TP 2

$f_1(X) = \alpha x^2 + \beta y^2$ $\qquad \nabla f_1(X) = \begin{pmatrix} 2\alpha x \\ 2\beta y \end{pmatrix}$

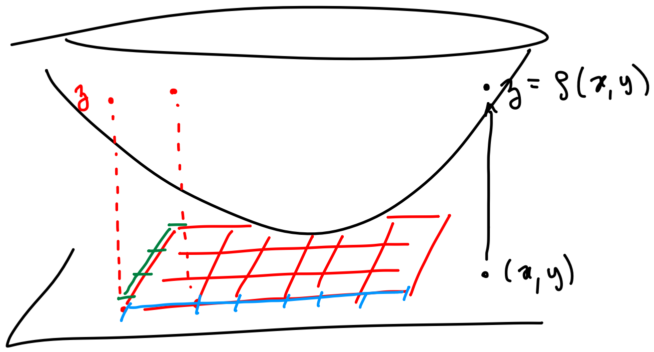
$X = \begin{pmatrix} x \\ y \end{pmatrix}$

$f_2(X) = \left(\sqrt{\alpha x^2 + \beta y^2}\right) \times \left(2 + \sin\left(\sqrt{x^2 + y^2}\right)\right)$

$$\nabla f_2(X) = \begin{pmatrix} 2\alpha x(2 + \sin(x^2+y^2)) + (\alpha x^2 + \beta y^2)(2x \cos(x^2+y^2)) \\ 2\beta y(2 + \sin(x^2+y^2)) + (\alpha x^2 + \beta y^2)(2y \cos(x^2+y^2)) \end{pmatrix}$$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

3D



display

[X, Y] = meshgrid(-1 : 0.1 : 1 , -1 : 0.1 : 1) $\qquad$ X, Y matrices

Z = f(X, Y) ← (*)

surface (X, Y, Z)
or
mesh (X, Y, Z)

f ! $\qquad$ ex : $\alpha x^2 + \beta y^2$

⚠ $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

arg for descent / min.

display
opt 1 (*) → f(x,y) $\qquad$ 2 args – $\mathbb{R}$

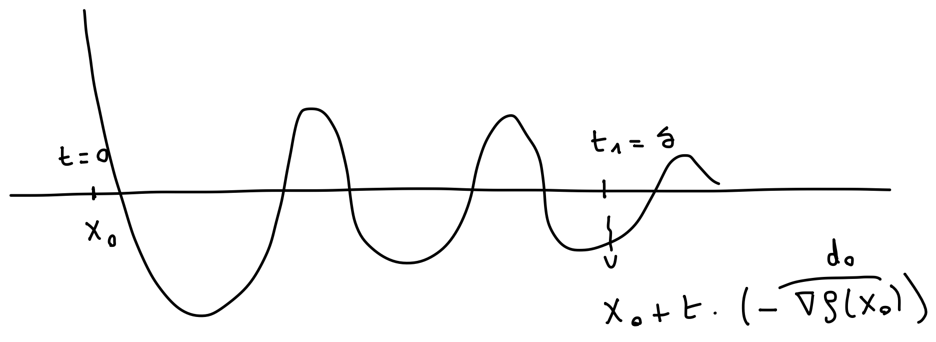
matrices
f = @(x,y) α * x.^2 + β * y.^2

~~opt 2~~ × → f(X) $\qquad$ 1 arg – vector

descent / minimization

`f1 = @(x,y) α*x.^2 + β*y.^2;` — style 1

`ff1 = @(X) f1(X(1),X(2));` — style 2



$$\alpha x^2 + \beta y^2 = X^t \cdot \underbrace{\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}}_{A} \cdot X$$

$$x^2 + y^2 = X^t \cdot I \cdot X$$

$$f_2(X) = \overbrace{X^t \cdot A \cdot X}\cdot \overbrace{\left(2 + \sin(X^t \cdot I \cdot X)\right)}$$

$$\nabla f_2(X) = \nabla(X^t \cdot AX) \cdot \left(2 + \sin(X^t \cdot I \cdot X)\right) +$$

$$X^t \cdot A \cdot X + \nabla\left(\quad '' \quad \right)$$

$$= 2AX\left(2 + \sin(X^t \cdot I \cdot X)\right) +$$

$$X^t \cdot A \cdot X \left(\underbrace{\sin'}_{} (X^t \cdot I \cdot X) \times \nabla(X^t \cdot I \cdot X) \right)$$

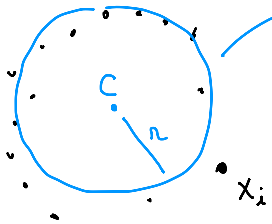
$$\cos(X^t \cdot X) \times 2IX$$

Ex. 2

Data $\rightsquigarrow X_i \quad i : 1 \to N$

1) Parameters / m

$$\underbrace{C, r}_{\mathbb{R}^3} \qquad m = 3$$

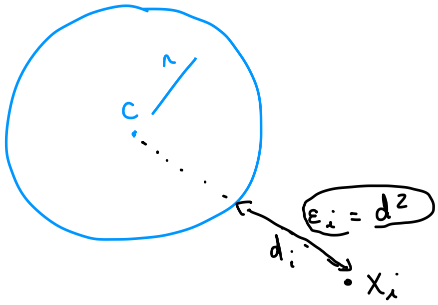


2) Function that has to be optimized

$$\varepsilon_i = \|X_i - C\|^2 - r^2$$

error at pt i

↓ Least squares optimization



$$\text{error} = \sum \varepsilon_i^2$$

$$= f(\dots)$$

$$f(\underbrace{C, r}_{X}) = \sum_{i=1}^{N} \left(\|X_i - C\|^2 - r^2 \right)^2 \longleftarrow$$

Best circle

↓

gradient descent of $f : \overset{\text{circles}}{\mathbb{R}^3} \longrightarrow \mathbb{R}$

$f(X)$

$$\nabla f(C, r) = \begin{pmatrix} \nabla_c f \\ \dots\dots \\ \frac{\partial f}{\partial r} \end{pmatrix} \qquad \nabla_c f = \begin{pmatrix} \frac{\partial f}{\partial c_x} \\ \frac{\partial f}{\partial c_y} \end{pmatrix}$$

quadratic, symmetrical

$$\nabla(X^t . A . X) \sim ax^2 \quad (1D)$$

$$= 2 . A . X \qquad \downarrow' \quad 2ax$$

linear

$$\nabla({}^tB . X) \sim \ell x \quad \downarrow' \quad \ell$$

$$= B$$

$$(f^2)' = 2 . f . f'$$

$$\nabla_c f = \nabla_c \left(\sum_i \left(\|C - x_i\|^2 - r^2 \right)^2 \right)$$

$$= \sum_i 2 . \left(\|C - x_i\|^2 - r^2 \right) . \nabla_c \left(\underbrace{\|C - x_i\|^2}_{{}^tX . \dots . X} - \cancel{r^2} \right)$$

$$\nabla_C \underbrace{\|C - X_i\|^2}$$

$$\|C - X_i\|^2 = \overbrace{(C - X_i)^t}^{C^t - X_i^t} \times (C - X_i)$$

$$= \underbrace{C^t . C} - \big(\underbrace{C^t . X_i}_{\mathbb{R}} + \underbrace{X_i^t . C}_{\mathbb{R}}\big) + X_i^t . X_i$$

$$\left\{ \begin{aligned} \|u\|^2 &= \langle u, u \rangle \\ &= \underbrace{u^t \times u} \\ \langle u, v \rangle &= u^t \times v \end{aligned} \right.$$

$$C^t . X_i \in \mathbb{R}$$
$$= (C^t . X_i)^t$$
$$= X_i^t . C$$

$$= \underbrace{\overset{\text{quad}}{C^t I C}} - \underbrace{\overset{\text{linear}}{2 X_i^t . C}} + X_i^t . X_i$$

$$\nabla_C \left(\|C - X_i\|^2 \right) = 2 . I . C - 2 X_i$$

Thus:

$$\nabla_C (g)(C, r) = \sum_i 2 \left(\|C - X_i\|^2 - r^2 \right) \left(2 (C - X_i) \right)$$

And

$$\frac{\partial g}{\partial r} = \frac{\partial}{\partial r} \left(\sum_{i=1}^{N} \left(\|X_i - C\|^2 - r^2 \right)^2 \right) = \frac{\partial}{\partial r} \left(\sum_i \left(r^2 - \|X_i - C\|^2 \right)^2 \right)$$

$$= \sum_i 2 \left(r^2 - \|X_i - C\|^2 \right) . 2r$$

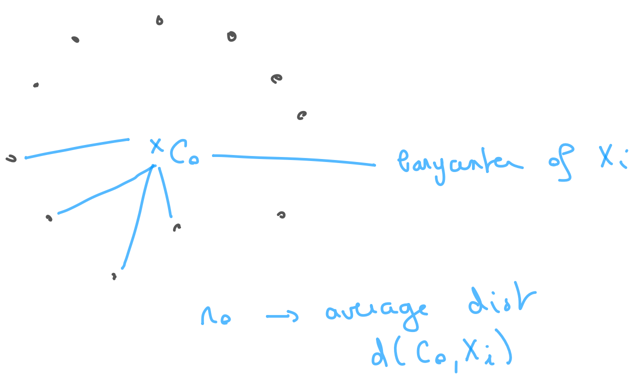
$$\left((x - a)^2 \right)' = 2 (x - a)$$
$$\|$$
$$\left((a - x)^2 \right)' = 2 (a - x) \times (-1)$$

Thus:

$$\nabla g (C, r) = \begin{pmatrix} \sum_i 2 \left(\|C - X_i\|^2 - r^2 \right) \left(2 (C - X_i) \right) \\ \sum_i 2 \left(r^2 - \|X_i - C\|^2 \right) . 2r \end{pmatrix}$$

X_0 ?
which initial circle

1) $\boxed{\text{First } X_0}$



Try also : 2) $\underbrace{X_0 = \begin{pmatrix}10\\10\\10\end{pmatrix}}_{\text{points far from the solution}}$ ...

3) Increase ε

$\downarrow$

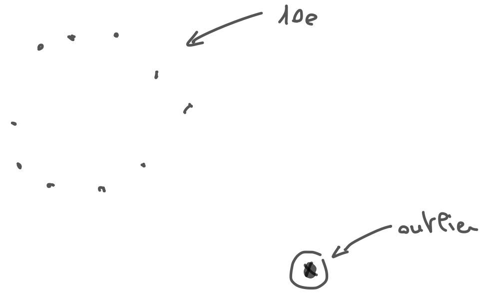
noise on X_i

$\boxed{\text{uniform}}$

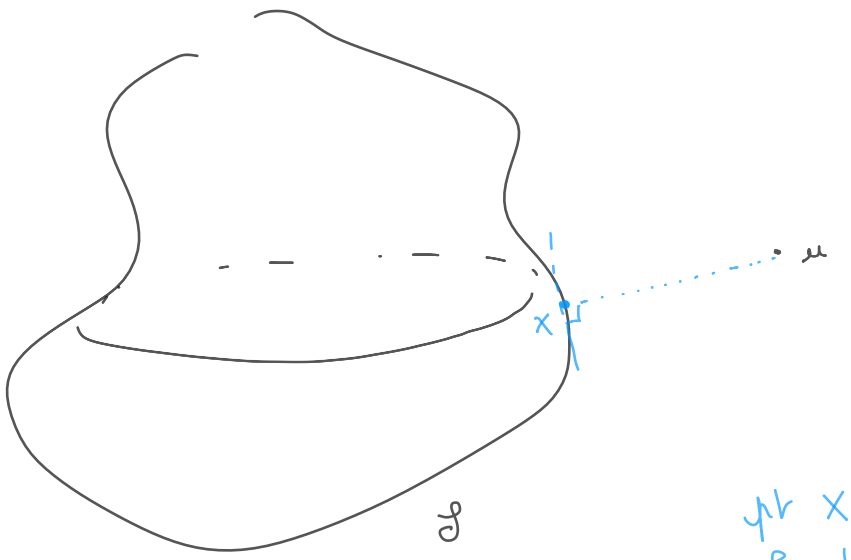
4) ⚠ Try to add a an outlier

$\downarrow$

1 pt far from the others



Ex. 3



$u \longrightarrow$ projection onto the surface

x

pt $x \in S$ closest to u

$\min_{x \in S} d(x, u)$

" $\vec{ux} \perp S$ "

How to compute it?

$\downarrow$

min $\equiv$ optim.

$n = 3$

$$\min_{x \in \mathbb{R}^3} d(x,u)^2 = \| x - u \|^2 \rightsquigarrow \mathcal{C}^\infty \text{ functions}$$

st $x \in S \longrightarrow$ constrained optimization

↓

" trick " to turn it into a non-constrained pb

Penalization

1) Constraint → equation $\quad X \in S$

$\boxed{S}$ → mesh (triangle)

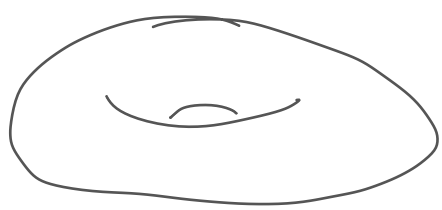
surface 3D

→ smooth models → parametric models

Beziers / NURBS / B-splines

$S : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$(u,v) \longmapsto S(u,v) \in 3D$ pt



smooth models → S

↓

implicit models

$S : \mathbb{R}^3 \longrightarrow \mathbb{R}$

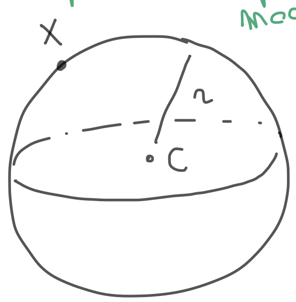
$S - \{X \, ; \, S(X) = 0\}$

Here

↓

$X \in S \iff S(X) = 0$

example of implicit model:



$\underbrace{\|X - C\|^2 - r^2}_{S(X)} = 0$

$$\min_{x \in S} \underbrace{\| X - \mu \|^2}_{J(x)} \quad \text{min}$$

$\Updownarrow$

$S(x) = 0$

constraint

0

$\xrightarrow{\text{penalization}}$ "trick"

small $\hookrightarrow$ $1/\alpha$ large $\}$ 10^{-2}

$$g(X) = J(X) + \frac{1}{\alpha} \cdot S(X)^2$$

$$\min_{X \in \mathbb{R}^3} g(X)$$

1D

$$(S^2)' = 2 \cdot S \cdot S'$$

So $\quad g(X) = \| X - u \|^2 + \dfrac{1}{\alpha} S(X)^2$

$$\nabla g(X) = \nabla\left(\| X - u \|^2 \right) + \frac{1}{\alpha} \cdot 2 \cdot S(X) \cdot \nabla S(X)$$

$\Downarrow$ see above

$$\nabla g(X) = 2(X - u) + \frac{1}{\alpha} \cdot S(X) \cdot \nabla S(X)$$

$$\nabla(\|X\|^2) = 2X$$

S $\leadsto$ depends on S

1) $S(X) = a x^2 + b y^2 + c z^2 - 1$ $\qquad \nabla S$?

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Downarrow$$

$$X^t \cdot \underbrace{\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}}_{A} \cdot X \quad -1$$

$$\nabla S(X) = 2 \cdot A \cdot X$$