Edge-disjoint Spanning Trees in Triangulated Graphs on Surfaces and application to node labeling [1]

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Abstract

In 1974, Kundu [4] has shown that triangulated (or maximal) simple toroidal graphs have three edge-disjoint spanning trees. We extend this result by showing that a triangulated graph on an orientable surface of genus g has at least three edge-disjoint spanning trees and so we can partition the edges of graphs of genus g into three forests plus a set of at most $6g - 3$ edges.

Key words: combinatorial problem, graphs on surface, edge partition, arboricity

1 Introduction

We will consider *orientable surface* of genus g denoted $\mathbb{S}_g$, i.e., the surfaces obtained from the sphere $\mathbb{S}_0$ by adding g handles. For instance $\mathbb{S}_1$ is the torus, $\mathbb{S}_2$ the double torus. A region of $\mathbb{S}_g$ is a 2-*cell* if it is homeomorphic to the unit disk. The *genus* of a graph G, denoted $g(G)$, is the minimum g such that G can be embedded in $\mathbb{S}_g$ without edge crossing. Such a embedding is a map of genus g. For the rest of he paper, we will consider simple graphs of genus g with no loops. The sets $V(G)$ and $E(G)$ are respectively the set of vertices and the set of edges of the graph G. We use the notion of arboricity defined by Nash-Williams [8]. The arboricity of a graph G is:

$$k = \max_{H \in A} \left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil$$

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[1] The author is supported by the project "GeoComp" of the ACI Masses de Données

where A is the set of all induced subgraphs of G which contains at least 2 vertices. The link between arboricity and the minimal number of forests needed to decompose the edges of a graph was given by the the Nash-Williams' theorem [8] which states that the edges of a k-arboricity graph can be decomposed into k forests. We define the *tree-density* of a graph G (denoted by $t(G)$) as the maximum number of edge-disjoint spanning trees contained in G.

The problem of finding the maximum number of edge-disjoint spanning is a long studied problem. In fact, this problem has numerous applications in network theory and combinatorics of graphs, for example for routing techniques using spanning trees such as wormhole routing. Using multiples trees [5,11] and so more edges permits to distribute more efficiently the flow of data on the edges of the network. The second application is for edge partition of graphs into trees. Actually, if we show that each graph G with n vertices and m edges has a tree-density of k then we can deduce an edge partition of G in k trees plus a set of $m - k(n-1)$ edges. This edge partition has application for adjacency labeling, i.e., labeling of vertices such that we can compute adjacency using only the labels of the two vertices. In fact, the familly of trees has an efficient adjacency labelling scheme with labels of size $\log n + O(\log^* n)$ [2] due to [1]. Numerous lower bounds for size of labels of adjacency labeling scheme are deduced from this scheme. For example, the best known upper bound for planar graph is $3\log + O(\log^* n)$ and is deduced from the fact that planar graphs have arboricity 3 or less.

The first known result about tree density is due to [12,7] and is related to contraction graph. Let $P = \{P_1, P_2, \cdots, P_r\}$ be a partition of $V(G)$ such that $\forall i, P_i$ induces a connected subgraph denoted H_i. We define the *contraction graph* of a graph G with the partition P denoted G_P as follows. Let v_i, $1 \leq i \leq r$, be the vertices of G_P in which each edge of G that joins a vertex of P_i to a vertex of P_j $(i \neq j)$ corresponds to an edge in G_P joining v_i to v_j. This is the only case in the paper in which we allow multiple edges. This result gives a lower bound of tree-density of a graph in terms of the number of edges of all its contraction graphs.

Theorem 1 *[12,7] A graph G has k pairwise edge-disjoint spanning trees if and only if the number of edges in any contraction graph G_P of G satisfies the following condition:*

$$|E(G_P)| \geq k(|V(G_P)| - 1).$$

Another important result about tree-density is the following theorem.

Theorem 2 *[4] Let G be k-edge-connected graph but not $(k+1)$-edge-connected.*

[2] All the logarithms are in base two and $\log^* n$ denotes the number of times log should be iterated to get a constant.

Then

$$\left\lfloor \frac{k}{2} \right\rfloor \leq t(G) \leq k.$$

In this paper, we show that all triangulated graph on an orientable surface of genus 1 or more have a tree-density of 3 or more. We can make several remarks on this result. First, it extends the result of Kundu [4] that says that maximal toroidal graphs have 3 edge-disjoint spanning trees. Secondly, this bound is tight in the sense that $\forall g \geq 1$, there are triangulated graphs of genus g that do not have 4 edge-disjoint spanning trees. We can construct these graphs by putting a vertex into a triangle of a triangulation and adding edges joining the new vertex to the three vertices of the triangle. Clearly, this graph can not have more than three edge-disjoint spanning trees because it has a vertex of degree 3. We can also remark that for G triangulated graph, if $|V(G)| > 6g(G) - 2$ then G can not have 4 edge-disjoint spanning trees due to Euler's formula. Indeed, G has $3|V(G)| + 6g(G) - 6$ edges, so $|E(G)| < 4(|V(G)| - 1)$ and therefore G cannot have 4 disjoint spanning trees because each spanning tree has $|V(G)| - 1$ edges. We can deduce from our main result a edge partition for graphs of genus g in 3 forests plus a set of $O(g)$ edges. We use this partition to obtain an adjacency labeling scheme for such graphs with labels of size $3 \log n + O(g) + 0(\log^* g)$.

In Section 2, we show the main result of the paper: the lower bound of 3 for tree-density of triangulated graphs on surfaces of genus $g \geq 1$. Then, in Section 3, we show as corollary of our main result an edge-partition of graphs of genus $g \geq 1$ into three forests plus a set of at most $6g - 3$ edges. Finally, in Section 4, we improve our result on edge partition by giving a necessary and sufficient condition on the set of $6g - 3$ edges such that the rest of the graph is of arboricity 3.

2 Lower bound for tree-density of triangulated graphs.

The following theorem is the main result of this paper. It gives a lower bound for tree-density of triangulated graphs on an orientable surface. This result can be used to find an edge-partition of such graphs into three forests plus a set of $6g(G) - 3$ edges. We can deduce a bound for size of universal graphs for family of graph of genus g with at most n vertices. We can also deduce a bound for size of labels in adjacency labelling scheme for the same family of graphs, as explained in Section **??**.

Theorem 3 *Let G be a triangulated graph on a fixed orientable surface of genus $g \geq 1$. Then $t(G) \geq 3$.*

In order to prove this theorem, we use the following lemma.

Lemma 4 *Let G_P be a contraction graph of G. Let H_i be the subgraphs of G induced by the i-th part of the partition. Then*

$$\sum_{i=1}^{r} g(H_i) \leq g(G).$$

Proof. We use a result of Battle, Harary, Kodama and Youngs [2] that says that if we make the union of two graphs with one and only one common vertex then the genus of the resulting graph is equal to the sum of the genera of the two original graphs. First, we construct a spanning tree T of G_P, taking only one edge in the case of parallel edges. We root the tree T at the vertex v_1 of G_P corresponding to the subgraph H_i. Then, we remove from G edges of $G_P - T$, obtaining the graph G' subgraph of G. The graph G' can be constructed starting from $G'' = H_1$ and recursively make the union with a subgraph H_i' composed of H_i plus the edge connecting H_i to G'' repeating this process as many times as necessary to obtain the graph G'. Clearly, G'' and H_i' have exactly one vertex in common. Hence, $g(G') = \sum_{i=1}^{i=r} g(H_i)$. Finally because G' is a subgraph of G, $g(G) \geq g(G') = \sum_{i=1}^{r} g(H_i)$. □

Proof of Theorem 3. Let G_P be a contraction of G corresponding to a partition $P = \{P_1, P_2, \cdots, P_r\}$ with $|P_1| \geq |P_2| \geq \cdots \geq |P_r|$. Let $n = |V(G)|$, $m = |E(G)|$, $g = g(G)$, $n_i = |V(H_i)|$, $m_i = |E(H_i)|$ and $g_i = g(H_i)$. From Euler's formula, we get $m = 3n + 6(g-1)$. We consider three different types of parts P_i in P:

- Parts of size at least 3: $1 \leq i \leq k$, $n_i \geq 3 \Rightarrow m_i \leq 3n_i + 6(g_i - 1)$
- Parts of size 2: $k+1 \leq i \leq k+q$, $n_i = 2 \Rightarrow m_i = 1$
- Parts of size 1: $k+q+1 \leq i \leq r$, $n_i = 1 \Rightarrow m_i = 0$

By definition, $|E(G_p)| = m - \sum_{i=1}^{r} m_i$. Therefore,

$$|E(G_p)| \geq 3n + 6(g-1) - \left\{ \sum_{i \leq k} (3n_i + 6(g_i - 1)) + q \right\}$$

$$|E(G_P)| - 3(r-1) \geq 2q + 3(k-1) + 6 \left\{ g - \sum_{i \leq k} g_i \right\}$$

which is non-negative for all k especially if $k = 0$, $\sum_{i \leq k} g_i = 0$ and $g > 0$. Due to Theorem 1, G has at least 3 edge-disjoint spanning trees. □

3 Corollaries and applications

The previous result of this paper has several applications, amongst other thing for multicast communication in wormhole-routed networks. In wormhole routing [9], each intermediate node forwards the unit of information transfer called worm to the desired output ports as soon as the head of the worm is received. Although one spanning tree is enough to construct a deadlock-free multicast routing algorithm in wormhole-routed networks [5,11], it is clear that finding multiple edge-disjoint spanning trees in the network allows us to use more edges of the graph and so to decrease communication latency. Another application is an edge-partition of graphs into forests.

Corollary 5 *Let G be a graph of orientable genus g. There is a partition of edges of G in 4 sets A_1, A_2, A_3 and A such that $|A| \leq 6g-3$ and for all $i = 1, 2, 3$, A_i induces a forest.*

Proof. First, we embed the graph on the orientable surface of genus g. We triangulate the map without adding loops or duplicated edges. We apply Theorem 3 on the triangulated graph and so we obtain an edge partition into 3 trees A_1, A_2 and A_3 plus a set A of at most $6g-3$ due to Euler's formula. □

The edge partition has direct application for coding adjacency using label in a graph. This is an implicit representation of graphs, i.e., a adjacency labelling scheme in which all vertices are assigned with a distinct label such that we can compute adjacency between any two vertices using only information contained in their labels. Recall that $\log^* n = \min\{i \mid \log^{(i)} n \leq 1\}$ where $\log^{(i)} n$ denotes the i-th iterated of $\log n$. All log are in base 2.

Corollary 6 *Let $A_{g,n}$ be the family of graphs with at most n vertices and orientable genus at most g. There is an adjacency labelling scheme for $A_{g,n}$ with label of length at most $3\log n + O(\log^* n) + O(\sqrt{g}\log g)$ bits.*

Proof. It is enough to use the result of Alstrup and Rauhe [1] that says that the family of forest of at most n nodes has an adjacency labelling scheme with label length bounded by $\log n + O(\log^* n)$ bits. So the edges of A_1, A_2 and A_3 can be coded in $3\log n + O(\log^* n)$ bits for each vertex. We can code A using $O(\sqrt{g}\log g)$ bits for each vertex because the graph induced by the edges of A is a graph of genus g and these graphs can be edge-partioned into at most $O(\sqrt{g})$ forests. Indeed, using the Euler's formula and the definition of arboricity [8], we can bound the arboricity by $O(\sqrt{g})$ because the ratio between the number of edges and the number of vertices is maximal for the greatest complete graph that can be embedded on $\mathbb{S}_g$ whose order and arboricity is in $O(\sqrt{g})$. □

Our result also has an application for induced-universal graphs. Let F be a finite familly of graph. The graph G is F *induced-universal* if $\forall H \in F$, H is an induced subgraph of G.

Corollary 7 *Let $A_{g,n}$ be the family of graphs with at most n vertices and of orientable genus at most g. There is an $A_{g,n}$ induced-universal graph G with at most $O(n^3).2^{O(\log^* n)+O(\sqrt{g}\log g)}$ vertices.*

Proof. We apply the result of Kannan, Naor and Rudich [3] that says that if a family F of graph has an adjacency labelling scheme with unique labels of length $k \log n$ then we can construct a universal graph for this family of size n^k. □

4 Characterization of the edge set A

First, we need to introduce some definitions. A map can be combinatorially defined as a *rotation system* $\pi = \{\pi_v \mid v \in V(G)\}$ with for each vertex v a cyclic permutation π_v of edges around v in the map. The following process determines a *facial walk* of a map, bounding the faces of the graph. We start with an arbitrary vertex v and an edge $e = uv$ incident to v. Then, we traverse the edge e from v to u. We continue the facial walk along the edge $e' = \pi_u(e)$. The facial walk is completed when the initial edge is encountered. Figure 1 gives an example of a facial walk on a map.

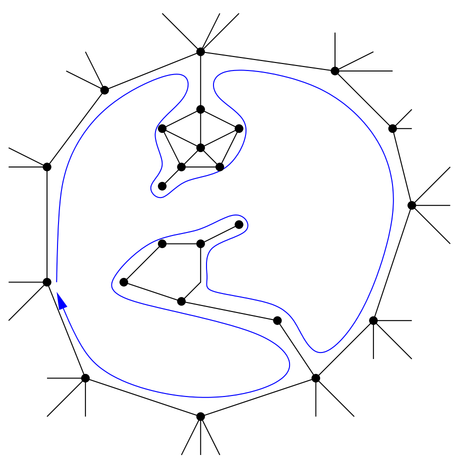

Figure 1. Example of a facial walk.

We can remark that each edge appears twice on the same facial walk or appears in exactly two facial walks. The edges that are contained (twice) in only one facial walk are called *singular*. We say that a facial walk is *singular* if it composed of only singular edges. Clearly, the subgraph induced by a singular facial walk is acyclic and reversely the subgraph induced by a non-singular facial walk contains a cycle. A graph is *triangulated* if all its faces are triangular, i.e., have three edges. For notions undefined here, the reader may consult [6].

Corollary 5 gives a partition of the graph G of genus g into three forests plus a set A of at most $6g-3$ edges. It is clear that A must fullfils some properties such that the arboricity of $G-3$ is 3. The following theorem gives, in a case of triangulated graphs, a characterization of all of these sets.

Theorem 8 *Let G be a triangulation of $\mathbb{S}_g$ with $g>0$ and let A be a set of $6g-3$ edges of G. The arboricity of $G-A$ is 3 if and only if: for each region R of $\mathbb{S}_g$ bounded by a set of cycles $C_1,\cdots,C_r$, R does not contain more than $6(h+s)+\sum_{i=1}^{i=r}(|C_i|-3)$ edges of A where $s+1$ is the number of connected components of G after removal of vertices in R and h is the difference between genus of G and sum of genera of these connected submaps.*

Proof. First, we show that if each region R respects the condition of the theorem, then the arboricity of $G-A$ is 3. Using Euler's formula on triangulation, we have that $|E(G-A)|=3(|V(G-A)|-1)$. So, by definition of arboricity, the arboricity of $G-A$ is at least 3. It remains to show that for every induced map H of G, we have $|E(H-A)|\leq 3(|V(H-A)|-1)$ and so the arboricity is at most 3. Let $F=\{F_1,\cdots,F_f\}$ be the set of faces of H. $\forall i, 1\leq i\leq f$, the face F_i is bounded by a set of facial walks $W_i=\{W_{1,1},\cdots,W_{1,l_i}\}$. We can remove from these sets singular facial walks because they are acyclic and not connected to the remainder of the graph. Consequently, their removal does not change the arboricity of the graph. So, each remaining facial walk contains a cycle. For each non 2-cell face F_i of H, we remove from the surface the corresponding region and $\forall j, 1\leq j\leq l_i$, we add into the facial walk W_i a 2-cell region obtaining a new face. By doing this, we obtain a set of maps $M=\{H_1,\cdots,H_s\}$ that are the connected components of H such that $\forall i, 1\leq i\leq s+1$ H_i is 2-cell embedded, i.e., each facial walk bounds a 2-cell region. $\forall i, 1\leq i\leq s+1$, let $F_i=\{F_{i,1},\cdots,F_{i,f_i}\}$ be the set of faces of H_i. Using Euler's formula and the well-known property of maps in compact 2-manifold surfaces that is: an edge is in exactly 2 faces or twice in the same face, we obtain:

$$
\begin{aligned}
|E(H_i)| &= 3|V(H_i)|+3f_i-2|E(H_i)|+6g(H_i)-6 \\
|E(H_i)| &= 3|V(H_i)|-\sum_{j=1}^{j=f_i}(|F_{i,j}|-3)+6g(H_i)-6
\end{aligned}
$$

For H, we have:

$$|E(H)| = \sum_{i=1}^{i=s+1} |E(H_i)|$$
$$|E(H)| = 3|V(H)| - \sum_{i=1}^{i=s+1} \sum_{j=1}^{j=f_i} (|F_{i,j}| - 3) - 6(\sum_{i=1}^{i=r} g(H_i) + (s+1))$$

$\forall i, 1 \leq i \leq f$, there can not be more than $6(h_i + s_i) + \sum_{j=1}^{j=l_i}(|W_{i,j}| - 3)$ edges of A in the region R_i of G corresponding to the face F_i. This is due to the fact that if the face contains more edges, then we can exhibit a region that does not respect the condition of the theorem because each facial walk contains a cycle. So, we have:

$$|E(H \cap A)| = |E(A)| - |E(A - H)|$$
$$|E(H \cap A)| \geq 6g(G) - 3 - 6\sum_{i=1}^{i=f}(h_i + s_i) - \sum_{i=1}^{i=f}\sum_{j=1}^{j=l_i}(|W_{i,j}| - 3)$$

By construction, we have:

$$\sum_{i=1}^{i=s}\sum_{j=1}^{j=f_i}(|F_{i,j}| - 3) = \sum_{i=1}^{i=f}\sum_{j=1}^{j=l_i}(|W_{i,j}| - 3)$$

Using the fact that $\sum_{i=1}^{i=f} h_i = g(G) - \sum_{i=1}^{i=r} g(H_i)$ and $\sum_{i=1}^{i=f} s_i = s$, we obtain:

$$|E(H \cap A)| \geq 6(\sum_{i=1}^{i=r} g(H_i) - s) - \sum_{i=1}^{i=s}\sum_{j=1}^{j=f_i}(|F_{i,j}| - 3) - 3$$

For $H - A$, we have:

$$|E(H - A)| = |E(H)| - |E(H \cap A)|$$
$$|E(H - A)| \leq 3|V(H - A)| - 3$$

And so the arboricity of G is 3.

Secondly, we show that if there is such a region that contains more than $6(h+s) + \sum_{i=1}^{i=r}(|C_i| - 3)$ edges of A then the arboricity of $G - A$ is greater than 3. Let R be such a region. By removing vertices inside this region from G, we obtain a submap H of G. Using the same reasoning and notation as in the previous part of the proof we obtain the following equality:

$$|E(H)| = 3|V(H)| - \sum_{i=1}^{i=s+1} (|C_i| - 3) + 6(g(H) - h - s - 1)$$

For $H \cap A$, we have:

$$|E(H \cap A)| < 6g(G) - 3 - (6(h+s) + \sum_{i=1}^{j=r}(|C_i| - 3))$$

For $H - A$, we have:

$$|E(H - A)| = |E(H)| - |E(H \cap A)|$$
$$|E(H - A)| > 3|V(H - A)| - 3$$

And so the arboricity of G is strictly greater than 3 and the implication is proved. □

5 Concluding remarks

Up to now, the most efficient algorithm that can be used to find the 3 spanning trees is the $O(m^2)$ time algorithm of Roskind and Tarjan [10] designed for finding the maximum number of edge-disjoint spanning trees in a graph with m edges. A possible improvement of our result would be to find a more efficient algorithm adapted to triangulated graphs on surfaces.

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