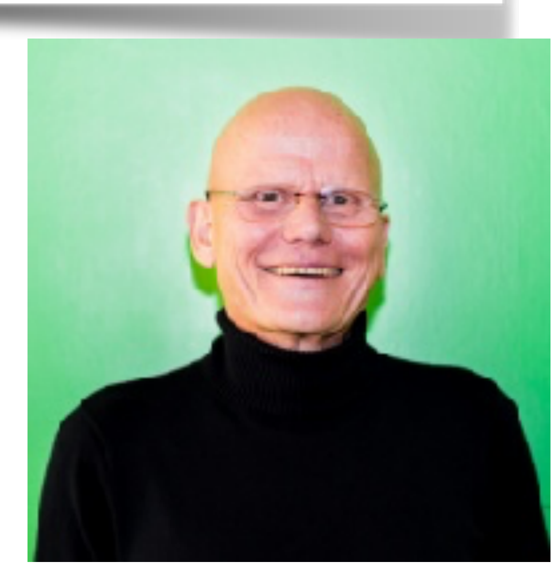

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PSPACE-complete… using algebra

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Decision procedure?… algebra is missing

A special case: the transducers

Functions {

- Two-way Deterministic Finite-State Transducers
- Functional One-way Finite-State Transducers
- MSOT (à la Courcelle)
- Copyless Streaming String Transducers (Alur et al)

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→ only finite valued relations…

Transduction as weights

- Desire: weight transitions with words… Difficult to equip A* with a semiring structure: how to combine several accepting runs?

- Works for deterministic or unambiguous automata: functional transducers

- For relations: semiring of languages

$$(2^{A^*}, \cup, \cdot, \varnothing, \{\varepsilon\})$$

Examples

$$\prod_x \left(P_x(a)?\{aa\} : \left(P_x(b)?\{bb\} : \varnothing\right)\right)$$

Examples

$\prod_x (P_x(a)?\{aa\} : (P_x(b)?\{bb\} : \varnothing))$ *aba* -> *aabbaa*

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Relation

Examples

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ababbaabb -> *aaaaccccc*

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Not comp. by 1-way
Func Transducer

ababbaabb -> *aaaaccccc*

Examples

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aba -> {*ε,a,b,ab,ba,aa,aba*}

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aba

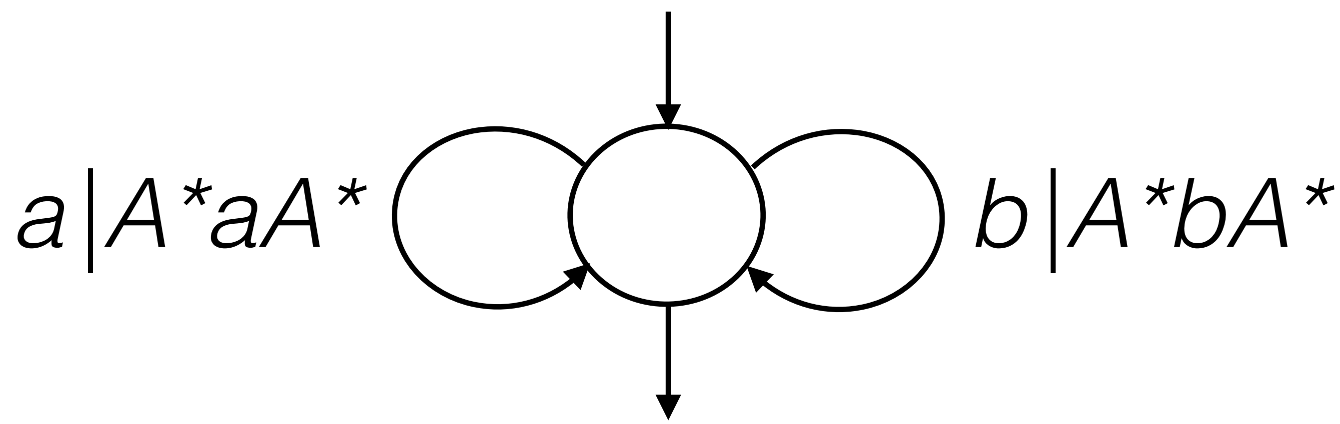
Infinitely-valued relation

$$\prod_x P_x(a)?\,A^* a A^*:(P_x(b)?\,A^* b A^*)$$

aba -> *A*aA*bA*aA**

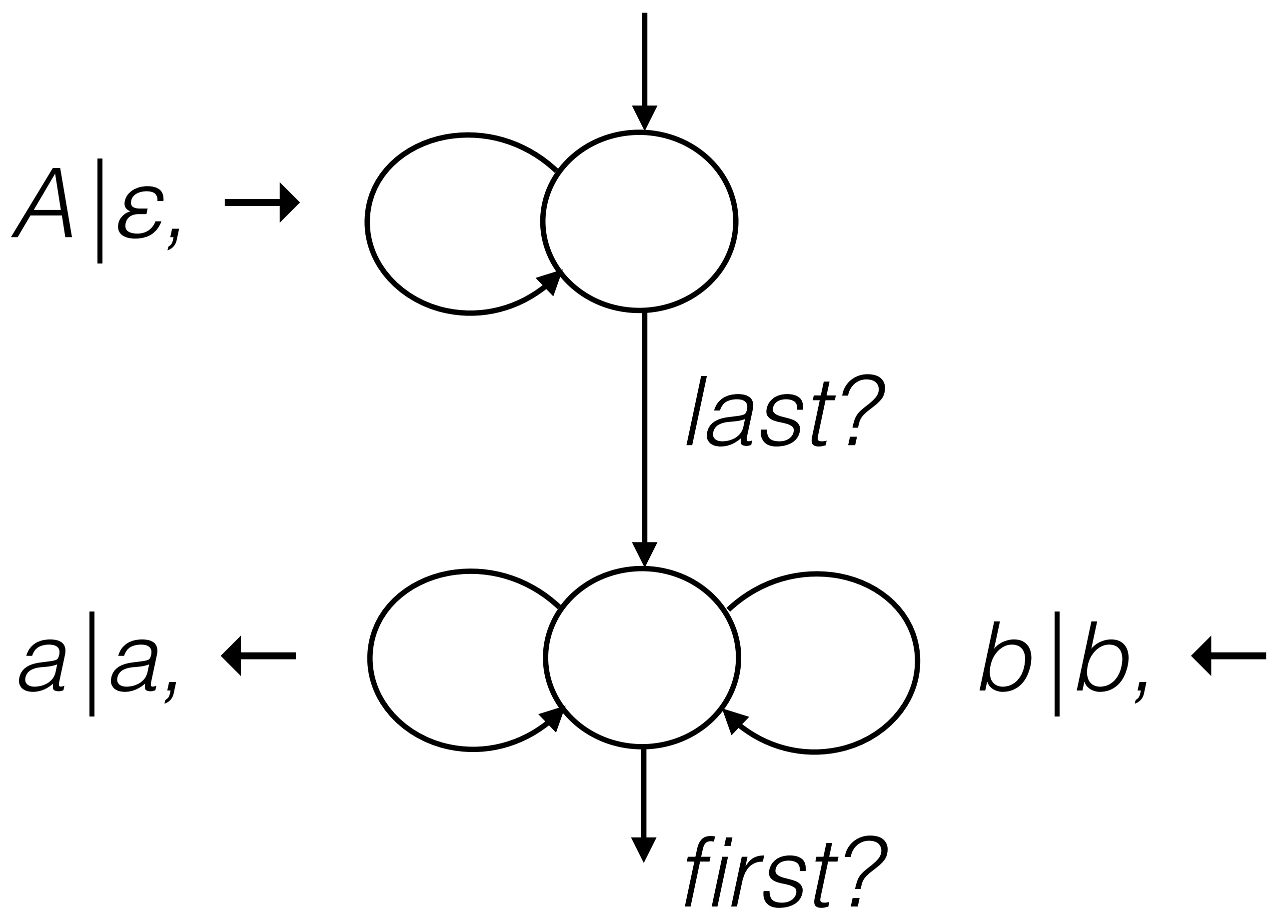
Transducers

$$\prod_x P_x(a)?\, A^* a A^* : (P_x(b)?\, A^* b A^*)$$



Infinite-valued, but <u>deterministic</u>

Reverse?



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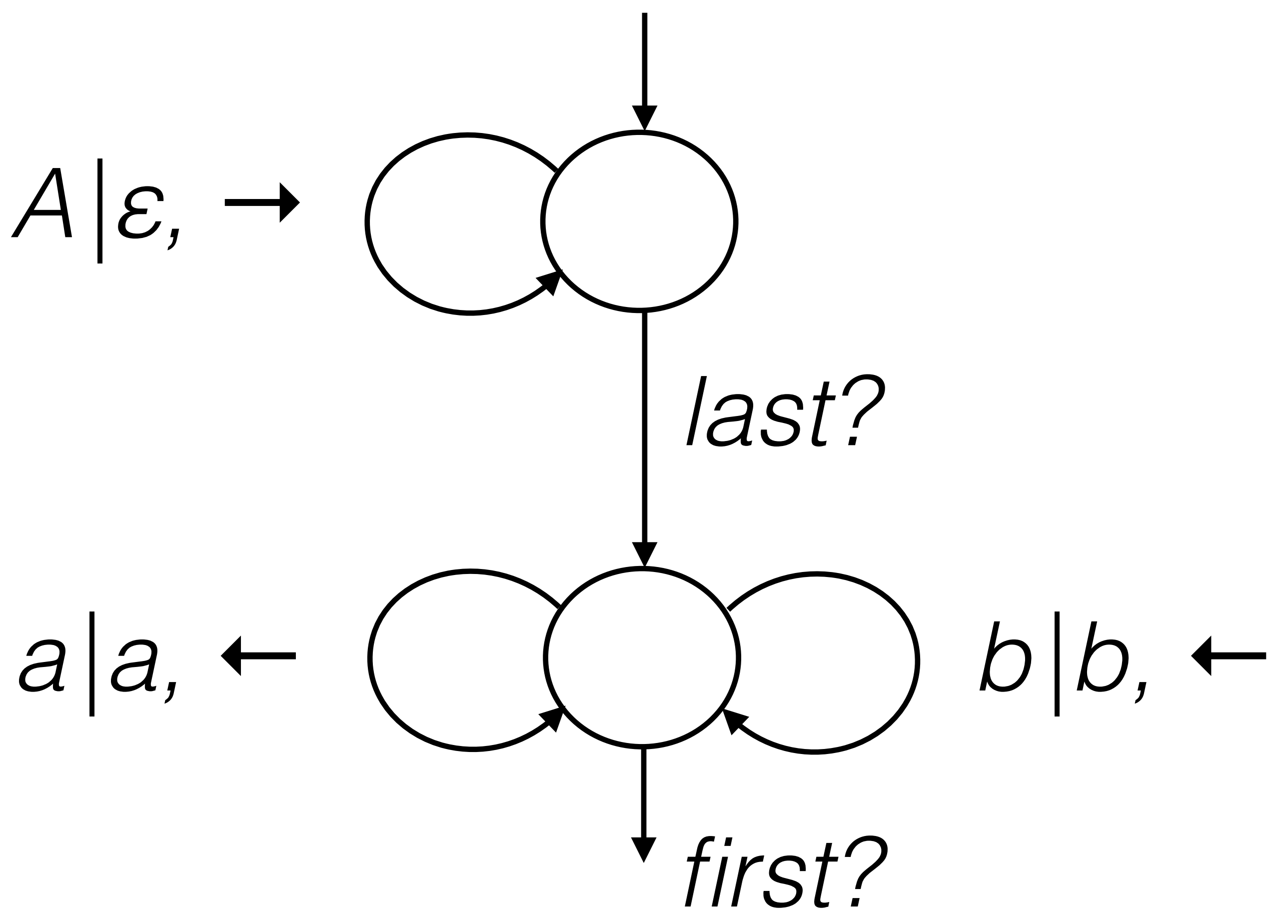

Impossible in FO…
… because of order of interpretation of product

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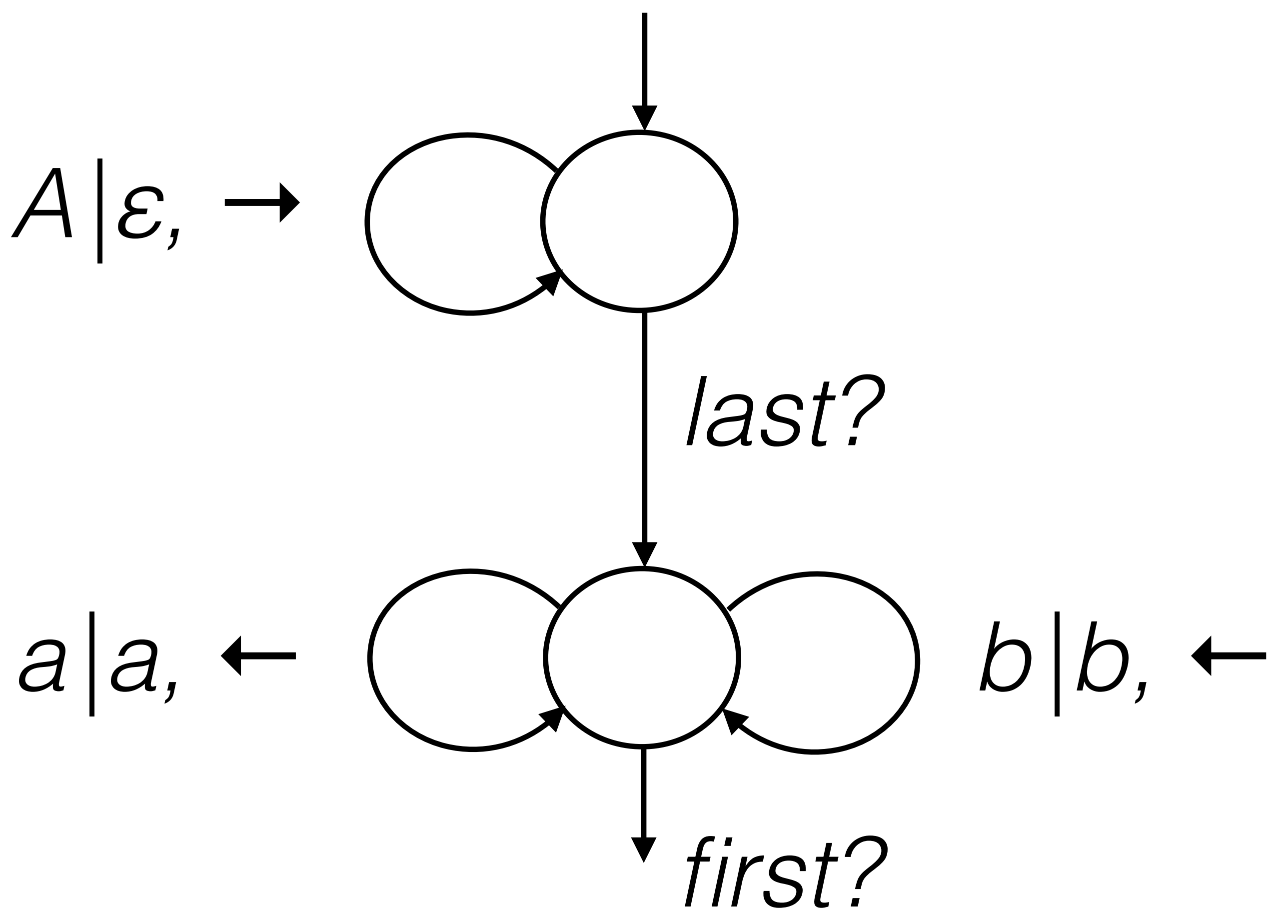

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Solution: in this non-commutative setting,
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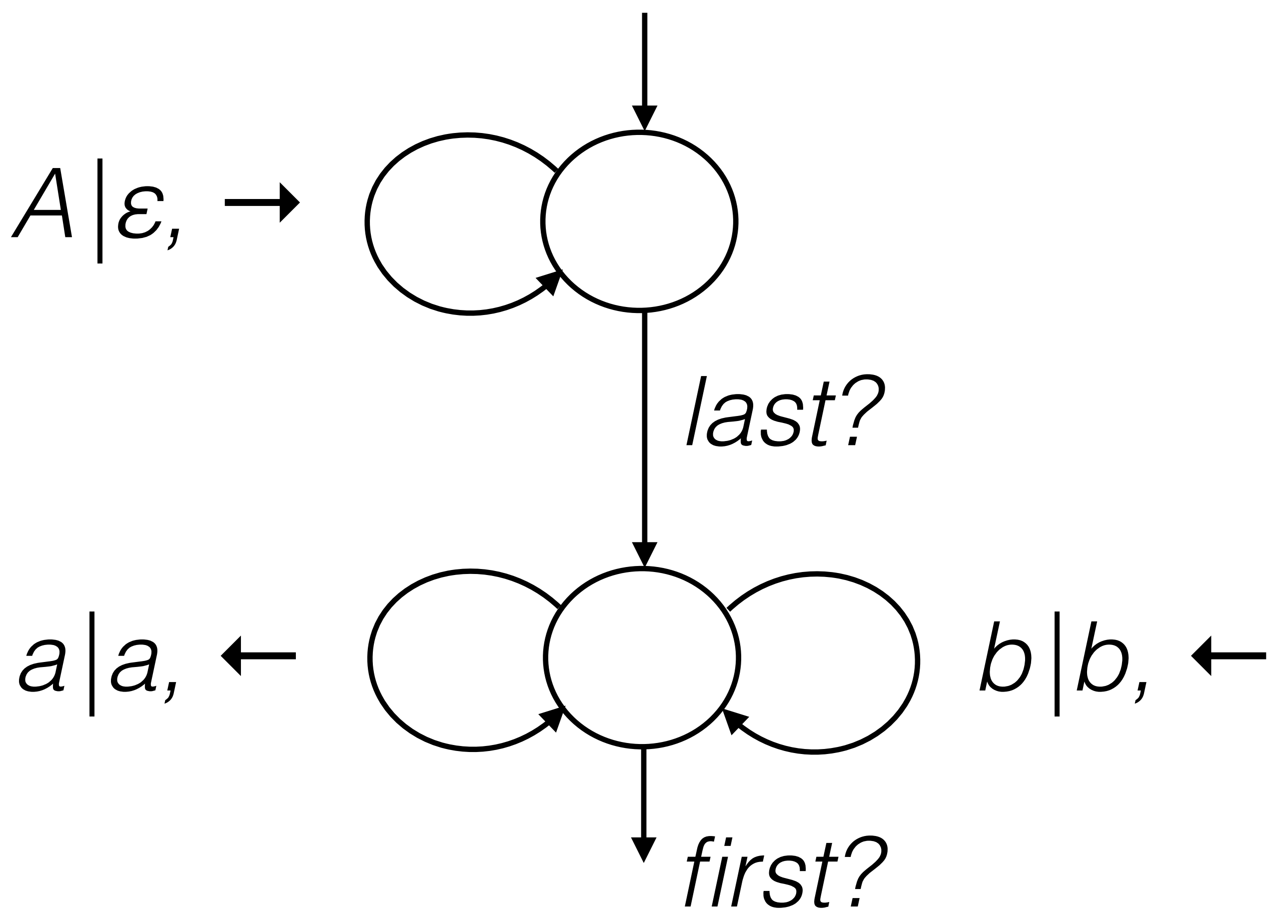

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Transitive closure

$$\varphi ::= \top \mid P_a(x) \mid x \leqslant y \mid x \in X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x\, \varphi$$

$$\Phi ::= L \mid \varphi ? \Phi : \Phi \mid \Phi + \Phi \mid \Phi \times \Phi \mid \sum_x \Phi \mid \prod_x \Phi \mid N\text{-}TC_{x,y} \Phi$$

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Regular language

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Regular language

Able to define right-to-left product

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Theorem: Pebble Transducers = wFO + bounded-TC

with regular language productions

linear transformation from logic to transducers

Functional transductions

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Theorem: Polyregular functions [Bojańczyk 2018]
Deterministic pebble automata

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= wFO without + and $\sum_x$, all weights being words, underlying boolean logic being MSO

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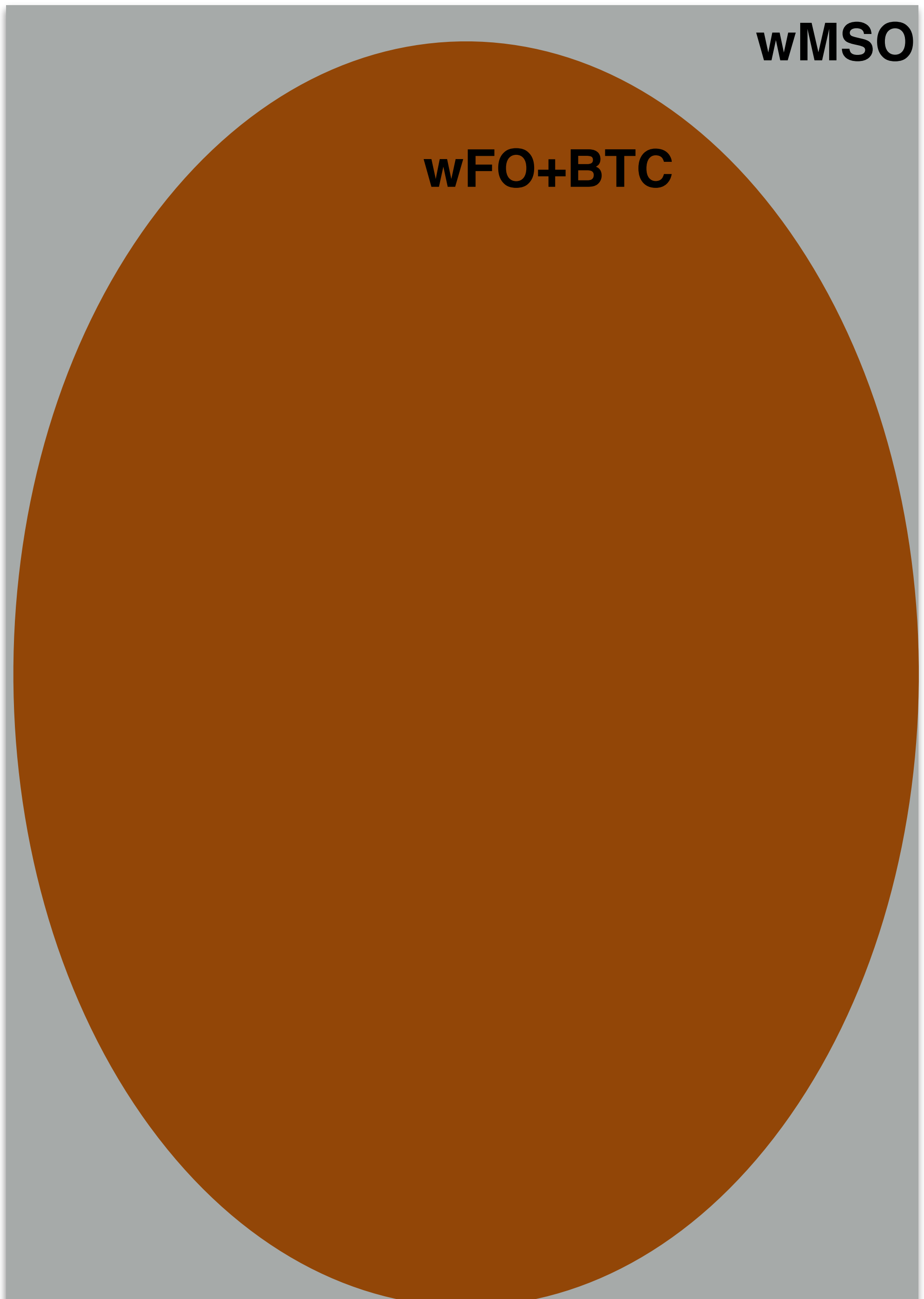
Similar characterizations for relational transductions / weighted functions ?

Summary

Summary

wMSO

Summary



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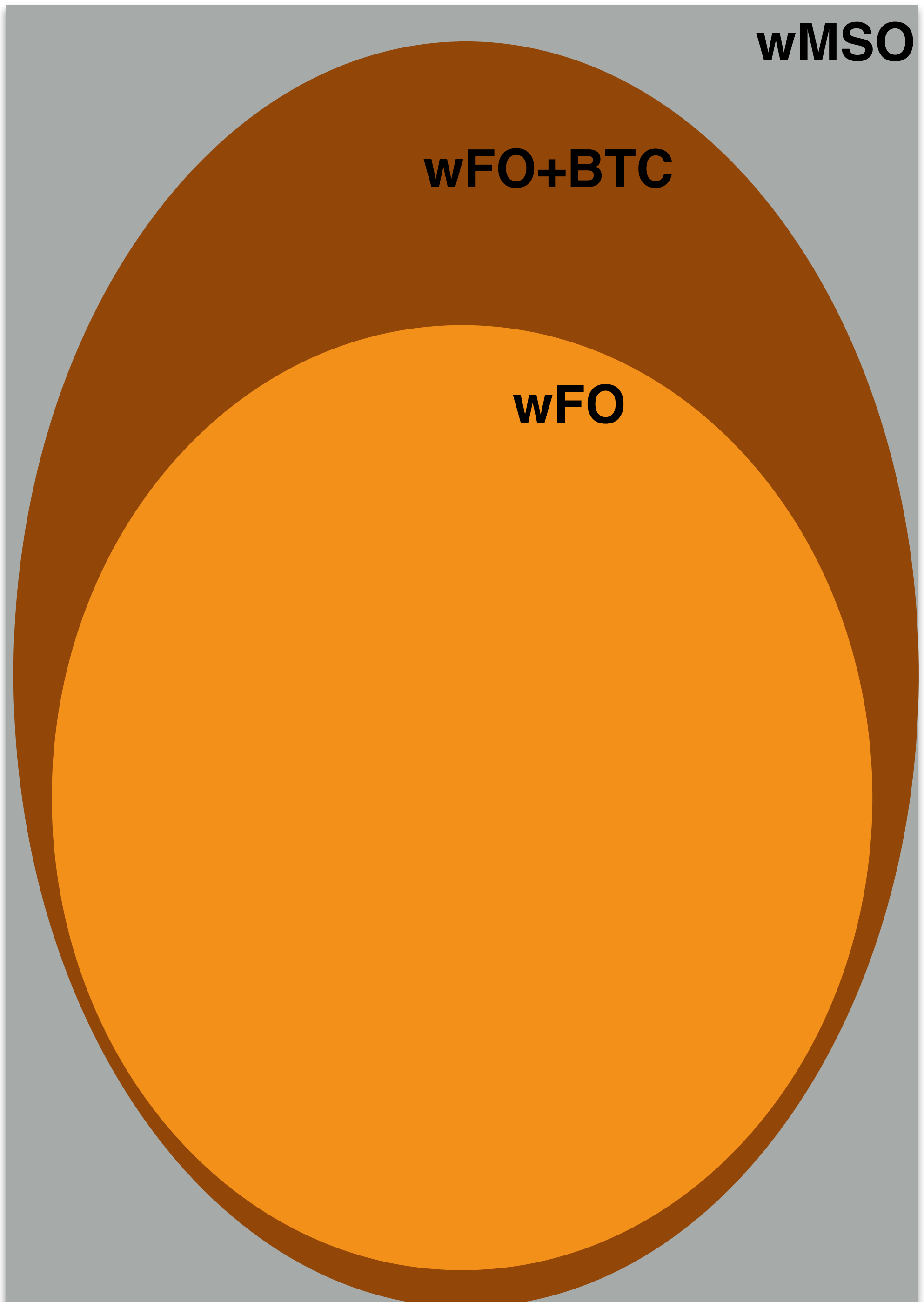

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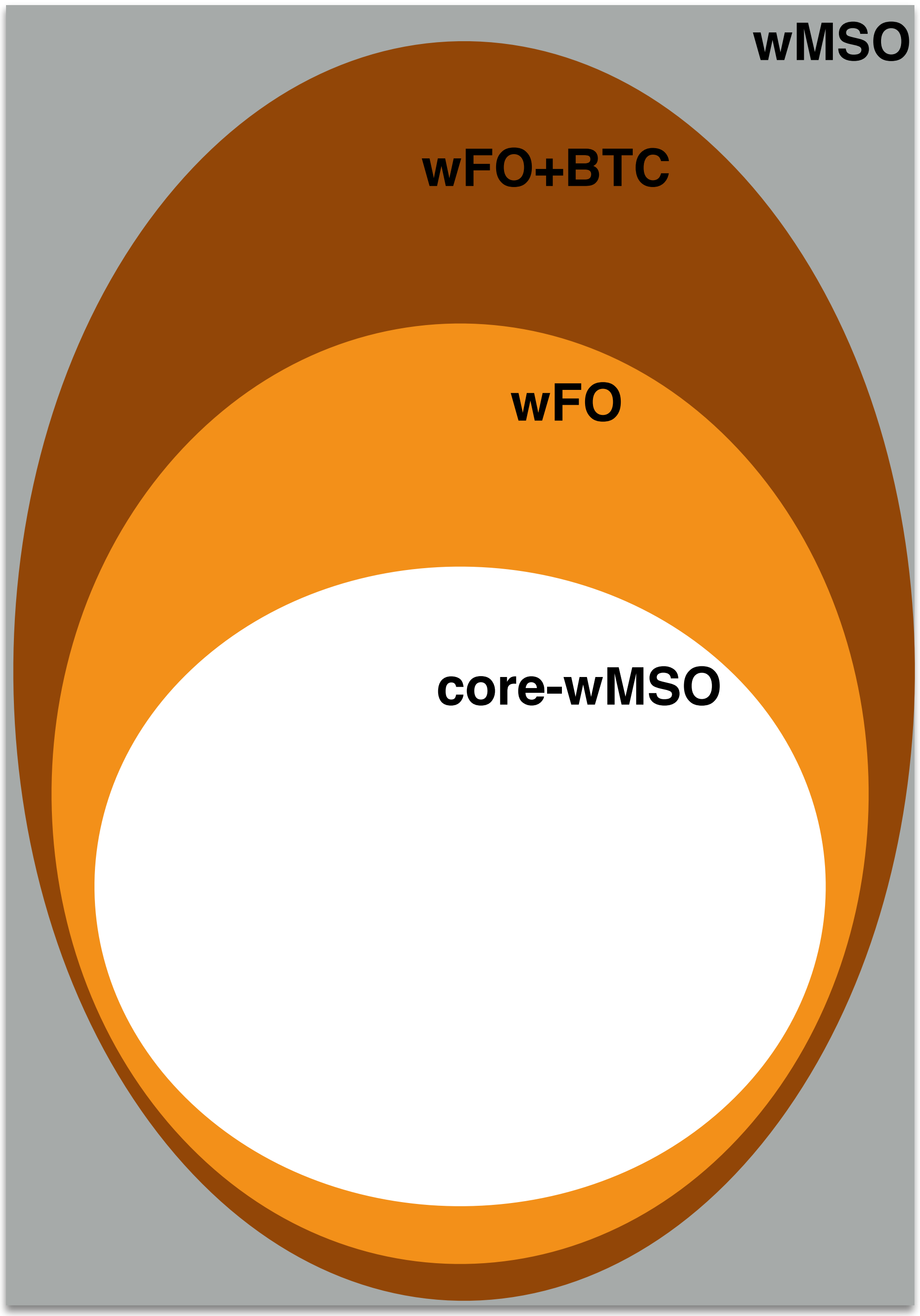

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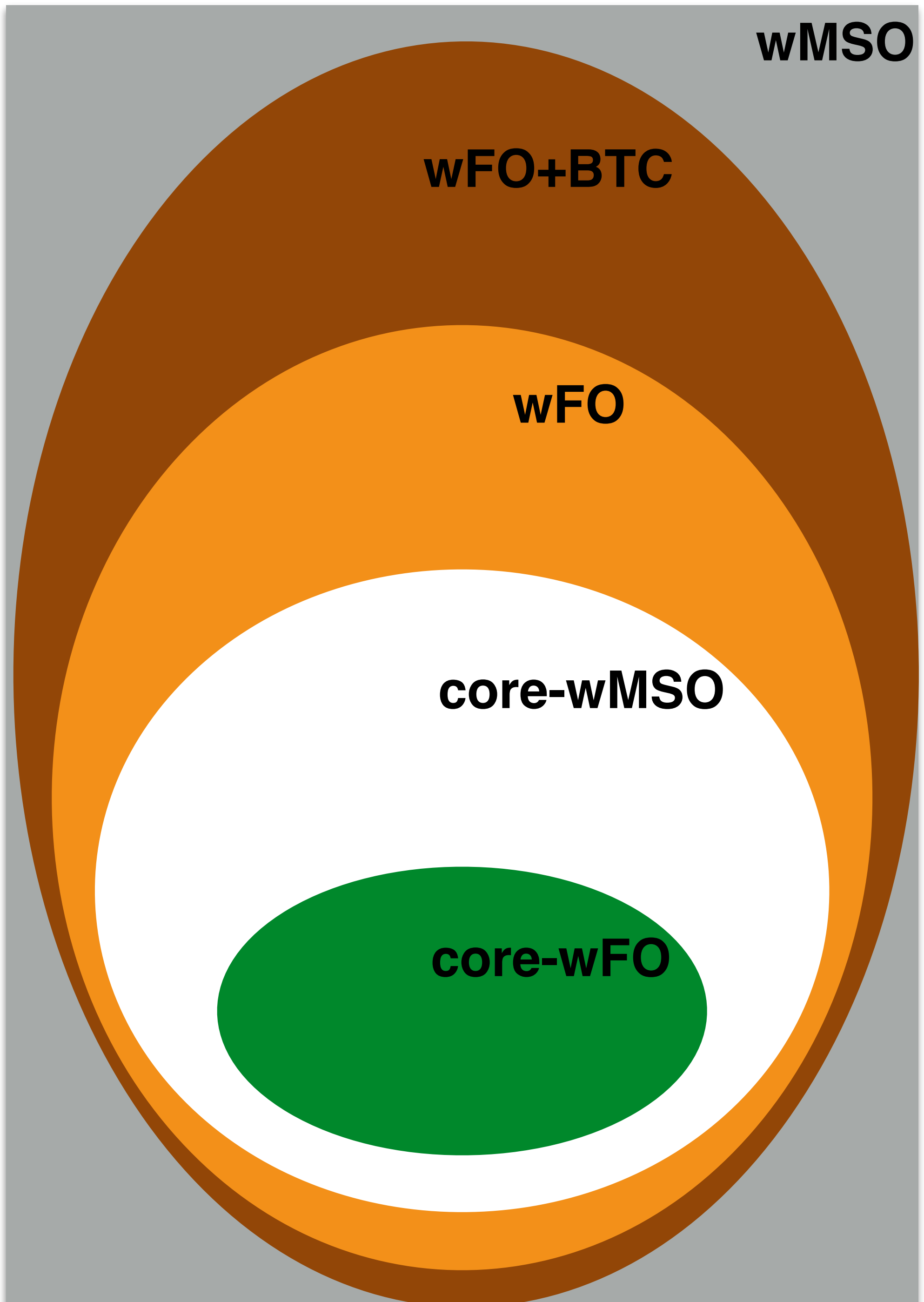

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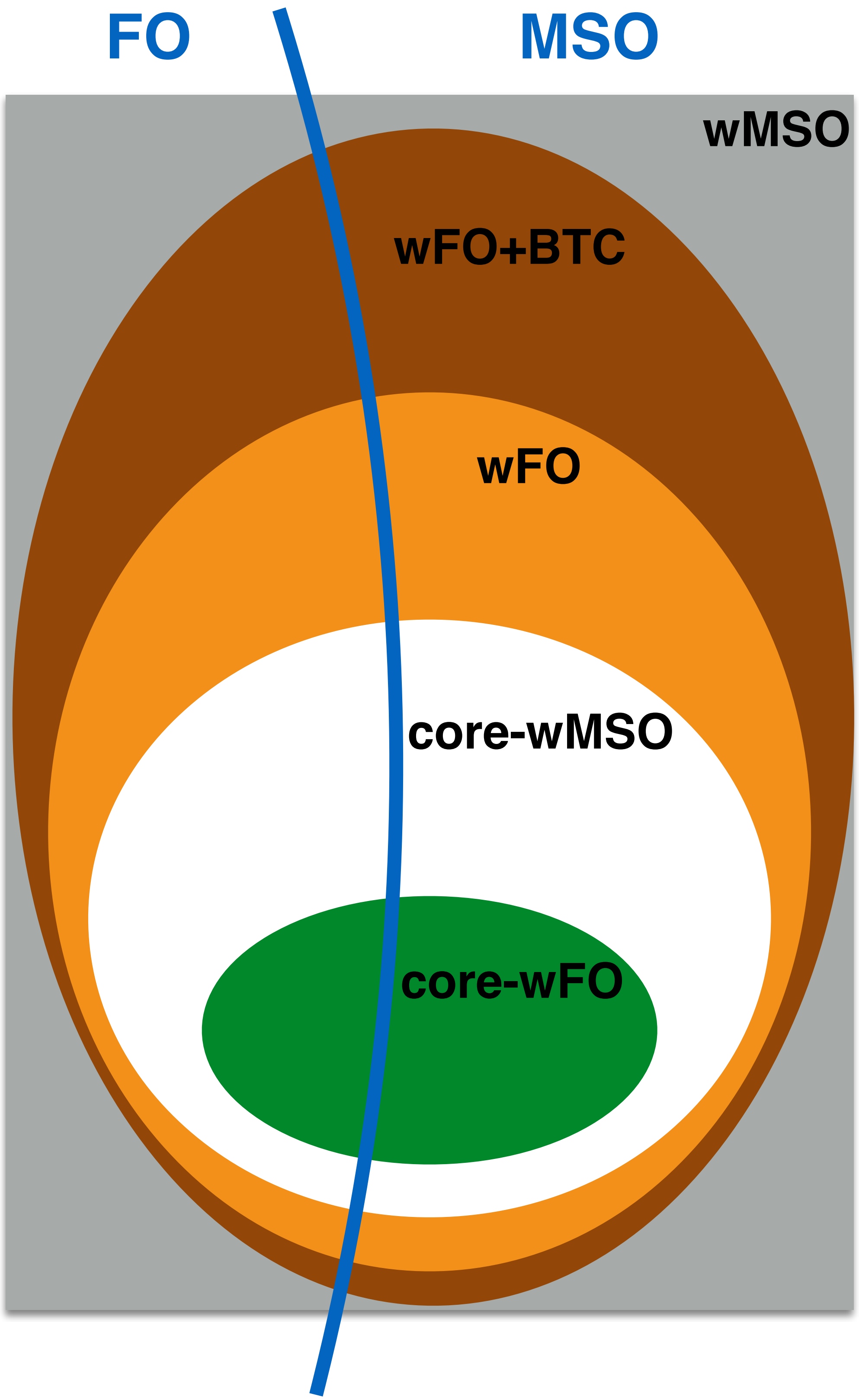

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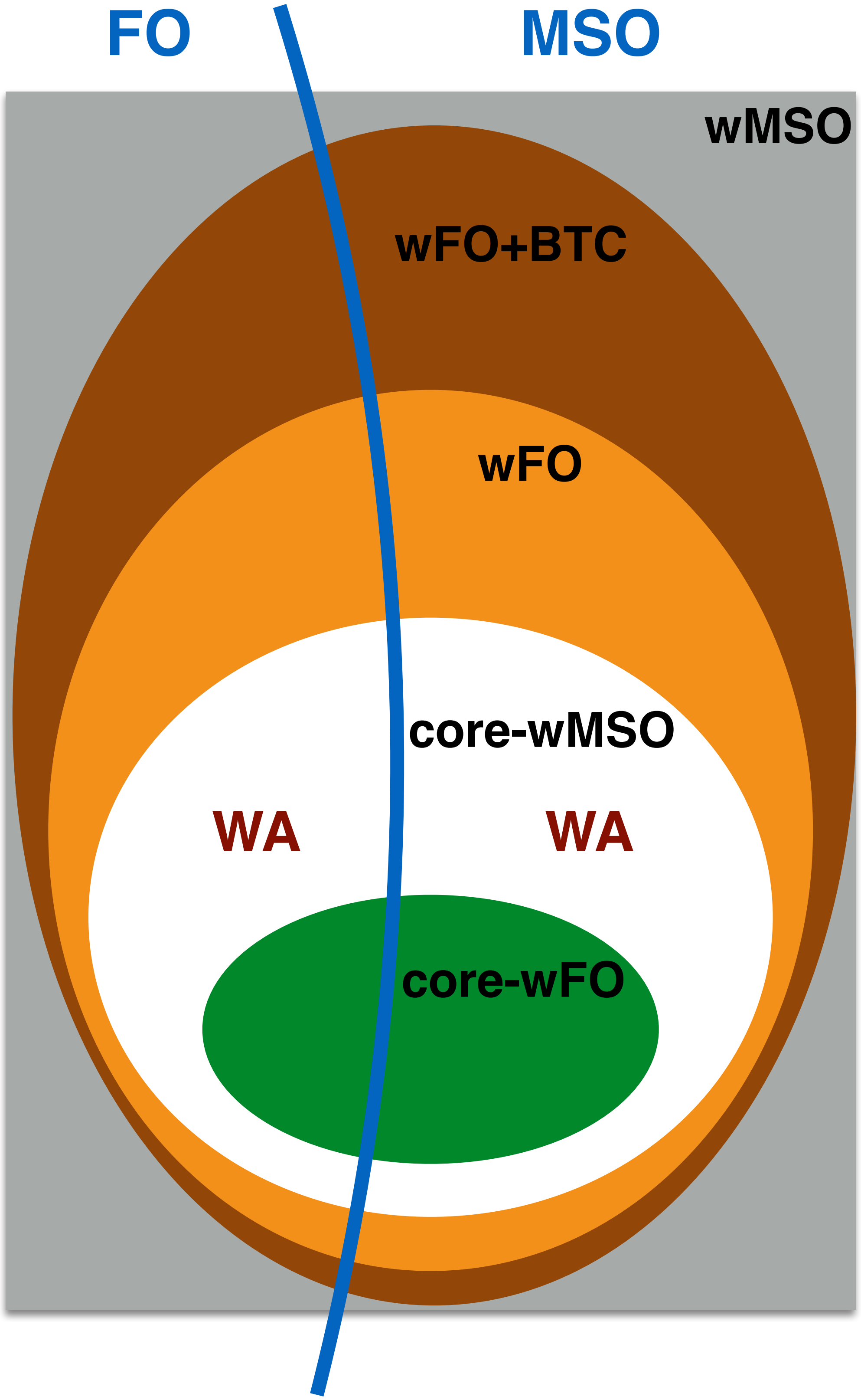

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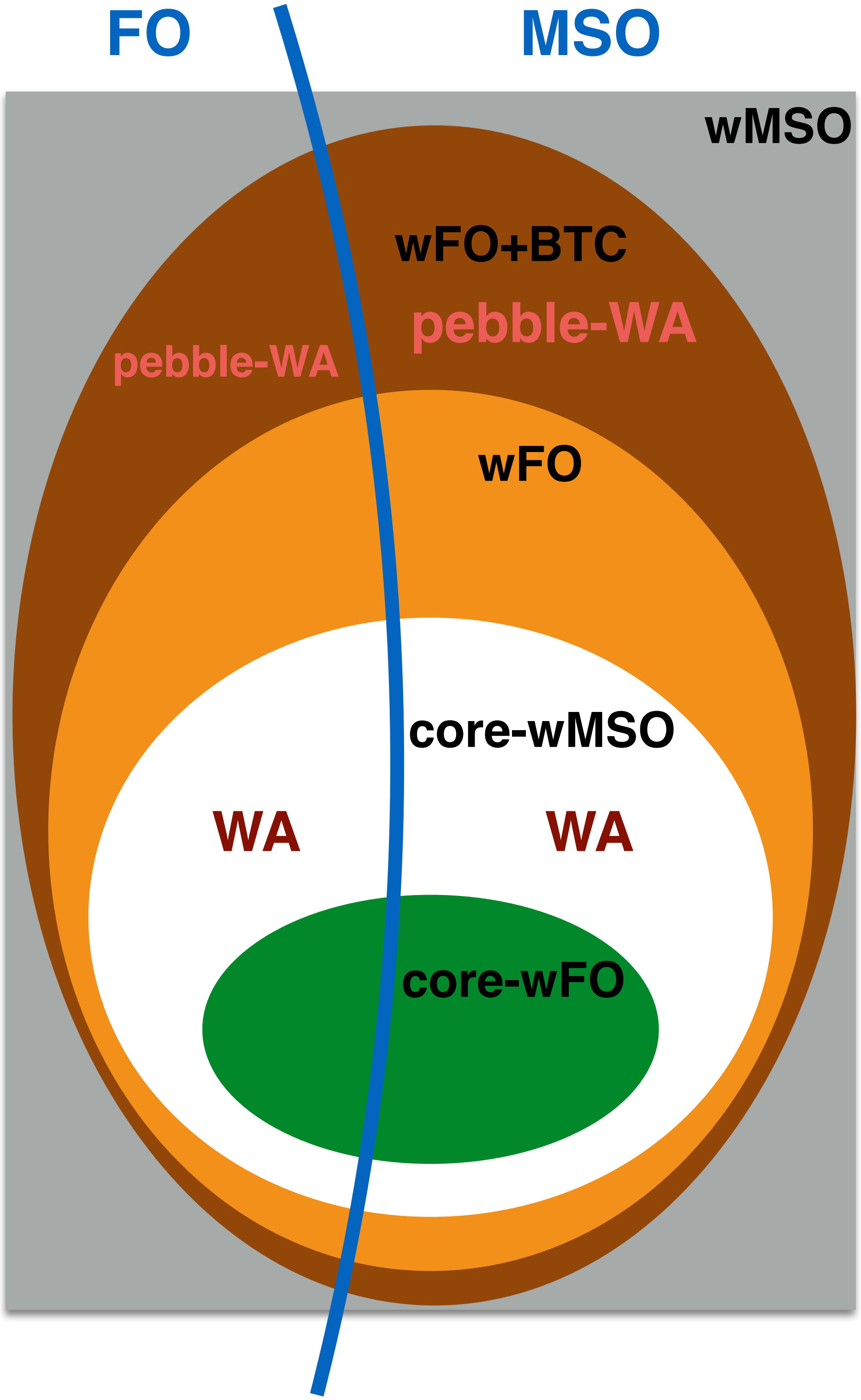
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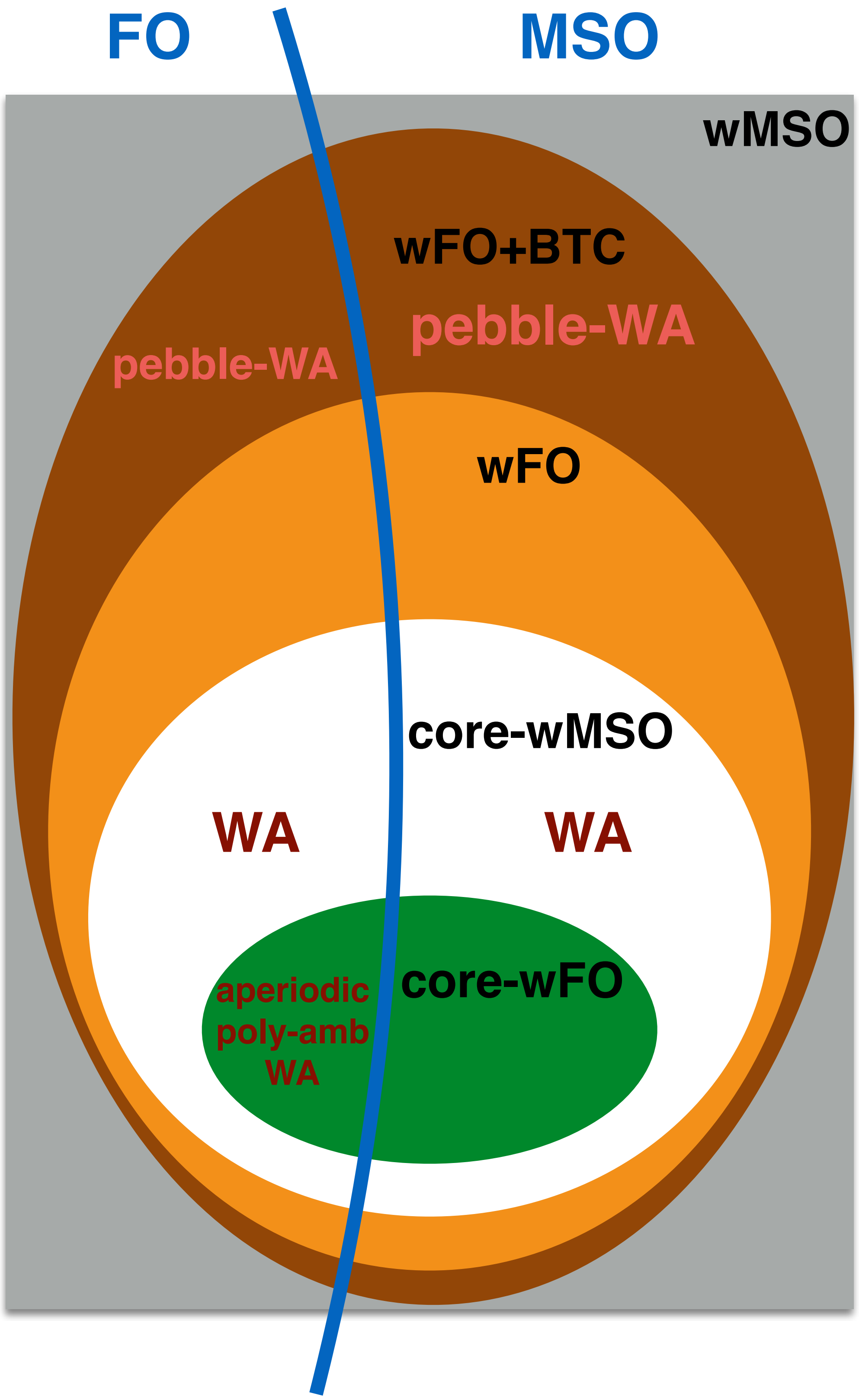
wMSO
wFO+BTC
pebble-WA
pebble-WA
wFO
core-wMSO
WA
WA
core-wFO

Summary



FO
MSO
wMSO
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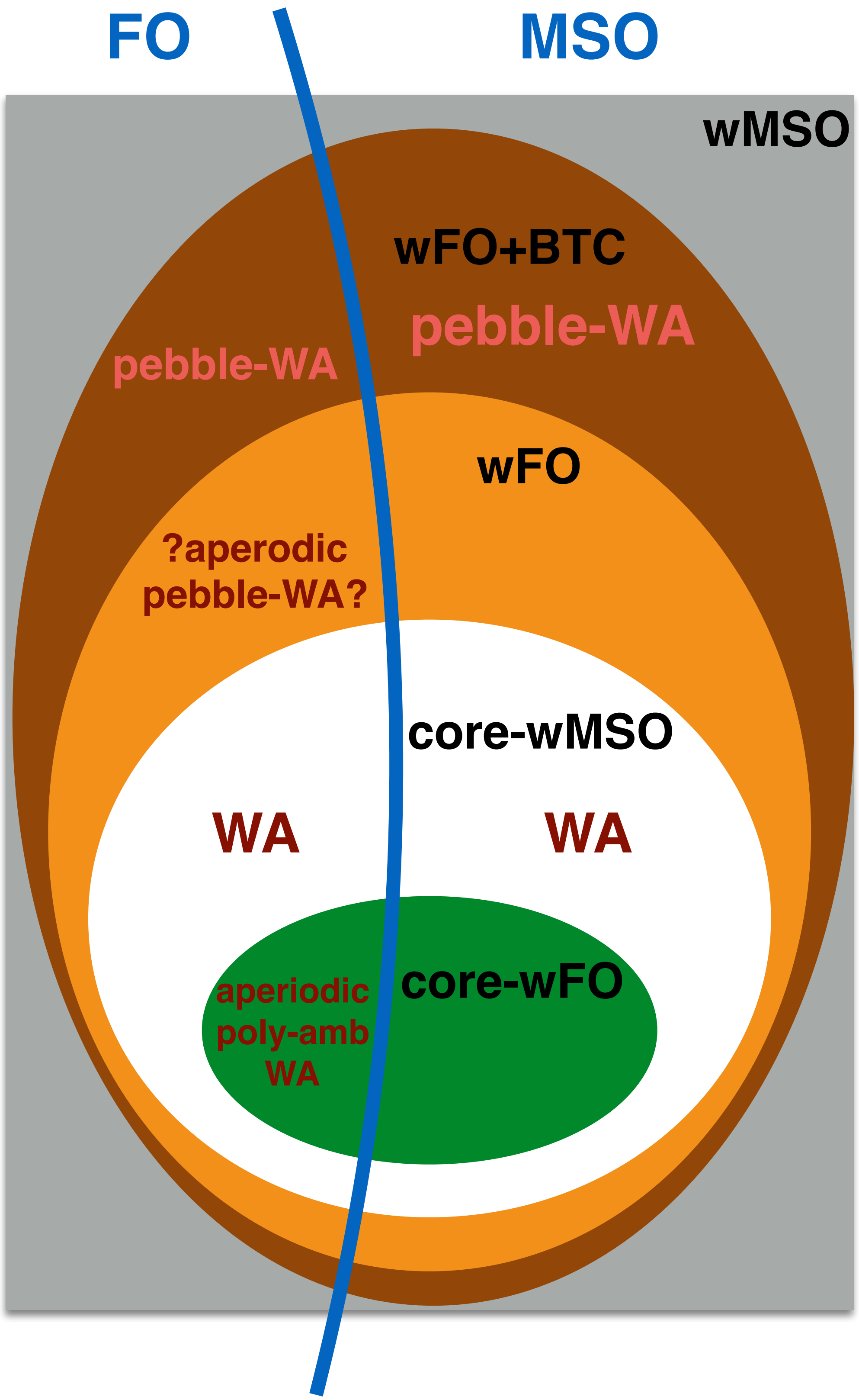
pebble-WA
wFO
core-wMSO
WA
WA
core-wFO
aperiodic
poly-amb
WA

Summary



FO
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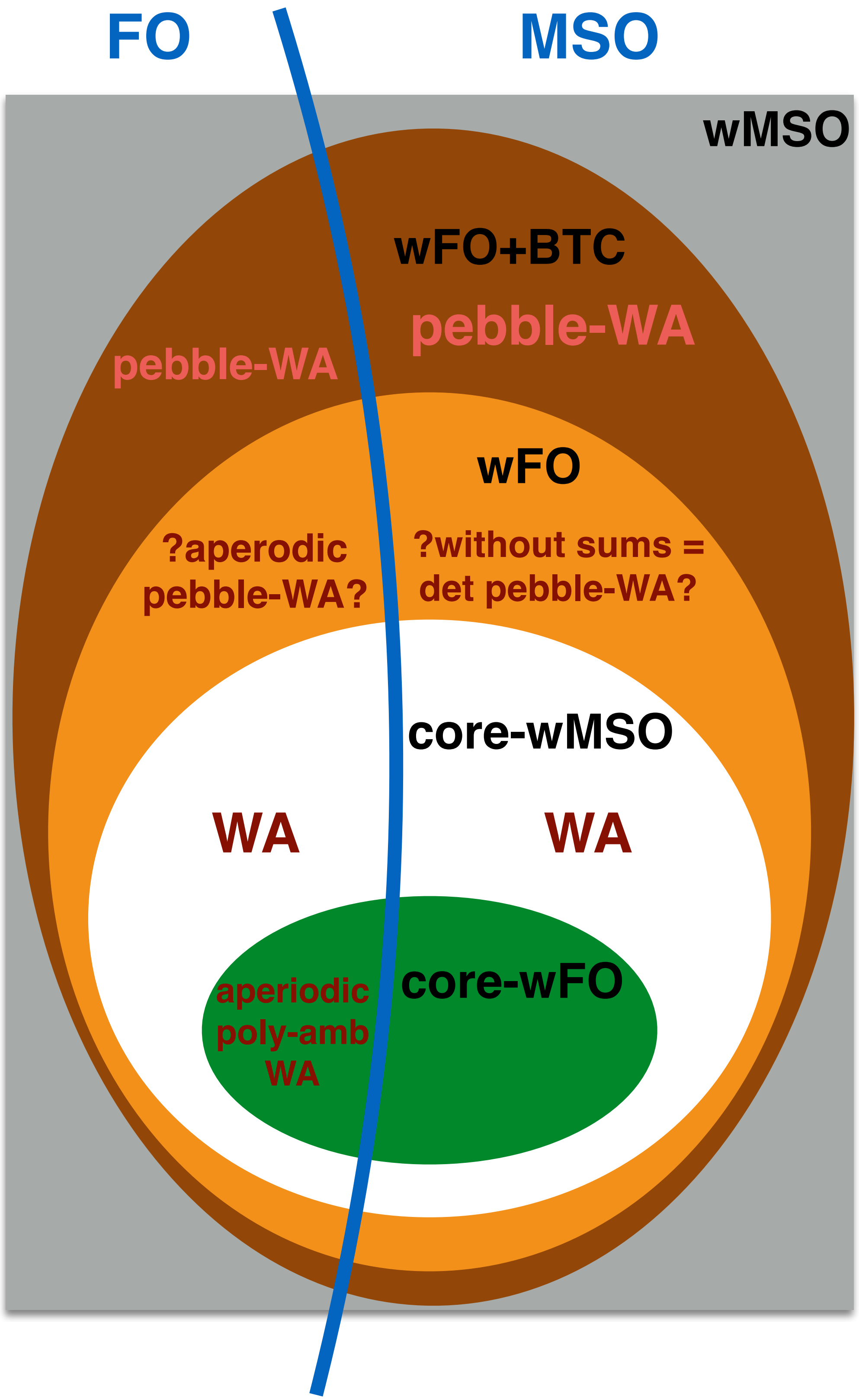

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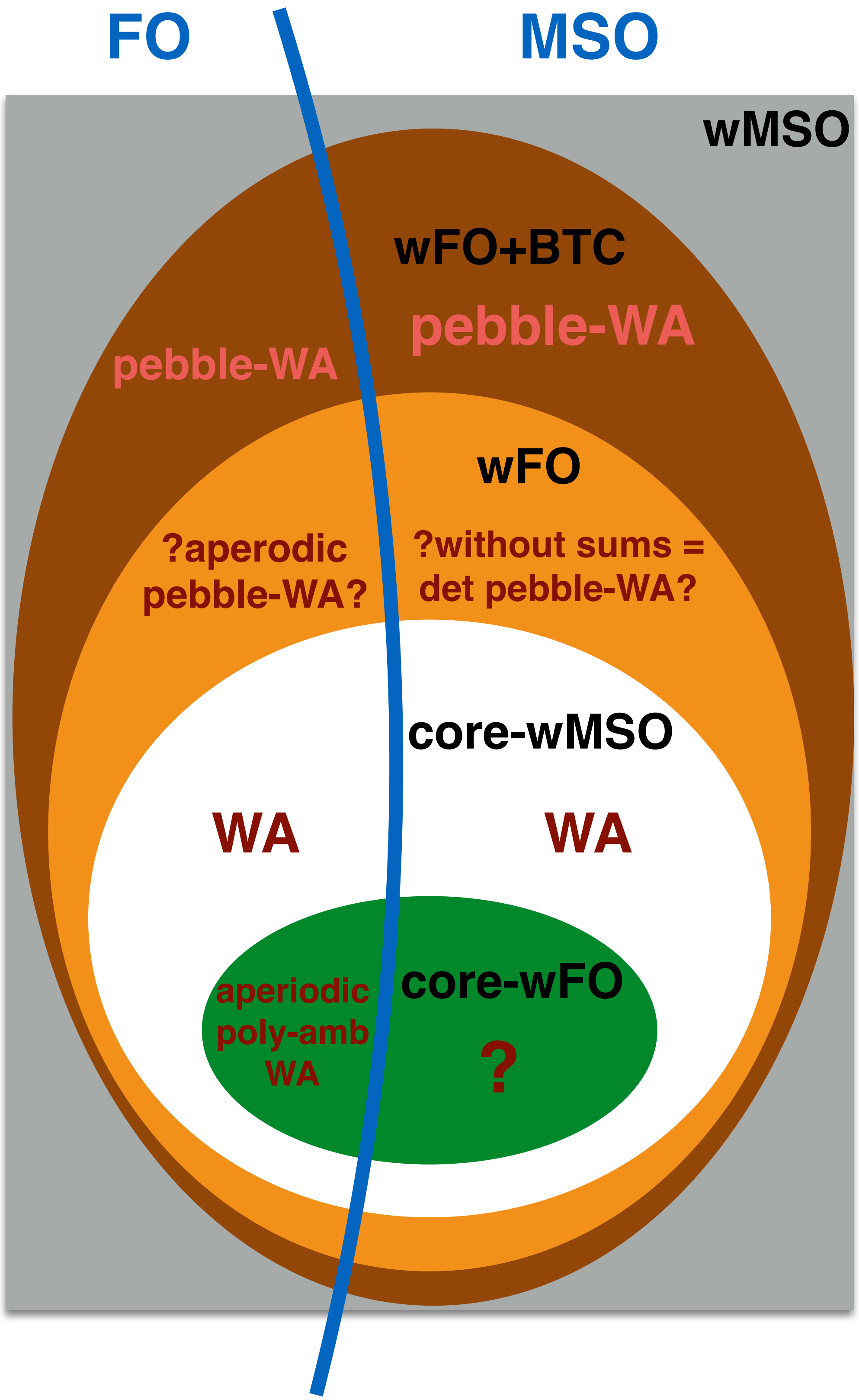

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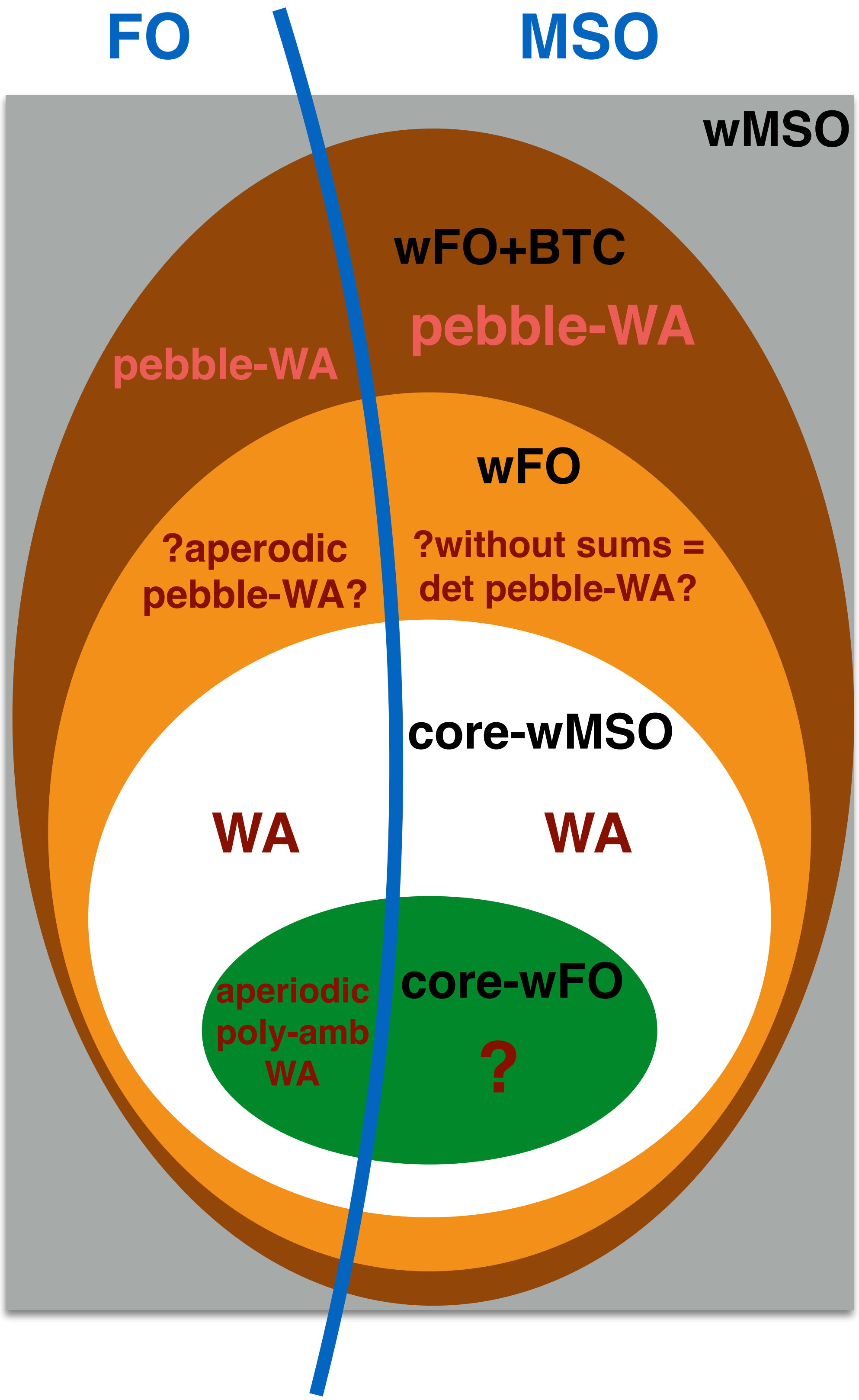
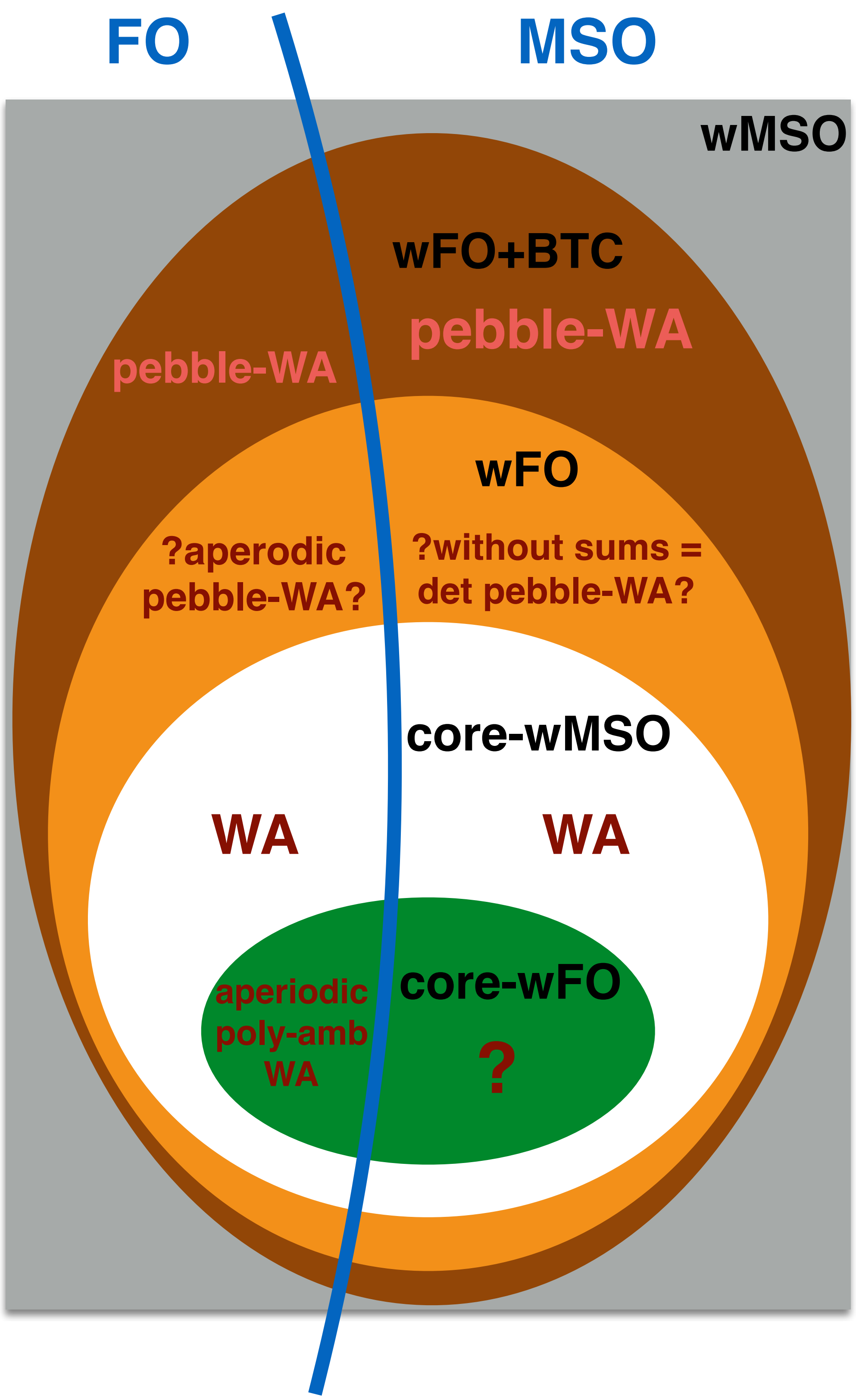


Equivalences between **logics** and **automata**

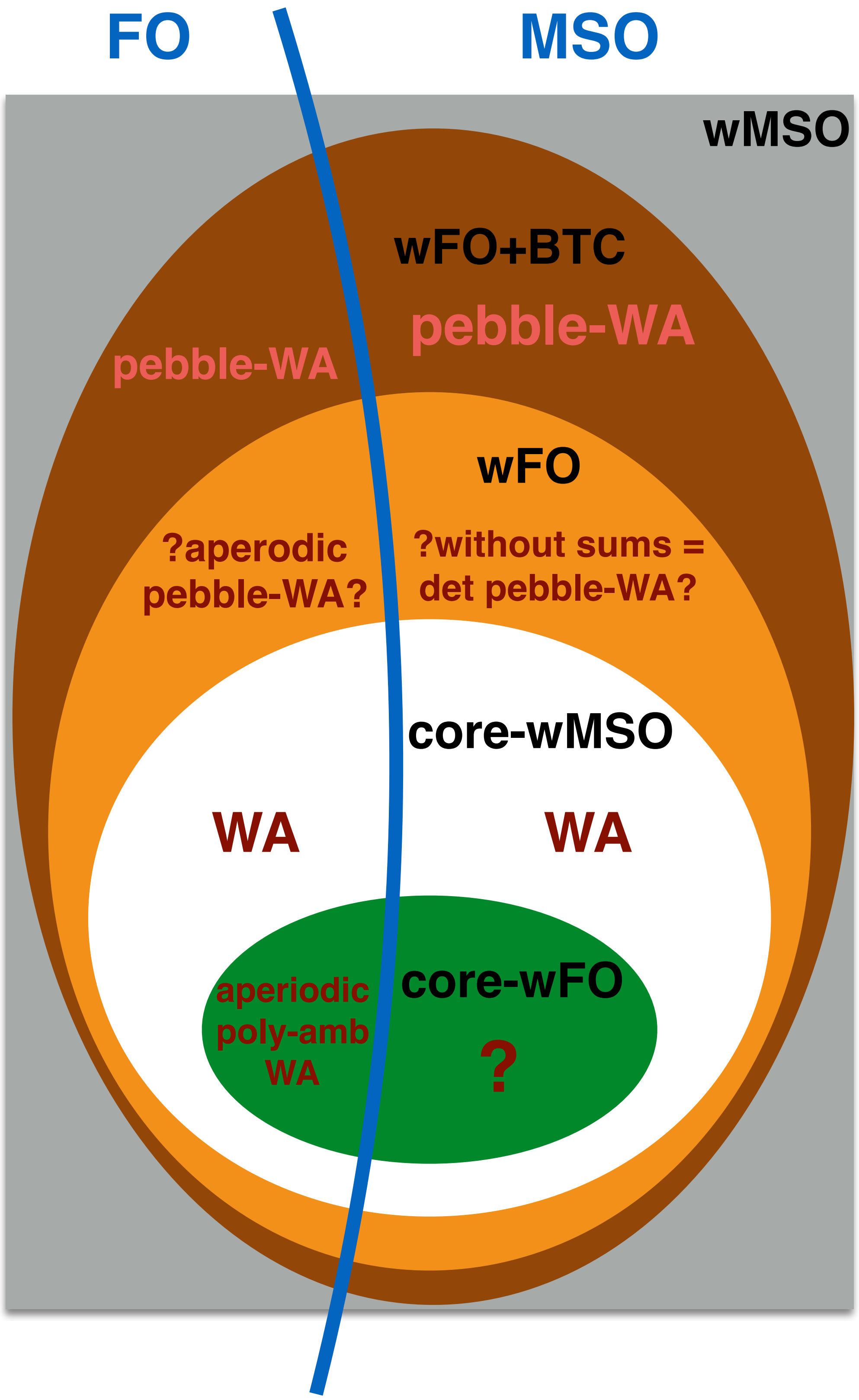
Summary



Equivalences between **logics** and **automata**

Evaluation in $\mathcal{O}(|\varphi| . |w|^{\#\text{vars}})$

Summary

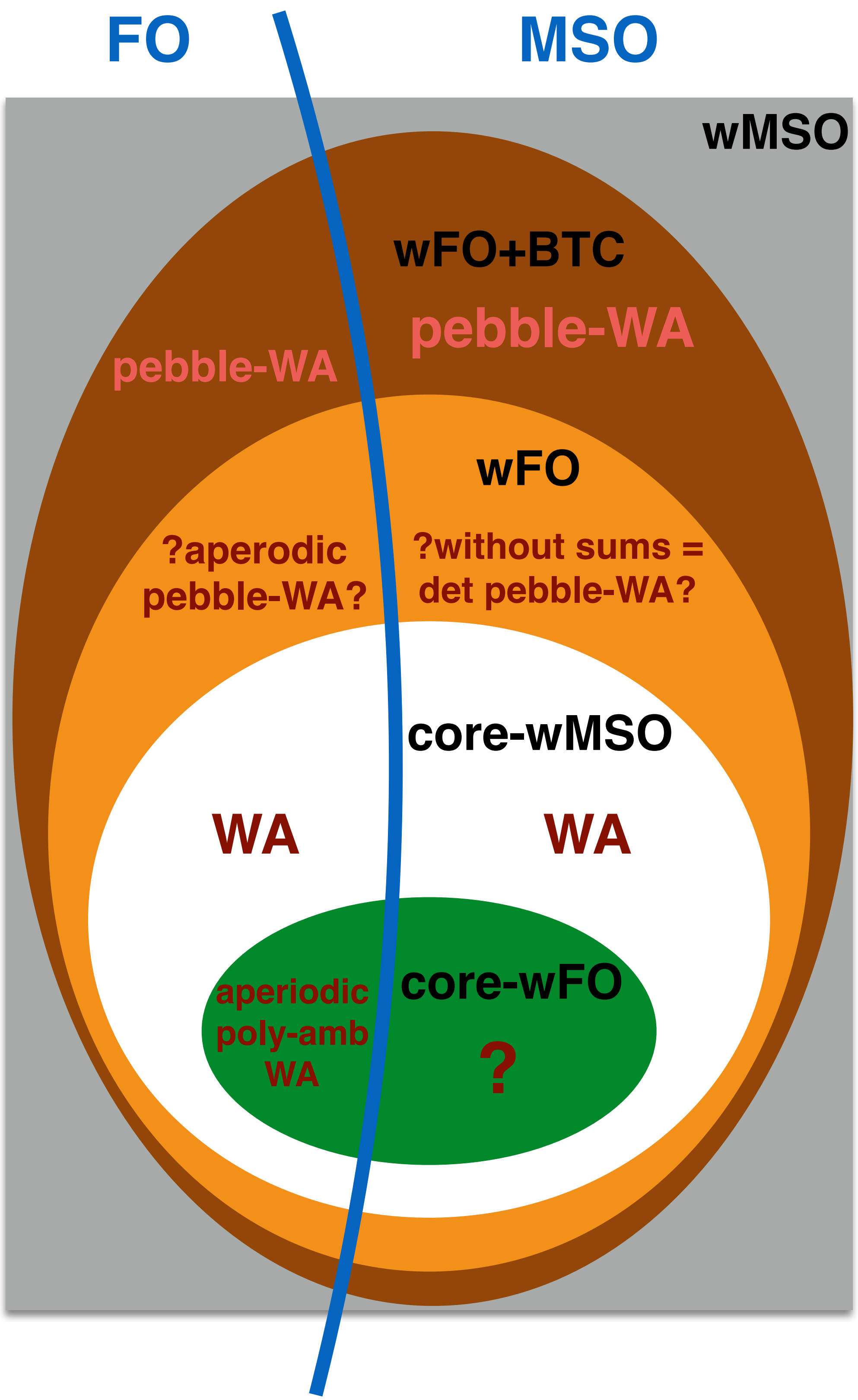


Equivalences between **logics** and **automata**

Evaluation in $\mathcal{O}(|\varphi|.|w|^{\#\text{vars}})$

Decidability procedures:
equivalence,
minimisation,
boundedness

Summary



Equivalences between **logics** and **automata**

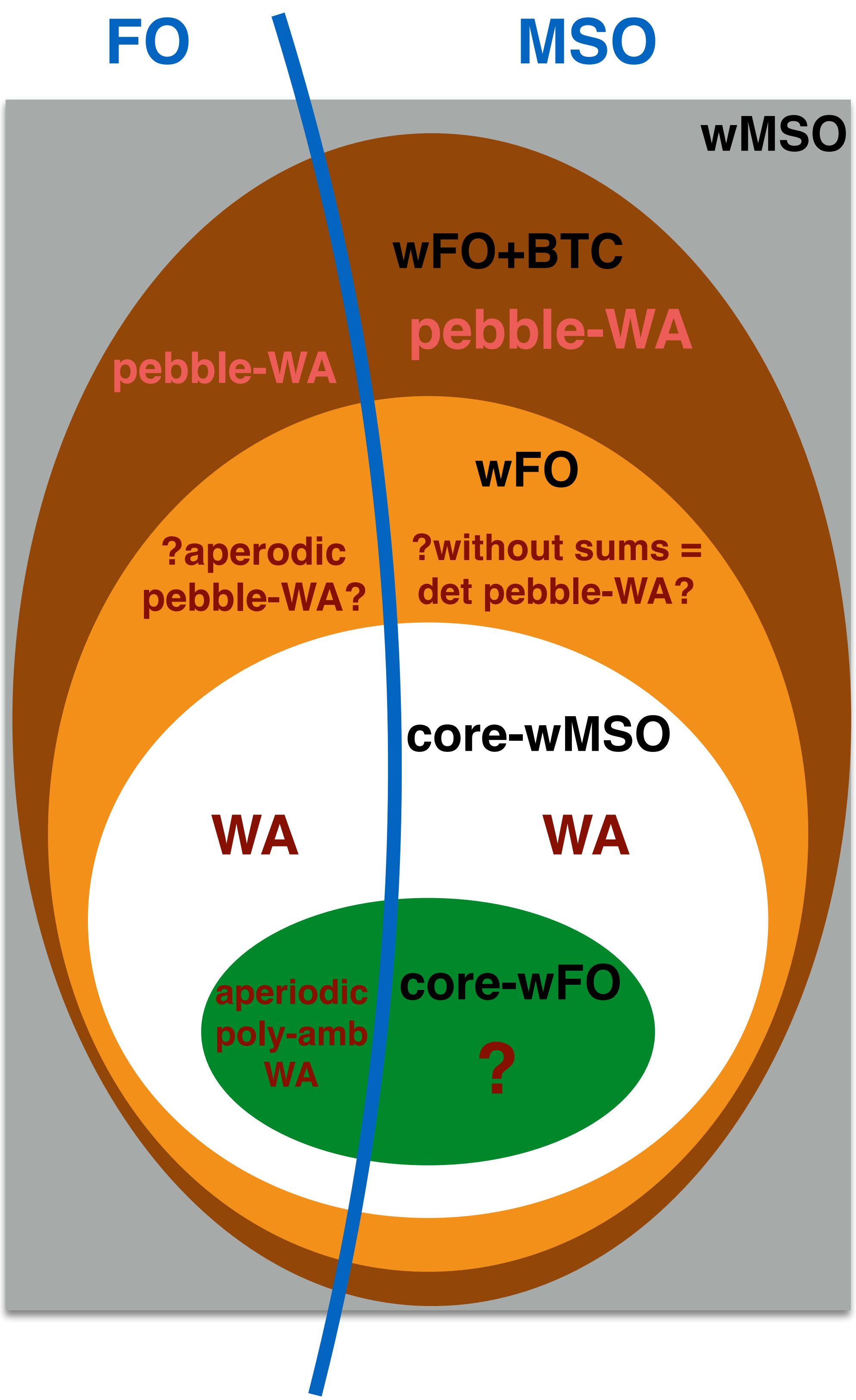
Evaluation in $\mathcal{O}(|\varphi|.|w|^{\#\text{vars}})$

Independant of weight structures

Decidability procedures: equivalence, minimisation, boundedness

Very dependant of weight structures

Summary



Equivalences between **logics** and **automata**

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Independant of weight structures

Link with register models?
[Douéneau&Filiot&Gastin 2018]
marbles/invisible-pebbles: fragments of logic?

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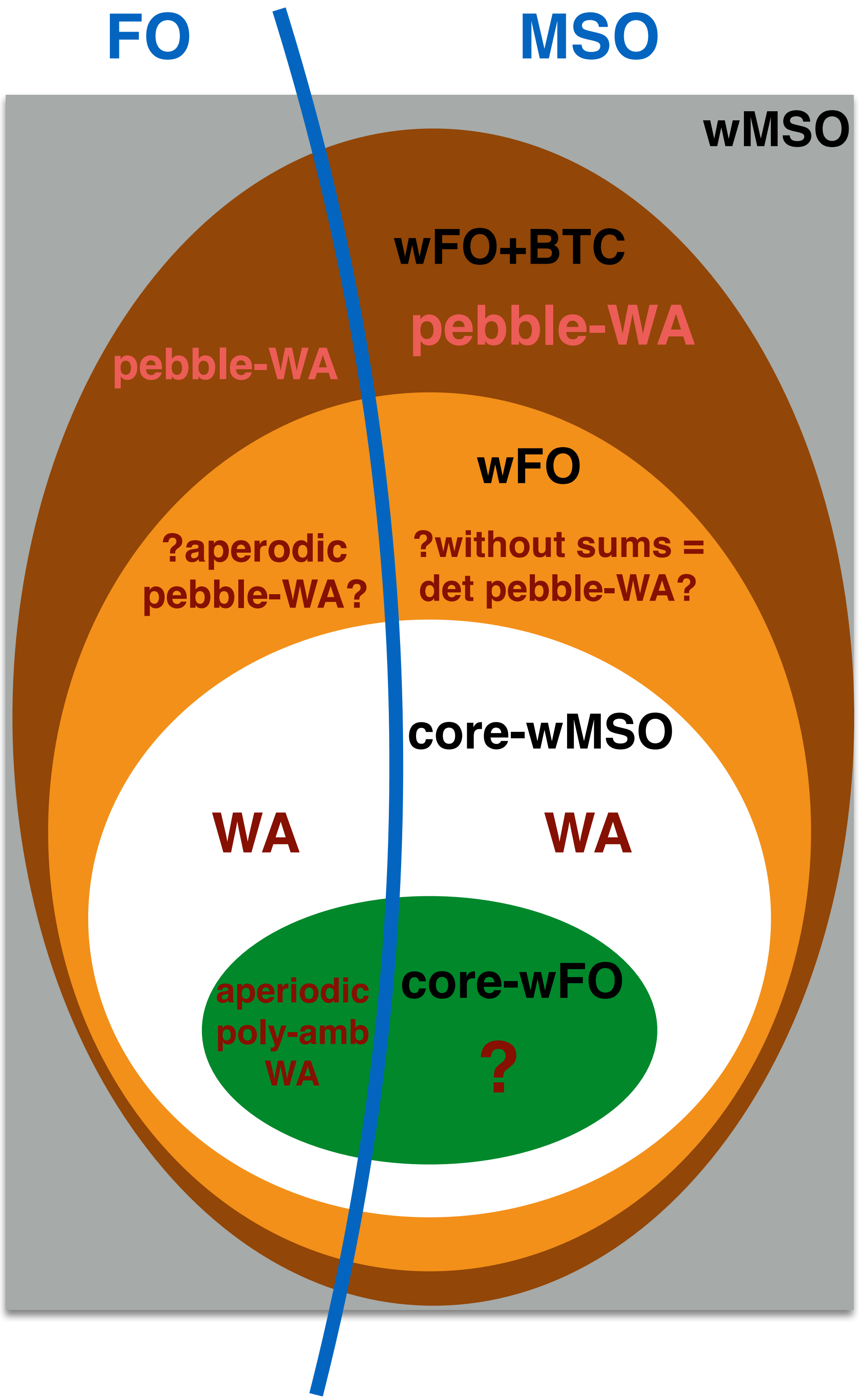

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2-way→1-way? EXPSPACE for functional transducers
[Filiot&Gauwin&Reynier&Servais 2013,
Baschenis&Gauwin&Muscholl&Puppis 2017+Jecker 2018]
partially-commutative weight structure?
with/without pebbles?

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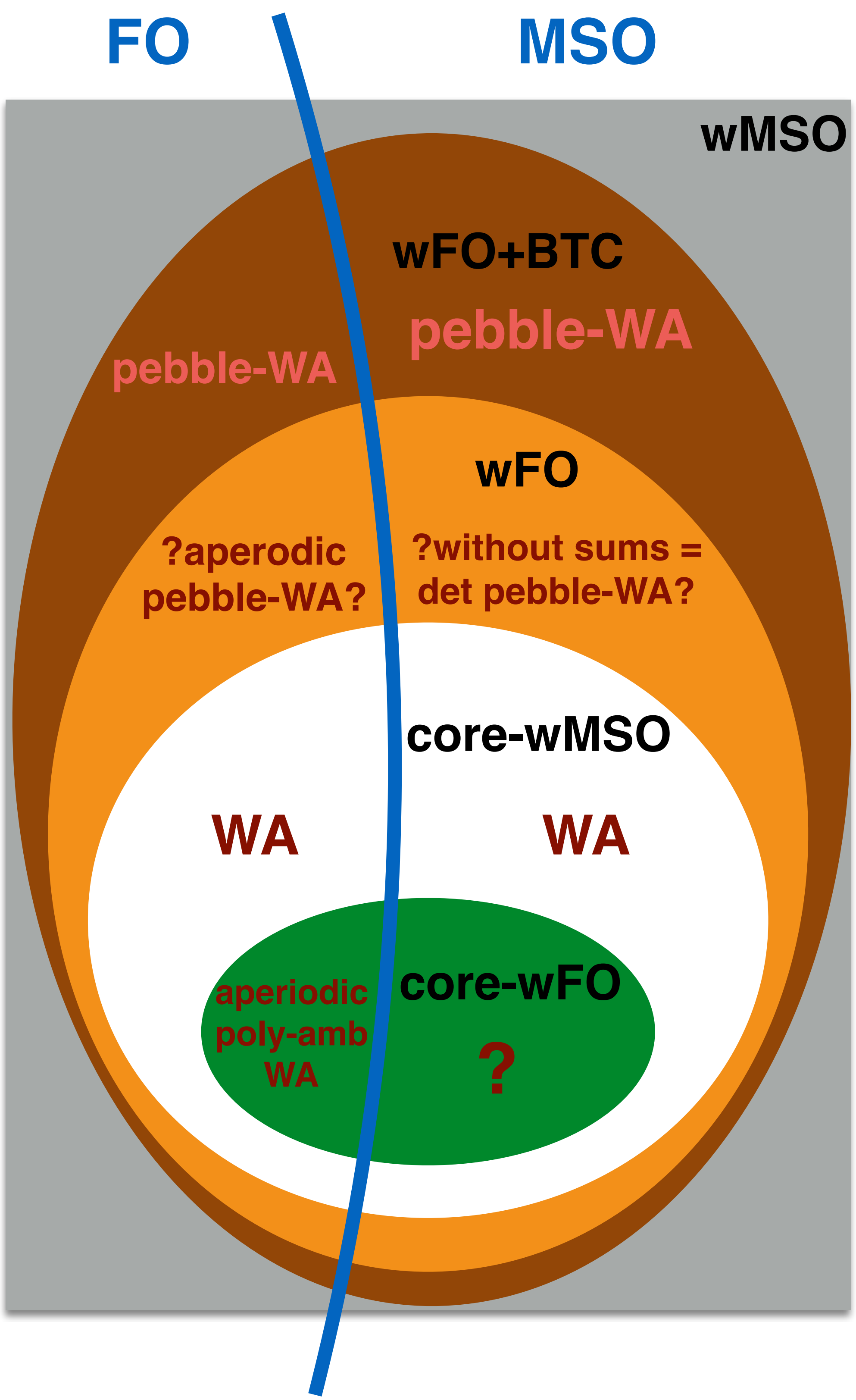

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Thank you!