Dynamics on Games: Simulation-Based Techniques and Applications to Routing

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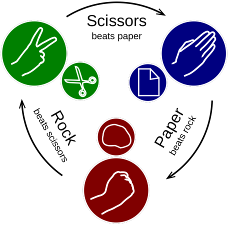
FSTTCS 2019

Static approach



$$\begin{array}{c} \\ R \\ P \\ S \end{array} \begin{array}{c} \begin{array}{ccc} R & P & S \end{array} \\ \begin{pmatrix} (0,0) & (-1,1) & (-1,1) \\ (1,-1) & (0,0) & (-1,1) \\ (-1,1) & (1,-1) & (0,0) \end{pmatrix} \end{array}$$

Classical game theory

Players are

- Clever: they reason perfectly;
- Rational: they want to maximize their payoff;
- Selfish: they only bother about their own payoff.

Notions of equilibrium (Nash Equilibria, Subgame Perfect Equilibria...)

Dynamic approach

If we discover a new game

- Find immediately a good strategy is concretely impossible.

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→ Learning in games (e.g. fictitious play)

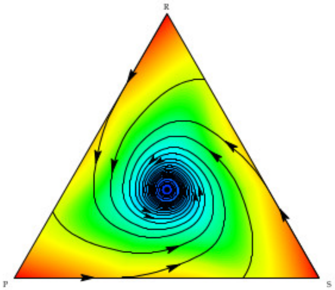
→ Strategy improvement (e.g. in parity games)

→ Evolutionary game theory (continuous time)

Equivalence

Static approach — Dynamic approach

Equilibria ⟷ Stable Points



Picture taken from *Evolutionnary game theory* by W. H. Sandholm

Equivalence

Static approach | **Dynamic approach**

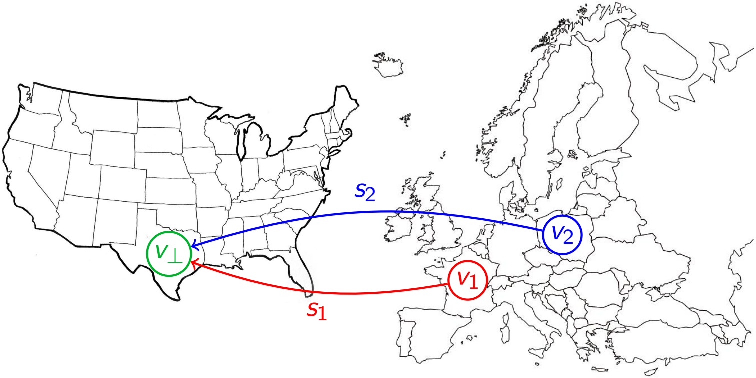
Equilibria ⟷ Stable Points

Our Goal

- Apply this idea of improvement on games played on graphs
- Prove termination via reduction/minor of games
- Show some links with Interdomain routing

Interdomain routing problem

Two service providers: v_1 and v_2 want to route packets to $v_\perp$.



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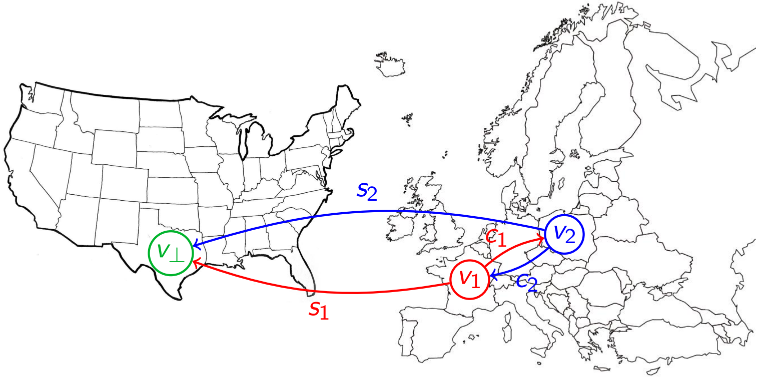

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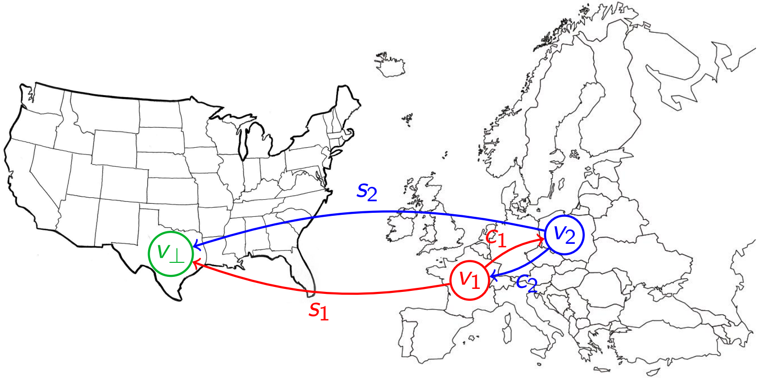

v_1 prefers the route $v_1 v_2 v_\perp$ to the route $v_1 v_\perp$ (preferred to $(v_1 v_2)^\omega$)

v_2 prefers the route $v_2 v_1 v_\perp$ to the route $v_2 v_\perp$ (preferred to $(v_2 v_1)^\omega$)

Interdomain routing problem as a game played on a graph

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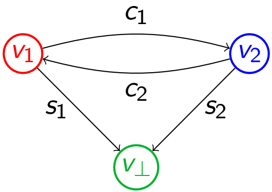

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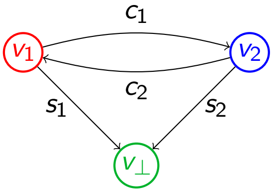
$$v_1 v_\perp \prec_1 v_1 v_2 v_\perp \quad \text{and} \quad v_2 v_\perp \prec_2 v_2 v_1 v_\perp$$

Games played on a graph – The strategic game approach

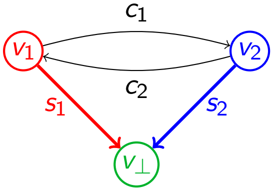


	c_2	s_2
c_1	$(0,0)$	$(2,1)$
s_1	$(1,2)$	$(1,1)$

We have two Nash equilibria: (c_1, s_2) and (s_1, c_2).

Static vision of the game: players are perfectly informed and supposed to be **intelligent**, **rational** and **selfish**

Games played on a graph – The evolutionnary approach



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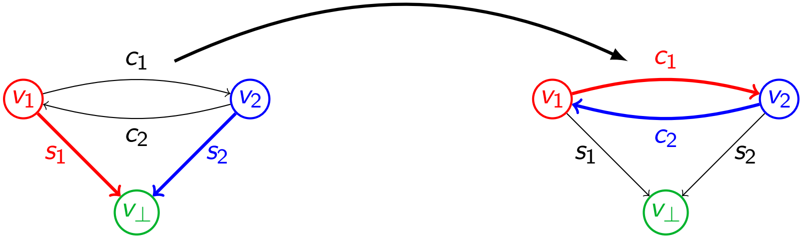

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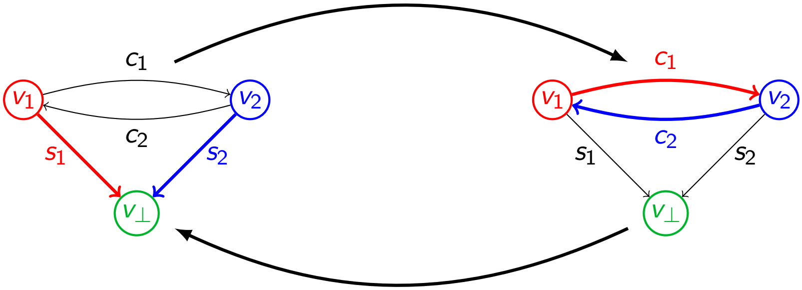
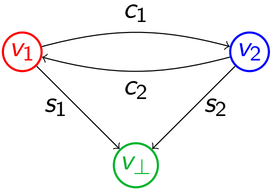
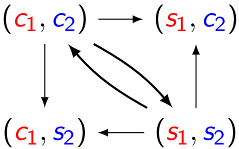


Asynchronous nature of the network could block the packets in an undesirable cycle...

Interdomain routing problem - open problem



The game **G**



The graph of the dynamics: $\mathbf{G}\langle \rightarrow \rangle$

Identify necessary and sufficient conditions on **G** such that $\mathbf{G}\langle \rightarrow \rangle$ has no cycle.

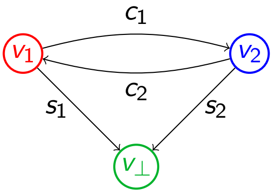
Ideally, the conditions should be algorithmically simple, locally testable...

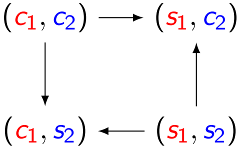
Numerous interesting partial solutions are proposed in the literature.

Daggitt, Gurney, Griffin. Asynchronous convergence of policy-rich distributed Bellman-Ford routing protocols. 2018

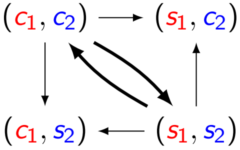
Games played on a graph – The evolutionnary approach

Different dynamics





D_1 with no cycle



D_2 with a cycle

Positional 1-step dynamics $\xrightarrow{P1}$

$$\text{profile}_1 \xrightarrow{P1} \text{profile}_2$$

if:

- a single player changes at a single node
- this player improves his own outcome

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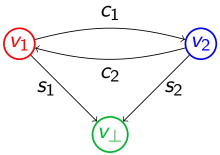
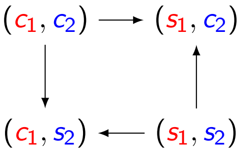


$\mathbf{G}\langle \xrightarrow{\text{P1}} \rangle$:



Positional Concurrent Dynamics $\xrightarrow{\text{PC}}$

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- one or several players change at a single node
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- but synchronous changes may result in worst outcomes...

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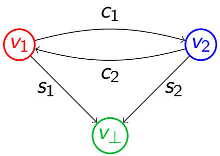

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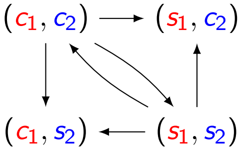

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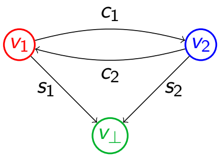

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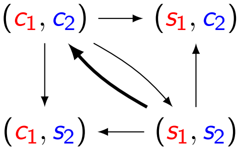

both players **intend** *to reach their best outcome ($v_1 v_\perp \prec_1 v_1 v_2 v_\perp$ and $v_2 v_\perp \prec_2 v_2 v_1 v_\perp$), even if they do not manage to do it (as the reached outcome is $(v_1 v_2)^\omega$ and $(v_2 v_1)^\omega$)*

Questions

What condition **G** should satisfy to ensure that

$$\mathbf{G}\langle \rightarrow \rangle \text{ has no cycle,} \qquad \text{i.e. dynamics } \rightarrow \text{ terminates on } \mathbf{G}?$$

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What relations $\rightarrowtail_1$ and $\rightarrowtail_2$ should satisfy to ensure that

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What should $\mathbf{G}_1$ and $\mathbf{G}_2$ have in common to ensure that

$$\mathbf{G_1}\langle \rightarrowtail \rangle \text{ has no cycle if and only if } \mathbf{G_2}\langle \rightarrowtail \rangle \text{ has no cycle?}$$

Simulation relation on dynamics graphs

G simulates G' ($G' \sqsubseteq G$) if **all that G' can do, G can do it too**.

$$\forall\ \text{profile}'_1 \longrightarrow \forall\ \text{profile}'_2$$

$$\sqsupseteq \qquad\qquad \sqsupseteq$$

$$\forall\ \text{profile}_1$$

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Folklore

If $\mathbf{G}_1\langle \rightarrow_1 \rangle$ simulates $\mathbf{G}_2\langle \rightarrow_2 \rangle$ and the dynamics $\rightarrow_1$ terminates on $\mathbf{G}_1$, then the dynamics $\rightarrow_2$ terminates on $\mathbf{G}_2$.

Relation between games

$\mathbf{G}'$ is a minor of $\mathbf{G}$ if it is obtained by a succession of operations:

- deletion of an edge (and all the corresponding outcomes);
- deletion of an isolated node;
- deletion of a node v with a single edge $v \to v'$ and no predecessor $u \to v$ such that $u \to v'$.

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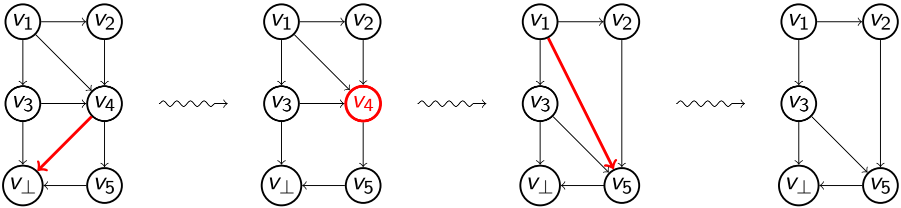

Relation between simulation and minor

Theorem

If $\mathbf{G}'$ is a minor of $\mathbf{G}$, then $\mathbf{G}\langle \xrightarrow{P1} \rangle$ simulates $\mathbf{G}'\langle \xrightarrow{P1} \rangle$. In particular, if $\xrightarrow{P1}$ terminates for $\mathbf{G}$, it terminates for $\mathbf{G}'$ too.

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Theorem

If $\mathbf{G}'$ is a minor of $\mathbf{G}$, then $\mathbf{G}\langle\xrightarrow{\text{PC}}\rangle$ simulates $\mathbf{G}'\langle\xrightarrow{\text{PC}}\rangle$. In particular, if $\xrightarrow{\text{PC}}$ terminates for $\mathbf{G}$, it terminates for $\mathbf{G}'$ too.

Remark: $\mathbf{G}\langle\xrightarrow{\text{P1}}\rangle \sqsubseteq \mathbf{G}\langle\xrightarrow{\text{PC}}\rangle$

More realistic conditions

Adding fairness

- Termination might be too strong to ask in interdomain routing...
- Every router that wants to change its decision will have the opportunity to do it in the future...
- Study of *fair termination*

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More realistic dynamics

Consider *best reply* variants $\xrightarrow{\text{bP1}}$ and $\xrightarrow{\text{bPC}}$ of the two dynamics, where each player that modifies its strategy changes in the best possible way

What results?

Previous theorem

If $\mathbf{G}'$ is a minor of $\mathbf{G}$, then $\mathbf{G}\langle\xrightarrow{\text{PC}}\rangle$ simulates $\mathbf{G}'\langle\xrightarrow{\text{PC}}\rangle$. In particular, if $\xrightarrow{\text{PC}}$ terminates for $\mathbf{G}$, it terminates for $\mathbf{G}'$ too.

- Becomes false for best reply dynamics $\xrightarrow{\text{bP1}}$ and $\xrightarrow{\text{bPC}}$: the best reply dynamics could terminate in $\mathbf{G}$ but not in the minor $\mathbf{G}'$
- Does not apply to fair termination: the dynamics could fairly terminate for $\mathbf{G}$ (and not *terminate*) but not for $\mathbf{G}'$
- The reciprocal does not hold...

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Theorem

If $\mathbf{G}'$ is a *dominant minor* of $\mathbf{G}$, then $\xrightarrow{\text{bPC}} / \xrightarrow{\text{bP1}}$ fairly terminates for $\mathbf{G}$ if and only if it fairly terminates for $\mathbf{G}'$.

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- Use of simulations that are partially invertible...

Application to interdomain routing

- Particular case of game with one target for all players (reachability game) and players owning a single node (router)

Theorem [Sami, Shapira, Zohar, 2009]

If $\mathbf{G}$ is a one-target game for which $\xrightarrow{\text{bPC}}$ fairly terminates, that it has exactly one *equilibrium*.

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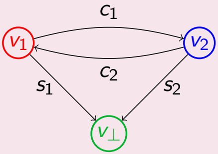
Theorem [Griffin, Shepherd, Wilfong, 2002]

There exists a pattern, called *dispute wheel*, that is a "circular set of conflicting rankings between nodes" such that if **G** is a one-target game that has no dispute wheels, then $\xrightarrow{\text{bPC}}$ fairly terminates.

Application to interdomain routing

Theorem

- There exists a stronger pattern, called *strong dispute wheel*, such that if $\xrightarrow{\text{PC}}$ terminates for **G**, then **G** has no strong dispute wheel.
- Moreover, if two paths having the same next-step are equivalent in the preferences (locality condition), then $\xrightarrow{\text{PC}}$ fairly terminates for **G** if and only if **G** has no strong dispute wheel.
- Finding a strong dispute wheel in **G** can be tested by searching whether **G** contains the following game as a minor:



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- Different possible dynamics
- Conditions for (fair) termination
- Use of game minors and graph simulations
- In the article, non-positional strategies are also considered

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- Consider games with imperfect information: model of malicious router
- A better model of asynchronicity?
- Model fairness using probabilities?

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Thank you!