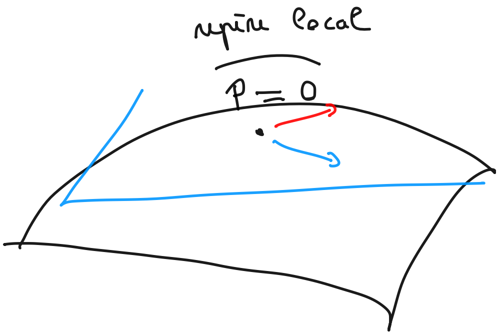
TD 4



$$z = a_0x^2 + a_1xy + a_2y^2 + a_3x + a_4y$$

$$\Downarrow$$

modèle paramétrique : f $\quad \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$(x,y) \longmapsto \begin{pmatrix} x \\ y \\ a_0x^2 + a_1xy + a_2y^2 + a_3x + a_4y \end{pmatrix}$$

Courbures :

1) Base du plan tangent

$$\left(P,\ \underbrace{\frac{\partial f}{\partial x}}_{\frac{\partial f}{\partial u}\ (\text{ici } u = x)},\ \underbrace{\frac{\partial f}{\partial y}}_{\frac{\partial f}{\partial v}\ (\text{ici } v = y)}\right)$$

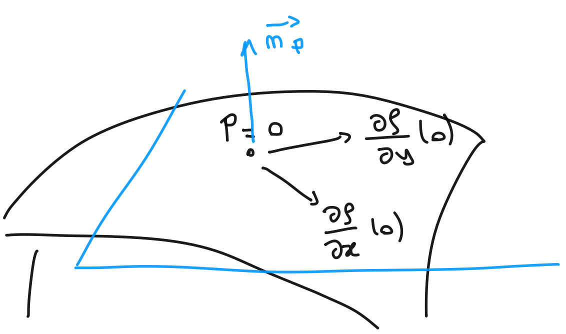
⚠ si les 2 vects sont étirés .....

$$\frac{\partial f}{\partial x} = \begin{pmatrix} 1 \\ 0 \\ 2a_0x + a_1y + a_3 \end{pmatrix} \qquad \frac{\partial f}{\partial y} = \begin{pmatrix} 0 \\ 1 \\ a_1x + 2a_2y + a_4 \end{pmatrix}$$

↳ Au pt P = 0 (repère local)

Donc en P = 0, les vects tangents sont : $\frac{\partial f}{\partial x}(\vec{0}) = \begin{pmatrix} 1 \\ 0 \\ a_3 \end{pmatrix}$

$$\frac{\partial f}{\partial y}(\vec{0}) = \begin{pmatrix} 0 \\ 1 \\ a_4 \end{pmatrix}$$



$$z = a_0x^2 + a_1xy + a_2y^2 + a_3x + a_4y$$

$$\vec{m}_P = \underbrace{\begin{pmatrix} 1 \\ 0 \\ a_3 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ a_4 \end{pmatrix}}_{= \begin{pmatrix} -a_3 \\ -a_4 \\ 1 \end{pmatrix}} \Big/ \| \ldots \|$$

$(\sim Oz)$

norme : $\underbrace{\sqrt{1 + a_3^2 + a_4^2}}_{g}$

$\Rightarrow$ a_3, a_4 petits

$$\boxed{\vec{m}_P = \frac{1}{g}\begin{pmatrix} -a_3 \\ -a_4 \\ 1 \end{pmatrix}}$$

2) Première forme fondamentale

$$I_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad \text{ici } (u = x,\ v = y)$$

$p = 0$

Dans la base du plan tangent $\left(0, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

$$E = \left\| \frac{\partial f}{\partial x}(0) \right\|^2 = \left\| \begin{pmatrix} 1 \\ 0 \\ a_3 \end{pmatrix} \right\|^2 = 1 + a_3^2$$

$$G = \left\| \frac{\partial f}{\partial y}(0) \right\|^2 = \left\| \begin{pmatrix} 0 \\ 1 \\ a_4 \end{pmatrix} \right\|^2 = 1 + a_4^2$$

$$F = \frac{\partial f}{\partial x}(0) \cdot \frac{\partial f}{\partial y}(0) = \begin{pmatrix} 1 \\ 0 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ a_4 \end{pmatrix} = a_3 a_4$$

$$\boxed{I_p = \begin{pmatrix} 1+a_3^2 & a_3 a_4 \\ a_3 a_4 & 1+a_2^2 \end{pmatrix}} \sim I$$

si on a bien $m_p \sim Oz$
a_3, a_4 petits

3) Deuxième forme fondamentale

$$II_p = \begin{pmatrix} l & m \\ m & n \end{pmatrix} \qquad l = \underbrace{\frac{\partial^2 f}{\partial x^2}} \cdot m_p \qquad m_p = \frac{1}{g}\begin{pmatrix} -a_3 \\ -a_4 \\ 1 \end{pmatrix}$$

ici $u = x$
$v = y$

$$\frac{\partial f}{\partial x} = \begin{pmatrix} 1 \\ 0 \\ 2a_0 x + a_1 y + a_3 \end{pmatrix}$$

$$\frac{\partial f}{\partial y} = \begin{pmatrix} 0 \\ 1 \\ a_1 x + 2a_2 y + a_4 \end{pmatrix}$$

csr ici ....

$$\frac{\partial^2 f}{\partial x^2} = \begin{pmatrix} 0 \\ 0 \\ 2a_0 \end{pmatrix} = \frac{\partial^2 f}{\partial x^2}(0)$$

$$\frac{\partial^2 f}{\partial y^2} = \begin{pmatrix} 0 \\ 0 \\ 2a_2 \end{pmatrix}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \begin{pmatrix} 0 \\ 0 \\ a_1 \end{pmatrix}$$

$\partial y\, \partial x$

$$l = \frac{\partial^2 f}{\partial x^2}(0) \cdot \vec{m_p} = \frac{1}{g} 2a_0$$

$$m = \frac{\partial^2 f}{\partial x \partial y}(0) \cdot \vec{m_p} = \frac{1}{g} a_1$$

$$n = \frac{\partial^2 f}{\partial y^2}(0) \cdot \vec{m_p} = \frac{1}{g} 2a_2$$

$$\boxed{\mathrm{II}_P = \frac{1}{g}\begin{pmatrix} 2a_0 & a_1 \\ a_1 & 2a_2 \end{pmatrix}}$$

4) Tenseur de courbure en $P = 0$:

$$K_P = \begin{pmatrix} 1+a_3^2 & a_3a_4 \\ a_3a_4 & 1+a_4^2 \end{pmatrix}^{-1} \times \frac{1}{g}\begin{pmatrix} 2a_0 & a_1 \\ a_1 & 2a_2 \end{pmatrix} \longrightarrow \text{eig} \dots \longrightarrow K_1, K_2 \; ; \; d_1, d_2$$

$$\det = (1+a_3^2)(1+a_4^2) - a_3^2a_4^2 = 1 + a_3^2 + a_4^2 + \cancel{a_3^2a_4^2} - \cancel{a_3^2a_4^2}$$

$$\text{com } A = \begin{pmatrix} \oplus\, 1+a_4^2 & \ominus\, a_3a_4 \\ \ominus\, a_3a_4 & \oplus\, 1+a_3^2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det A}(\text{com } A)^t$$

$$I_P^{-1} = \frac{1}{\underbrace{1+a_3^2+a_4^2}_{= g^2}}\begin{pmatrix} 1+a_4^2 & -a_3a_4 \\ -a_3a_4 & 1+a_3^2 \end{pmatrix}$$

$$\boxed{\underset{0}{K_P} = \frac{1}{g^3}\underbrace{\begin{pmatrix} 1+a_4^2 & -a_3a_4 \\ -a_3a_4 & 1+a_3^2 \end{pmatrix}}_{\sim I}\begin{pmatrix} 2a_0 & a_1 \\ a_1 & 2a_2 \end{pmatrix}}$$

si a_3, a_4 petits ...

$g \sim 1$

$$\wr\quad \begin{pmatrix} 2a_0 & a_1 \\ a_1 & 2a_2 \end{pmatrix} \longrightarrow \text{val. propres } K_1, K_2$$

base $(\vec{d}_1, \vec{d}_2)$

base de diag

la mat devient $\rightsquigarrow$ $\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} = \underline{\underline{B}}$

Tr: $2(a_0+a_2)$

$\det = 4a_0a_2 - a_1^2$

$H = \dfrac{K_1+K_2}{2} = \dfrac{\mathrm{Tr}(B)}{2}$ | $K = K_1 \times K_2 = \det(B)$

Invariants par ch. base

$K \simeq 4a_0a_2 - a_1^2$

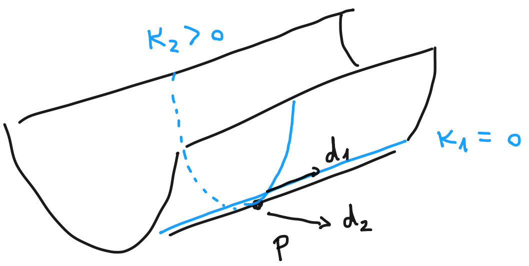
$H \simeq a_0 + a_2$

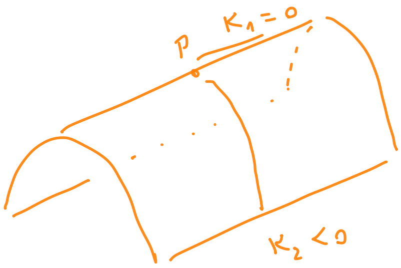
K_1, K_2

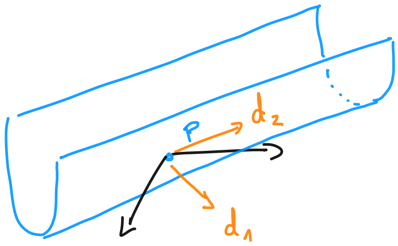
d_1, d_2

$\begin{matrix} K_p \\ 2 \times 2 \end{matrix}$ $\xrightarrow{\text{calcul}}$ vecteurs propres ⇝ $\overbrace{d_1, d_2}^{\text{dir associées}}$

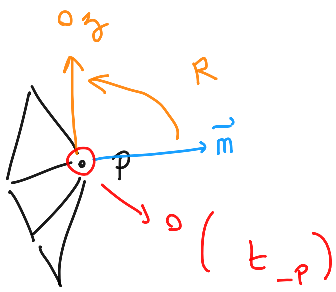
valeurs propres ⇝ $\underbrace{K_1, K_2}_{\text{courbures extr.}}$

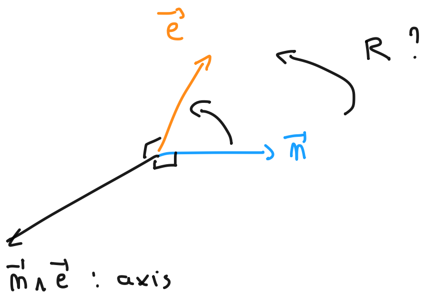




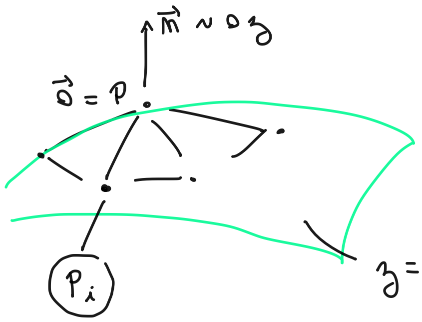


Base plan tangent





ch_base



← pi_e pts P_i ds repère local.

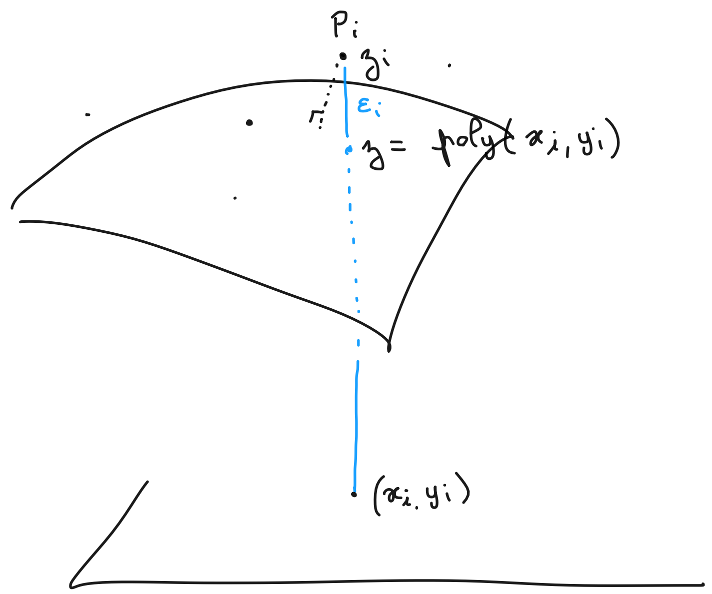
$$z = a_0 x^2 + a_1 xy + a_2 y^2 + a_3 x + a_4 y$$

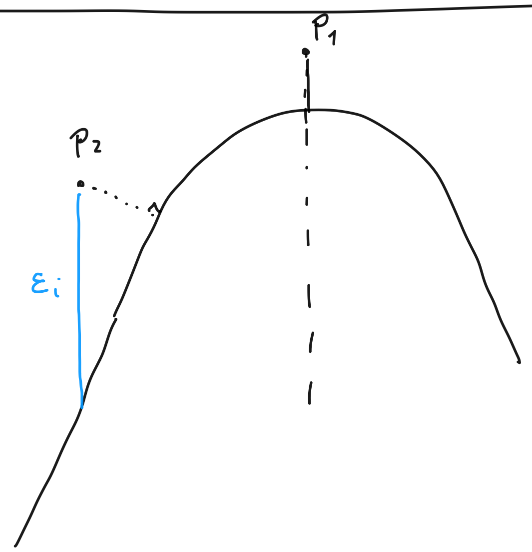
Ajustement aux moindres carrés

1) <u>Paramètres</u> $a_0 \cdots a_4$ $\quad n = 5$

2) Fonction à minimiser :

$$\sum \varepsilon_i^2 \qquad \boxed{\begin{array}{c}\varepsilon_i \\ \shortparallel \\ a_0 x_i^2 + a_1 x_i y_i + a_2 y_i^2 + a_3 x_i - z_i\end{array}}$$





$$J(a_0 \ldots a_4) = \sum \varepsilon_i^2 = \sum \Big(\underbrace{a_0 x_i^2 + a_1 x_i y_i + a_2 y_i^2 + a_3 x_i + a_4 y_i - z_i}_{\varepsilon_i} \Big)^2$$

ε_i lin. / $a_0 \ldots$

⇒ eq. norm / House / $\boxed{SVD}$ TP

A, B ?

$$\vec{\varepsilon} = \begin{pmatrix} \vdots \\ \varepsilon_i \\ \vdots \end{pmatrix} = \begin{pmatrix} a_0 x_i^2 + a_1 x_i y_i + a_2 y_i^2 + a_3 x_i + a_4 y_i - z_i \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ x_i^2 & x_i y_i & y_i^2 & x_i & y_i \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_0 \\ \vdots \\ a_4 \end{pmatrix}}_{X} + \underbrace{\begin{pmatrix} \vdots \\ +z_i \\ \vdots \end{pmatrix}}_{B}$$

code ⇒

⟶ SVD par A et +B