Moindres carrés ← Optim ss contraintes

$f(x) = \sum_{i=1}^{N} f_i(x)^2$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ "générales"

algo

descente grad.
grad. conjugués.
+ quasi-Newton.

tant que $(\|X_{i+1} - X_i\| > \varepsilon)$

⋮

ftq

algo. it. converge ε

f_i

linéaires (*) — moindres carrés linéaires

¬ lin — moindres carrés non lin

Gauss-Newton

Levenberg-Marquardt

syst. linéaire

f_i linéaires

algo qui donne la solution.

$\exists$ matrice A — $N \times m$
vect B — N

tq $f(X) = \|AX + B\|^2$

mod. linéaire

méth ① — Eq. normales syst linéaire $m \times m$

$A^t A X + A^t B = 0$

cond m — instabilité

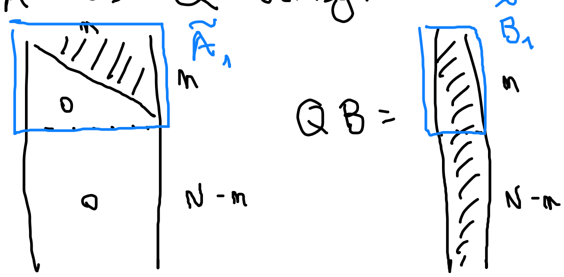
② Householder (QR)

transfo QR de A ⟶ Q orthogonale

$Q^{-1} = Q^t$
$Q \cdot Q^t = I$

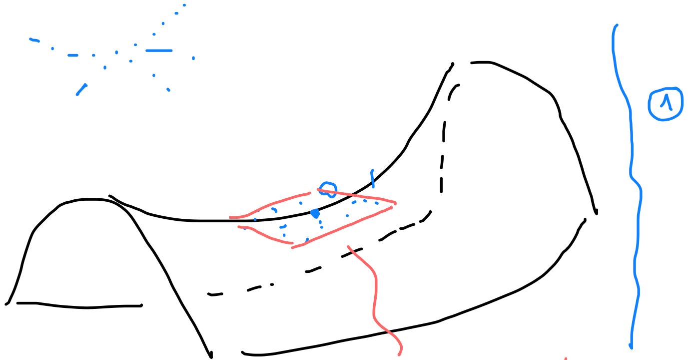
$QA =$ triang sup

$QB =$



min : X solution de $\tilde{A}_1 X + \tilde{B}_1 = 0$ ($m \times m$)

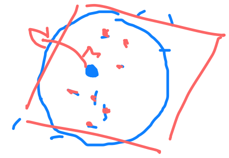
Matlab... $X = A \backslash B$ (rect.)



∼ plan tangent.

Moindres carrés !

① Plan moyen.

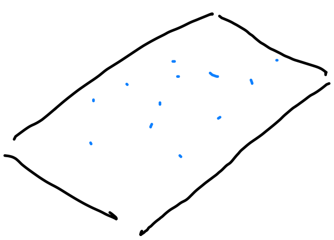


② Moindres carrés surf. quad.

$z = f(x,y)$

poly d° 2

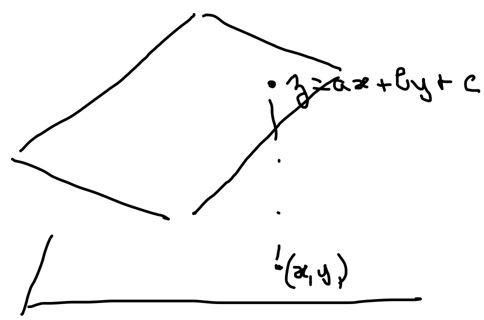
① Approx. par un plan



$X_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$ $i = 1 \dots N$

$\cancel{ax + by + \underbrace{c}_{\neq 0} z + d = 0}$

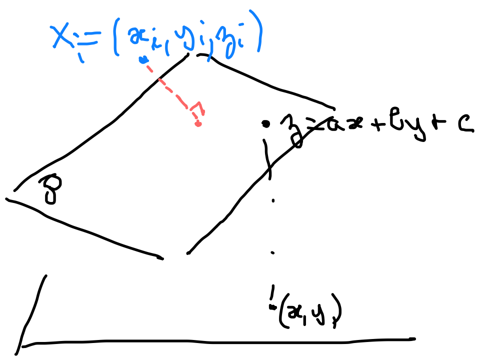
$z = ax + by + c$



1) Param. du plan ? a, b, c

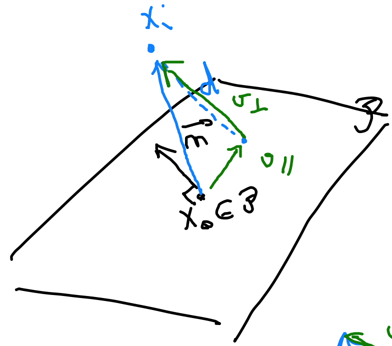
$n = 3$

2) Fct à minimiser ⇝ erreur plan — pts données.



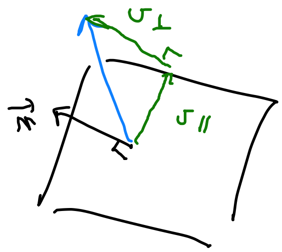
Erreur au pt i ...

$\varepsilon_i = d(X_i, \mathcal{P})^2$



$$\overrightarrow{X_0X_i} = v_\parallel + \underbrace{\overset{\|v_\perp\|}{\overbrace{v_\perp}}}_{\substack{\equiv \\ ?}}$$

d



$$\overrightarrow{X_0X_i} \cdot \vec n = \underbrace{\cancel{v_\parallel \cdot \vec n}}_{0} + \underbrace{v_\perp \cdot \vec n}$$

$$v_\perp \parallel \vec n$$

$$\Downarrow$$

$$v_\perp \cdot \vec n = \underbrace{\|v_\perp\|}_{\substack{\| \\ d}} \cdot \|\vec n\|$$

$$\underset{\substack{\| \\ d(x_i, \mathcal{P})}}{d} = \frac{X_0X_i \cdot \vec n}{\|\vec n\|}$$

$\vec n, x_0$?

$$z = ax + by + c$$

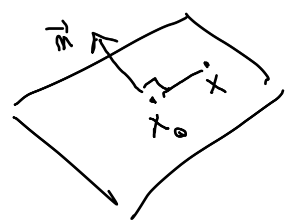
$$\hookrightarrow X_0 = (0, 0, c)$$

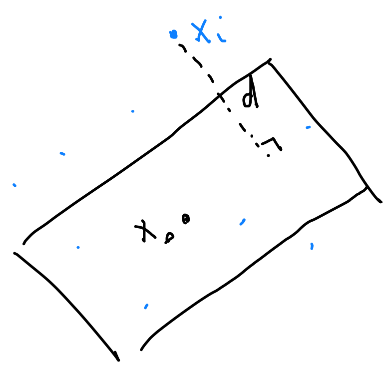
$$\Updownarrow \quad \Leftrightarrow$$

$$ax + by - (z - c) = 0.$$

$$\underbrace{\begin{pmatrix} a \\ b \\ -1 \end{pmatrix}}_{\vec n} \cdot \left(\underset{\substack{\| \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix}}}{\begin{pmatrix} X \end{pmatrix}} - \underset{\substack{\| \\ \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}}}{\begin{pmatrix} X_0 \end{pmatrix}} \right) = 0 \quad \longleftarrow \quad X \text{ tq}$$

$$\vec n \cdot (X - X_0) = 0$$





$$z = ax + by + c$$

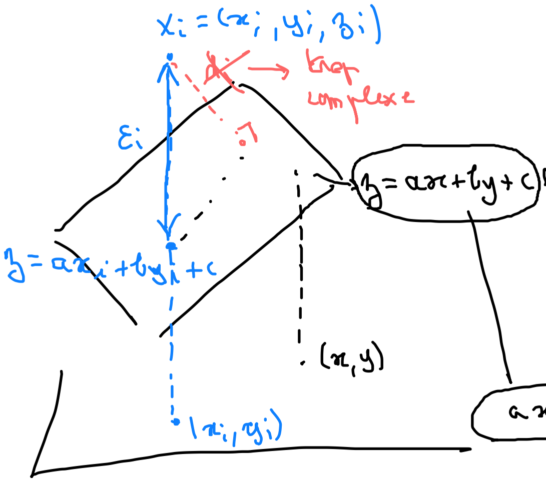
$$d = \frac{\overrightarrow{X_0X_i} \cdot \vec n}{\|\vec n\|} \qquad \vec n = \begin{pmatrix} a \\ b \\ -1 \end{pmatrix}, \quad \|\vec n\| = \sqrt{a^2 + b^2 + 1}$$

$$\|$$

Param : a, b, c

$$\varepsilon_i = \underbrace{\left(X_i - \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}\right) \cdot \begin{pmatrix} a \\ b \\ -1 \end{pmatrix} \Big/ \sqrt{a^2 + b^2 + 1}}_{f_i}$$

/ non linéaire -



$z = ax + by + c$ min ?

$ax + by + c - z = 0$

$$f(X) = \sum \varepsilon_i^2$$

$\varepsilon_i \geqslant d_i$

$$\boxed{\varepsilon_i = ax_i + by_i + c - z_i}$$

fct à ⇝ min

$$f(\underbrace{a, b, c}_{X}) = \sum_i \varepsilon_i^2 = \sum_{i=1}^{N} \big(\underbrace{\boxed{ax_i + by_i + c - z_i}}_{f_i(a,b,c)}\big)^2$$

f_i — linéaire.

$f_i(a,b,c)$ — ε_i

$$\min_{X \in \mathbb{R}^3} f(X)$$

③ Modélisation linéaire ! $f(X) = \| \overset{?}{A} X + \overset{?}{B} \|^2$

⇝ vect. d'erreur

$$\vec{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \vdots \\ \varepsilon_N \end{pmatrix} = \begin{pmatrix} ax_1 + by_1 + c - z_1 \\ \vdots \\ ax_i + by_i + c - z_i \\ \vdots \\ ax_N + by_N + c - z_N \end{pmatrix} = \begin{array}{c} \\ i \\ \\ \end{array} \overset{A}{\begin{pmatrix} \vdots & \vdots & \vdots \\ x_i & y_i & 1 \\ \vdots & \vdots & \vdots \end{pmatrix}} \overset{X}{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} + \overset{B}{\begin{pmatrix} \vdots \\ -z_i \\ \vdots \end{pmatrix}}$$

$$f(X) = \|\vec{\varepsilon}\|^2 = \|AX + B\|^2 \quad ✓$$

↓ QR ...

Matlab : $X = A \backslash (-B)$ → param modèle $X = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$f(x, y) = 2x^2 - y^2$$

rand − 0,5   pts −0,5 → 0,5

N pts 2D

$$M = \begin{pmatrix} \text{———} \ x_i \ \text{———} \\ \text{———} \ y_i \ \text{———} \\ \text{———} \ z_i \ \text{———} \end{pmatrix}$$

$z_i = f(x_i, y_i)$

Tracer

→ fcts

→ surf. implt.

$z = f(x,y)$

```
[XX, YY] = meshgrid(-1:.1:1, -1:.1:1);
ZZ = f(XX,YY);
surface(XX,YY,ZZ);
```

g = @(x,y) a*x+b*y+c

X_i ⇝ ds M par colonnes

$$M = \begin{array}{c} \\ \begin{pmatrix} \text{———} & x_i & \text{———} \\ \text{———} & y_i & \text{———} \\ \text{———} & z_i & \text{———} \end{pmatrix}_N \end{array} \qquad \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \sim \text{ones}(n,m).$$

↓ X_i

L_1 $M(:,i) \sim X_i$

$$f(X) = \|AX+B\|^2 \xrightarrow{QR} \tilde{A}_1 X + \tilde{B}_1 = 0 \longrightarrow \tilde{A}_1 X = -\tilde{B}_1$$

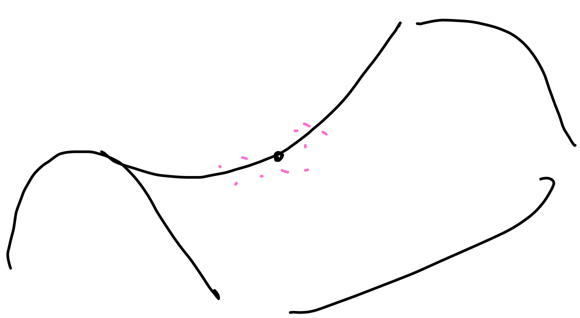
↓

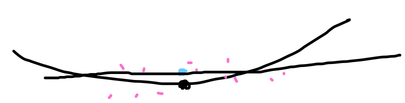
$X = A\backslash(-B)$

plan

↓

tangente.





eq. plan passe par O

$z = ax + by \cancel{+c}$

$z = ax + by + \frac{1}{2}c$

2) Approx → surf. quad.

(surf. init
$z = 2x^2 - y^2$ — quad...)

$z = f(x,y)$ — ~~Plan~~

$$\boxed{z = a_1 x^2 + a_2 y^2 + a_3 xy + a_4 x + a_5 y + a_6}$$

→ $n = 6$ param.

$A = N \times 6$

$B : N$

$X = A \backslash (-B)$

↓

vect 6

$a_1 \cdots a_6$

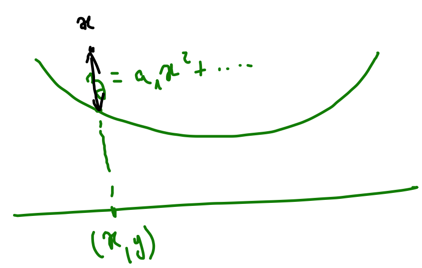
$N \rightsquigarrow A \rightsquigarrow \text{cond}(A)$

Plan $\quad z = a_1 x + a_2 y + a_3 \quad \rightsquigarrow \quad N$ min : 3 ⟶ $N = 50$

Quad $\quad z = a_1 x^2 + \cdots \; a_6 \quad \rightsquigarrow \quad N$ min : 6 ⟶ $N = 50$.

$$\underset{z_i}{z} = \overbrace{a_1 x^2 + a_2 y^2 + a_3 xy \cdots + a_6}^{x_i,\, y_i}$$

$$\varepsilon_i = (a_1 x_i^2 + a_2 y_i^2 + \cdots + a_6) - z_i$$



Plan ⟶ $z = ax + by + \underline{c}$ — tous les plans sauf plans verticaux.

éq. spécifique

$c \neq 0$

éq. gén. ⟶ $\boxed{ax + by + cz + d = 0.}$ — tous les plan.

Pb ...

$\mathcal{P}$ fixé $\xrightarrow{\text{éq}}$ $ax + by + cz + d = 0$.

$2ax + 2by + 2cz + 2d = 0$

$(\times \lambda)$

$(\lambda a)x + (\lambda b)y + (\lambda c)z + \lambda d = 0$

$\boxed{\text{éq. pas unique}}$

incompatible avec optim ! — infinité de minima !

Eq. générale $\quad ax + by + cz + d = 0$

$\oplus$ cond $\quad \boxed{a^2+b^2+c^2+d^2} \quad \frac{a}{c}x + \frac{b}{c}y + \frac{c}{c}z + \frac{d}{c}$, $\frac{c}{c} = 1$

~~si $c \neq 0$ ↓ plans verticaux ...~~

a, b, c, d non tous nuls

$\Downarrow$

$$\boxed{a^2+b^2+c^2+d^2 \neq 0}$$

plan d'éq. $\quad ax+by+cz+d=0$

tq $\quad \sqrt{a^2+b^2+c^2+d^2} = 1 \iff \|X\|^2 = 1.$

$\sqrt{a^2+b^2+c^2+d^2} = \|X\|$

$X = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ params.

$$f(X) = \sum \varepsilon_i^2 = \sum_i (ax_i + by_i + cz_i + d)^2$$

ajuster plan $\boxed{\text{général.}}$

$\Downarrow$

$\boxed{\text{optim. sous contraintes}}$

$$\min_{X \in \mathbb{R}^4} f(X) \quad \text{tq } \|X\|^2 = 1$$

moindres carrés linéaires

$\boxed{\text{contrainte}}$

$\|X\| = 1$

---

surf init $\xrightarrow{\text{rand}}$ pts bruit $\longrightarrow$ surf. reconstruite f

écart ?

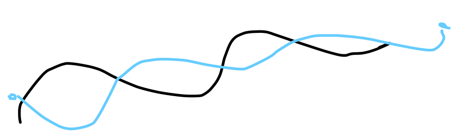
pts $\longrightarrow$ f'

écart!

augmenter le bruit

pts $\longrightarrow$ f''

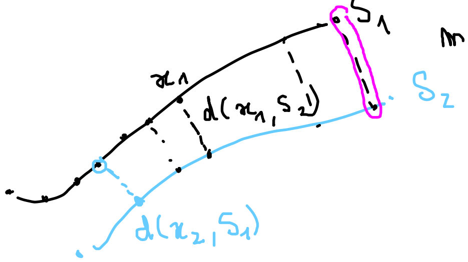
2 surfaces $\xleftrightarrow{\text{écart}}$ ens pts.



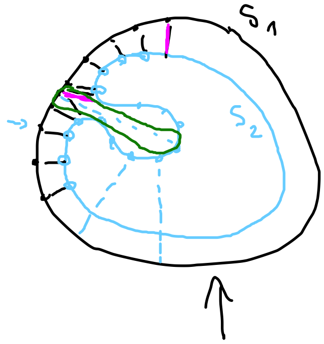
Mesures d'écarts entre ensembles

distances entre ensembles.

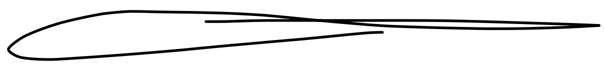
→ dist. de Hausdorff



$$\max\left(\max_{x_1\in S_1} \left(d(x_1,S_2)\right) \; - \; \max_{x_2\in S_2}\left(d(x_2,S_1)\right) \right)$$



$$\mathrm{Hauss}(S_1,S_2) = \max\left(\max_{x_1\in S_1} d(x_1,S_2) \, , \, \max_{x_2\in S_2} \overbrace{d(x_2,S_1)}^{\min_{x\in S_1} d(x_2,x)} \right)$$

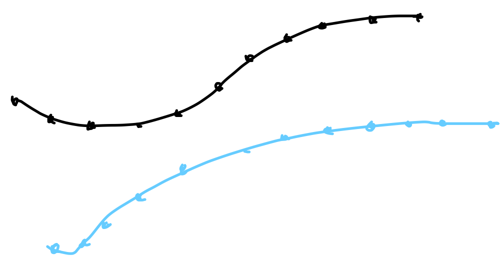


Surfaces S_1, S_2

→ discrétiser → pts → M_1

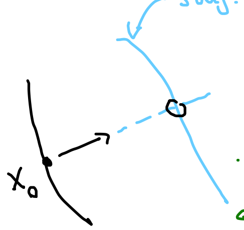
→ discrétiser → pts → M_2

$M_1 \overset{\text{Hauss}}{\longleftrightarrow} M_2$

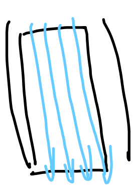


(XX, YY, ZZ)

surf. implicite — TD2 (optim. sous contraintes)



surf. param — optim. sans contr → descente grad.

max  $= (\cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot)$

poly d° 2 en x, y

$$z = \underbrace{a_1 x^2 + a_2 y^2 + a_3 xy + a_4 x + a_5 y + a_6}$$

modèle

linéaire

$$\varepsilon_i = (a_1 x_i^2 + a_2 y_i^2 + a_3 x_i y_i \dots - z_i)$$

$$X = \begin{pmatrix} a_1 \\ \vdots \\ a_6 \end{pmatrix}$$

param

ε_i

A, B

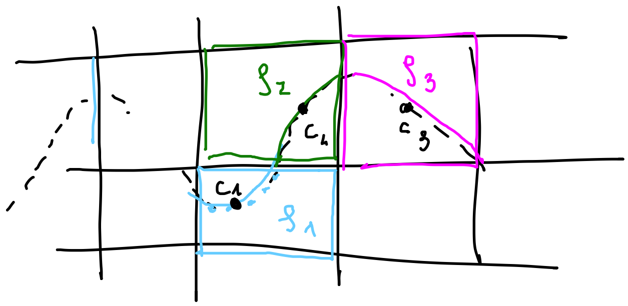
tq

$$\vec{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_6 \end{pmatrix} = AX + B$$

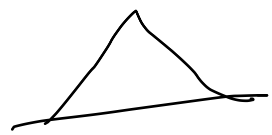
A, B

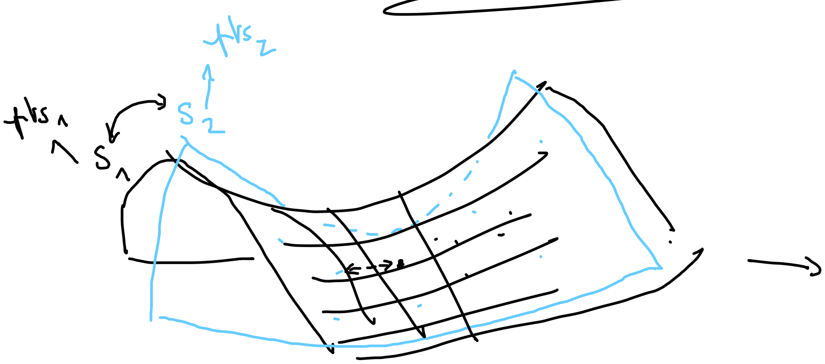
$$A \times \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{pmatrix}$$

Dhtrake



$$f = \sum f_i \cdot \underbrace{\Phi(X - c_i)}_{\text{fct mélange}}$$



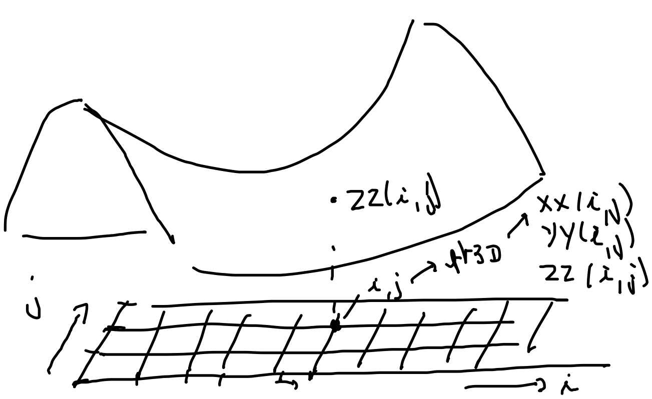


$\rightarrow (XX, YY, ZZ)$

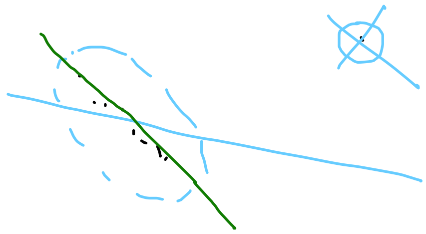
$x \times x$ $\quad$ $x \times x$ $\quad$ $x \times x$

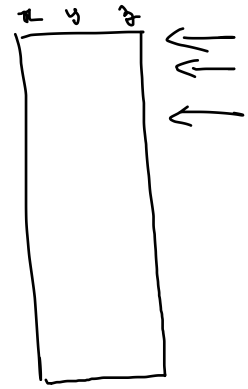
liste de pts ....

```
liste = []
for i = 1 : size(XX,1)
  for j = 1 : size(XX,2)
     liste = [liste ;
              XX(i,j),
              YY(i,j),
              ZZ(i,j)]
```



XX(i,j)





pts2 → S_2

image pts → reconstruire

f

test

dist pts — f

suppression outliers

$$f(x) = \sum \varepsilon_i^2$$

$$= \|AX+B\|^2$$

min X^*

RMSE

$$\sqrt{f(X^*)/N}$$

RMSE
root mean square error.