A journey through negatively-weighted timed games: undecidability, decidability, approximability

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Motivation: quantitative aspects of real-time synthesis

$$\boxed{\text{Environment}} \quad \| \quad \boxed{\text{Controller??}} \quad \models \quad \text{Spec}$$

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Two-player game

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Two-player **weighted** timed game

Additional difficulty: **negative weights**

$\Longrightarrow$ to model production/consumption of resources

Modelling via weighted timed games

	Peak-hour 	Offpeak-hour 
	15 c€/kWh	12 c€/kWh
rate:	total power $\times$ 15 c€/h	total power $\times$ 12 c€/h

states to record which device is on/off: computation of the total power

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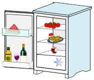
states to record which device is on/off: computation of the total power

Power consumption:

-  100W (1.5 c€/h in peak-hour, 1.2 c€/h in offpeak-hour)
-  2500W (37.5 c€/h in peak-hour, 30 c€/h in offpeak-hour)
-  2000W (24 c€/h in offpeak-hour)

Modelling via weighted timed games

	Peak-hour 	Offpeak-hour 	Solar panels 
	15 c€/kWh	12 c€/kWh	Reselling: 20 c€/kWh
rate:	total power $\times$ 15 c€/h	total power $\times$ 12 c€/h	-0.5×20 c€/h

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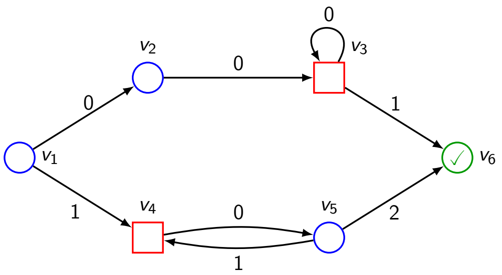
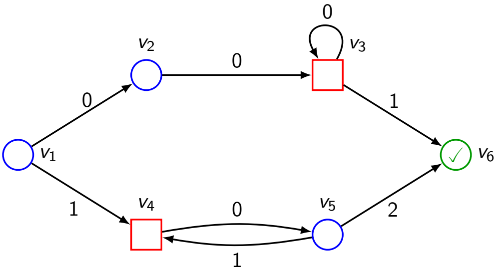
Solution 1 : discretisation of time, resolution via a *weighted game*
Solution 2 : thin time behaviours, resolution via a *weighted timed game*

Weighted games



Weighted automaton with vertices partition between 2 players
+ reachability objective

Weighted games



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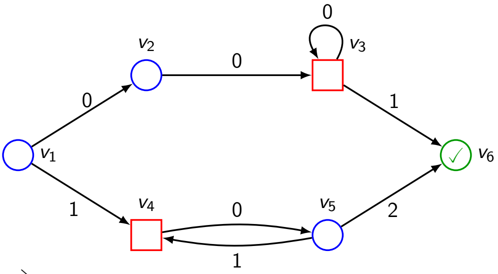
v_1

Weighted games



$v_1 \xrightarrow{\searrow} v_4$

Weighted automaton with vertices partition between 2 players
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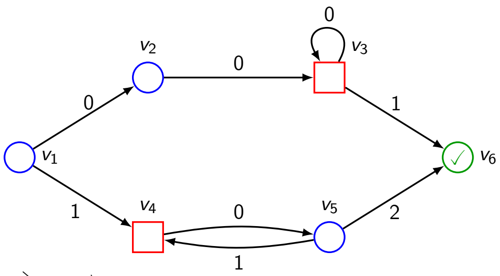
Weighted games



$v_1 \xrightarrow{\searrow} v_4 \xrightarrow{\rightarrow} v_5$

Weighted automaton with vertices partition between 2 players
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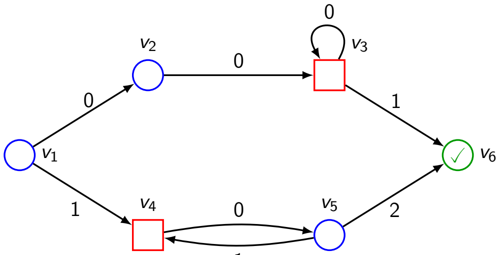
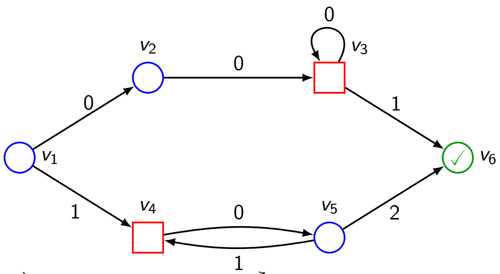
Weighted games



$v_1 \xrightarrow{\searrow} v_4 \xrightarrow{\rightarrow} v_5 \xrightarrow{\leftarrow} v_4 \xrightarrow{\rightarrow} v_5 \xrightarrow{\nearrow} \checkmark$

Weighted automaton with vertices partition between 2 players + reachability objective

Weighted games



$v_1 \xrightarrow{\searrow} v_4 \xrightarrow{\rightarrow} v_5 \xrightarrow{\leftarrow} v_4 \xrightarrow{\rightarrow} v_5 \xrightarrow{\nearrow} \checkmark$

$1 \quad +1 \quad +2 \quad = 4$

Weighted automaton with vertices partition between 2 players
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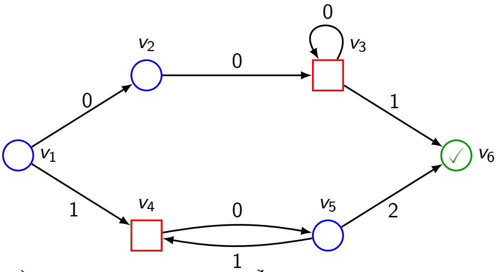
Weighted games



$v_1 \xrightarrow{\searrow} v_4 \xrightarrow{\rightarrow} v_5 \xrightarrow{\leftarrow} v_4 \xrightarrow{\rightarrow} v_5 \xrightarrow{\nearrow} \checkmark$

$1 \quad +1 \quad +2 \quad = 4$

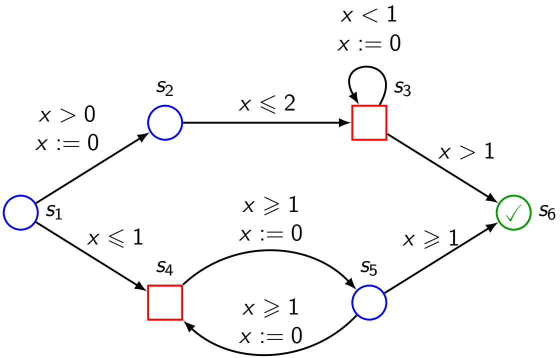
$v_1 \xrightarrow{\nearrow} v_2 \xrightarrow{\rightarrow} v_3 \xrightarrow{\circlearrowleft} v_3 \xrightarrow{\circlearrowleft} v_3 \quad \cdots$

$\cdots \quad = +\infty (\checkmark$ not reached$)$

Weighted automaton with vertices partition between 2 players
$+$ reachability objective

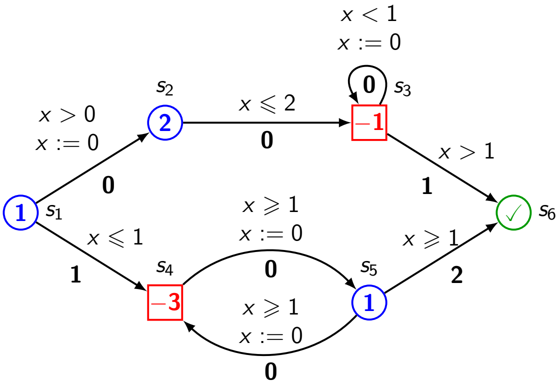
Weight of a path: $\begin{cases} +\infty & \text{if } \checkmark \text{ not reached} \\ \text{total weight until } \checkmark & \text{otherwise} \end{cases}$

Weighted timed games



Timed automaton
with state partition between
2 players
+ reachability objective

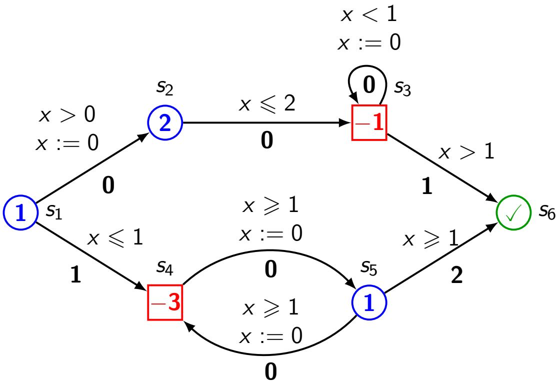
Weighted timed games



Timed automaton
with state partition between
2 players
+ reachability objective
+ linear rates on states
+ discrete weights on
transitions

$(s_1, 0)$

Weighted timed games



$(s_1, 0) \xrightarrow{0.4, \searrow} (s_4, 0.4)$

Timed automaton with state partition between 2 players
+ reachability objective
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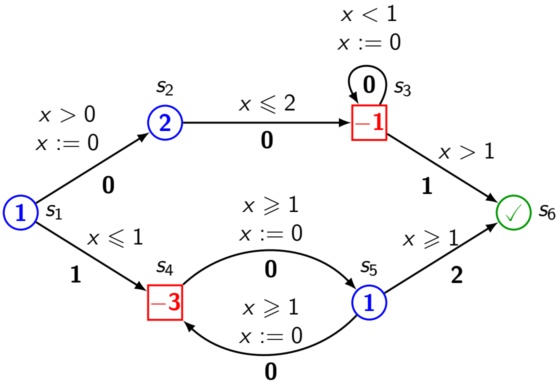
Weighted timed games



$(s_1, 0) \xrightarrow{0.4, \searrow} (s_4, 0.4) \xrightarrow{0.6, \rightarrow} (s_5, 0)$

Timed automaton
with state partition between
2 players
+ reachability objective
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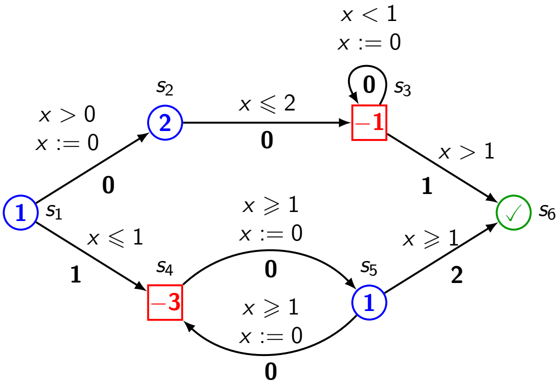
Weighted timed games



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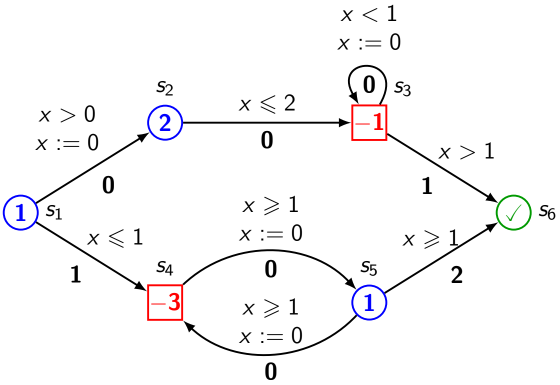
$$(s_1, 0) \xrightarrow{0.4,\searrow} (s_4, 0.4) \xrightarrow{0.6,\rightarrow} (s_5, 0) \xrightarrow{1.5,\leftarrow} (s_4, 0) \xrightarrow{1.1,\rightarrow} (s_5, 0) \xrightarrow{2,\nearrow} (\checkmark, 2)$$

Weighted timed games



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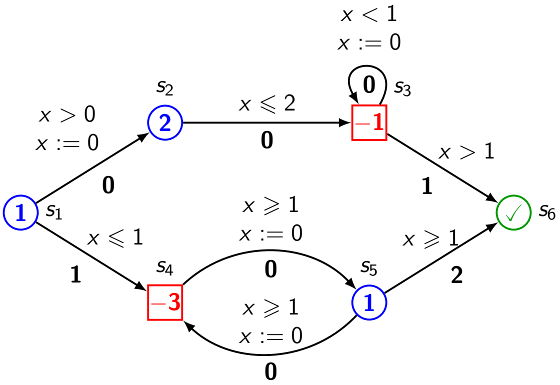
$$\mathbf{1} \times 0.4 + \mathbf{1} \quad -\mathbf{3} \times 0.6 + \mathbf{0} \quad +\mathbf{1} \times 1.5 + \mathbf{0} \quad -\mathbf{3} \times 1.1 + \mathbf{0} \quad +\mathbf{1} \times 2 + \mathbf{2} \quad = 1.8$$

Weighted timed games



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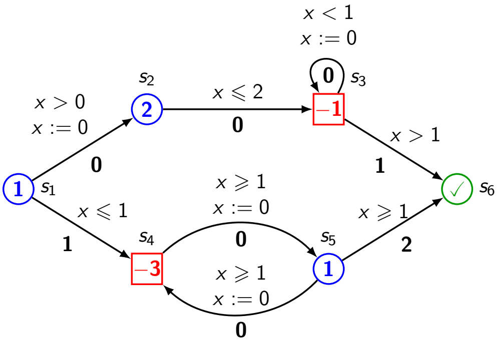
$$\mathbf{1}\times 0.4+\mathbf{1} \quad -\mathbf{3}\times 0.6+\mathbf{0} \quad +\mathbf{1}\times 1.5+\mathbf{0} \quad -\mathbf{3}\times 1.1+\mathbf{0} \quad +\mathbf{1}\times 2+\mathbf{2} \quad = 1.8$$

$$(s_1, 0) \xrightarrow{0.2, \nearrow} (s_2, 0) \xrightarrow{0.9, \rightarrow} (s_3, 0.9) \xrightarrow{0.2, \circlearrowleft} (s_3, 0) \xrightarrow{0.9, \circlearrowleft} (s_3, 0) \quad \cdots$$

$$\mathbf{1}\times 0.2+\mathbf{0} \quad +\mathbf{2}\times 0.9+\mathbf{0} \quad -\mathbf{1}\times 0.2+\mathbf{0} \quad -\mathbf{1}\times 0.9+\mathbf{0} \quad \cdots \quad = +\infty$$

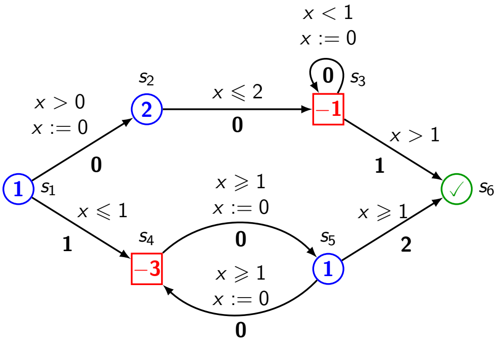
$$\text{Weight of an execution : } \begin{cases} +\infty & \text{if } \checkmark \text{ not reached} \\ \text{total weight until } \checkmark & \text{otherwise} \end{cases}$$

Strategies and objectives



Strategy for a player: map finite executions to a delay and a transition

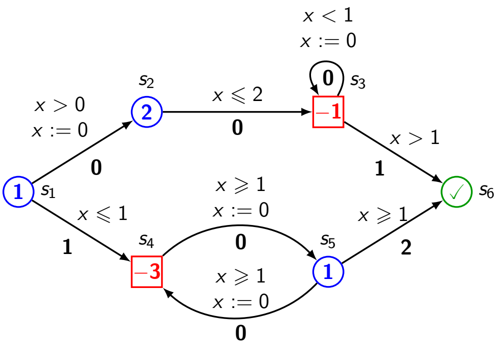
Strategies and objectives



Strategy for a player: map finite executions to a delay and a transition

Objective of player $\bigcirc$: reach $\checkmark$ **and** minimise the weight
Objective of player $\square$: avoid $\checkmark$ **or, if not possible,** maximise the weight

Strategies and objectives



Strategy for a player: map finite executions to a delay and a transition

Objective of player ○: reach ✓ **and** minimise the weight
Objective of player □: avoid ✓ **or, if not possible,** maximise the weight

Main object of interest:
$$\mathsf{Val}(s)(\nu) = \inf_{\sigma_{\mathsf{Min}} \in \mathsf{Strat}^{\mathsf{Min}}} \sup_{\sigma_{\mathsf{Max}} \in \mathsf{Strat}^{\mathsf{Max}}} \mathsf{Weight}(\mathsf{Exec}(s, \nu, \sigma_{\mathsf{Min}}, \sigma_{\mathsf{Max}})) \in \overline{\mathbb{R}}$$
What weight can players guarantee? Following which strategies?

Part I : Weighted games

State of the art: weighted games (shortest-path objective)

$\mathsf{F}_{\leqslant K}\checkmark$: $\exists$ a strategy in the weighted game for player $\bigcirc$ reaching $\checkmark$ with a cost $\leqslant K$?

- one-player: shortest path in a weighted graph... polynomial algo.
- two players, non-negative weights only: polynomial algo.
  "Dijkstra algorithm on 2 players games" (Khachiyan et al., 2008)

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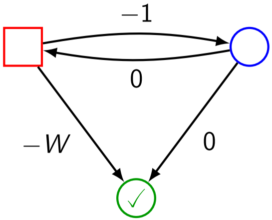

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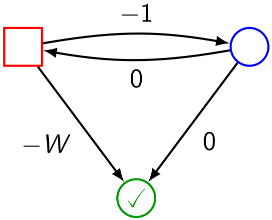


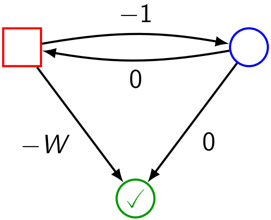
$\bigcirc$ needs memory!
Value $-\infty$: detection is as hard as solving parity games ($\mathbf{NP} \cap \mathbf{co\text{-}NP}$)

Pseudo-polynomial algorithm to solve weighted games

Joint work with Thomas Brihaye, Gilles Geeraerts and Axel Haddad (Brihaye et al., 2016)

Value iteration algorithm: compute $\mathcal{F}^i(+\infty)$...

$$\mathcal{F}(\boldsymbol{x})_v = \begin{cases} \min\limits_{e=(v,a,v')\in E} \big(\mathsf{Weight}(e) + \boldsymbol{x}_{v'}\big) & \text{if } v \in V_{\mathsf{Min}} \\ \max\limits_{e=(v,a,v')\in E} \big(\mathsf{Weight}(e) + \boldsymbol{x}_{v'}\big) & \text{if } v \in V_{\mathsf{Max}} \end{cases}$$



	□	○
horizon 0:	$+\infty$	$+\infty$

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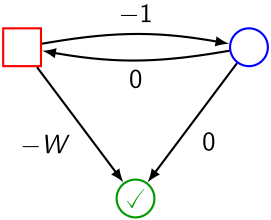

	□	○
horizon 0:	$+\infty$	$+\infty$
horizon 1:	$+\infty$	0

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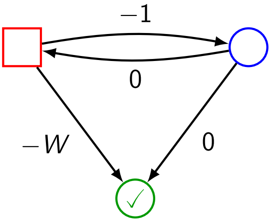

	□	○
horizon 0:	$+\infty$	$+\infty$
horizon 1:	$+\infty$	0
horizon 2:	-1	0

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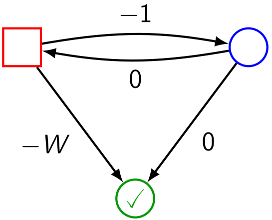

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horizon 0:	$+\infty$	$+\infty$
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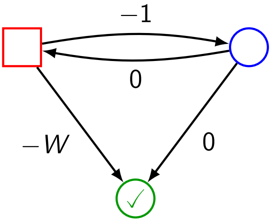

	□	○
horizon 0:	$+\infty$	$+\infty$
horizon 1:	$+\infty$	0
horizon 2:	-1	0
horizon 3:	-1	-1
horizon 4:	-2	-1

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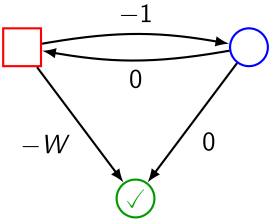

	$\square$	$\bigcirc$
horizon 0:	$+\infty$	$+\infty$
horizon 1:	$+\infty$	0
horizon 2:	-1	0
horizon 3:	-1	-1
horizon 4:	-2	-1
	...	...
horizon $2W+1$:	$-W$	$-W$

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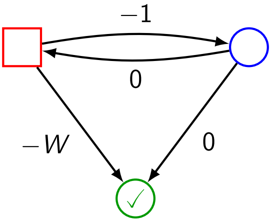


	□	○
horizon 0:	$+\infty$	$+\infty$
horizon 1:	$+\infty$	0
horizon 2:	-1	0
horizon 3:	-1	-1
horizon 4:	-2	-1
	...	...
horizon $2W+1$:	$-W$	$-W$
horizon $2W+2$:	$-W$	$-W$

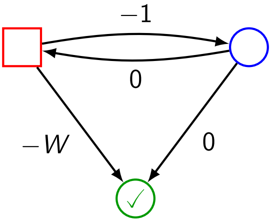
strategy of ○

Pseudo-polynomial algorithm to solve weighted games

Joint work with Thomas Brihaye, Gilles Geeraerts and Axel Haddad (Brihaye et al., 2016)

Value iteration algorithm: compute $\mathcal{F}^i(+\infty)$...

$$\mathcal{F}(\boldsymbol{x})_v = \begin{cases} \min\limits_{e=(v,a,v')\in E} \big(\mathrm{Weight}(e) + \boldsymbol{x}_{v'}\big) & \text{if } v \in V_{\mathrm{Min}} \\ \max\limits_{e=(v,a,v')\in E} \big(\mathrm{Weight}(e) + \boldsymbol{x}_{v'}\big) & \text{if } v \in V_{\mathrm{Max}} \end{cases}$$



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horizon $2W+2$:	$-W$	$-W$

strategy of ○

Theorem:

We can compute in pseudo-polynomial time the value of a weighted game, as well as optimal strategies for both players: ○ may require (pseudo-polynomial) memory to play optimally (but has counter strategies), □ has optimal memoryless strategy.

Large polynomial fragment: divergent weighted games

Joint work with Damien Busatto-Gaston and Pierre-Alain Reynier (Busatto-Gaston et al., 2017)

Divergence property (in the underlying graph):
Every cycle has total weight either $\leqslant -1$ or $\geqslant 1$

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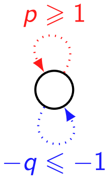
Theorem:

Deciding if a weighted game is divergent is in PTIME.

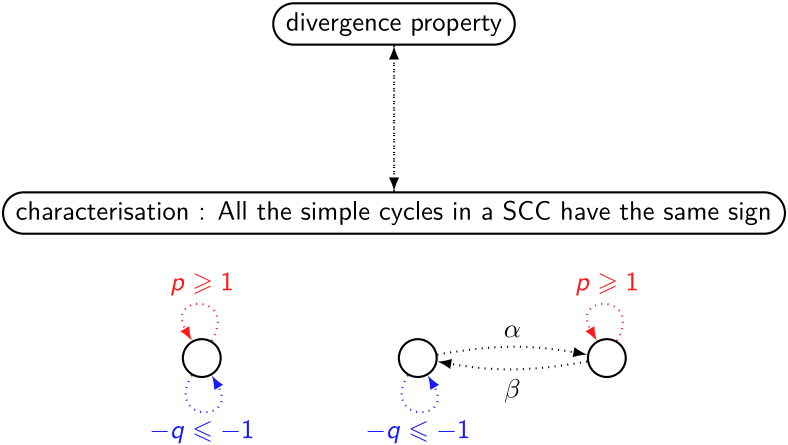
Divergent weighted games analysis

divergence property

characterisation :



Divergent weighted games analysis



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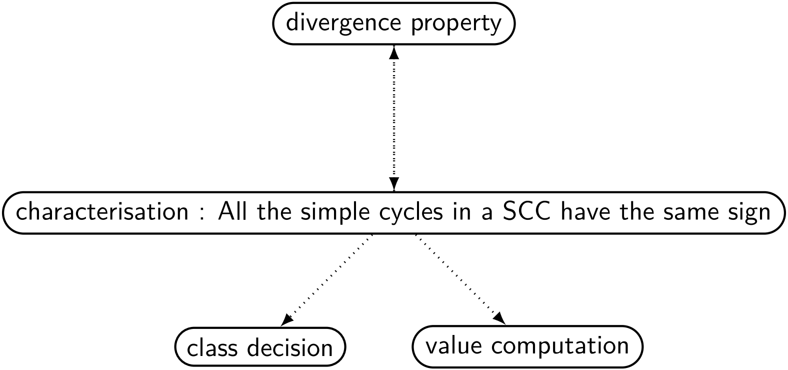

Value computation in a divergent weighted game

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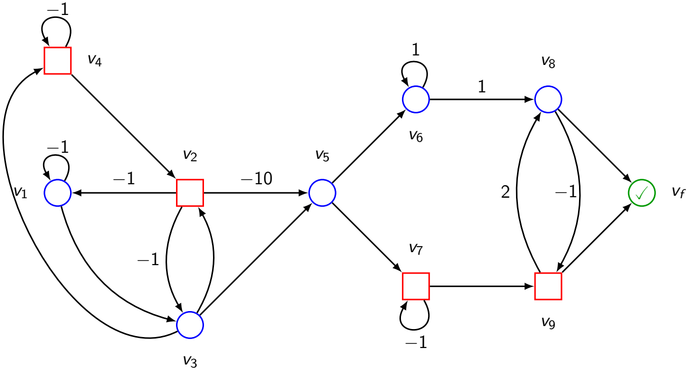
negative SCC

- Outside of the attractor of player $\square$ toward $\checkmark$ $\Rightarrow -\infty$
- The "value iteration" algorithm converges in linear time with initialisation at $-\infty$

Example



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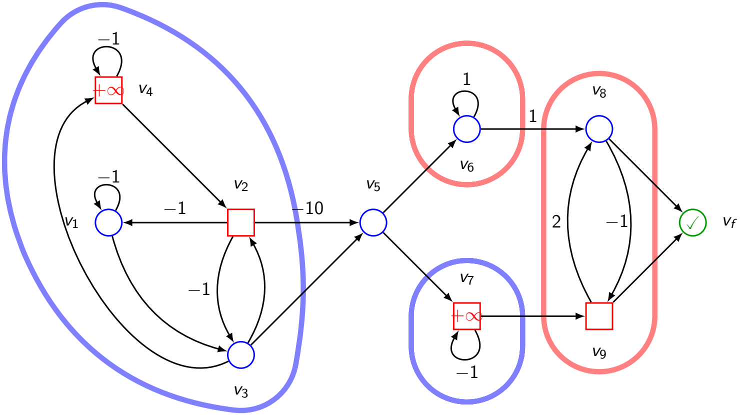

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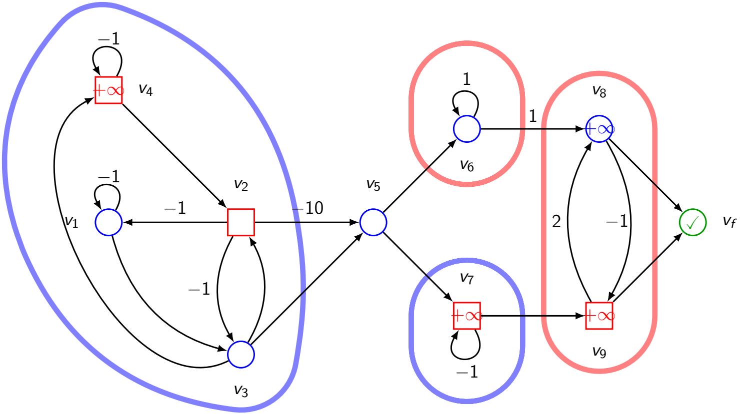
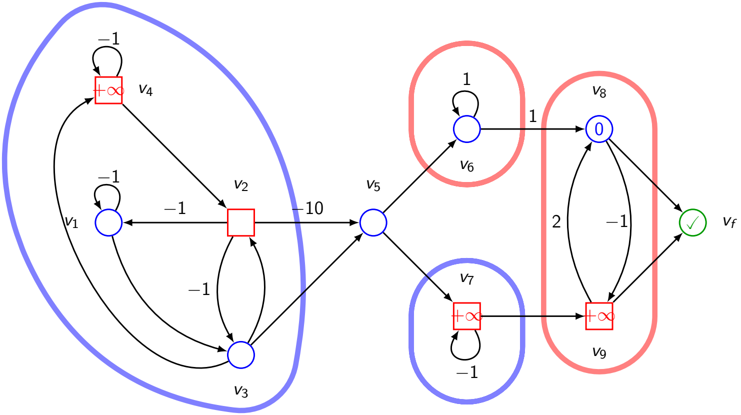
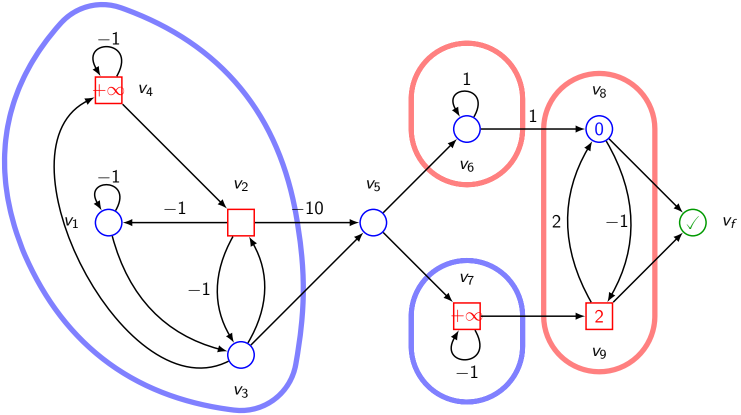


−1
$+\infty$
v_4
−1
v_1
−1
v_2
−10
v_5
−1
v_3
1
1
v_6
v_8
$+\infty$
2
−1
v_f
v_7
$+\infty$
−1
$+\infty$
v_9

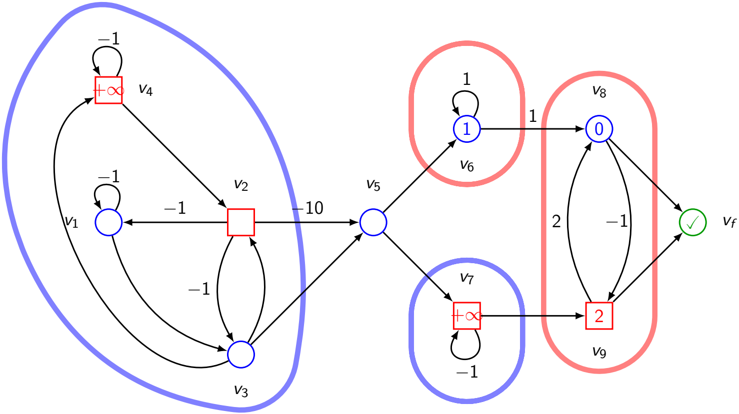
Example



Example



Example



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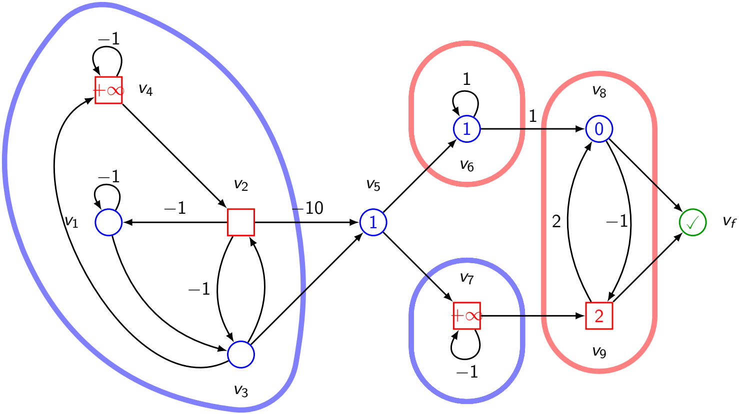
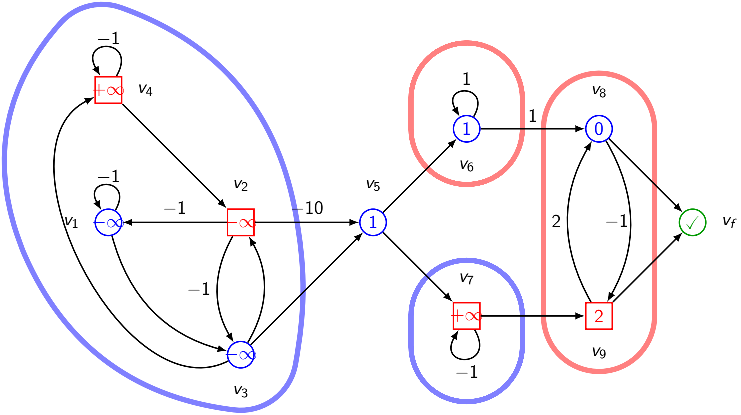


-1
$+\infty$
v_4
-1
v_1
-1
v_2
-10
v_5
1
-1
v_3
1
1
v_6
1
v_7
$+\infty$
-1
v_8
0
2
-1
2
v_9
$\checkmark$
v_f

Example



-1
$+\infty$
v_4
-1
v_1
$-\infty$
-1
v_2
$-\infty$
-10
-1
$-\infty$
v_3
v_5
1
1
1
v_6
1
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0
2
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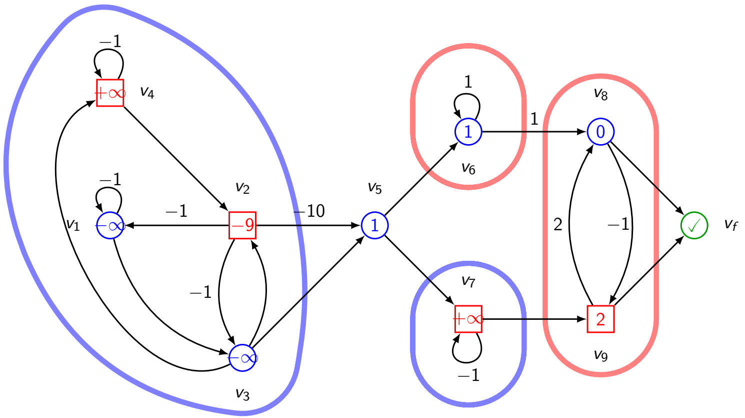
v_9
$\checkmark$
v_f

Example



-1
$+\infty$
v_4
-1
v_1
$-\infty$
-1
v_2
-9
-10
-1
$-\infty$
v_3
v_5
1
1
1
v_6
1
v_8
0
2
-1
v_f
v_7
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2
v_9

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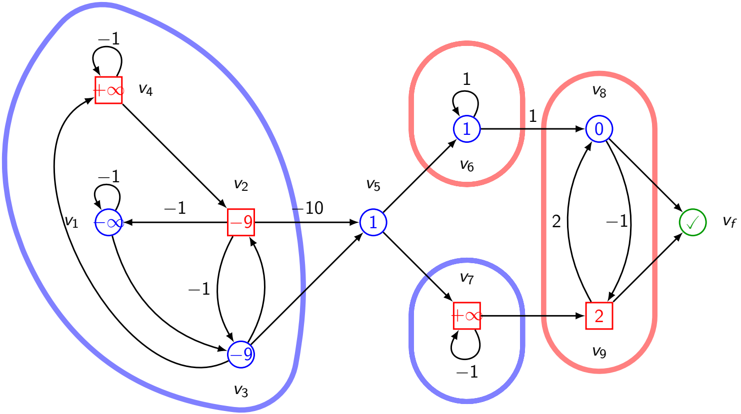

Part II : Weighted **timed** games

State of the art

$\mathsf{F}_{\leqslant K}\checkmark$: $\exists$ a strategy in the WTG (weighted timed game) for player $\bigcirc$ reaching $\checkmark$ with a cost $\leqslant K$?

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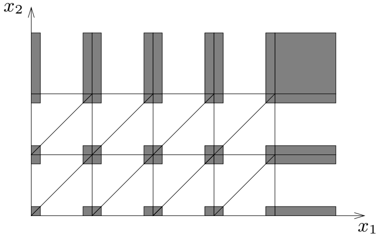
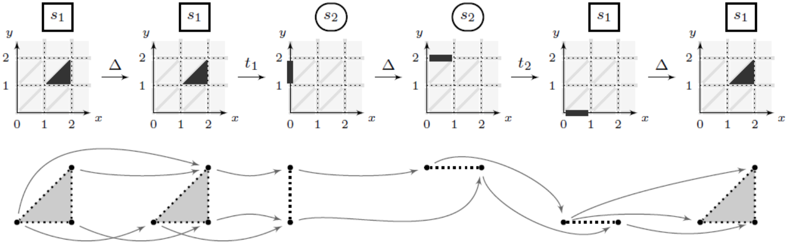
- Decidability results for WTGs with arbitrary weights?

One-player case: weighted timed automata

- Main tool: refinement of regions via corner point abstraction / ε-graph (Bouyer et al., 2004a, 2007)





One-clock Bi-Valued WTGs (1BWTGs)

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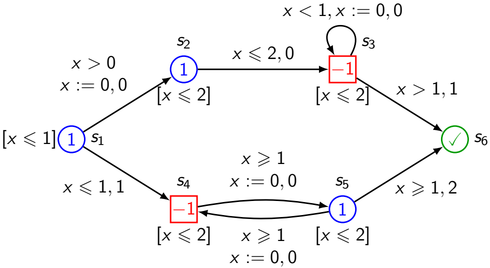
Joint work with Thomas Brihaye, Gilles Geeraerts, Shankara Krishna Narayanan, Lakshmi Manasa and Ashutosh Trivedi (Brihaye et al., 2014)

Assumption: rates of locations $\{p^-, p^+\}$ included in $\{0, +d, -d\}$ ($d \in \mathbb{N}$) (no assumption on costs of transitions)



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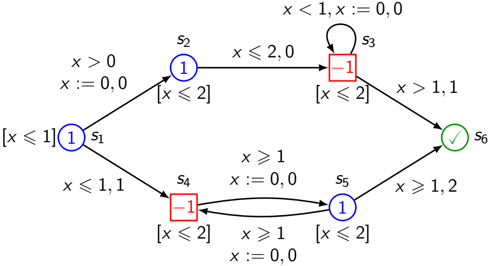

regions: $\{0\}, (0,1), \{1\}, (1,2), \{2\}, (2,+\infty)$

regions refined with corner information:

$\{0\}, (0,\eta), (1-\eta, 1), \{1\}, (1, 1+\eta), (2-\eta, 2), \{2\}, (2, +\infty)$

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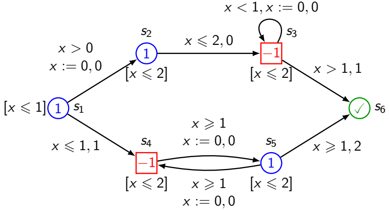

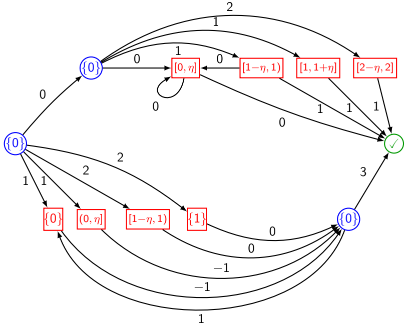

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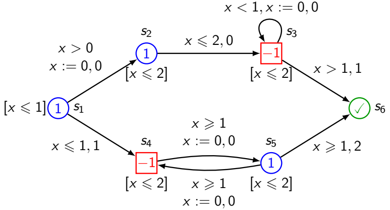

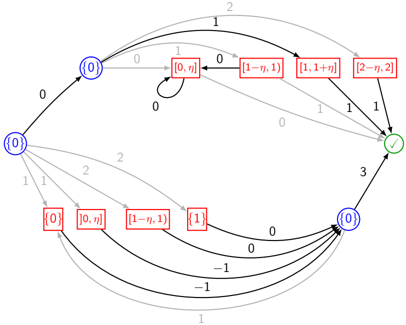

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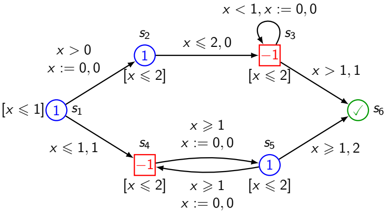

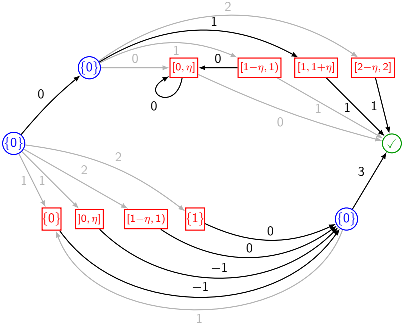

Theorem:

Computation of the value $\overline{\mathsf{Val}}(s, v)$ of states of a 1BWTG and synthesis of ε-optimal strategies for $\bigcirc$ in pseudo-polynomial time

- Only non-negative costs $\Longrightarrow$ polynomial time

Crucial tool: symmetrize the viewpoint

Value for player $\bigcirc$:
$$\overline{\mathsf{Val}}(s, v) = \inf_{\sigma_{\mathsf{Min}} \in \mathsf{Strat}^{\mathsf{Min}}} \sup_{\sigma_{\mathsf{Max}} \in \mathsf{Strat}^{\mathsf{Max}}} \mathsf{Weight}(\mathsf{Exec}((s, v), \sigma_{\mathsf{Min}}, \sigma_{\mathsf{Max}}))$$
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How to compare them? $\underline{\mathsf{Val}}(s, v) \leqslant \overline{\mathsf{Val}}(s, v)$

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Value for player □:

$$\underline{\mathsf{Val}}(s,v) = \sup_{\sigma_{\mathsf{Max}} \in \mathsf{Strat}^{\mathsf{Max}}} \inf_{\sigma_{\mathsf{Min}} \in \mathsf{Strat}^{\mathsf{Min}}} \mathsf{Weight}(\mathsf{Exec}((s,v),\sigma_{\mathsf{Min}},\sigma_{\mathsf{Max}}))$$

How to compare them? $\underline{\mathsf{Val}}(s,v) \leqslant \overline{\mathsf{Val}}(s,v)$

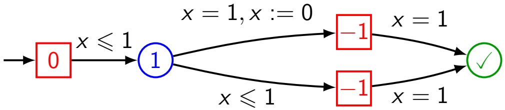
Theorem: Minmax theorem

- 1BPGs (and even all WTGs (Brihaye et al., 2015)) are determined, i.e., $\underline{\mathsf{Val}}(s,v) = \overline{\mathsf{Val}}(s,v)$
- Synthesis of ε-optimal strategies for player □ in pseudo-polynomial time (and polynomial in case of non-negative weights)

1BWTG: maximal fragment for corner-point abstraction

Generalisation by Engel Lefaucheux: two rates $\{p^-, p^+\}$ included in $\{0, +d, -c\}$ ($d, c \in \mathbb{N}$)
In more general settings, players may need to play far from corners...

- With 3 weights in $\{-1, 0, +1\}$: value $1/2$...



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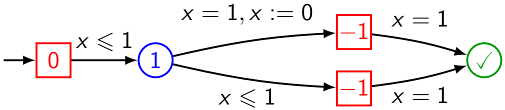
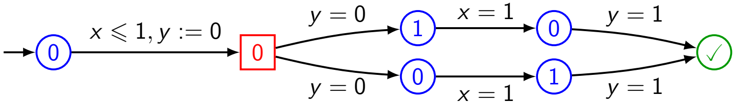
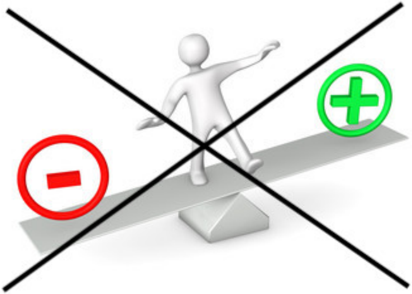


- With 2 weights in $\{-1, 0, +1\}$ but 2 clocks: value $1/2$...



- **How to push further the resolution of WTGs?**



One-clock WTG... Almost!

Related work: 1-clock, non-negative weights

(Hansen et al., 2013): strategy improvement algorithm

(Bouyer et al., 2006b; Rutkowski, 2011): iterative elimination of locations

- precomputation: polynomial-time cascade of simplification of 1-clock WTGs into simple 1-clock WTGs (SWTGs)
  - clock bounded by 1, no guards/invariants, no resets

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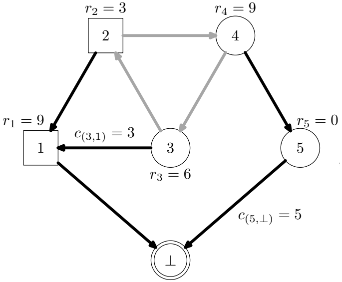
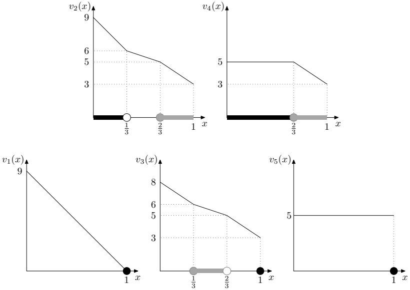
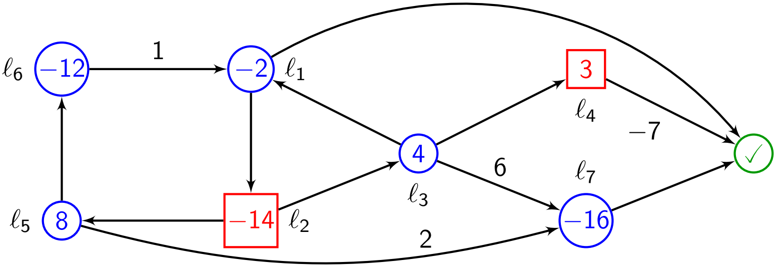
- for SWTGs: compute value functions $\overline{\mathsf{Val}}(\ell, x)$.





SWTGs with arbitrary weights

Joint work with Thomas Brihaye, Gilles Geeraerts, Axel Haddad and Engel Lefaucheux (Brihaye et al., 2015)



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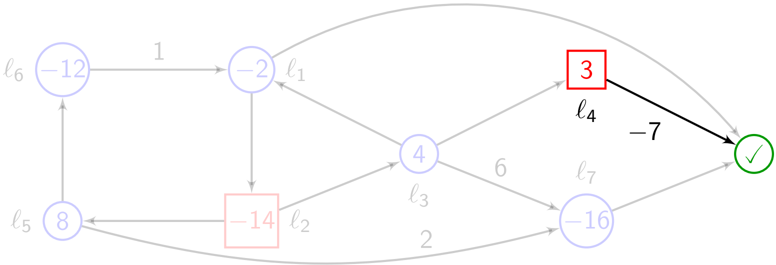
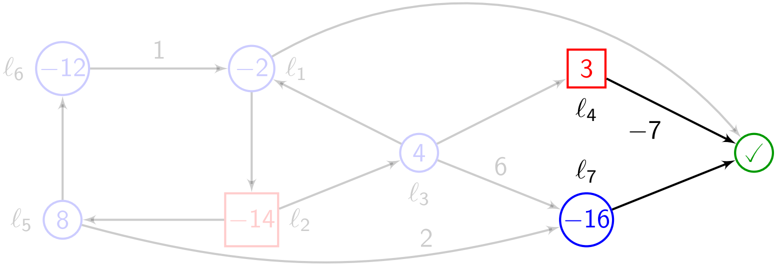


$$\mathsf{Val}(\ell_4, x) = \sup_{0 \leqslant t \leqslant 1-x} 3t - 7 = 3(1-x) - 7 = -3x - 4$$

SWTGs with arbitrary weights

Joint work with Thomas Brihaye, Gilles Geeraerts, Axel Haddad and Engel Lefaucheux (Brihaye et al., 2015)



$$\mathsf{Val}(\ell_4, x) = -3x - 4, \qquad \mathsf{Val}(\ell_7, x) = -16(1-x)$$

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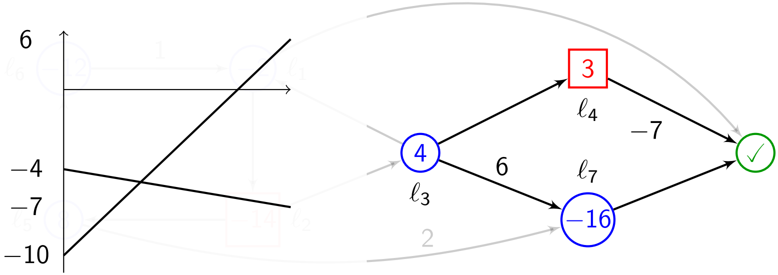

$$\mathsf{Val}(\ell_4, x) = -3x - 4, \qquad \mathsf{Val}(\ell_7, x) = -16(1-x),$$
$$\mathsf{Val}(\ell_3, x) = \inf_{0 \leqslant t \leqslant 1-x}[4t + \min(-3(x+t) - 4, 6 - 16(1-(x+t)))] = \min(-3x - 4, 16x - 10)$$

Recursive elimination of states

- Player ○ prefers to stay as long as possible in locations with **minimal rate**: add a final location allowing him to stay until the end, and make the location urgent

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- Player ○ prefers to stay as long as possible in locations with **minimal rate**: add a final location allowing him to stay until the end, and make the location urgent
- Player □ prefers to leave as soon as possible in locations with **minimal rate**: make the location urgent

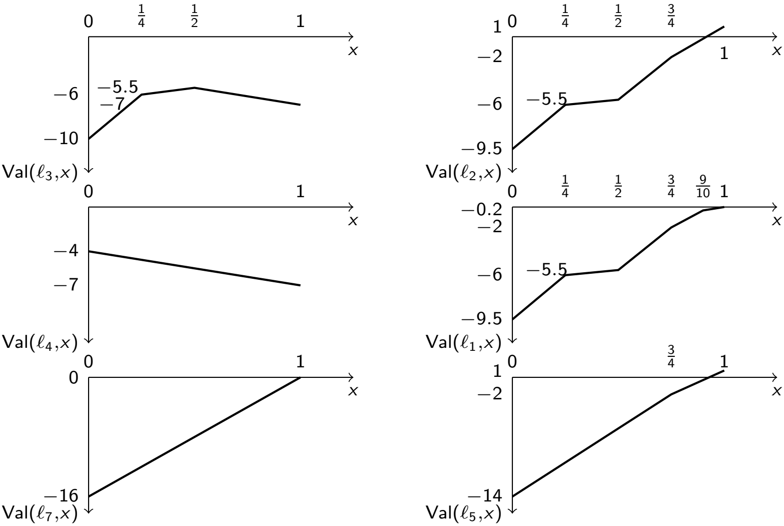
Theorem:

For every SWTG, all value functions are piecewise affine, with at most an exponential number of cutpoints (in number of locations).

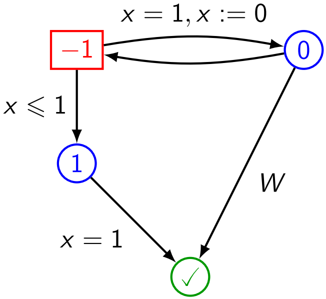
For general 1-clock WTGs?

- removing guards and invariants: previously used techniques work!
- removing resets: previously, bound the number of resets...

Solving SWTGs with arbitrary weights



Bounding the number of resets needed is not possible



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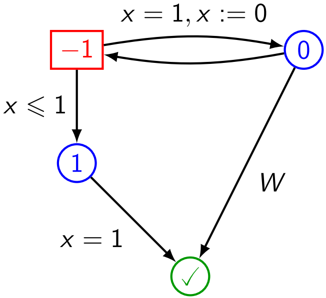

Player ○ can guarantee (i.e., ensure to be below) value ε for all $\varepsilon > 0$...

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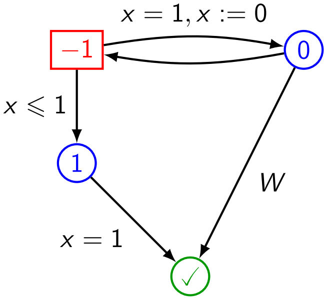

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... but cannot obtain 0: hence, no optimal strategy...

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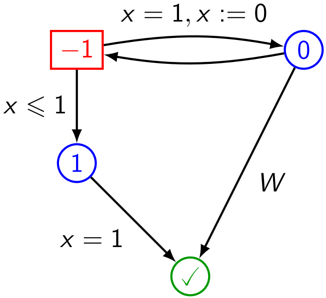

Player $\bigcirc$ can guarantee (i.e., ensure to be below) value ε for all $\varepsilon > 0$...

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... moreover, to obtain ε, $\bigcirc$ needs to loop at least $W + \lceil 1/\log \varepsilon \rceil$ times before reaching $\checkmark$!

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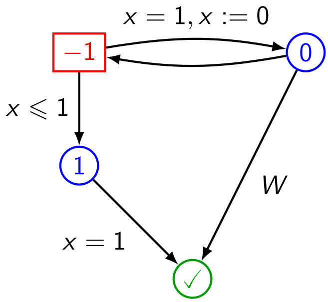

Player ○ can guarantee (i.e., ensure to be below) value ε for all $\varepsilon > 0$...

... but cannot obtain 0: hence, no optimal strategy...

... moreover, to obtain ε, ○ needs to loop at least $W + \lceil 1/\log \varepsilon \rceil$ times before reaching ✓!

Best we can do: exponential time algorithm for reset-acyclic 1-clock WTGs with arbitrary weights

Finally several clocks...

More than one clock?

non-negative weights and strictly non-Zeno-cost cycles:
2-exponential algorithm (Bouyer et al., 2004c; Alur et al., 2004b)

Value iteration algorithm: compute $\mathcal{F}^i(+\infty)$...

$$\mathcal{F}(\boldsymbol{x})_{(s,\nu)} = \begin{cases} \sup\limits_{(s,\nu)\xrightarrow{d,t}(s',\nu')} \big(d \times \mathrm{Weight}(s) + \mathrm{Weight}(t) + \boldsymbol{x}_{(s',\nu')}\big) & \text{if } s \in S_{\mathrm{Max}} \\ \inf\limits_{(s,\nu)\xrightarrow{d,t}(s',\nu')} \big(d \times \mathrm{Weight}(s) + \mathrm{Weight}(t) + \boldsymbol{x}_{(s',\nu')}\big) & \text{if } s \in S_{\mathrm{Min}} \end{cases}$$

Stabilises after a number of iterations at most exponential in the size of the game (because of the number of regions)

Extension to negative weights

Joint work with Damien Busatto-Gaston and Pierre-Alain Reynier (Busatto-Gaston et al., 2017)

Divergence property (of the underlying timed automaton):
Every execution following a cycle of the region automaton has a total weight either $\leqslant -1$ or $\geqslant 1$

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Theorem:

The value problem on divergent weighted timed games is in 2-**EXP**, and is **EXP**-hard.

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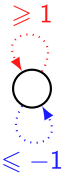
Theorem:

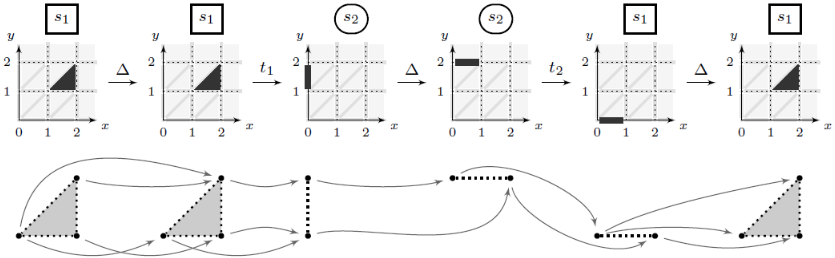
Deciding if a weighted timed game is divergent is **PSPACE**-complete.

Weighted timed games analysis

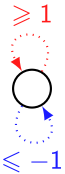


divergence property

characterisation :

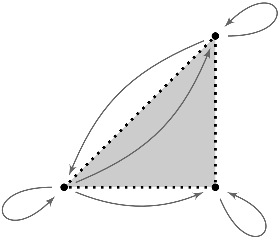


Weighted timed games analysis

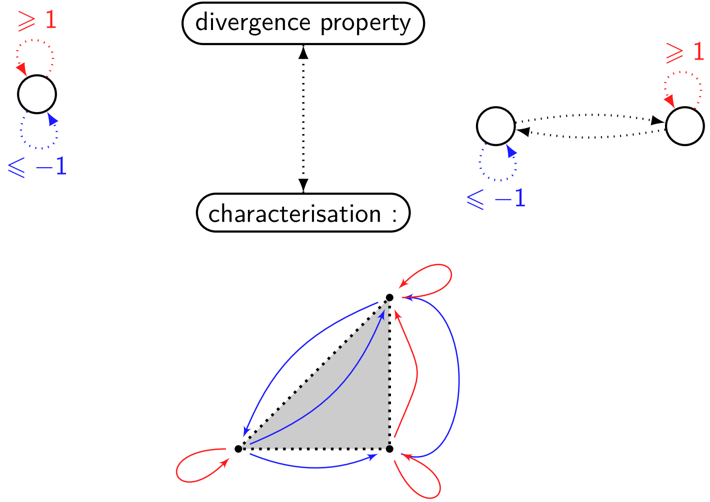


divergence property

characterisation :



Weighted timed games analysis



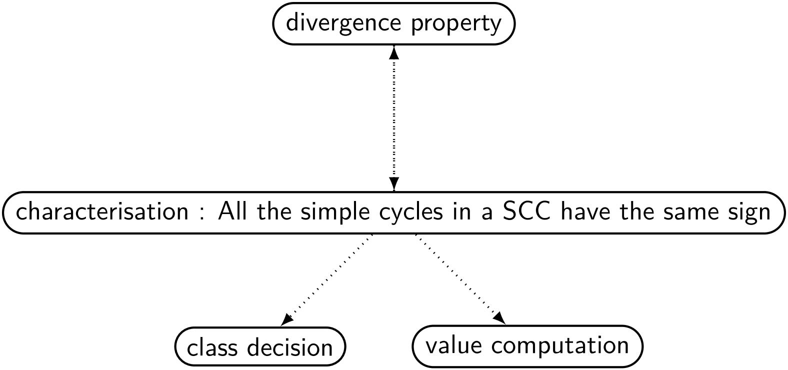
Weighted timed games analysis

divergence property

$\Updownarrow$

characterisation : All the simple cycles in a SCC have the same sign

Weighted timed games analysis



Value computation in divergent weighted timed games

- Remove $+\infty$ states
- SCC decomposition
- Value computation SCC after SCC, bottom-up

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negative SCC

- Outside of the attractor of player $\square$ toward $\checkmark \Rightarrow -\infty$
- The iterative algorithm converges on the other states in a number of steps linear with the region automaton's size, with $-\infty$ initialisation

What about cycles of weight $= 0$?

- Adding cycles of weight $= 0$ to divergent WTG $\Longrightarrow$ **Undecidable!**

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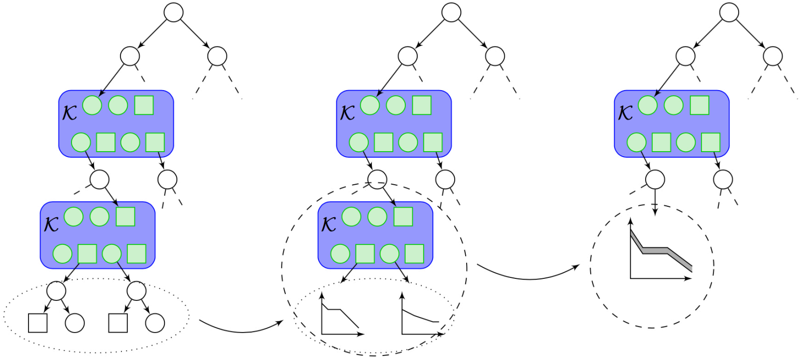
- Adding cycles of weight $= 0$ to divergent WTG $\Longrightarrow$ **Undecidable!**
- Already with only non-negative weights (Bouyer et al., 2015): but possible to approximate the value (with elementary complexity)...



Extension in the negative case?

Ongoing work with Damien Busatto-Gaston and Pierre-Alain Reynier

Almost-divergent WTG: every SCC of the region automaton is

$$\text{either } (\geqslant 1 \text{ or } = 0), \qquad \text{or } (\leqslant -1 \text{ or } = 0)$$

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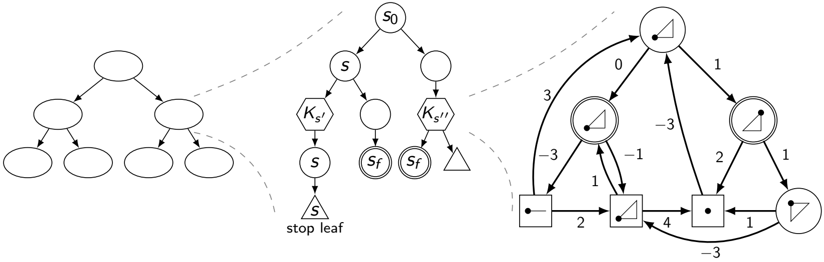

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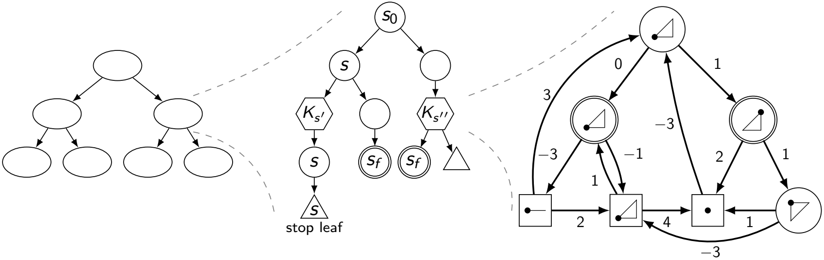


Theorem:

Approximation is decidable (with elementary complexity) for almost-divergent WTGs: (semi-)symbolic algorithm that does not rely on an a-priori discretisation of the regions with a fixed granularity $1/N$ (as in (Bouyer et al., 2015))

- circumvent the need for an SCC decomposition?

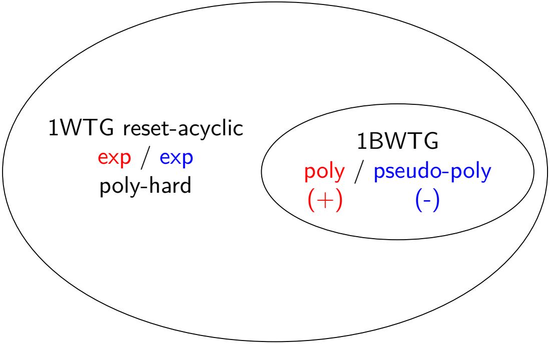
Conclusion

1BWTG

poly / pseudo-poly

(+) (-)

Conclusion



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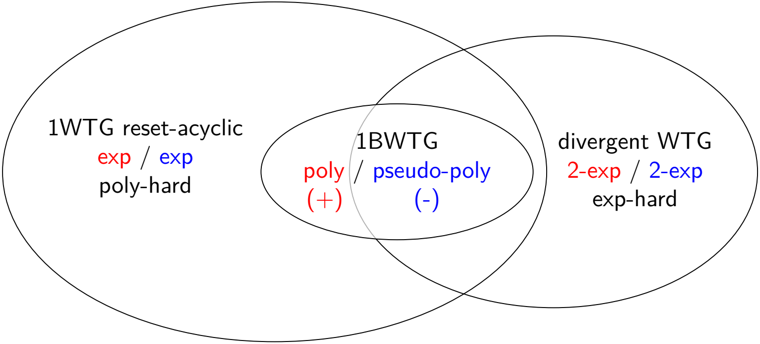
1WTG reset-acyclic
exp / exp
poly-hard
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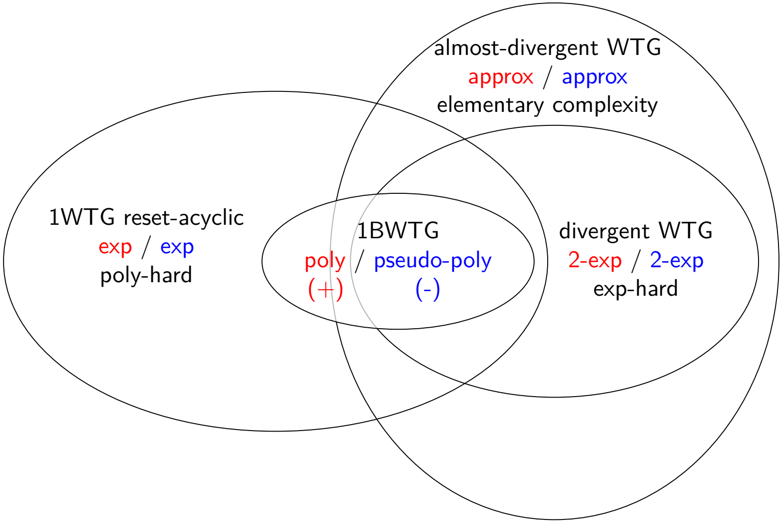

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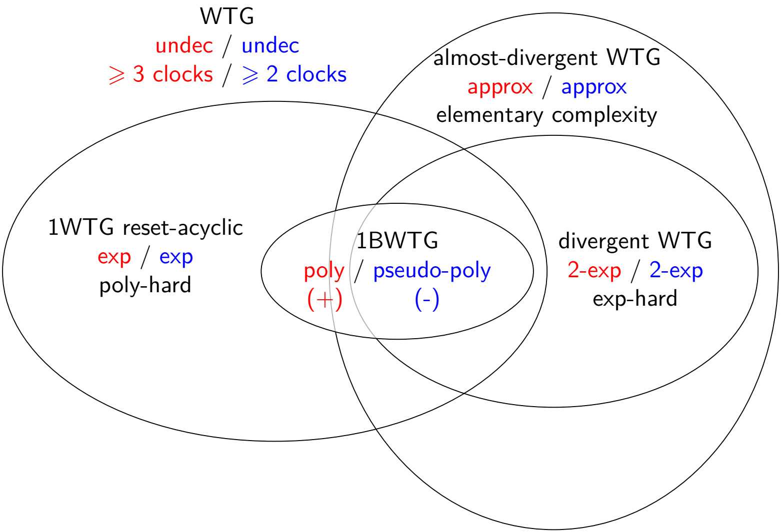
WTG
undec / undec
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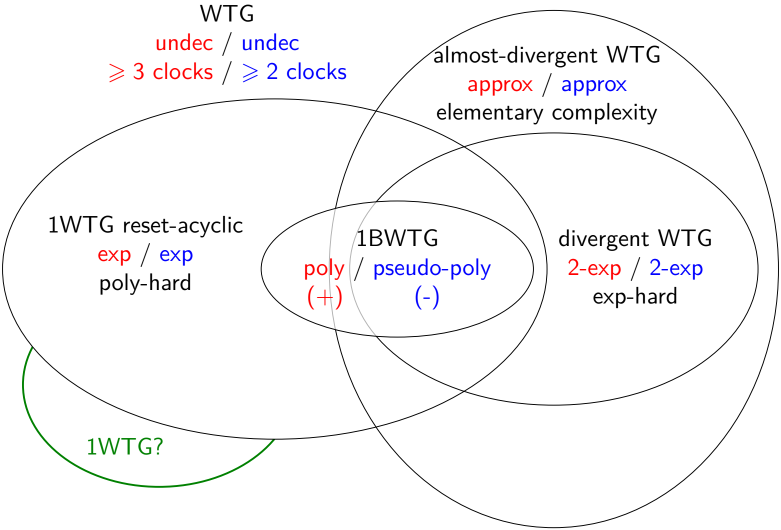
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1WTG?

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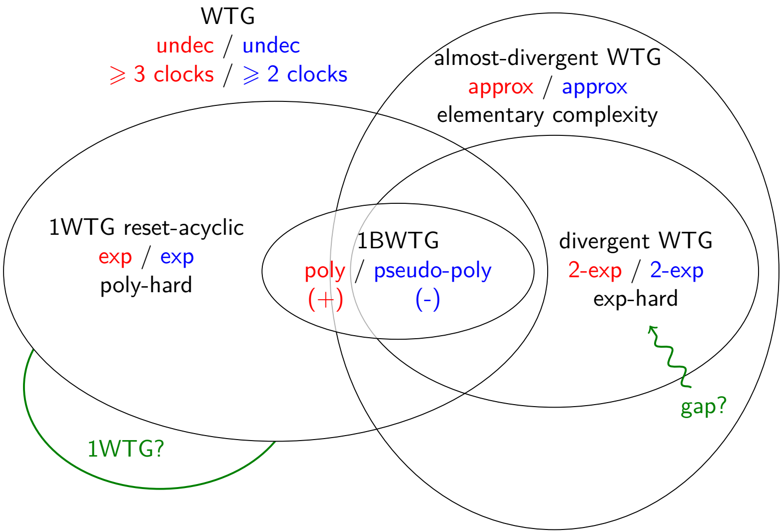

WTG
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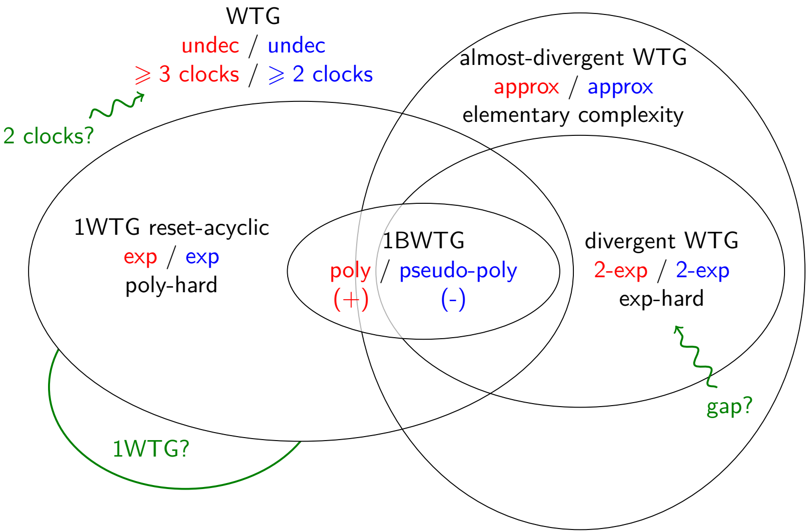
gap?
1WTG?

Conclusion


WTG
undec / undec
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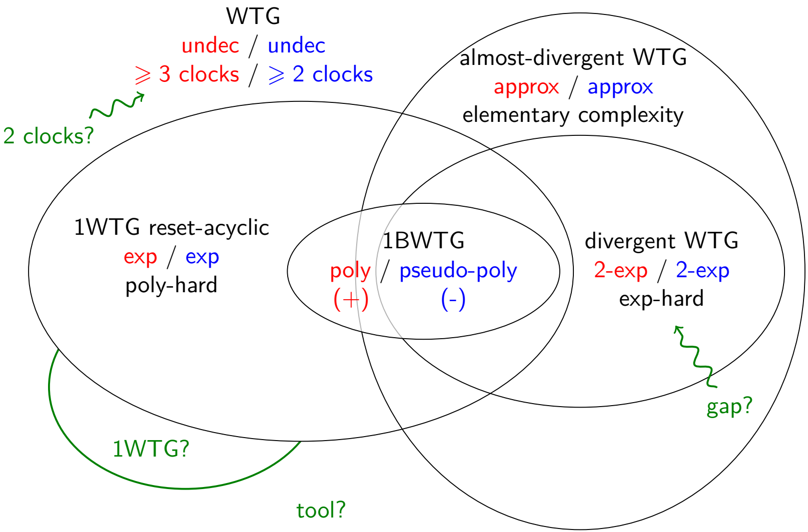
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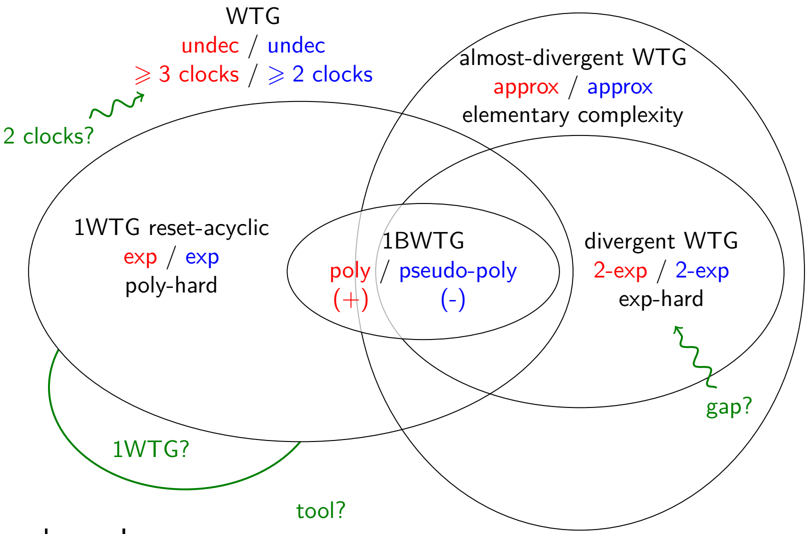
tool?

Conclusion



Thank you!

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Sketch of proof for 1BWTG

1. **Reduce the space of strategies in the 1BWTG**
   - restrict to uniform strategies w.r.t. timed behaviours

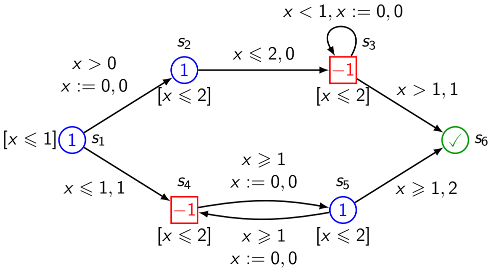
2. **Build a finite weighted game $\mathcal{G}$**
   - based on a refinement of the region abstraction

3. **Study $\mathcal{G}$**

4. **Lift results of $\mathcal{G}$ to the original 1BWTG**

1. Reduce the space of strategies

Intuition: no need for both players to play far from borders of regions



Regions:
$\{0\}, (0,1), \{1\}, (1,2), \{2\}, (2,+\infty)$

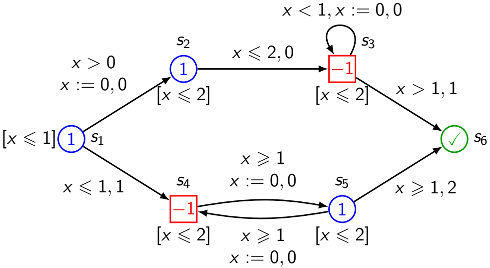
Player ○ wants to leave as soon as possible a state with rate p^+, and wants to stay as long as possible in a state with rate p^-: so, he will always play η-close to a border...

Lemma:

Both players can play arbitrarily close to borders w.l.o.g.: for every η

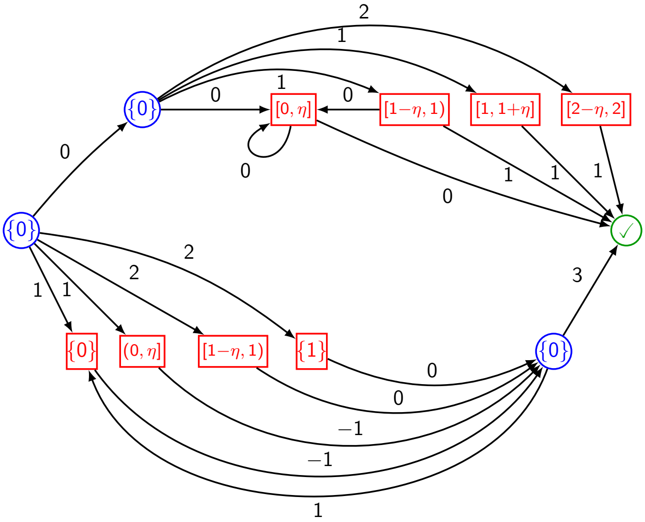
$$\underline{\mathsf{Val}}^{\eta}(s,v) \leqslant \underline{\mathsf{Val}}(s,v) \quad \leqslant \quad \overline{\mathsf{Val}}(s,v) \leqslant \overline{\mathsf{Val}}^{\eta}(s,v)$$

2. Finite weighted game abstraction



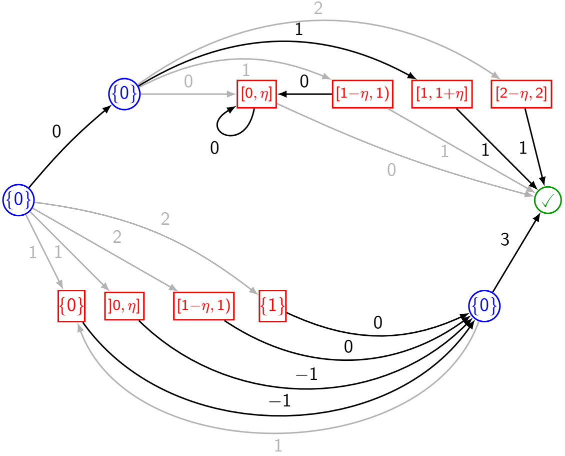
η-regions: $\{0\}, (0, \eta), (1 - \eta, 1), \{1\}, (1, 1 + \eta), (2 - \eta, 2), \{2\}, (2, +\infty)$

2. Finite weighted game abstraction



3. Study $\mathcal{G}$: values, optimal strategies of a min-cost reachability game

(Brihaye et al., 2016)



Optimal value: $\mathsf{Val}_{\mathcal{G}}(s_1, \{0\}) = +2$ (for both players)

4. Lift results to the original 1BWTG

Reconstruct strategies in the 1BWTG from optimal strategies of $\mathcal{G}$

Lemma:

For all $\varepsilon > 0$, there exists $\eta > 0$ such that:

$$\mathsf{Val}_{\mathcal{G}}(s, \{0\}) - \varepsilon \leqslant \underline{\mathsf{Val}}^{\eta}(s, 0) \leqslant \underline{\mathsf{Val}}(s, 0) \leqslant \overline{\mathsf{Val}}(s, 0) \leqslant \overline{\mathsf{Val}}^{\eta}(s, 0) \leqslant \mathsf{Val}_{\mathcal{G}}(s, \{0\}) + \varepsilon$$

4. Lift results to the original 1BWTG

Reconstruct strategies in the 1BWTG from optimal strategies of $\mathcal{G}$

Lemma:

For all $\varepsilon > 0$, there exists $\eta > 0$ such that:

$\mathsf{Val}_{\mathcal{G}}(s, \{0\}) - \varepsilon \leqslant \underline{\mathsf{Val}}^{\eta}(s, 0) \leqslant \underline{\mathsf{Val}}(s, 0) \leqslant \overline{\mathsf{Val}}(s, 0) \leqslant \overline{\mathsf{Val}}^{\eta}(s, 0) \leqslant \mathsf{Val}_{\mathcal{G}}(s, \{0\}) + \varepsilon$

- So $\underline{\mathsf{Val}}(s, 0) = \overline{\mathsf{Val}}(s, 0)$, i.e., determination
- ε-optimal strategies for both players
  - Finite memory for player $\bigcirc$ (finite memory in finite weighted games)
  - Infinite memory for player $\square$ (even though memoryless in finite weighted games), because it needs to ensure convergence of its differences between the 1BWTG and $\mathcal{G}$
- Overall complexity: pseudo-polynomial (polynomial if non-negative costs) in the size of $\mathcal{G}$, which is polynomial in the 1BWTG (because 1 clock)