Internship Report: Queue layouts of graphs and posets

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1 Introduction

A key question in Very Large Scale Integration (VLSI) design is circuit layout: given a logical circuit, how can one best draw it on a wafer of silicon? By best, one usually means a conjunction of various minimization constraints. A prominent such constraint is minimizing layout area, since area impacts cost, reliability, and complexity of testing. Another classical constraint is the minimization of propagation delay, either by decreasing wire lengths or by increasing transistor sizes, or minimizing the number of wire crossings; see [6] for instance.

In order to minimize wire crossings, Bernhart and Kainen introduced the notion of book embeddings of graphs in 1979 [5]. A book embedding of a graph G is a finite collection of pages which are half-planes, with the same line as boundary. All the vertices are placed on the boundary line, while the edges are drawn on pages, and only intersect at their endpoints: see Figure 1.

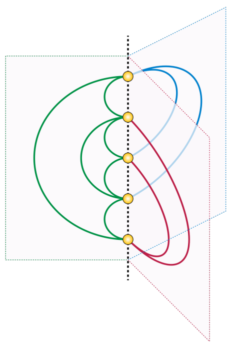


Figure 1: Example of three-page book embedding of the complete graph K_5 [By David Eppstein - Own work, CC0, https://commons.wikimedia.org/w/index.php?curid=33302655].

The main question on book embeddings is minimizing the number of books for a given graph. Berhart and Kainen originally called book thickness of a graph G the minimal number of books needed for a book embedding of G. Denominations have shifted, and one talks about stack layouts and stack-number instead of book embeddings and book thickness, nowadays: the order of the vertices on the boundary line is last-in first-out (LIFO, or stack) order for each page.

Heath and Rosenberg generalized those embeddings in 1992 [15]. Notably, if one replaces the LIFO order in stack layouts by a first-in first-out (FIFO, or queue) order, one obtains queue layouts and queue-numbers. We will give precise definitions below.

Outside VLSI design, queue and stack layouts have applications in fault-tolerant computing, scheduling parallel processes, sorting with networks of queues and stacks, and matrix computation [15].

In the latter case, it is more realistic to model the application domain with directed acyclic graphs (DAGs), or with finite partially ordered sets (posets). Indeed, the direction of arcs imposes some restrictions on the vertex orders that one is allowed [15]. This led to extending queue and stack layouts from undirected graphs to DAGs and posets.

The domain is young, and many problems are yet unsolved. For a sampler, let us cite a few problems left open by Heath, Leighton, Rosenberg, and Pemmaraju in 1992 and 1997 [13–15]. To start off, they conjecture that the queue-number of any planar poset with at most n elements is $O(\sqrt{n})$. The current best known upper bound is $\frac{n}{2}$ [14]. Next, and of more central importance to us, Heath and Rosenberg conjectured that planar graphs have bounded queue-numbers, namely that the queue-number of all planar graphs is bounded from above by a fixed constant. This remained unsolved for 27 years. Buss and Shor had proved that planar graphs have bounded stack-number [7]; Yannakakis gave an upper bound of 4 in 1989 [19], which was shown to be tight by Bekos, Kaufmann, Klute, Pupyrev, Raftopoulou, and Ueckerdt in 2020 [4]. For queue-numbers, the conjecture was finally settled in 2020 by Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood: planar graphs indeed have bounded queue-number.

During my internship, I focused on the problem of finding best possible upper bounds of queue-numbers of graphs and posets. This is an extraordinarily complex question, and only recently have the first upper bounds been proved for the planar graphs [10]. The ultimate goal of my internship was to try and improve upon the best known upper bounds, which is 42 for planar graphs [3], and $3w - 2$ for planar posets of width w [16]. That was probably too much to be expected, and I spent most of my time reading and understanding the recent literature on the subject. My aim in this report is to explain recent findings on this subject in the most leisurely way.

Outline. We set out all preliminary notions in Section 2. The notion of layered width is particularly important, and will be the keystone to Dujmović, Morin, and Wood's proof that the queue-number of planar graphs is at most 49, which we explain in Section 3. The distinction between intra or inter layer and intra or inter-bag edges is also the keypoint of the improvement of the previous bound to 42 by Bekos, Gronemann, and Raftopoulou in Section 3. The latter bound allows us to derive upper bounds on queue-numbers of planar posets of bounded height in Section 4. In Section 4, we present upper bounds of the queue-number of posets in function not only of their height but also in function of their width and their number of elements.

2 Preliminaries

2.1 Queue and stack layouts

We will use standard definitions of total order, partially ordered set (poset), and graphs [8]. All our graphs are undirected and simple (where there cannot exist several edges for the same pair of vertices). For a graph G, we will write $V(G)$ for its set of vertices and $E(G)$ for its edges. We will write an edge between u and v: uv. We now define all the important notions we need in this report.

Definition 2.1 (Vertex Order) A *vertex order* $\prec$ of a simple undirected graph G is a total order of its vertices. We say that such that for any two vertices u and v of G u precedes v if $u \prec v$.

Definition 2.2 (Queue Layout) Let G be a graph and $\prec$ be a vertex order of G. We say that the edges $uv, u'v' \in E(G)$ are *nested* with respect to $\prec$ if $u \prec u' \prec v' \prec v$ or $u' \prec u \prec v \prec v'$. We say that the edges $u_1v_1, u_2v_2, ..., u_kv_k$ of G form a *rainbow* of size k if $u_1 \prec u_2 \prec ... \prec u_k \prec v_k \prec ... \prec v_1$ (see Figure 2).

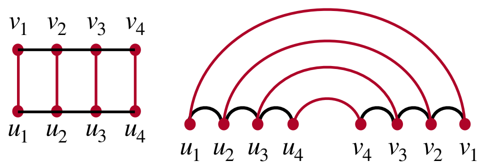


Figure 2: Example of a 4-rainbow in the graph at the left with respect to the order: $u_1 \prec u_2 \prec u_3 \prec u_4 \prec v_4 \prec v_3 \prec v_2 \prec v_1$. The edges of the graph are in black and red; those in red are the one that form the 4-rainbow.

Given a graph G and a vertex order $\prec$ on G, a *queue (with respect to* $\prec$*)* is a subset of the edges of G that does not contain any pair of nested edges with respect to $\prec$. A *queue layout* of a graph G consists of a vertex order $\prec$ of G and of a partition of its edges into queues with respect to that vertex order. The minimum number of queues needed in a queue layout of a graph G is called its *queue-number* and denoted by $\mathrm{qn}(G)$. Note that minimization occurs over all partitions into queues, but also over all vertex orders. If we keep the vertex order fixed, we have the following key lemma.

Proposition 2.3 ([15]) *Given a graph G and a vertex order $\prec$, the edges of G can be partitioned into k queues with respect to $\prec$ if and only if there is no rainbow of size $k+1$ in G with respect to $\prec$.*

Example 2.4 See Figure 3.

Those definitions extend to posets, through the following construction.

Definition 2.5 (Posets and Queue Layouts) Two elements a, b of a poset are called *comparable* if $a < b$ or $b < a$ where $<$ is the relation of the poset, and *incomparable*, denoted $a||b$, otherwise. All our posets are finite and non-empty.

Posets are visualized by their *Hasse diagrams*: Elements are placed as points in the plane and whenever $a < b$ in the poset, and there is no element c with $a < c < b$, there is a curve from a to b going upwards. This case is denoted by $a \prec b$.

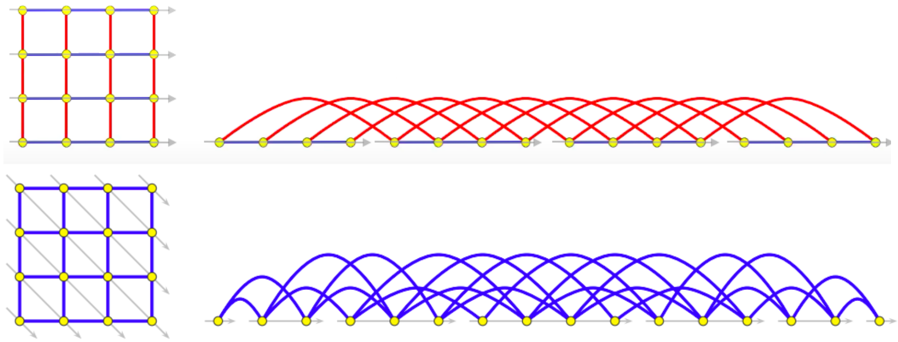


Figure 3: Example of two different vertex orders for a 4×4 grid graph: the first one is ordered horizontally, the second one diagonally [10].
The queues are represented with different colors, the first queue is blue, the second one is red (which exists only in the first example). With the first order there are rainbows of size two, it is impossible to obtain a queue layout with only one queue like with the second order. Better vertex orders yield better bounds on queue-numbers.

The *cover relations* are the relations which are essential in the sense that they are not implied by transitivity. The diagram represents those relations. The undirected graph implicitly defined by such a diagram is the *cover graph* G_P of the poset P.

Given a poset P, a *linear extension* L of P is a linear order on the elements of P such that $x <_P y$ implies $x <_L y$.

The *queue-number of a poset* P, denoted $\mathrm{qn}(P)$, is the smallest k such that there is a linear extension L of P for which the resulting linear layout of G_P contains no $(k+1)$-rainbow. Clearly we have $\mathrm{qn}(G_P) \leq \mathrm{qn}(P)$, i.e., the queue-number of a poset is at least the queue-number of its cover graph.

Definition 2.6 (Width and Height of a Poset) A *chain* of a poset is a non empty set of pairwise comparable elements, for a chain $C = \{c_1, c_2, ...c_k\}$ of length k, we can assume that the elements are ordered such that $c_1 < c_2 < ... < c_k$. An *antichain* of a poset is a non-empty set of pairwise incomparable elements. For chains and antichains, we will talk about *length* to talk about their cardinalities.

The *width* of a poset P is the maximum length of an antichain of P. The *height* of a poset P is the maximum length of a chain of P.

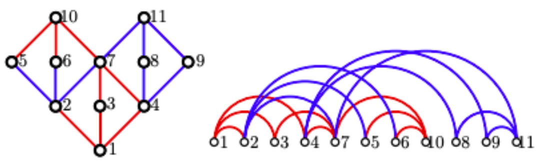


Figure 4: Example of a poset of width 5 and queue-number 2.

Definition 2.7 (Chain decomposition) A *chain decomposition* is a set of chains which partition the edge set of the graph.

An important theorem about the queue-number of posets, is Dilworth's Theorem [9]:

Theorem 2.8 *In any finite partially ordered set, the largest antichain has the same length as the smallest chain decomposition.*

Hence, by reformulating, the width of a poset P coincides with the smallest natural number w such that P can be decomposed into w pairwise disjoint chains of P.

2.2 Layering, Treewidth and Layered Treewidth

Definition 2.9 (Layering) A *layering* of a graph G is an ordered partition $(V_0, V_1, ...)$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$, then $|i-j| \leq 1$.
If $i = j$ then vw is an *intra-layer* edge. If $|i-j| = 1$ then vw is an *inter-layer* edge.

If r is a vertex in a connected graph G and $V_i := \{v \in V(G) : \mathrm{dist}_G(r, v) = i\}$ for every $i \geq 0$, then $(V_0, V_1, ...)$ is called a *breadth-first search (BFS) layering* of G *rooted at* r, where dist_G is the usual distance in a graph G [8].

Definition 2.10 (H-decomposition) For graphs H and G, an *H-decomposition* of G consists a collection of subsets B_x of $V(G)$ (the *bags*), one for each vertex x of H, with the following properties:

- for every vertex v of G, the set $\{x \in V(H) : v \in B_x\}$ induces a non-empty connected subgraph of H, and
- for every edge vw of G, there is a vertex $x \in V(H)$ for which $v, w \in B_x$.

The *width* of such an *H-decomposition* is $\max\{|B_x| : x \in V(H)\} - 1$. The elements of $V(H)$ are called *nodes* while the elements of $V(G)$ are called *vertices*.

Definition 2.11 (Treewidth) A *tree-decomposition* is a T-decomposition for some tree T. The *treewidth* of a graph G is the minimum width of a tree-decomposition of G.

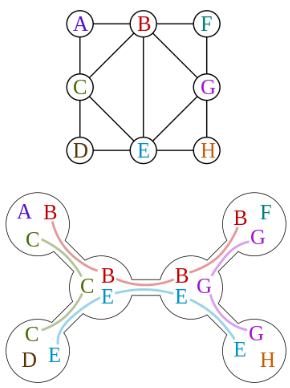


Figure 5: A graph G with eight vertices, and a tree decomposition of it onto a tree T with six nodes. Each tree node lists at most three vertices, so the width of this decomposition is two. [By David Eppstein - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=3011976].

It measures how similar a given graph is to a tree. In 2005, Dujmović, Morin, and Wood proved that the queue-number is bounded in a graph of bounded treewidth [11], their bound was improved by Wiechert in 2016 [18]:

Theorem 2.12 ([11, 18]) *Every graph with treewidth k has queue-number at most $2^k - 1$.*

The case $k = 3$ was improved in 2018 by Alam, Bekos, Gronemann, Kaufmann, and Pupyrev [2]:

Lemma 2.13 *Every planar graph with treewidth at most 3 has queue-number at most 5.*

Definition 2.14 (Partition) A *vertex partition*, or simply *partition*, of a graph G is a set $\mathcal{P}$ of non-empty set of vertices in G such that each vertex of G is in exactly one element of $\mathcal{P}$. Each element of $\mathcal{P}$ is called a *part*. The *quotient* (sometimes called the *touching pattern*) of $\mathcal{P}$ is the graph, denoted by $G/\mathcal{P}$, with a vertex set $\mathcal{P}$ where distinct parts $A, B \in \mathcal{P}$ are adjacent in $G/\mathcal{P}$ if and only if some vertex in A is adjacent in G to some vertex in B. A partition of G is *connected* if the subgraph induced by each part is connected.

A partition $\mathcal{P}$ of a graph G is called an *H-partition* if H is a graph that contains a spanning subgraph isomorphic to the quotient $G/\mathcal{P}$. Alternatively, an *H-partition* of a graph G is a partition $\{A_x : x \in V(H)\}$ of $V(G)$ indexed by the vertices of H, such that for every edge $vw \in E(G)$, if $v \in A_x$ and $w \in A_y$ then $x = y$ (and vw is called an *intra-bag* edge) or $xy \in E(H)$ (and vw is called an *inter-bag* edge). The *width* of such an H-partition is $\max\{|A_x| : x \in V(H)\}$. Note that a layering is equivalent to a path-partition.

A *tree-partition* is a T-partition for some tree T.

With the aim of showing that planar graphs have bounded queue-number, those results are only partial answers; indeed, planar graphs do not have bounded treewidth (already grids, see Figure 3). The next idea was found by Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood in 2020, in the form of the notion of *layered width* [10], a layered variant of partitions (analogous to layered treewidth being a layered variant of treewidth, that is a notion that Dujmović, Morin, and Wood introduced in 2017 [12]).

Definition 2.15 The *layered width* of a partition $\mathcal{P}$ of a graph G is the minimum integer ℓ such that for some layering $(V_0, V_1, ...)$ of G, each part in $\mathcal{P}$ has at most ℓ vertices in each layer V_i.

Example 2.16 In the $n \times n$ grid graph G (like the Figure 3 which is the 4×4 grid graph): the columns determine a partition $\mathcal{P}$ of layered width 1 with respect to the layering determined by the rows. The quotient $G/\mathcal{P}$ is an n-vertex path.

Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood showed in 2020 that if one does not care about the exact treewidth bound, then it suffices to consider partitions with layered width 1 [10].

Lemma 2.17 *If a graph G has an H-partition of layered width ℓ with respect to a layering $(V_0, V_1, ...)$, for some graph H of treewidth at most k, then G has an H'-partition of layered width 1 with respect to the same layering, for some graph H' of treewidth at most $(k + 1)\ell - 1$.*

Definition 2.18 (ℓ-Blowup) Let $\ell \geq 1$. An *ℓ-blowup* of a graph H is any graph G built as follows. The vertex set $V(G)$ of G is the disjoint union of sets B_v of cardinality at most ℓ, one for each vertex v of H. The sets B_v are the *blocks*. There is an edge between $x \in B_v$ and $y \in B_w$ in G if and only if vw is an edge of H.

In other words, we blow up each vertex v of H into at most ℓ pairwise distinct vertices in G, and for each vertex vw in H, we connect each vertex in block B_v to every vertex in block B_w. The following lemma is a key argument in the proof of Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood. Bekos, Gronemann, and Raftopoulou for Lemma 2.20 [10]:

Lemma 2.19 *Let H be a graph with a 1-queue layout with respect to some vertex order $\preceq$. Let $\ell \geq 1$, and G be any ℓ-blowup of H. Let also B_v denote the blocks of G, one for each vertex v of H. One obtains a vertex order on G by letting every vertex of B_v be smaller than every vertex of B_w whenever $v \prec w$, and by linearly ordering the elements of each B_v in an arbitrary way. With any such vertex order, G has an ℓ-queue layout.*

The proof is immediate, and is omitted.

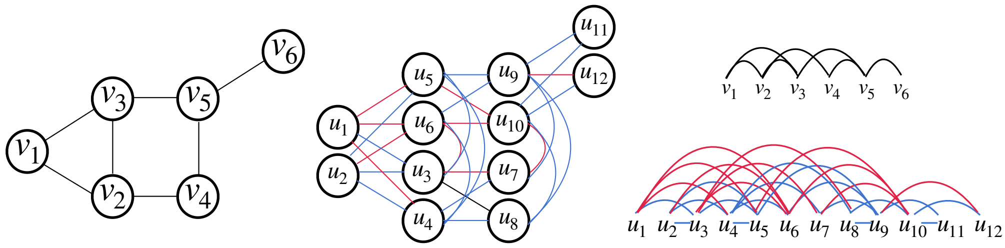


Figure 6: Left: a graph H with 6 vertices. Middle: the 2-blowup graph G of H, for which $\forall i \in [1,6], B_i = \{u_{2i-1}, u_{2i}\}$. Right: Queue-layouts of H and G. Since $\ell = 2$ here, we present a 2-queue layout of G: one queue is in blue, the other is in red.

The following lemma is the key argument in the proof that every planar graph has bounded queue-number. It is due to Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood. Bekos, Gronemann, and Raftopoulou improved it for one case in Lemma 3.13.

Given any vertex order $<$ on a graph $G = (V, E)$, one can sort the vertices of G in increasing $<$-order, yielding an *enumeration* $\vec{V}$ of the elements of V, namely, a tuple listing each element of V once exactly. Conversely, every enumeration $\vec{V}$ of V defines a unique vertex order on G. We will switch freely between the two views. This will also apply to subsets of V: enumerations $\vec{V}'$ of a subset V' of V are in one-to-one correspondence with vertex orders on the subgraph of G induced by V'.

Given a partition of V into pairwise disjoint subsets $V_0, ..., V_n$, and given corresponding enumerations $\vec{V}_0, ..., \vec{V}_n$, the concatenation $\vec{V}_0; ...; \vec{V}_n$ defines an enumeration of V, hence a vertex order on G.

Lemma 2.20 *Let H be a graph with a k-queue layout. Let G be a graph with a layering $V_0, ..., V_n$, and with an H-partition of layered width ℓ with respect to that layering. Then G has a $(3\ell k + \lfloor \frac{3}{2}\ell \rfloor)$-queue layout using the vertex order $\vec{V}_0; ...; \vec{V}_n$, where each $\vec{V}_i$ is some enumeration of V_i. In particular,*

$$\mathrm{qn}(G) \leq 3\ell\mathrm{qn}(H) + \lfloor \frac{3}{2}\ell \rfloor.$$

Proof Let $\{A_x : x \in V(H)\}$ be an H-partition of G of layered width l with respect to some layering $(V_0, V_1, ..., V_n)$ of G; that is $|A_x \cap V_i| \leq \ell$ for all $x \in V(H)$ and $i \geq 0$. We remember that this means that $|A_x \cap V_i| \leq \ell$ for every vertex x of H and for every $i, 0 \leq i \leq n$. Let the vertex order on H enumerate its vertices as $x_1, ..., x_h$. We fix a k-queue layout $QL(H)$ of H, with queues $E_1, ..., E_k$. For each i, $0 \leq i \leq n$, and for each j, $1 \leq j \leq h$, we fix an arbitrary enumeration $\vec{V}_{i_j}$ of $A_{x_j} \cap V_i$, and we define the required enumeration $\vec{V}_i$ as the concatenation $\vec{V}_{i_1}; ...; \vec{V}_{i_h}$. In other words, $\vec{V}_i$ lists the elements of V_i by putting the elements of $A_{x_1} \cap V_i$ first, then those of $A_{x_2} \cap V_i$, ..., and finally those of $A_{x_h} \cap V_i$.

The enumeration $\vec{V}_0; ...; \vec{V}_n$ now defines a vertex order $\leq$ on the set V of vertices of G. We recall that the sets V_i are the *layers*, and that the sets A_{x_j} are the *bags*. We call the sets $A_{x_j} \cap V_i$ are the *parts*. Every vertex of G lies in a unique part.

We will use the following property, which follows directly from the definition of $\leq$: if u and u' are two vertices of G such that $u \leq u'$, if u is in part $A_{x_j} \cap V_i$, and if u' is in part $A_{x'_j} \cap V'_i$, then (i, j) is lexicographically smaller than or equal to (i', j'). This gives us two other properties:

1. Two intra-layer intra-bag edges uv and $u'v'$ are nested so that $u \leq u' \leq v \leq v'$ only if u, u', v', v belong to the same part.

   Indeed, by definition of intra-layer intra-bag edges, u and v are in the same part $A_{x_j} \cap V_i$, and u' and v' are in the same part $A_{x_{j'}} \cap V_{i'}$. Since $u \leq u'$, and by definition of $\leq$, (i, j) is lexicographically smaller than or equal to (i', j'). Similarly, $v' \leq v$ entails that (i', j') is lexicographically smaller than or equal to (i, j). Hence $i = i'$ and $j = j'$.

2. For any two inter-layer edges uv and $u'v'$, nested so that $u \leq u' \leq v' \leq v$, there is a natural number i, $0 \leq i < n$, such that u and u' both belong to V_i and v and v' both belong to V_{i+1}.

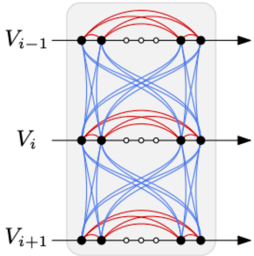
   Indeed, we remember that uv, with $u \in V_i$ and $v \in V_j$, is an inter-layer edge if $|i - j| = 1$. Hence $j = i + 1$ or $i = j + 1$. However, $u \leq v$ can only hold in the first case, so $j = i + 1$. Similarly, u' is in some layer $V_{i'}$ and v' is in the next layer $V_{i'+1}$, for some index i'. Since $u \leq u'$, and by definition of $<$, $i \leq i'$. Similarly, $v' \leq v$ entails $i' \leq i$, so $i = i'$.

3. Two inter-layer intra-bag edges uv and $u'v'$ are nested so that $u \leq u' \leq v \leq v'$ only if u, u', v', v belong to the same bag and for some $0 \leq i < n, u, u' \in V_i$, $v, v' \in V_{i+1}$.

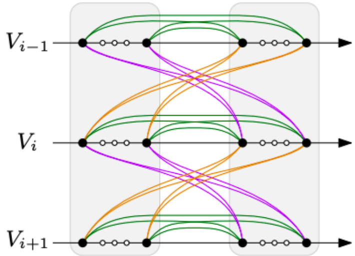
   Indeed, let uv and $u'v'$ be two nested edges such that $u \leq u' \leq v' \leq v$, where $u \in A_{x_j} \cap V_i$, $v \in A_{x_j} \cap V_{i+1}$, $u \in A_{x'_j} \cap V'_i$, and $v' \in A_{x'_j} \cap V_{i'+1}$. By Property (2), since uv and $u'v'$ are two nested inter-layer edges, we have $i = i'$. Since $u \leq u'$, by definition of $\leq$, (i, j) is lexicographically smaller than or equal to (i, j'), so $j \leq j'$. Similarly, since $v' \leq v$, by definition of $<$, (i, j') is lexicographically smaller than or equal to (i, j), so $j' \leq j$. Hence $j = j'$. So two inter-layer intra-bag edges uv and $u'v'$ nest only if $uv, u'v'$ belong to the same bag and for some $0 \leq i < n, u, u' \in V_i$, $v, v' \in V_{i+1}$.

It remains to construct a partition of the edges of G into queues so that the constraints of the lemma are satisfied. We will proceed as follows. We will first separate the edges of G into four categories, namely intra vs. inter layer and intra vs. inter bag edges, and we will partition the edges of each category into queues.

- **intra-layer intra-bag edges.** For all i and j, let G_{ij} be the subgraph of G induced by the part $A_{x_j} \cap V_i$. We will also call G_{ij} itself a part. Let also G' be the disjoint union of all the subgraphs $G_{ij}, 0 \leq i \leq n, 1 \leq j \leq h$. The edges of G' are exactly the edges of G that connect two vertices in the same layer V_i and in the same bag A_{x_j}. We note that the vertex order $\leq$ is also a vertex order on G'. With respect to that vertex order, let us consider any pair of edges uv and $u'v'$ of G', nested so that $u \leq u' \leq v' \leq v$. It must be the case that u, v, u' and v' are in the same part, because of Property (1) of intra-layer intra-bag edges. It follows that any m-rainbow in G', for any $m \geq 1$, must be an m-rainbow inside a unique part G_{ij}: all its edges must lie in the same part. Now an m-rainbow involves



(a) Intra-bag edges



(b) Inter-bag edges

Figure 7: Illustration of (a) Intra-bag edges; the intra-layer ones are red, while the inter-layer ones are blue, and (b) inter-bag edges; the intra-layer ones are green, while the inter-layer ones are purple (forward) and orange (backward) [3].

exactly $2m$ vertices. By assumption $|A_{x_j} \cap V_i| \leq \ell$, namely, each part G_{ij} contains at most ℓ vertices. This implies that $m \leq \lfloor \frac{\ell}{2} \rfloor$, and therefore, there is no $(\lfloor \frac{\ell}{2} \rfloor + 1)$-rainbow in G'. By Proposition 2.3, the edges of G' can be partitioned into $\lfloor \frac{\ell}{2} \rfloor$ queues with respect to $\leq$. This gives us our first group of queues.

- **intra-layer inter-bag edges.** For $i \in [0, n], j \in [1, k]$, let G_{ij} be the subgraph of G formed by those edges $vw \in E$ such that $v \in A_x \cap V_i$ and $w \in A_y \cap V_i$ for some edge $xy \in E_j$. We remember that E_j is a queue from the k-queue layout of H, $QL(H)$. Let Z_j be the 1-queue layout of the subgraph $(V(H), E_j)$ of H. We observe that G_{ij} is a subgraph of the graph isomorphic to the ℓ-blowup of Z_j. By Lemma 2.19, G_{ij} admits an ℓ-queue layout.

  Let G_j be the disjoint union of all the subgraphs $G_{ij}, 0 \leq i \leq n$. Let G' be the disjoint union of all the subgraphs $G_j, 1 \leq j \leq k$. We claim that ℓ queues suffice for G_j. Indeed we show that if uv and $u'v'$ are two intra-layer inter-bag edges with $u, u' \in A_{x_a}$ and $v, v' \in A_{x_b}$ for some edge $x_a x_b \in E_j$, then they nest only if u, u', v, v' are in the same layer. Let i, i' be two natural numbers such that $u \in A_{x_a} \cap V_i$, $v \in A_{x_b} \cap V_i$, $u' \in A_{x_a} \cap V_{i'}$ and $v' \in A_{x_b} \cap V_{i'}$. We assume that uv and $u'v'$ nest such that $u \leq u' \leq v' \leq v$. Since $u \leq u'$, by definition of $\leq$, (i, a) is lexicographically smaller than or equal to (i', a), so $i \leq i'$. Similarly, since $v' \leq v$, by definition of $\leq$, (i', b) is lexicographically smaller than or equal to (i, b), so $i' \leq i$. Hence $i = i'$. Since, in G_{ij}, there are only intra-layer inter-bag edges, two edges $uv \in G_{ij}$ and $u'v' \in G_{i'j'}$ with $(i, j) \neq (i', j')$ cannot nest. Since G' is partitioned into k subgraphs, G_j for $1 \leq j \leq h$, ℓk queues suffice for G'. This gives us our second group of queues.

- **inter-layer intra-bag edges.** Let G_{ij} be the subgraph of G formed by those edges $vw \in E$ such that $v \in A_{x_j} \cap V_i$ und $w \in A_{x_j} \cap V_{i+1}$ for some $1 \leq j \leq h$ and $0 \leq i \leq n-1$. Let G be the disjoint union of all the G_{ij}. We remember that there are no inter-layer intra-bag edges in G that are not in G', since by definition of layering, when $v \in V_i$ and $w \in V_j$, we have $|i - j| = 1$, and without loss of generality we can assume that $j = i + 1$.

  Using Property (3), it is immediate to see that two edges $uv \in E(G_{ij})$ and $u'v' \in E(G_{i'j'})$ cannot nest if $i \neq i'$ or $j \neq j'$. It follows that any m-rainbow in G', for any $m \geq 1$, must be an m-rainbow inside a unique part G_{ij}.

  Moreover, $V(G_{ij}) = (A_{x_j} \cap V_i) \cup (A_{x_j} \cap V_{i+1})$. So $|V(G_{ij})| \leq 2\ell$. Similarly to the intra-layer intra-bag case, G_{ij} can be partitioned into $\lfloor \frac{2\ell}{2} \rfloor = \ell$-queues with respect to $<$. Thus, ℓ queues suffice for G'. This gives us our third group of queues.

- **inter-layer inter-bag edges**: Let uv be an inter-layer inter-bag edge with $u \in A_x \cap V_i$ and $v \in A_y \cap V_{i+1}$, for some $i, 0 \leq i < n$. Then uv is *forward*, if $x <_H y$ holds in H, see the purple edges in Figure 8b; otherwise, it is *backward*, see the orange edges in Figure 8b. We will show that ℓk queues suffice for forward edges, the case of backward edges being similar.

  For $i \in [0, n], j \in [0, k]$, let G_{ij} be the subgraph of G formed by the forward edges $vw \in E$ such that $v \in A_x \cap V_i$ and $w \in A_y \cap V_{i+1}$ for some edge xy of H in E_j. Similarly as the intra-layer inter-bag case, ℓ-queues suffice for each G_{ij}. Let G_j by the disjoint union of G_{ij} for $0 \leq i \leq n$. By Property (2), two edges $uv \in G_{ij}$ and $u'v' \in G_{i'j}$ cannot nest when $i \neq i'$. So ℓ-queues suffice for G_j.

  Let G' be the disjoint union of the G_j for $1 \leq j \leq k$. So ℓk queues suffice for G'. The case of backward edges, being similar, they also only requires at most ℓk queues. In the end, $2\ell k$ queues suffice for inter-layer intra-bag edges. This gives us our last group of queues.

In total $3\ell k + \lfloor \frac{3}{2}\ell \rfloor$ suffice.

Our proof of Lemma 2.20 corrects a mistake from Bekos, Gronemann, and Raftopoulou [3]. They claim that any two intra-bag edges are nested if and only if they belong to the same bag. This is wrong, as one realizes by considering the 2×3 grid graph on six vertices (see Figure 7) ordered as $u_1 < u_2 < u_3 < v_1 < v_2 < v_3$. If we consider the layers $V_0 = \{u_1, u_2, u_3\}$ and $V_1 = \{v_1, v_2, v_3\}$, and the bags $A_i = \{u_i, v_i\}$ for $i \in \{1, 2, 3\}$, the two edges u_1v_1 and u_2u_3 are nested, but belong to different bags. The claim holds if the two edges are both intra-layer or both inter-layer. We fix this mistake in our proof by establishing Properties (1) and (3) instead.

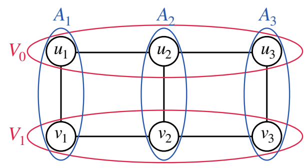


Figure 8: 2×3 grid graph. We represent a layering V_0, V_1 and bags A_1, A_2, A_3.

Lemma 2.20 has the following corollary.

Corollary 2.21 *If a graph G has a partition P of layered width ℓ such that G/P has treewidth at most k, then G has queue-number at most $3\ell(2^k - 1) + \lfloor \frac{3}{2}\ell \rfloor$.*

Remark: In the case $k = 3$, with Lemma 2.13, this bound is reduced to $15\ell + \lfloor \frac{3}{2}\ell \rfloor$.

3 Queue-Number of Planar Graphs

In 1992, Heath, Leighton, and Rosenberg made the conjecture that planar graphs had bounded queue-number [13]. They also showed a property for a subclass of planar graphs:

Definition 3.1 (Arched leveled-planar graph) Consider the normal cartesian (x, y) coordinate system for the plane. For i an integer, let l_i be the vertical line defined by $l_i = \{(i, y) | y \in$ **Reals**$\}$. A graph $G = (V, E)$ is *leveled-planar* if V can be partitioned into levels $V_1, V_2, ..., V_m$, in such a way that

- G has a planar embedding in which all vertices of V_i are on the line l_i;

- Each edge in E is embedded as a straight-line segment wholly between l_i and l_{i+1} for some i.

Such a planar embedding is called a *leveled-planar embedding.*

A leveled-planar graph augmented by (zero or more) arches is called an *arched leveled-planar graph.*

The following lemma of theirs will be instrumental in the 2020 solution of the conjecture:

Lemma 3.2 *A graph G is a 1-queue graph if and only if G is an arched leveled-planar graph.*

It is only recently, in 2020, that Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood proved the conjecture of Heath, Leighton and Rosenberg [10]:

Theorem 3.3 *The queue-number of planar graphs is bounded.*

They first proved that it is bounded by 766, and in the same paper they improved this bound to 49. In 2021, Bekos, Gronemann, and Raftopoulou improved this bound to 42 [3]. It is still unknown whether it is the best upper bound.

3.1 The 766 Bound

Lemma 3.4 ([10]) *The queue-number of planar graphs is at most 766.*

Every theorem in this section is from the paper *Planar graphs have bounded queue-number* by Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [10]. Lemma 3.4 is a direct consequence of Corollary 2.21 and the following theorem:

Lemma 3.5 ([10]) *Every planar graph G has a connected partition P with layered width 1 such that G/P has treewidth at most 8. Moreover, there is such a partition for every BFS layering of G.*

To prove this theorem they have been inspired by Pilipczuk and Siebertz [17] and they strengthened their result:

Lemma 3.6 *Let T be a rooted spanning tree in a connected planar graph G. Then G has a partition P into vertical paths in T such that G/P has treewidth at most 8.*

To prove Lemma 3.6 they used a variant of Sperner's Lemma [1]. We will not explain more details about the proof of this lemma here, since the proof for the 49 bound is different from this point, so it is not fundamental to understand the 49 bound.

3.2 The 49 Bound

Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood improved the bound 766 to 49 [10], by proving a theorem similar to Lemma 3.5 where the partition has a layered width 3 instead of the previously known value 8:

Lemma 3.7 *Every planar graph G has a partition P with layered width 3 such that G/P is planar and has treewidth at most 3. Moreover, there is such a partition for every BFS layering of G.*

Remark: In Lemma 3.7, the partition is not necessarily connected contrarily to the partition in the Lemma 3.5.

Thus, with Corollary 2.21:

Lemma 3.8 *The queue-number of planar graphs is at most 49.*

To prove the Lemma 3.7, they proved a theorem similar to the Lemma 3.6.

Lemma 3.9 *Let T be a rooted spanning tree in a triangulation G. Then, G has a partition P into tripods in T such that G/P has treewidth at most* 3.

To prove this bound, they used the following notion of tripods:

Definition 3.10 (Tripod) In a rooted spanning tree T of a graph G, a *tripod* consists of up to three pairwise disjoint vertical paths in T whose lower endpoints form a clique in G.

Lemma 3.11 *Let G^+ be a plane triangulation, let T be a spanning tree of G^+ rooted at some vertex r on the boundary of the outer-face of G, and let $P_1, ..., P_k$, for some $k \in \{1, 2, 3\}$, be pairwise disjoint bipods such that $F = [P_1, ..., P_k]$ is a cycle in G^+ with r in its exterior. Let G be the near triangulation consisting of all the edges and vertices of G^+ contained in F and the interior of F.*

Then, G has a partition P into tripods such that $P_1, ..., P_k \in P$, and the graph $H := G/P$ is planar and has a tree-decomposition in which every bag has size at most 4 and some bag contains all the vertices of H corresponding to $P_1, ..., P_k$.

3.3 The 42 Bound

Lemma 3.12 ([3]) *The queue-number of planar graphs is at most 42.*

To prove this bound, Bekos, Gronemann, and Raftopoulou improved the bound found in the Lemma 2.20 in the case where the H-partition of the graph G has treewidh at most 3, hence a 5-queue layout, with the Lemma 3.13. We recall that for this case, Lemma 2.20 proved that:

Let H be a graph with a 5-queue layout. Let G be a graph with a layering $V_0, ..., V_n$, and with an H-partition of layered width 3 with respect to that layering. Then $\lfloor \frac{3}{2} \rfloor = 1$ queue suffices for intra-layer intra-bag edges; 15 queues suffice for intra-layer inter-bag edges; 3 queues suffice for inter-layer intra-bag edges; and 30 queues suffice for inter-layer intra-bag edges. So, Bekos, Gronemann, and Raftopoulou improved the following statements:

- In G, no 4 intra-bag inter-layer edges of G form a 4-rainbow;
- and the inter-bag edges of G do not form a 46-rainbow.

Lemma 3.13 *Let H be a graph with a 5-queue layout. Let G be a graph with a layering $V_0, ..., V_n$, and with an H-partition of layered width 3 with respect to that layering. There is a queue layout such that*

- *no three intra-bag inter-layer edges of G form a 3-rainbow.*
- *the inter-bag edges of G do not form a 40-rainbow.*

Since the upper bound of the number of queues needed by the intra-bag intra-layer edges (which is 1) is not improved by Bekos, Gronemann, and Raftopoulou, there is no 43-rainbow, so that proves Lemma 3.12.

The best known corresponding lower bound is 4 due to Alam, Bekos, Gronemann, Kaufmann, and Pupyrev [2]. The exact queue-number of planar graphs is still unknown.

4 Queue-Number of Posets

4.1 Posets of bounded width

We recall that the cover graph G_P of a poset P is the undirected graph implicitly defined by the Hasse diagram of P. The queue-number $\mathrm{qn}(P)$ of a poset P is the smallest k such that there is a linear extension L of P for which the resulting linear layout of G_P contains no $(k+1)$-rainbow. We recall that $\mathrm{qn}(G_P) \leq \mathrm{qn}(P)$, i.e., the queue-number of a poset is at least the queue-number of its cover graph.

In 1997, Heath and Pemmaraju made the following conjecture [14]:

Conjecture 1 *Every poset of width w has queue-number at most w.*

Currently the conjecture has only been proved for some subclasses of planar posets. Nevertheless, some improvements have been made concerning planar posets of bounded width compared with general ones, but the conjecture has only been proved asymptotically.

4.1.1 In General

We do not know a lot about the general case, for now, the best upper bound was proved by Knauer, Micek, and Ueckerdt [16]:

Lemma 4.1 *For every poset P, if width$(P) \leq w$ then* $\mathrm{qn}(P) \leq w^2$.

To be more precise, they even proved:

Corollary 4.2 *Every poset of width w has queue-number at most $w^2 - 2\lfloor \frac{w}{2} \rfloor$.*

It is still unknown if Conjecture 1 can be approached asymptotically for the general case.

4.1.2 Planar Posets with 0 and 1

Even though Conjecture 1 hasn't been completely proved for planar posets (as we will see in the next subsection), it has been proved for a subclass of planar posets:

Definition 4.3 (Planar Posets with 0 and 1) A *planar posets with 0 and 1* is a planar poset $(P, \leq)$ where there exist two elements $0, 1 \in P$ such that $\forall x \in P, 0 \leq x$ and $x \leq 1$. We call 0 the *minimal element* and 1 the *maximal element*.

In fact, the conjecture has been proved for a larger subclass, and the fact that it is true for planar posets with 0 and 1 is only a corollary. The proof is on the posets that do not contain any embedded P_k for $k > 2$, where P_k is defined by:

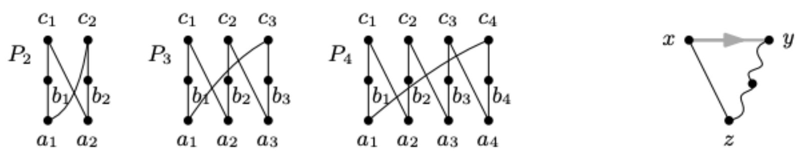


Figure 9: Left: The posets P_2, P_3, and P_4. Right: The existence of an element z with cover relation $z < x$ and non-cover relation $z < y$ gives rise to a gray edge from x to y [16].

Definition 4.4 (Subdivided *k*-Crown) For a natural number $k > 2$ we define a *subdivided k-crown* as the poset P_k. The elements of P_k are $\{a_1, ..., a_k, b_1, ..., b_k, c_1, ..., c_k\}$ and the cover relations are given by $a_i < b_i$ and $b_i < c_i$ for $i = 2, ..., k, a_i < c_{i-1}$ for $i = 1, ..., k-1$, and $a_1 < c_k$; see the left of Figure 9. We refer to the covers of the form $a_i < c_j$ as the diagonal covers and we say that a poset P has an embedded P_k if P contains $3k$ elements that induce a copy of P_k in P with all diagonal covers of that copy being covers of P.

Theorem 4.5 ([16]) *If P is a poset that for no $k > 2$ has an embedded P_k, then the queue-number of P is at most the width of P.*

Corollary 4.6 *For any planar poset with 0 and 1 P of width w we have* $\mathrm{qn}(P) \leq w$.

As we will see in Section 4.1.3, the question of the upper bounds of queue-number on arbitrary planar posets reduces to posets with 0 and 1.

Also, this bound is tight:

Lemma 4.7 ([16]) *For each w there exists a planar poset Q_w with 0 and 1 of width w and queue-number w.*

Proof By induction on w. For $w = 1$, Q_1 can be any chain. Then, for the inductive step, by taking P and P', two copy of Q_{w-1}, and three other points a, b, c as in Figure 10, it constructs Q_w. Indeed, for any vertex order of Q_w, either we have $b < x'$, where x' is the minimal element of P', and it adds the edge bc to a $w-1$-rainbow in P', either we have $x' < b$ and it adds the edge ab to a $w-1$-rainbow in P. In the end, there is at least one w-rainbow, and no $w+1$-rainbow, which proves this lemma according to Proposition 2.3.

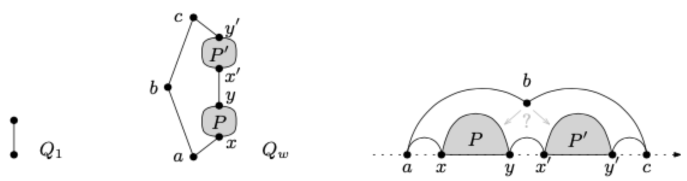


Figure 10: Recursively constructing planar posets Q_w of width w and queue-number w. Left: Q_1 is a two-element chain. Middle: Q_w is defined from two copies P, P' of Q_{w-1}. Right: The general situation for a linear extension of Q_w [16].

4.1.3 Planar Posets

In 1997, Heath and Pemmaraju showed that the largest queue-number among planar posets of width w lies between $\sqrt{w}$ and $4w - 1$ [14]. Knauer, Mice, and Ueckerdt proved better bounds in 2018:

Theorem 4.8 ([16]) *Every planar poset of width w has queue-number at most* $3w - 2$.
Moreover, there are planar posets of width w and queue-number w.

The proof relies on how to add edges to the planar poset P such that the new constructed poset P' is a planar poset with 0 and 1. But if there are two edges ab and cb in P and we add an edge ac in P' then the relation $a \prec b$ is assured by transitivity since $a < c < b$ with $<$ the partial order of P' (the edges of a poset representing exactly its cover relations), so some edges of P are not in P'. Hence it is not possible to take the same queue layout for P and P'. They show that there are no more than $2w-2$ edges that are in P and not in P'.

Hence they proved that every planar poset of width w has queue-number $O(w)$, but it remains to know if it is possible to improve the bound or to found a counter example of the conjecture 1.

My supervisor and I made several attempts at improving the previous bound. One of our ideas was to think about the inverse problem: from a planar poset of queue-number n, can we find a lower bound of it width in function of n? Our best result is when there exists a specific type of n-rainbow in the poset.

Let $(P,\leq)$ be a planar poset of queue-number n. According to Proposition 2.3, there is a vertex order $\prec$ such that there is at least one n-rainbow and no $n+1$-rainbow in P. Let B be the sub-poset composed by the vertices of n-rainbow of P. We write $a_1,...,a_n$ for the *bottom nodes* of the rainbow, and $b_1,...,b_n$, the *top nodes*, such that the $a_1b_1, a_2b_2, ..., a_nb_n$ form an n-rainbow (so $V(B)=\{a_1,...,a_n,b_1,...,b_n\}$). We made several assumptions concerning the edges of B (see Figure 11), we will write them the assumptions (*):

- on the Hasse diagram of P, the edges of the rainbow respect the left-to-right order, *ie.* when scanning the diagram from left to right, we encounter first the edge a_1b_1, and for every $i\in[1,n-1]$, we encounter a_ib_i before $a_{i+1}b_{i+1}$;
- We can partition B into $k\in[1,n]$ non-empty *blocks* $B_1,...,B_k$, such that $\exists(i_j)_{j\in[1,k+1]}\subseteq[1,n]$, such that $i_1=1$, $i_{k+1}=n+1$, and for all $j\in[1,k]$, $V(B_j)=\{a_{i_j},...,a_{i_{j+1}-1},b_{i_j},...,b_{i_{j+1}-1}\}$
- For all $j\in[1,k]$ there exists a path from a_{i_j} to $a_{i_{j+1}-1}$ going through each a_s with $s\in(i_j,i_{j+1})$;
- There is no path between other pairs of elements in $a_1,...,a_n$;
- Let $i,j\in[1,n]$ there is no path from a_i to b_j that does not begin with the arrow a_ib_i.

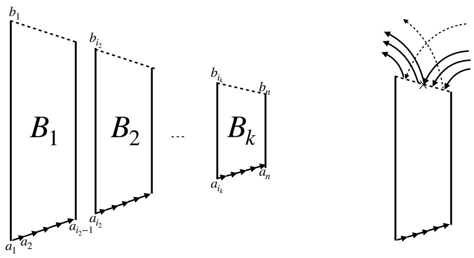


Figure 11: Left: The blocks $B_1,B_2,...,B_k$. Right: example of a block with arrows among top vertices; the dotted arrows are arrows that cannot exist according to Proposition 4.9.

We do not make any explicit assumption on comparability between the top vertices. Some properties about the comparability of top vertices nonetheless hold.

Proposition 4.9 *Let P be a planar posets of queue-number n. Let $a_1b_1,a_2b_2,...,a_nb_n$ form an n-rainbow in P such that $a_1,a_2,...,a_n$ are the top vertices, and $b_1,b_2,...,b_n$, the bottom vertices that verify the assumptions (*).*

1. *In each block B_j, all the top vertices $b_{i_j},...,b_{i_{j+1}-1}$ are not comparable.*
2. *If b_i is a top vertex in the block B_j then: $b_{i'}\prec b_i$ or $b_i||b_{i'}$ for all top vertex b'_i in a block $B_{j'}$ with $j'>j$; and $b_i\prec b_{i'}$ or $b_i||b_{i'}$ for all top vertex b'_i in a block $B_{j'}$ with $j'<j$.*

3. *For every top vertex b_b in a block B_j such that there are two other top vertices b_a, b_c that verify $b_a \prec b_b \prec b_c$, for all $b \in B_j$: if $b \prec b_b$, there is no top vertex b' such that $b \prec b'$; otherwise if $b_b \prec b$, there is no top vertex b' such that $b' \prec b$.*

4. *In each block, there is at most one top vertex b_b such that there are two other top vertices b_a, b_c that verify $b_a \prec b_b \prec b_c$.*

Proof 1. Let us assume that there there are in a block, two top vertices b and b' such that $b \prec b'$. For the two corresponding bottom vertices a and a' such that $ab, a'b'$ are in the rainbow at the origin of B. So $a' \prec a$, and a and a' are bottom vertices of the same block. So there is a path p from a' to a. Since $b \prec b'$ there is also a path p' from b to b'. Hence, the concatenation of $p; ab; p'$ is a path from a' to b'. So the edge $a'b'$ doesn't exist by transitivity, which is impossible.

2. We assume that there exists a top vertex $b_{i'}$ in a block B'_j with $j' > j$ such that $b_i \prec b_{i'}$. Since $j < j'$, $a_{i_j} \prec a_{i_{j'}}$ and $b_{i_{j'}} \prec b_{i_j}$, and since the edges $a_x b_y$ of the rainbow respect the left-to-right order, $\forall x, y$ if b_x is in the block B_y then $b_{i_y - 1} \prec b_x \preceq b_{i_y}$. So $b_{i'} \preceq b_{i_{j'}} \preceq b_{i_j - 1} \prec b_i \preceq b_{i_j}$, and $b_{i'} \prec b_i$, which is a contradiction with the hypothesis that $b_i \prec b_{i'}$. The second case when $j' < j$ and $b'_i \prec b_i$ is similar.

3. We prove only the case $b \prec b_b$ since the other case is symmetrical. Let us assume that b' exists. Because of the property 1, b' is a top vertex of a block $B_{j'}$ such that $j' < j$. So for any path from b to b', it cross the path from b_a to b_b. By planarity they cross at a vertex x, and $b \prec x$ and $x \prec b_b$. By transitivity, $b \prec b_b$, which is impossible according to the previous property.

4. It is immediate from the previous property.

We define nb_{double}, the number of top vertices b_b such that there exist two other top vertices b_a, b_c that verify $b_a \prec b_b \prec b_c$; nb_{in} the number of top vertices b_b such that there exists another top vertices b_a that verify $b_a \prec b_b$; nb_{out} the number of top vertices b_b such that there exists another top vertices b_a that verify $b_b \prec b_c$; nb_{nothing} the number of top vertices incomparable with all the other top vertices. With these notations, we have: $nb_{\text{out}} + nb_{\text{in}} - nb_{\text{double}} + nb_{\text{nothing}} = n$.

Construction of an antichain. Those properties enable us to build an antichain $\mathcal{A}$ of length at least $\frac{n}{2}$ as follows. Let $\mathcal{A} = \{\}$ be the antichain at the beginning, we add vertices with the following method:

- If $nb_{\text{in}} + nb_{\text{nothing}} \geq \frac{n}{2} + nb_{\text{double}}$: let b_b be a top vertex such that there is no other top vertex b_c such that $b_b \prec b_c$. There are at least $\frac{n}{2}$ such vertices b_b, we add all of them to $\mathcal{A}$.
- Otherwise, we know that $nb_{\text{out}} \leq \frac{n}{2}$. Let b_b be a top vertex such that there is no other top vertex b_a such that $b_a \prec b_b$. There are at least $\frac{n}{2} - nb_{\text{double}}$ such vertices b_b, we add all of them to $\mathcal{A}$. Also, for every $j \in [2, k]$ such that $b_{i_j - 1}$ is in $\mathcal{A}$, we add $a_{i_j - 1}$ to $\mathcal{A}$. There are at least nb_{double} such vertices.

In every case, at the end there are $\frac{n}{2}$ vertices in $\mathcal{A}$, and the properties of Proposition 4.9 and the assumptions (*) enable us to prove that is really an antichain.

This technique is the best that Piotr Micek and I have been able to achieve on the question of the lower bound of the length of the maximal antichain in a planar poset of queue-number n. The shape of rainbows is severely restricted, and there is no evidence that there actually are planar posets with queue-number n which would satisfy all the conditions we require. Also,

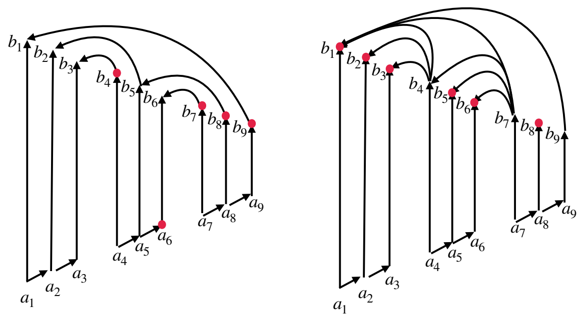


Figure 12: Two examples of 9-rainbows where we have constructed an antichain with with the method of section 4.1.3. The nodes of the antichain are represented by red points. Left: $nb_{\text{in}} = 5, nb_{\text{out}} = 5, nb_{\text{double}} = 1, nb_{\text{nothing}} = 0$, so $nb_{\text{in}} + nb_{\text{nothing}} < \frac{n}{2} + nb_{\text{double}}$. Right: $nb_{\text{in}} = 6, nb_{\text{out}} = 3, nb_{\text{double}} = 1, nb_{\text{nothing}} = 1$, so $nb_{\text{in}} + nb_{\text{nothing}} \geq \frac{n}{2} + nb_{\text{double}}$, in this one we remark that it is possible to add a_9 to the antichain too.

our conditions on rainbows fail to take into account the relations that hold between the bottom vertices, and between bottom and top vertices; adding constraints to maintain those relations tends to contradict the goal of keeping an $\frac{n}{2}$ antichain.

4.2 Of Bounded Height

Heath and Pemmaraju made a conjecture in 1997 about a bound of the queue-number of posets of bounded height [14] and it has been proven only partially for now:

Conjecture 2 *Every planar poset of height h has queue-number at most h.*

In 2018, Knauer, Micek, and Ueckerdt found a counter example [16]:

Lemma 4.10 *There is a planar poset of height 2 with queue-number at least 4.*

But it is still interesting to understand if there are subclasses of planar posets such that this conjecture still holds. Knauer, Micek, and Ueckerdt proved the conjecture for planar posets with 0 and 1 [16]:

Lemma 4.11 *Every planar poset with O and 1 of height h has queue-number at most $h - 1$.*

Also, it is still possible to show that it is true for planar posets asymptotically, by using the bound of planar graphs. In 1997, Heath and Pemmaraju proved a link between the queue-number of a bounded height poset and the one of its cover graph [14]:

Lemma 4.12 *For any poset P of bounded height h, if we note its cover graph $H(P)$, then:*

$$\mathrm{qn}(P) \leq 2(h-1)\mathrm{qn}(H(P)).$$

With Lemma 3.12, since the cover graph of a planar poset is planar, it is immediate that:

Corollary 4.13 *For any poset P of bounded height h:*

$$\mathrm{qn}(P) \leq 84(h-1).$$

Thus, planar posets of bounded height have queue-number $O(h)$. There is still the question whether the constant 84 can be reduced or whether it is tight.

4.3 Of Bounded Number of Elements

Heath and Pemmaraju made the following conjecture in 1997 [14]:

Conjecture 3 *For any n-element planar poset P,* $\mathrm{qn}(P) = O(\sqrt{n})$.

For now, apart from the trivial upper bound, no other upper bound have been proven:

Proposition 4.14 *Every planar poset on n elements has queue-number at most $\frac{n}{2}$.*

Proof Any two nested edges must have disjoint sets of vertices.

Compared to the situation with posets of bounded width or bounded height, bounding the number of elements of a planar poset gives much less information on the structure of the poset. My supervisor and I thought about proving a relaxed variant of Conjecture 3, which is the following conjecture:

Conjecture 4 *Every planar poset on n elements has queue-number in $O(n^\alpha)$ with $\alpha < 1$.*

We tried to define several cases depending on the width w and the height h of a planar posets on n elements. Let c be the constant, such that we want to prove that $\mathrm{qn}(P) \leq cn^\alpha$.

- if $w \leq \frac{cn^\alpha+2}{3}$ then it is immediate from Theorem 4.8;
- if $h \leq \frac{c}{84}n^\alpha + 1$ then it is immediate from Corollary 4.13;
- it remains the case where $w > \frac{cn^\alpha+2}{3}$ and $h > \frac{c}{84}n^\alpha + 1$. The idea was that if $\alpha \geq \sqrt{n}$ that from the decomposition of P into w chains of Theorem 2.8, it was not possible to have w distinct chains of length at least h since $wh \geq \alpha^2 \geq n$. One may then hope to partition the posets, by induction on n, into a finite number of sub-posets verifying one of the previous two conditions, such that the sum of the upper bounds of their queue-number would be less than cn^α. None of our attempts to achieve this succeeded.

Concerning the lower bound, it has been proven that it is possible to construct a planar poset with a $O(\sqrt{n})$-queue-number:

Lemma 4.15 *For each $n \geq 1$, there exists a planar poset P_n with $3n+3$ elements such that*

$$\lceil \sqrt{n+1} \rceil \leq \mathrm{qn}(P_n) \leq \lceil \sqrt{n} \rceil + 1$$

.

Proof We just need to prove that the poset of Figure 13 has the right queue-number. In fact, all the proof comes from the following lemma:

Lemma 4.16 (Erdös and Szekeres [14]) *Let $(x_i)_{i=1}^n$ be a sequence of distinct elements from a set X. Let δ be a total order on X. Then $(x_i)_{i=1}^n$ either contains a monotonically increasing subsequence of size $\lceil \sqrt{n} \rceil$ or a monotonically decreasing subsequence of size $\lceil \sqrt{n} \rceil$ with respect to δ.*

To give the intuition of the bound, in fact all (u_i) and (v_i) are always ordered the same way relatively to each other in every extension of the partial order. The number of queues depends only on the order of the (w_i), we apply the Erdös and Szekeres' lemma on this sequence, if the found a subsequence is increasing then there is a $\lceil \sqrt{n} \rceil$ rainbow between the (u_i) and (w_i), otherwise it is between the (v_i) and (w_i). $\square$

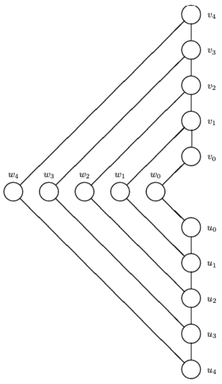


Figure 13: Representation of the planar posets constructed for $n = 5$, which can be extended to every n.

Conclusion

We have presented the state of the art on the problem of finding best possible upper bounds on planar graphs and posets of bounded width, height or number of elements.

Concerning planar graphs, after that Dujmović, Joret, Micek, Morin, and Ueckerdt proved that the queue-number was bounded, their bound has been improved twice, and now the current best upper-bound is 42, which was proved by Bekos, Gronemann, and Raftopoulou.

The upper-bound of the queue-number for posets of bounded width w in general has not experienced major improvement since the first bound proved by Heath and Pemmaraju in 1997, and remains in $O(w^2)$. Also their conjecture that planar posets of width at most w have a queue-number at most w was proved by Knauer, Micek, and Ueckerdt for planar posets with 0 and 1, but still remain open for planar posets, for which the current best bound is $3w - 2$.

The conjecture of Heath and Pemmaraju concerning the planar posets of height at most h that they have a queue-number at most h was refuted by Knauer, Micek, and Ueckerdt, but it is still true asymptotically; the question remains whether it is a tight bound or whether one can still prove a better bound. Another conjecture of Heath and Pemmaraju that planar posets with at most n elements have a queue-number $\sqrt{n}$ is still unknown, and the problem is still open.

We have also described a few ideas that we have explored in order to expand and improve on those recent results. While we have not succeeded, we hope that those attempts shed some light on the difficulty of the endeavour.

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I would like to thank my supervisor, Piotr Micek, for introducing me to the world of posets and queue-layouts, and also for the many opportunities he has given me to interact with the scientific community, to follow the seminars of *CanaDAM* and *Round the World Relay* or for inviting me to a workshop of *Sparse Coalition* about combinatorial reconfiguration, where I could think about the conjecture that *a* k*-colorable* P_5*-free graph is* $(k+1)$*-mixing* with other researchers.

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