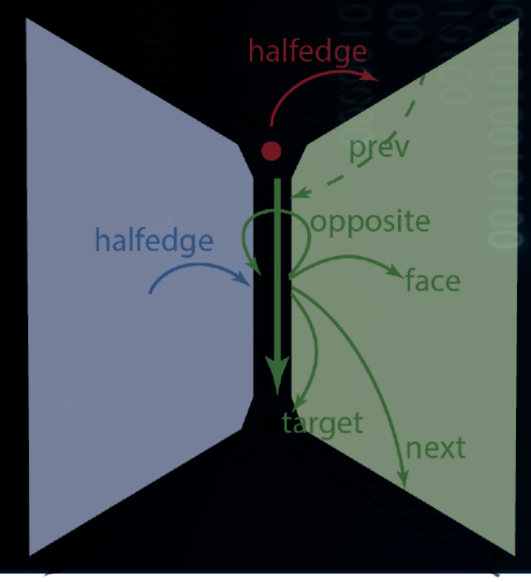
MODELISATION GEOMETRIQUE



Alexandra Bac

POLYTECH 4A INFORMATIQUE REVA

4 – GÉOMÉTRIE DES SURFACES

Certaines illustrations sont issues du livre « polygon mesh processing »

« MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES

Chapitre 2

MAILLAGES

Chapitre 4

GÉOMÉTRIE DES SURFACES

Chapitre 1

MODÉLISATION DES SURFACES

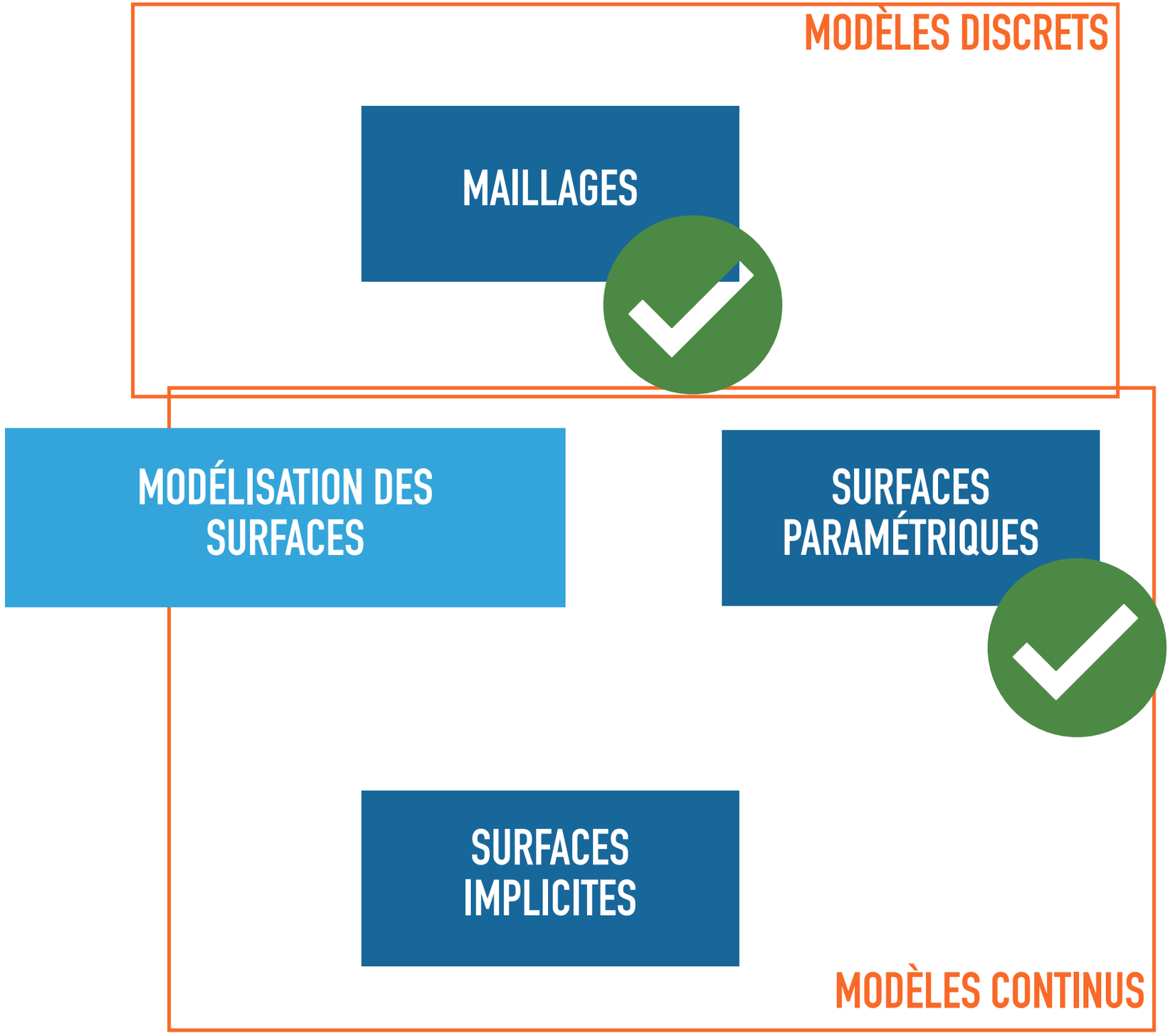
Chapitre 3 (+ 5A)

SURFACES PARAMÉTRIQUES

Chapitre 5

SURFACES IMPLICITES

« MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES

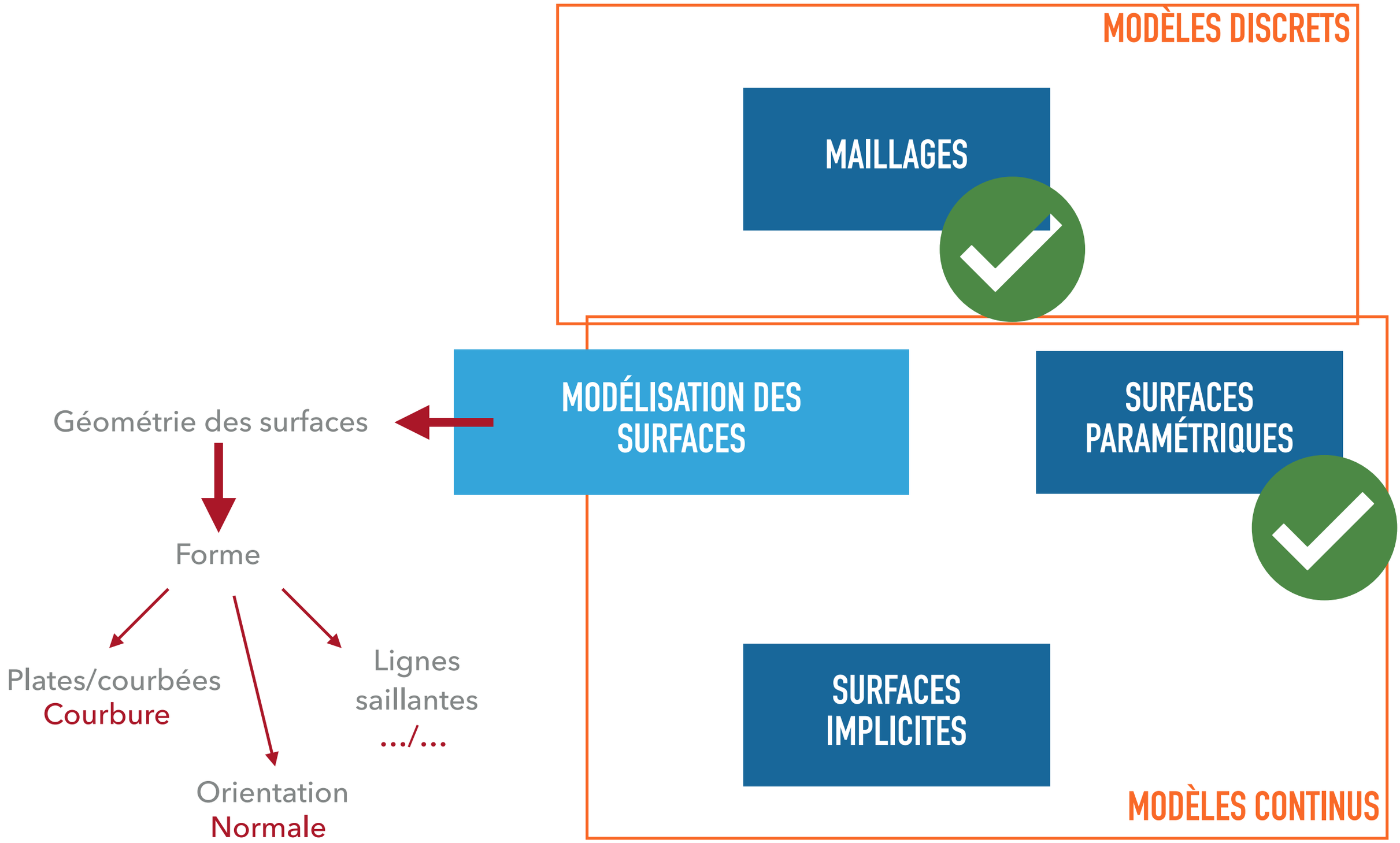


Courbure

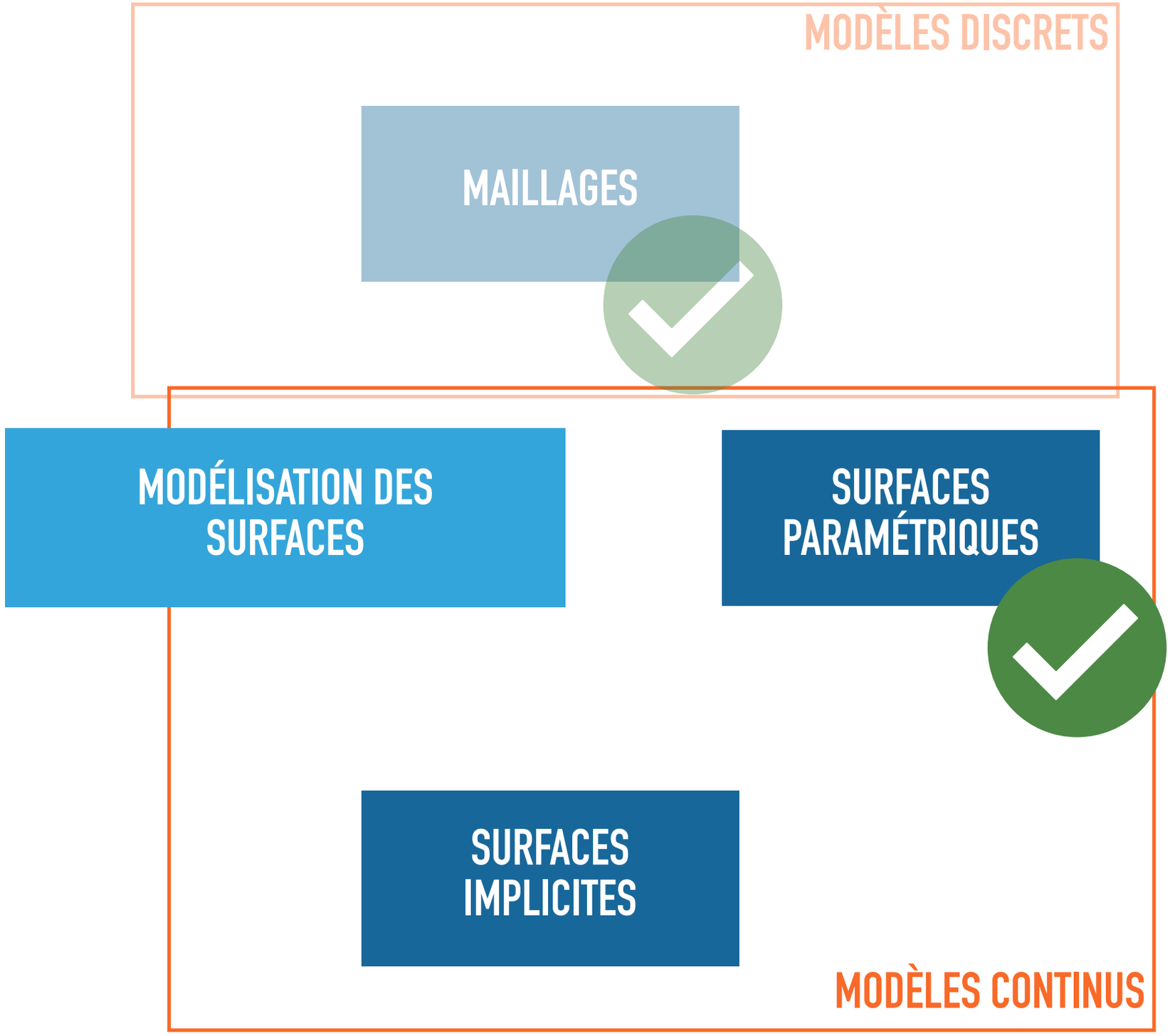
.../...

Normale

« MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES



« MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES

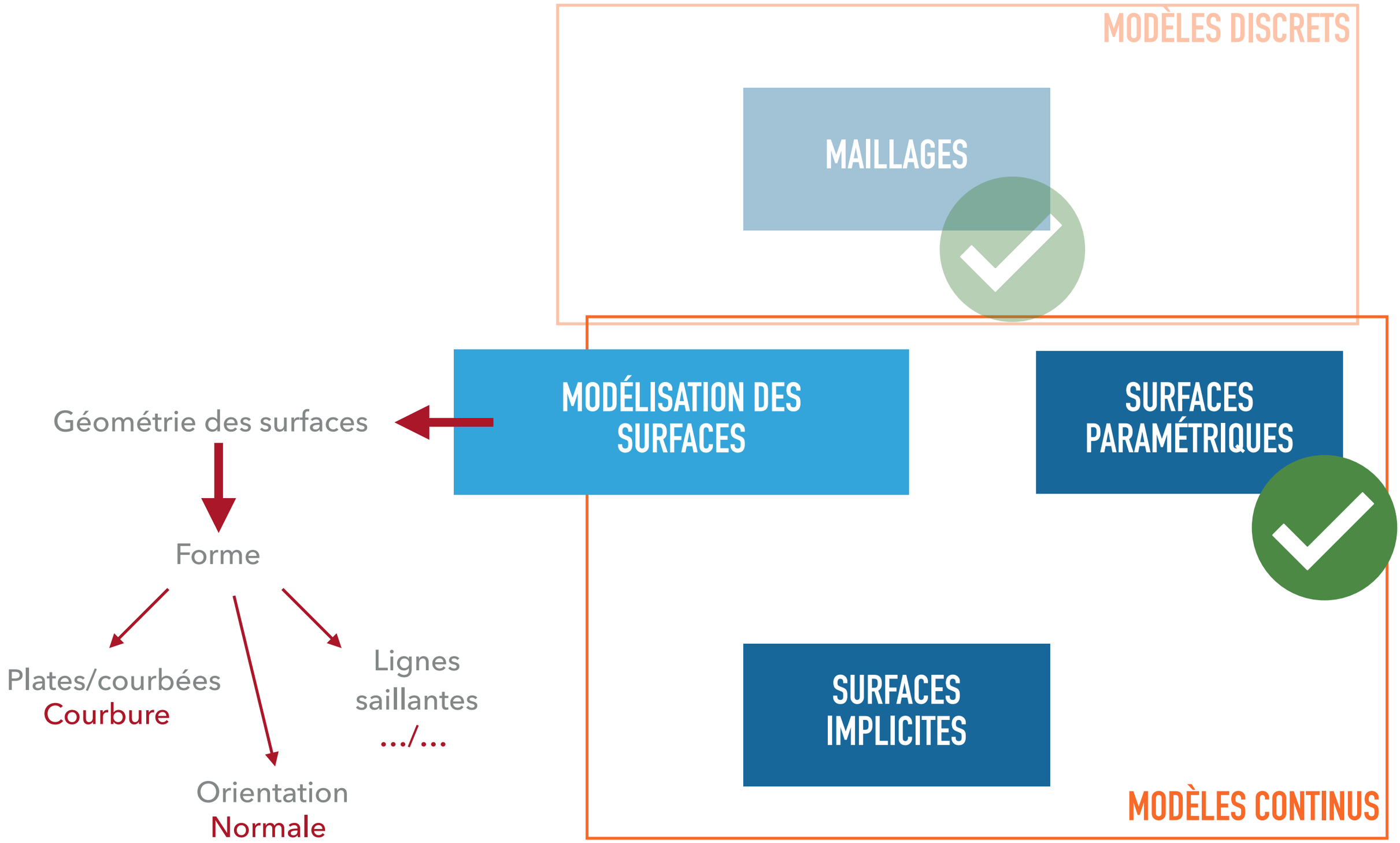


Courbure

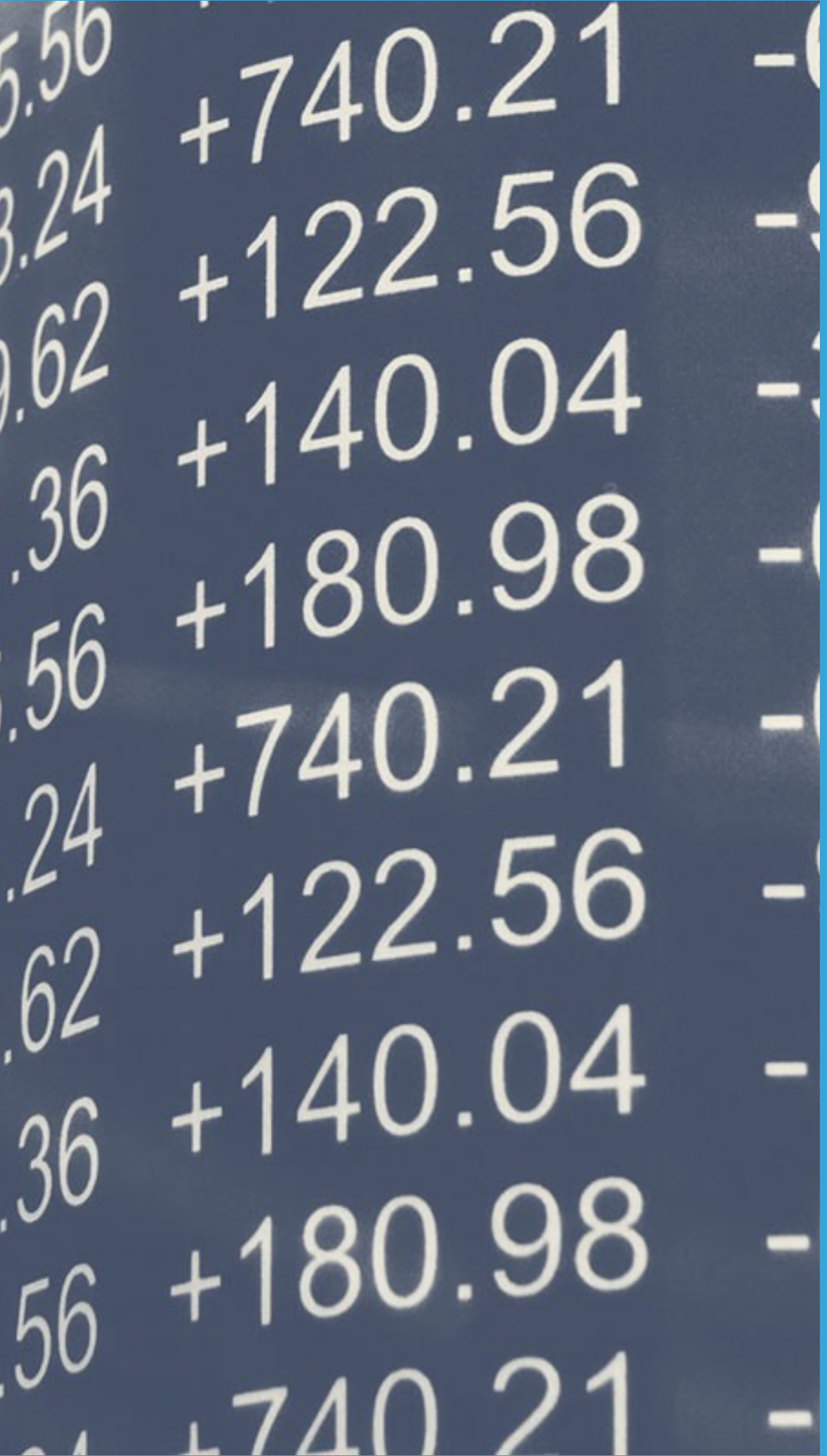
.../...

Normale

« MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES



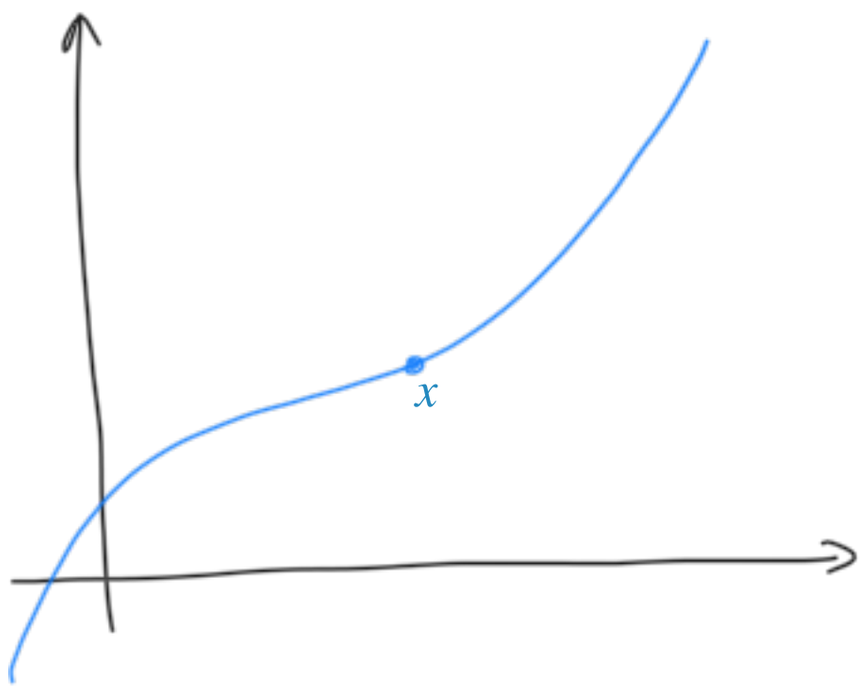
POUR L'INTUITION : GÉOMÉTRIE DES COURBES PARAMÉTRIQUES

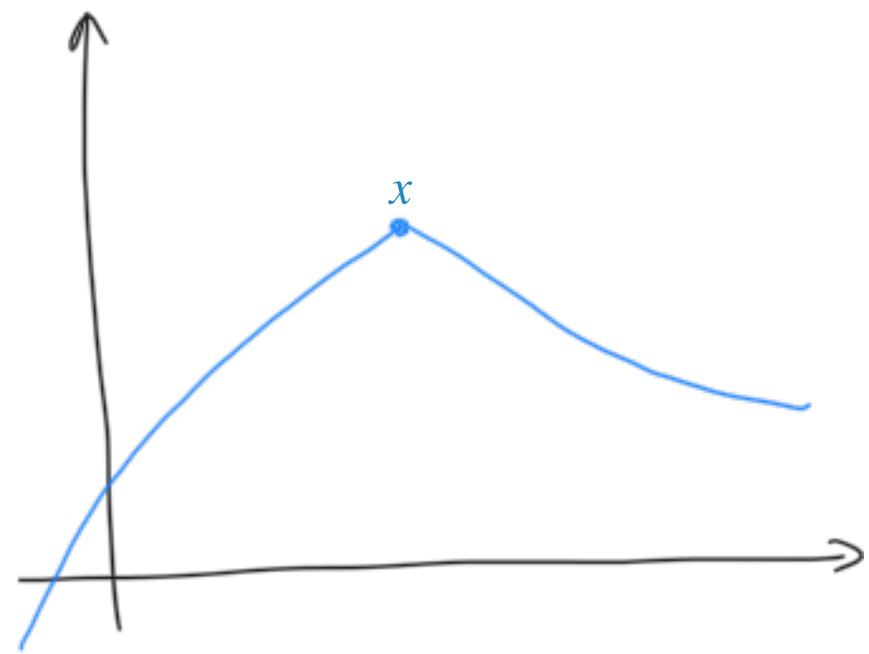


COURBES CARTÉSIENNES

Modélisées sous la forme $y = f(x)$

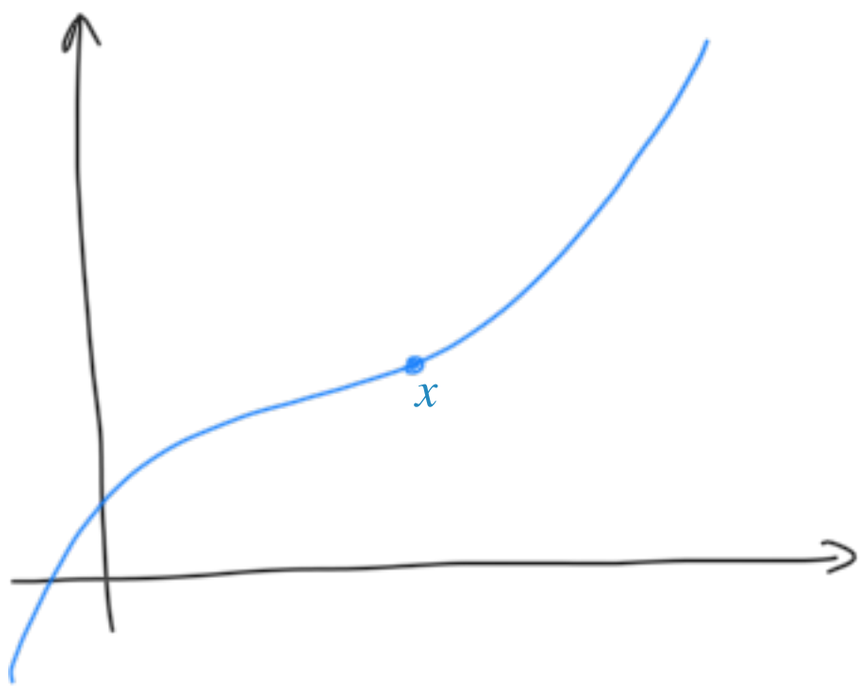
Connues depuis la terminale …

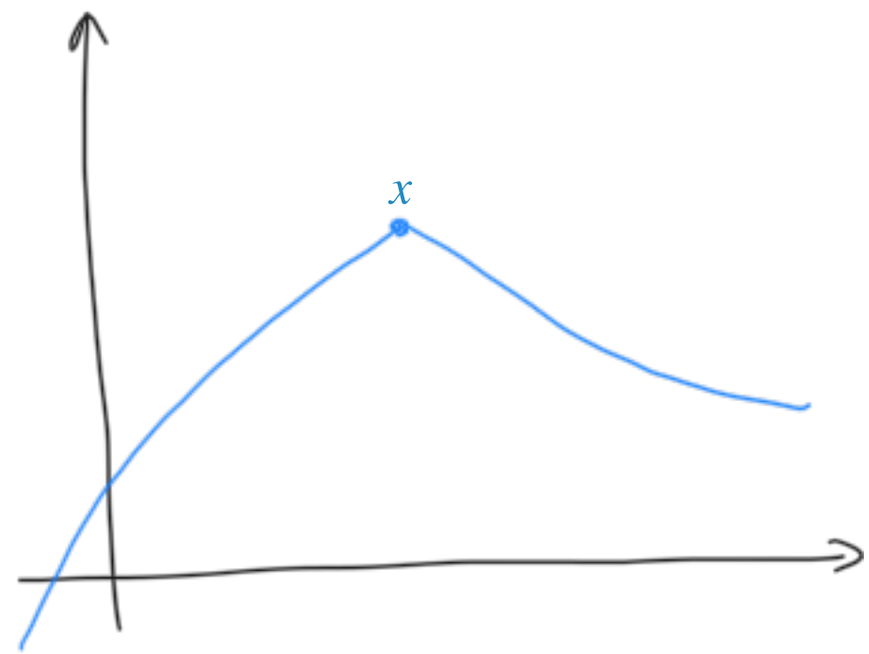




Modélisées sous la forme $y = f(x)$

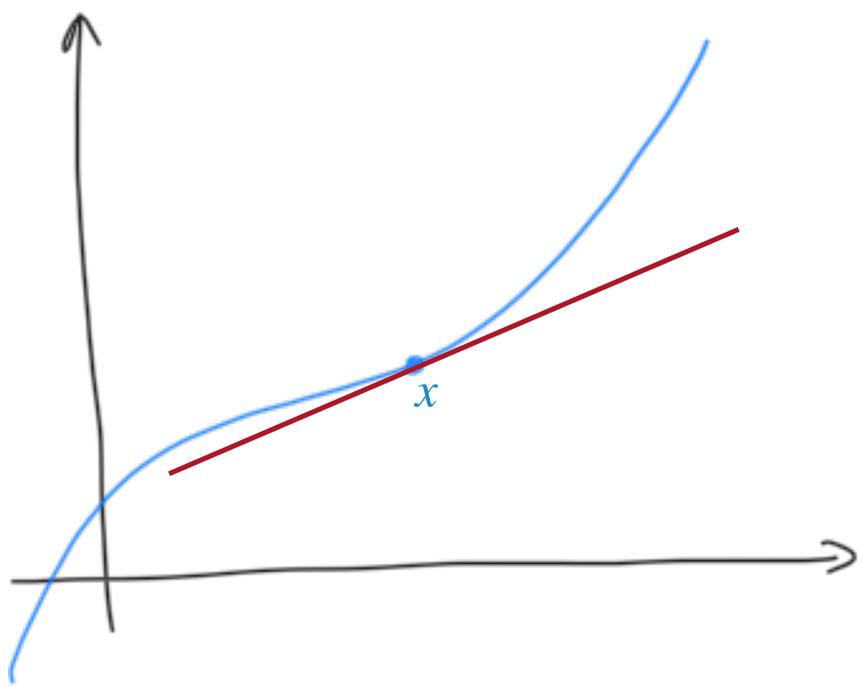
Connues depuis la terminale …

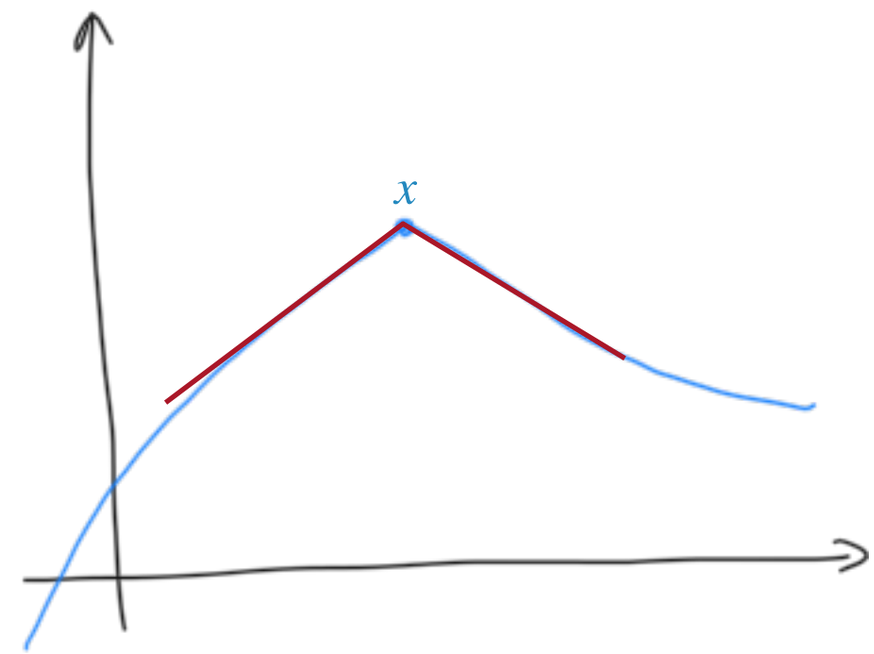




Modélisées sous la forme $y = f(x)$

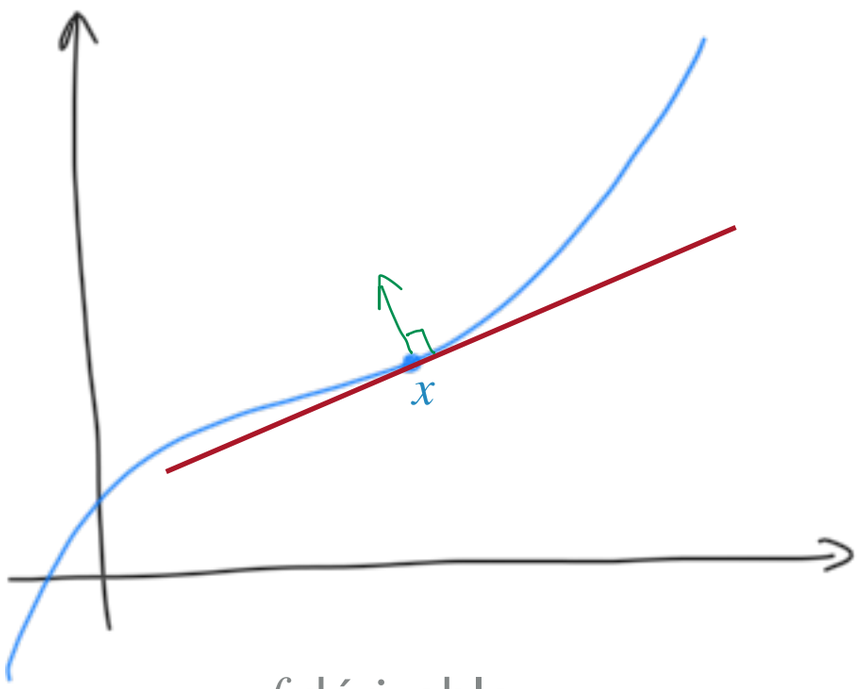
Connues depuis la terminale …



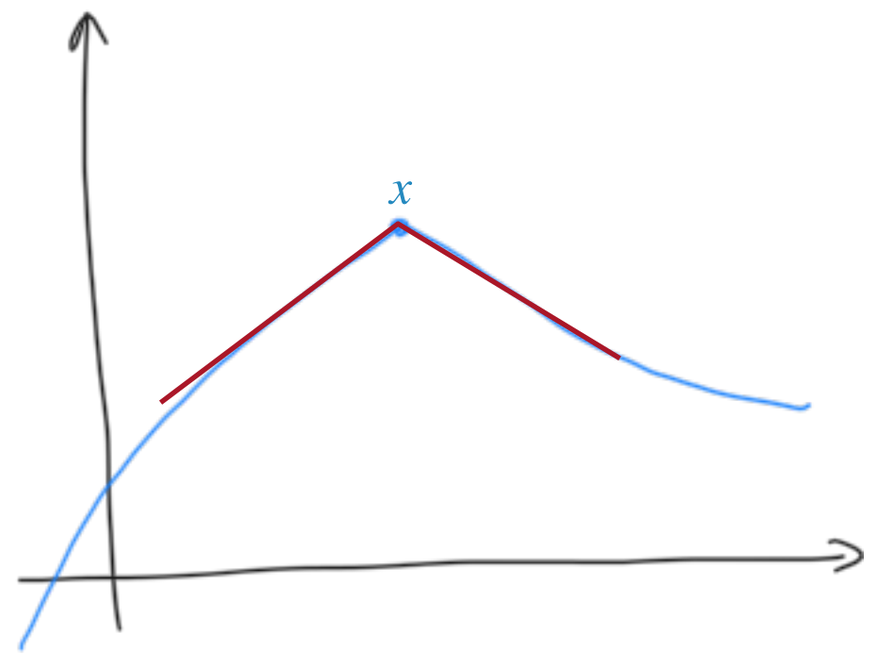


Modélisées sous la forme $y = f(x)$

Connues depuis la terminale …



f dérivable



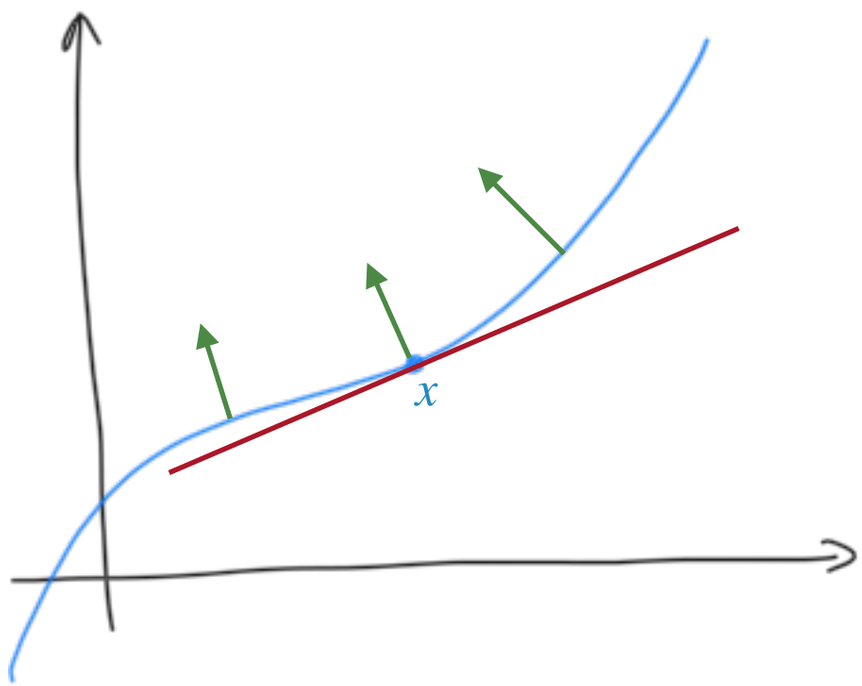
f non dérivable

Courbe proche d'une droite de pente $f'(x)$

Tangente

Modélisées sous la forme $y = f(x)$

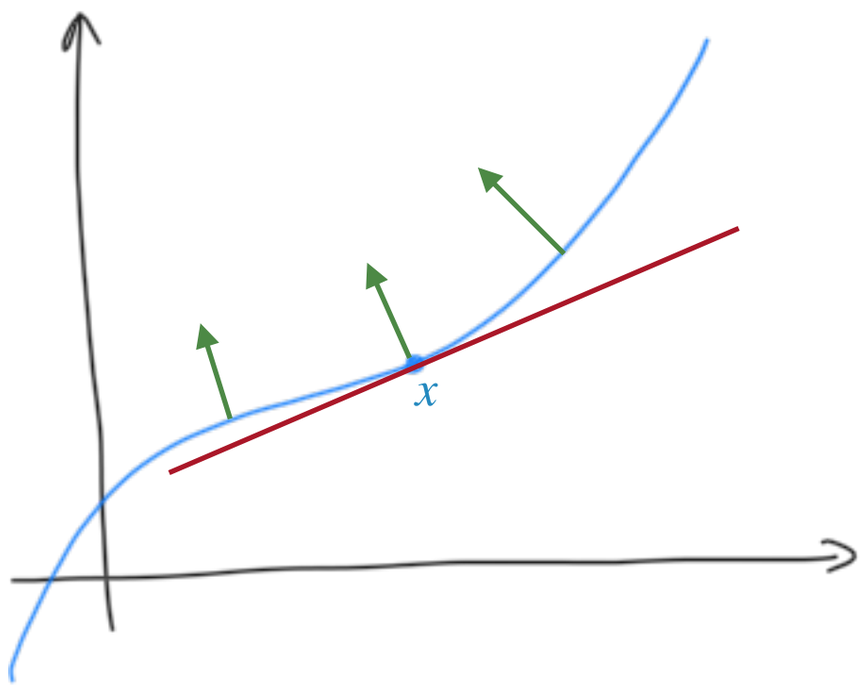
Connues depuis la terminale …



Normale :
orthogonale à la courbe/tangente

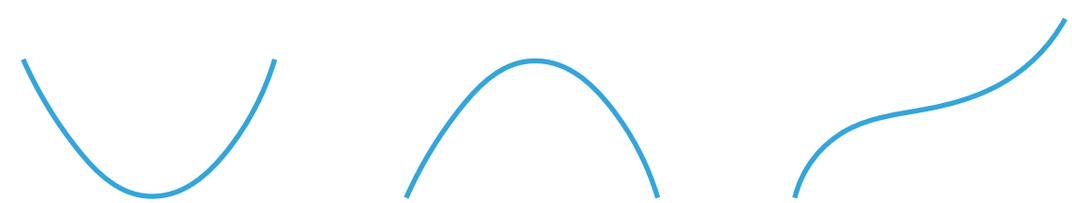
Modélisées sous la forme $y = f(x)$

Connues depuis la terminale …



Dérivées secondes :

$f''(x)$ indique la concavité de la courbe



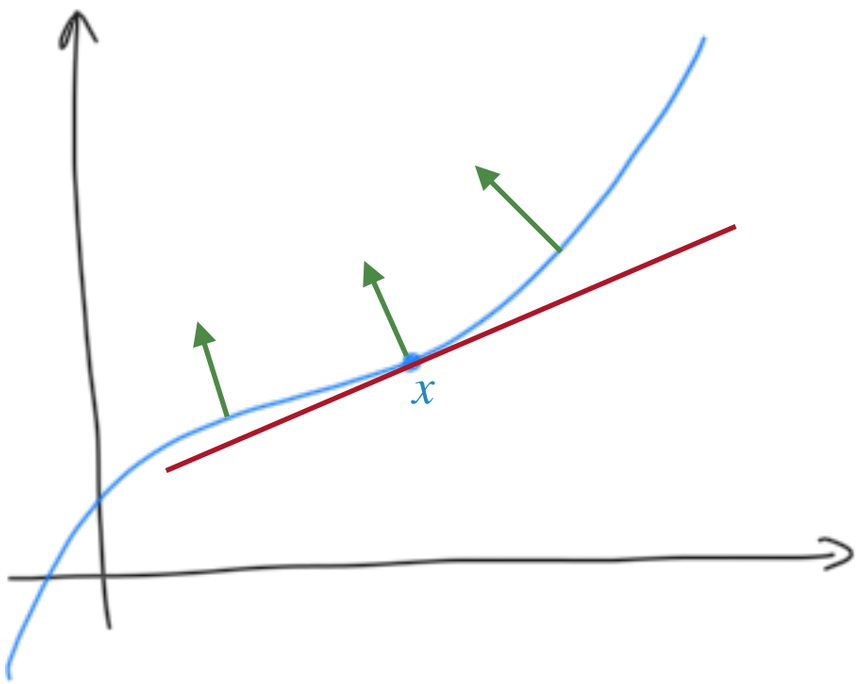
Concave
$f''(x) > 0$

Convexe
$f''(x) < 0$

Inflexion
$f''(x) = 0$

Modélisées sous la forme $y = f(x)$

Connues depuis la terminale …



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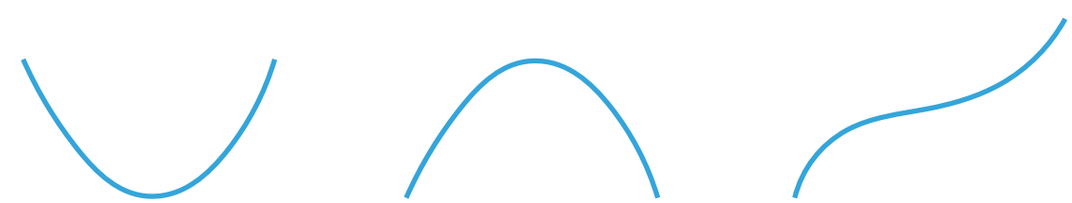


Concave
$f''(x) > 0$

Convexe
$f''(x) < 0$

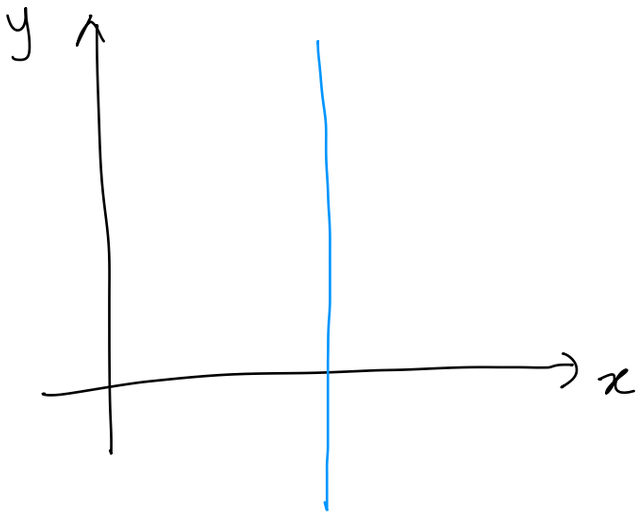
Inflexion
$f''(x) = 0$

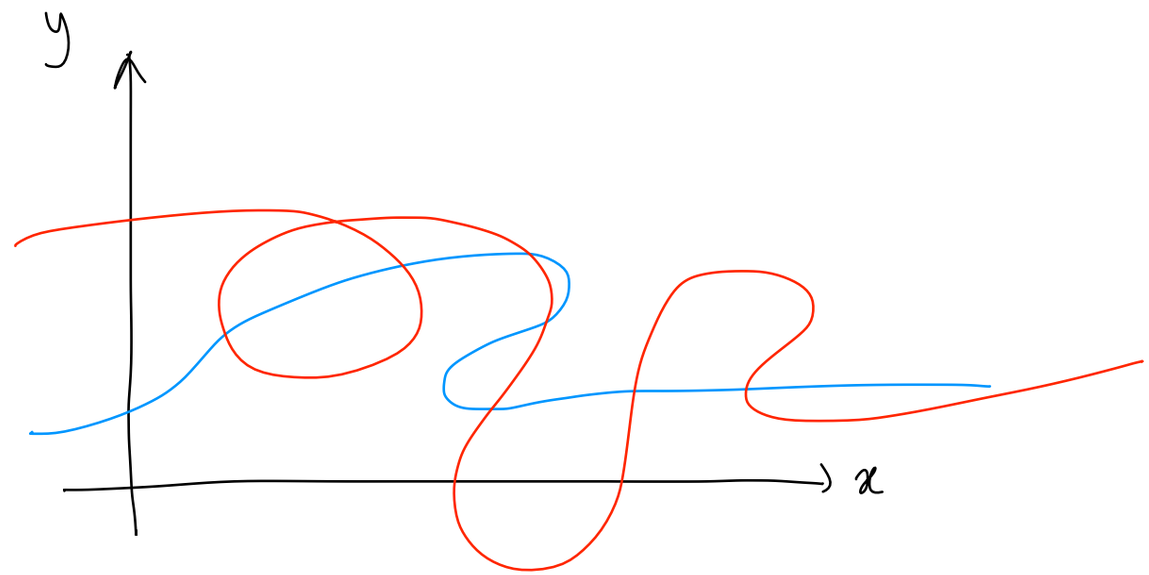
Courbure

Curves that cannot be modelled through a Cartesian equation:

$\downarrow$

$y = f(x)$





---

implicit model

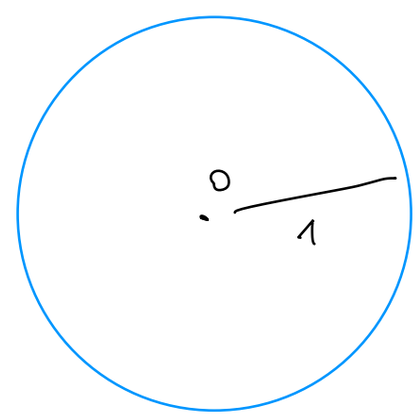
$x^2 + y^2 = 1$

$\downarrow$

$f(x,y) = 0 \iff (x,y) \in \text{circle}$

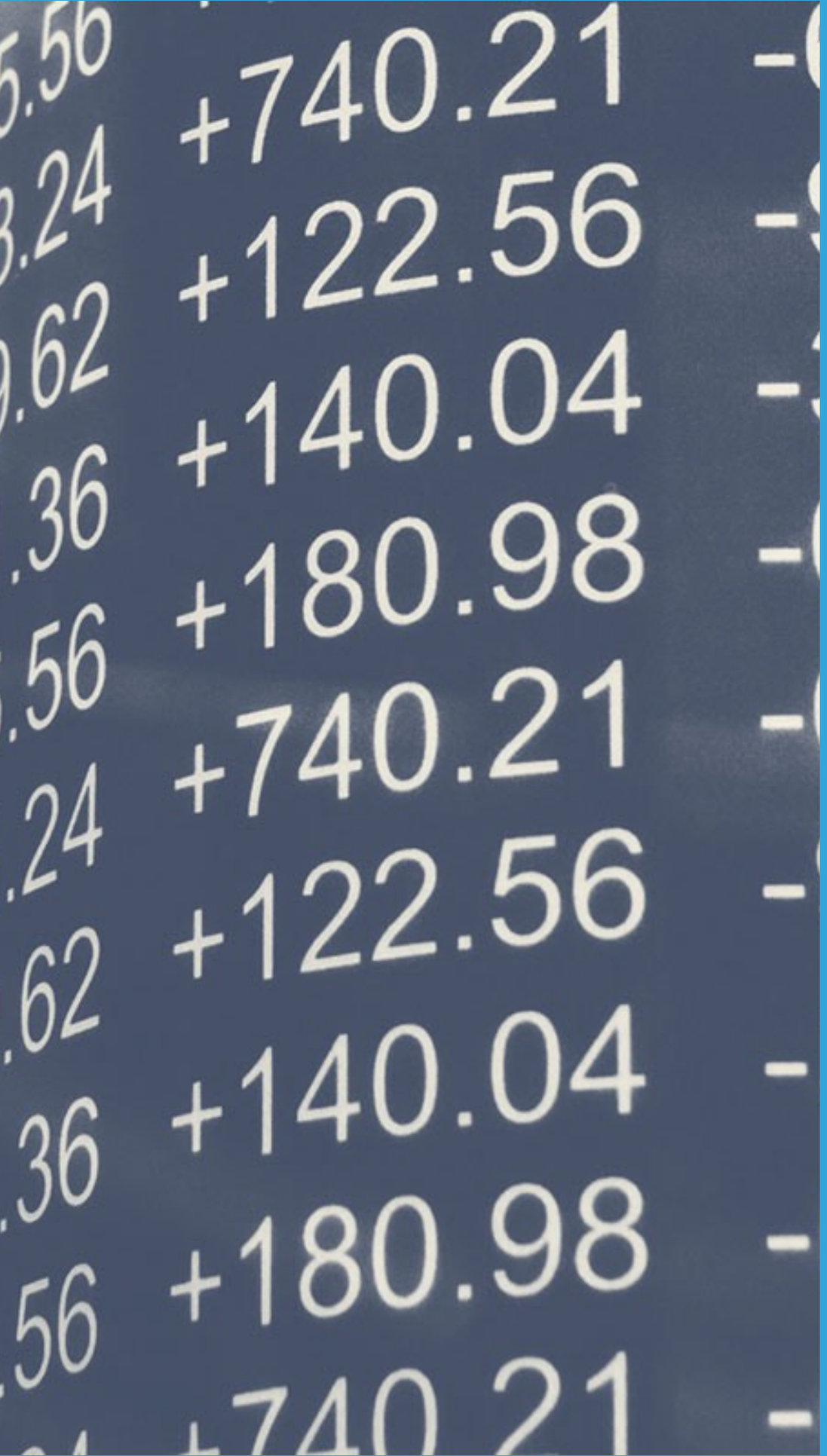
$\parallel$

$x^2 + y^2 - 1$



~~Cartesian~~

$$\left.\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases} \quad t \in [0, 2\pi]\right\} \text{general param. eq.}$$



COURBES PARAMÉTRIQUES GÉNÉRALES

Cf. Chapitre 3

COURBES

Courbe <u>finie</u> :

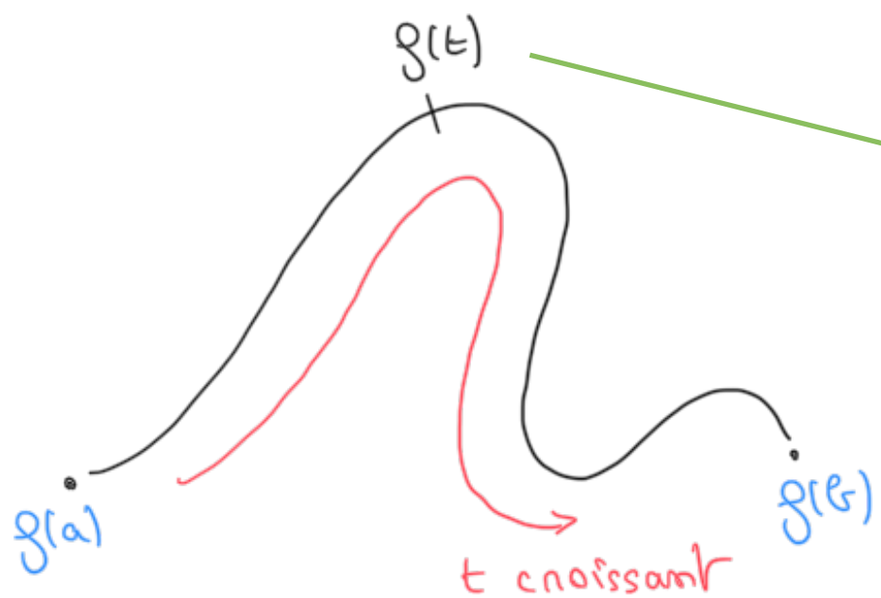
$f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^2$ (ou $\mathbb{R}^3$)

Cf. Chapitre 3

COURBES

Courbe <u>finie</u> :

$f\ :\ [a,b] \subseteq \mathbb{R} \to \mathbb{R}^2$ (ou $\mathbb{R}^3$)



$f(t) \in \mathbb{R}^2/\mathbb{R}^3$ donc :

$$f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ [f_3(t)] \end{pmatrix}$$

$$\updownarrow$$

$f_i\ :\ \mathbb{R} \to \mathbb{R}$

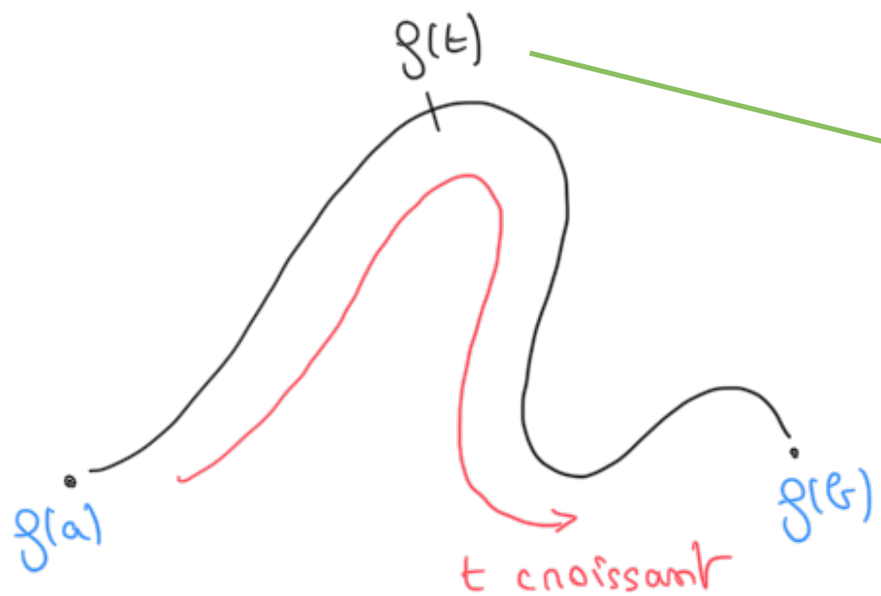
$\nabla g \qquad g : \mathbb{R}^n \to \mathbb{R}$

Cf. Chapitre 3

COURBES

Courbe <u>finie</u> :

$f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^2$ (ou $\mathbb{R}^3$)



$f(t) \in \mathbb{R}^2/\mathbb{R}^3$ donc :

$$f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ [f_3(t)] \end{pmatrix} \qquad g' = \begin{pmatrix} g_1' \\ g_2' \end{pmatrix}$$

$$\updownarrow$$

$f_i : \mathbb{R} \to \mathbb{R}$

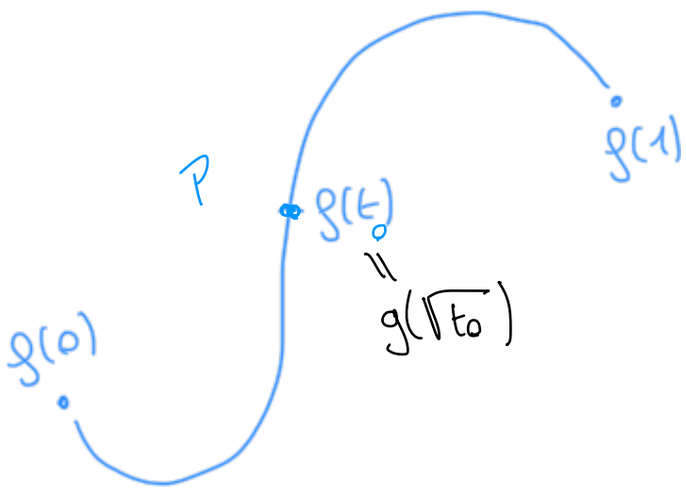
Tangente	Courbure
$f'(t)$	$f''(t)$

?

Cf. Chapitre 3

f / g 2 models of the same curve

tangent ⇝ same result ?



Comme vu au chapitre 3 :

$$g(t) = f(t^2)$$

décrit la même courbe parcourue à une vitesse différente

$t_0 \longrightarrow P = f(t_0)$

$$g'(t) = 2t \cdot f'(t^2)$$

Tangent at P :

→ with the f eq : $f'(t_0)$

→ with the g eq :

$f(t_0) = g(\sqrt{t_0})$

$\neq$ parallel ✓

$$g''(t) = 4t^2 \cdot f''(t^2) + 2 f'(t^2)$$

$$g'(\sqrt{t_0}) = 2\sqrt{t_0} \cdot \underbrace{f'(\sqrt{t_0}^2)}_{f'(t_0)}$$

$t = t_0 \xrightarrow{f} f''(t_0)$

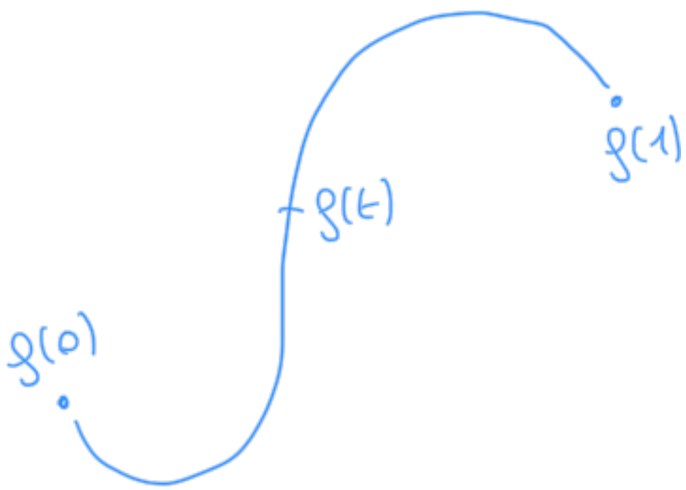
↕ proportional

$\xrightarrow{g} 4t_0 f''(t_0) + 2 f'(t_0)$ ....

$(t = \sqrt{t_0})$

scale pb ....

Cf. Chapitre 3



Comme vu au chapitre 3 :

$$g(t) = f(t^2)$$

décrit la même courbe parcourue à une vitesse différente

$g'(t) =$

$g''(t) =$

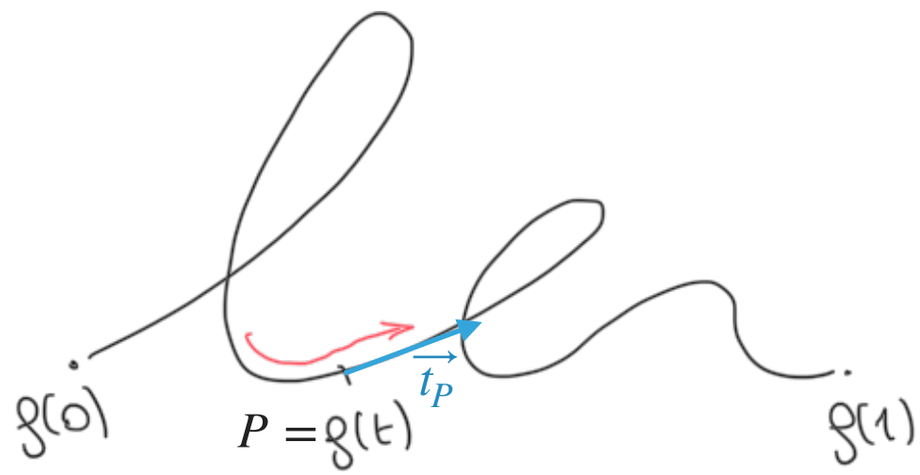
unique param : speed along the curve is constant

$\|g'\|$

Paramétrisation normale

Tangente	Courbure
$f'(s)$	$f''(s)$

TANGENTE / NORMALE



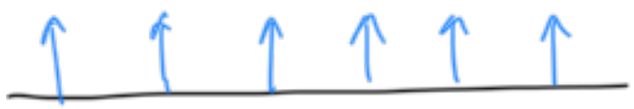
En <u>paramétrisation normale</u> :

$$\vec{t_P} = f'(s)$$

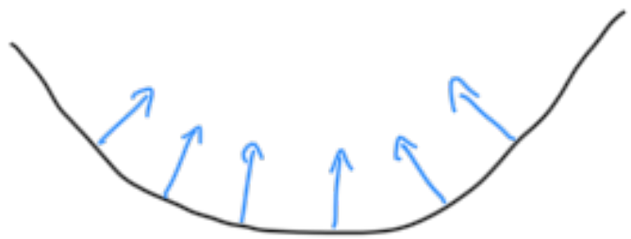
En <u>paramétrisation quelconque</u> :

$$\vec{t_P} = \frac{f'(t)}{\|f'(t)\|} \longleftarrow \text{speed}$$

COURBURE



Non courbé



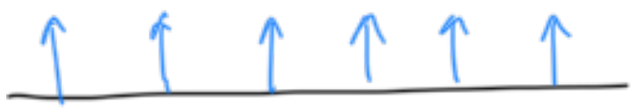
Moyennement courbé

Fortement courbé

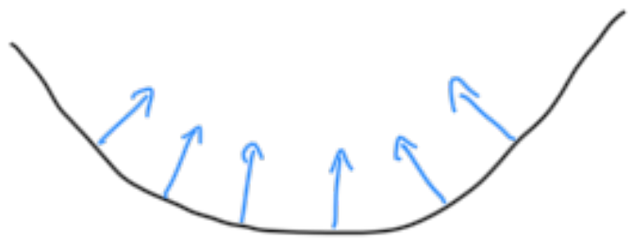
Normale/
tangente
constante

Variation
violente de la
normale/
tangente

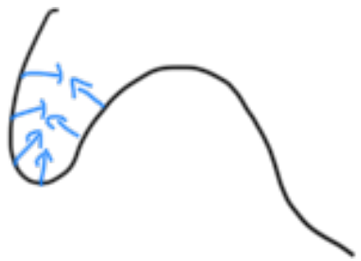
COURBURE



Non courbé



Moyennement courbé



Fortement courbé

Normale/ tangente constante

Variation violente de la normale/ tangente

Courbure ↔ variation de la normale

Moyennant la paramétrisation …

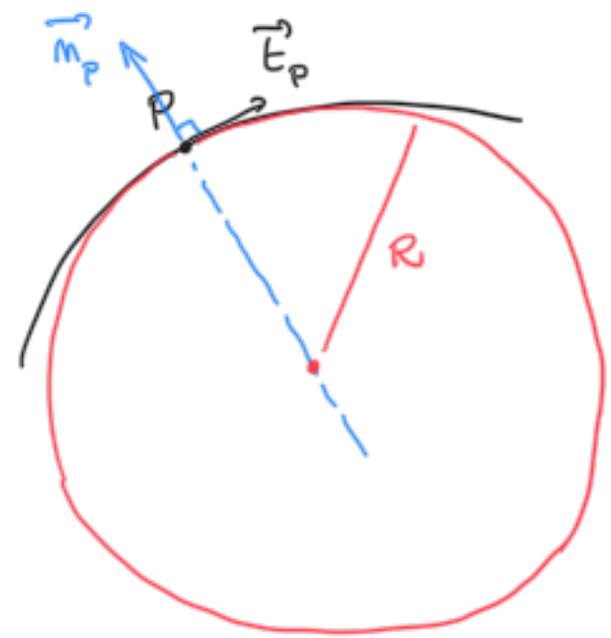
COURBURE

En paramétrisation normale, on montre que :

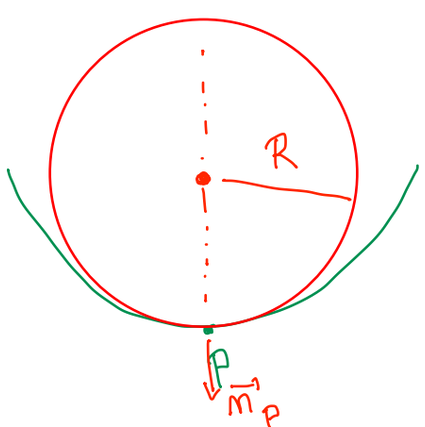
$$\boxed{\frac{\mathrm{d}\overrightarrow{t_P}}{\mathrm{d}s} = \frac{1}{R} \cdot \overrightarrow{n_P} = k_P \cdot \overrightarrow{n_P}}$$

R rayon de courbure

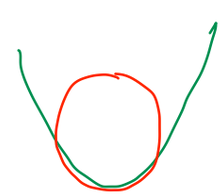
courbure au point P



$P = f(s)$

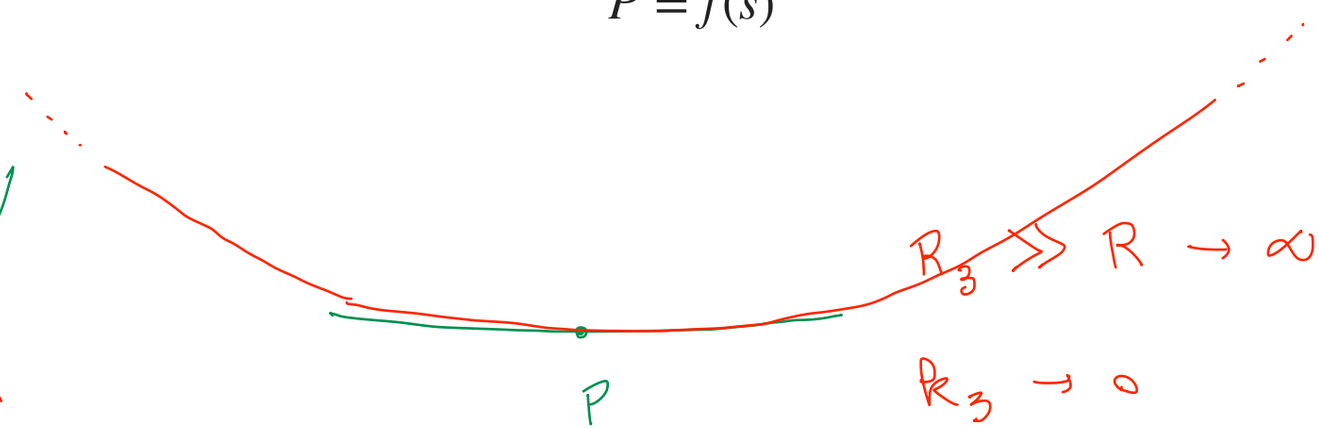


$\frac{1}{R} = k_P$



$R_2 < R$

$k_2 > k_P$



$R_3 \gg R \to \infty$

$k_3 \to 0$

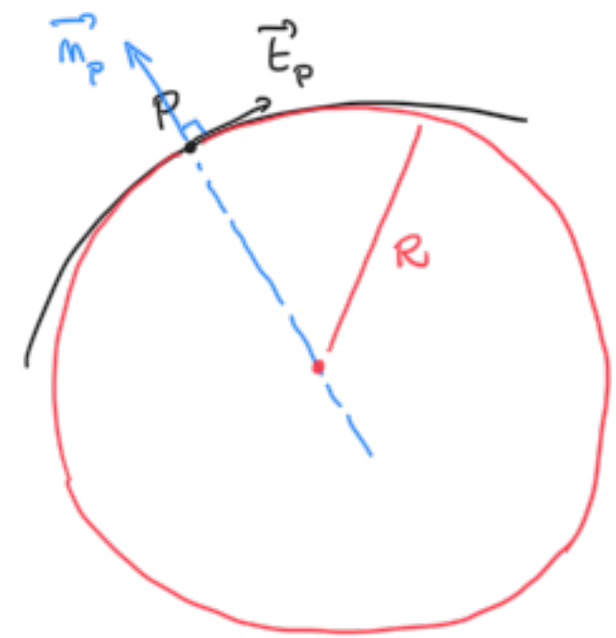
COURBURE

En paramétrisation normale, on montre que :

$$\boxed{\frac{\mathrm{d}\overrightarrow{t_P}}{\mathrm{d}s} = \frac{1}{R} \cdot \overrightarrow{n_P} = k_P \cdot \overrightarrow{n_P}}$$

R rayon de courbure

courbure au point P



$P = f(s)$

En paramétrisation quelconque :

$$\boxed{k_P \cdot \overrightarrow{n_P} = \frac{f'(t) \wedge f''(t)}{\|f'(t)\|^3}}$$

TRIÈDRE DE FRENET (COURBES GAUCHES)

On pose le <u>vecteur de la binormale</u> :
$\overrightarrow{b_P} = \overrightarrow{t_P} \wedge \overrightarrow{n_P}$

$$\frac{\mathrm{d}\overrightarrow{t_P}}{\mathrm{d}s} = \frac{1}{R} \cdot \overrightarrow{n_P} = k_P \cdot \overrightarrow{n_P}$$

R rayon de courbure

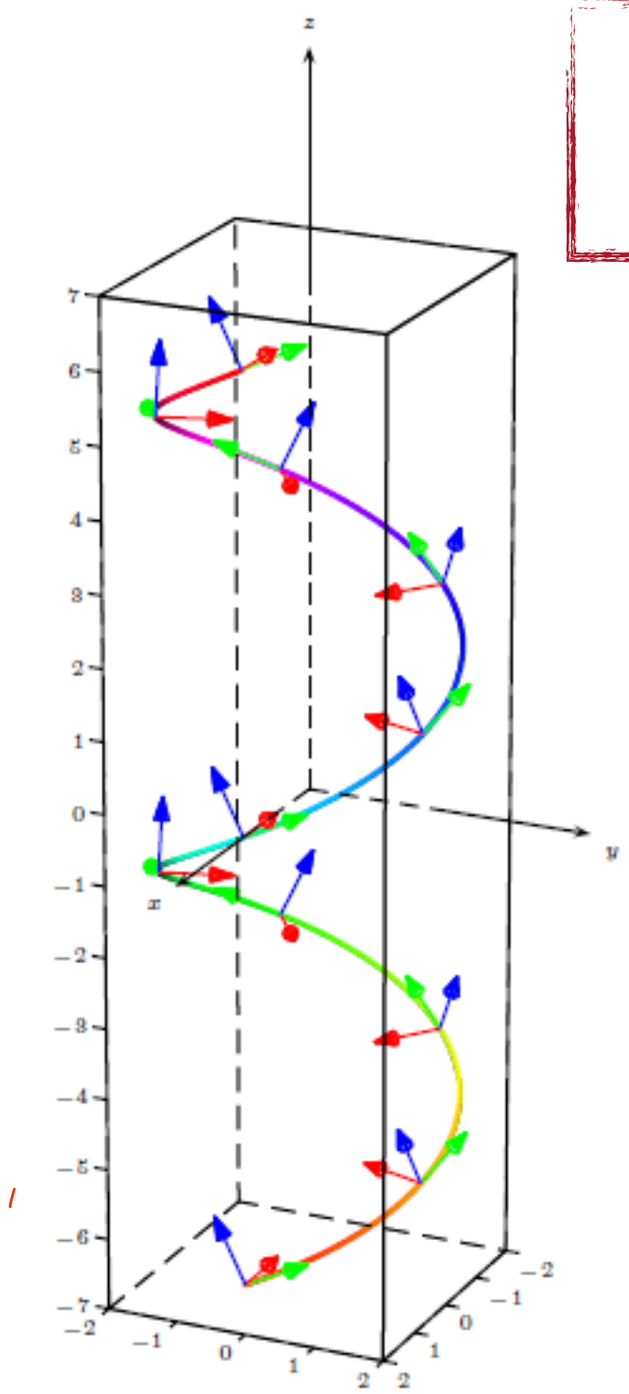
courbure

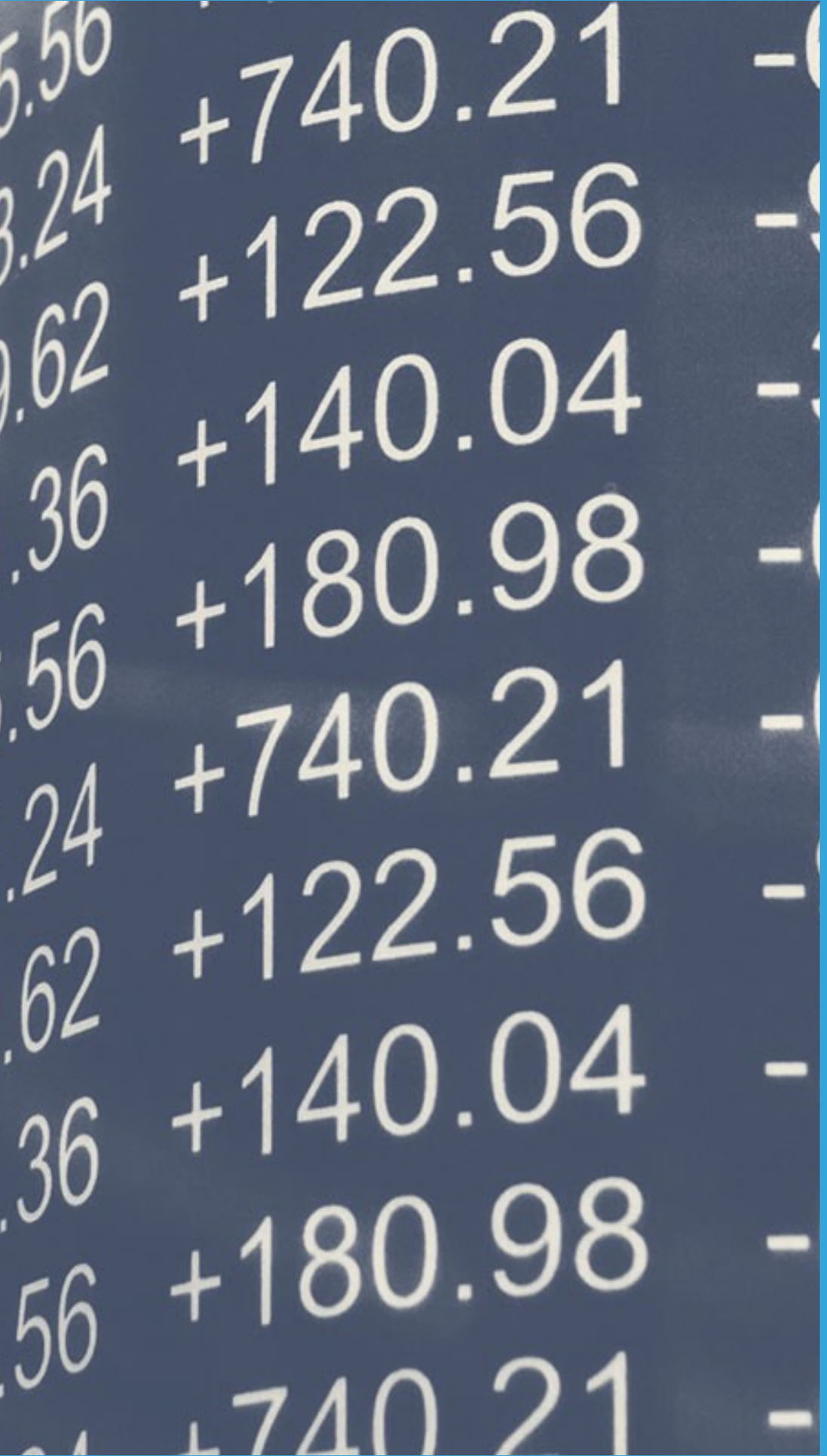
$$\frac{\mathrm{d}\overrightarrow{b_P}}{\mathrm{d}s} = \frac{1}{\tau_P} \cdot \overrightarrow{n_P}$$

rayon de torsion

$\frac{1}{\tau_P}$ torsion

<u>Repère de Frénet</u> :
$(P, \overrightarrow{t_P}, \overrightarrow{n_P}, \overrightarrow{b_P})$





BONUS : FORMES QUADRATIQUES

FORMES QUADRATIQUES ET MATRICES

Forme quadratique sur $\mathbb{R}^n$:

Φ : polynôme de degré 2 exactement (n variables)

$\updownarrow$

Codable par une matrice Ω de taille $n \times n$:

$$\Phi(X) = X^t \cdot \Omega \cdot X$$

ex : $\mathbb{R}^3$

$$\Phi(x,y,z) = x^2 - y^2 + 2z^2 + \underline{\underline{xy}} - 4yz$$

$\cancel{2x}$ $\cancel{x}$ $\downarrow$ $\searrow$ yx

xy

$$\Omega = \begin{array}{c|ccc} & x & y & z \\ \hline x & 1 & 1/2 & 0 \\ y & 1/2 & -1 & -2 \\ z & 0 & -2 & 2 \end{array}$$

symmetrical

square terms

FORMES QUADRATIQUES ET MATRICES

Forme quadratique sur $\mathbb{R}^n$:

Φ : polynôme de degré 2 exactement (n variables)

Sur $\mathbb{R}^3$:

$$\Phi(x, y, z) = x^2 - y^2 + 2z^2 - xy + 2yz$$

$\updownarrow$

Codable par une matrice Ω de taille $n \times n$:

$$\Phi(X) = X^t \cdot \Omega \cdot X$$

FORMES QUADRATIQUES ET MATRICES

Forme quadratique sur $\mathbb{R}^n$:

Φ : polynôme de degré 2 exactement (n variables)

Sur $\mathbb{R}^3$:

$$\Phi(x, y, z) = \boxed{x^2 - y^2 + 2z^2} \boxed{- xy} + \boxed{2yz}$$

Codable par une matrice Ω de taille $n \times n$:

$$\Phi(X) = X^t \cdot \Omega \cdot X$$

$$\Omega = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

GÉOMÉTRIE DES SURFACES PARAMÉTRIQUES

Cadre : surfaces paramétriques

$f : \mathbb{R}^2 \to \mathbb{R}^3$

Implicit surf: $g : \mathbb{R}^3 \to \mathbb{R}$

surf $\longleftrightarrow$ $\{X \in \mathbb{R}^3 \,;\, g(X) = 0\}$

∇g

g is $\mathcal{C}^2$

th. of local inversion

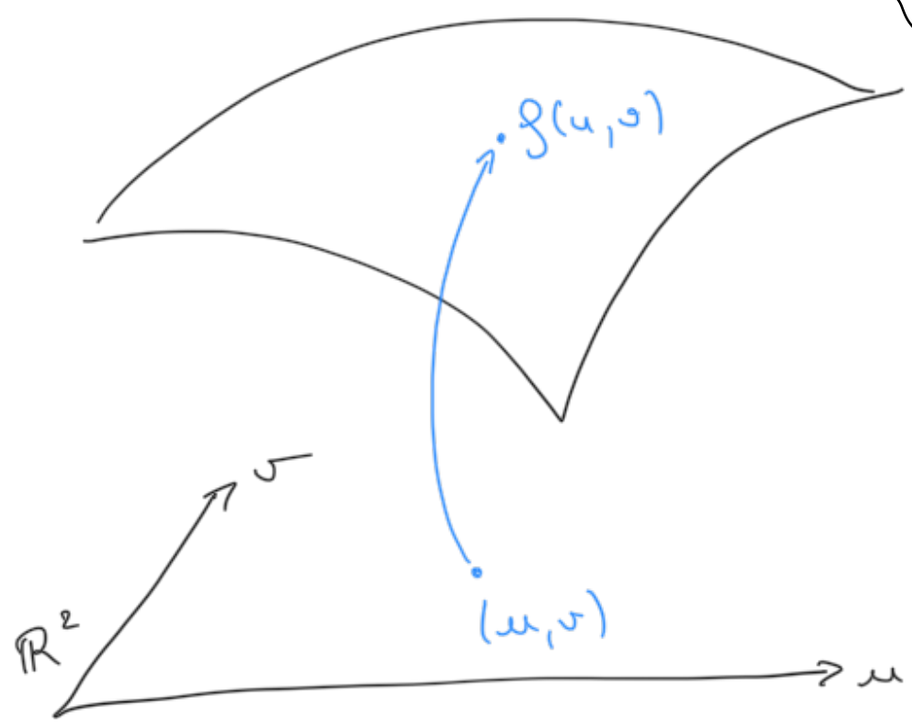
locally around each $X_0 \in$ surface

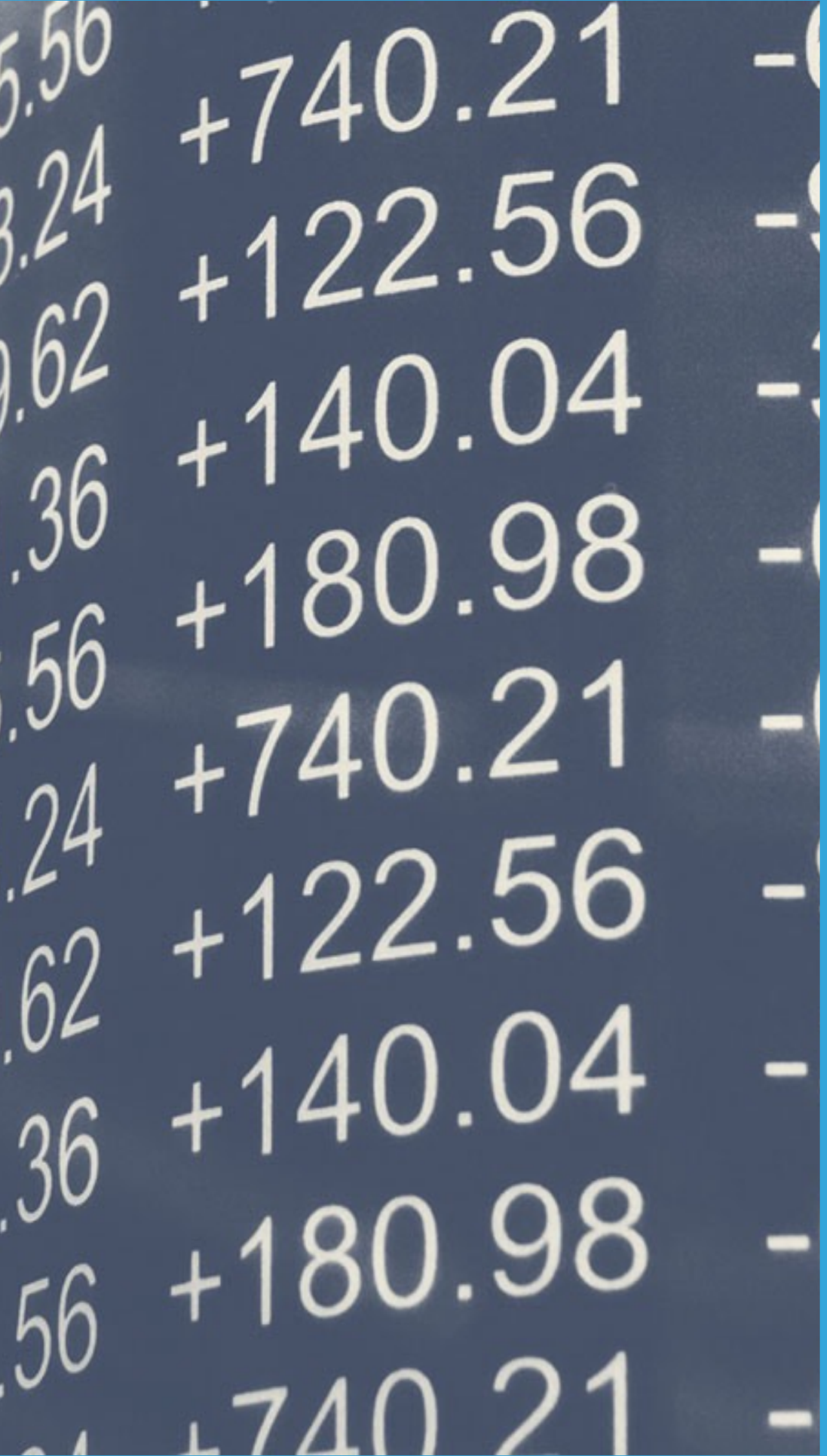
$\exists$ parametric model

$f : \mathbb{R}^2 \to \mathbb{R}^3$

$f(0) = X_0$

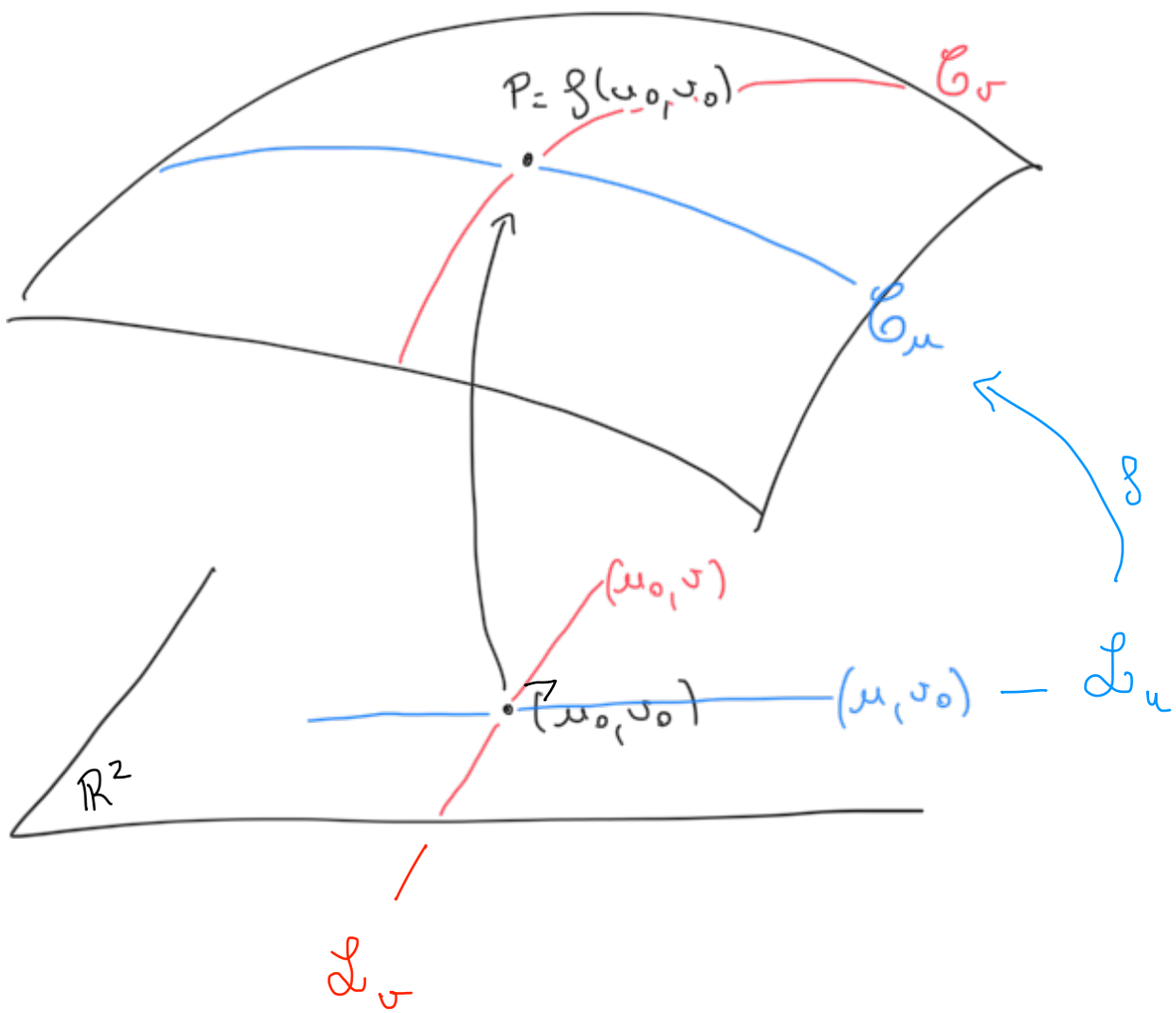
$\frac{\partial f}{\partial x_i}$





NORMALE

On obtient naturellement deux courbes sur la surface :



En fixant v_0 :

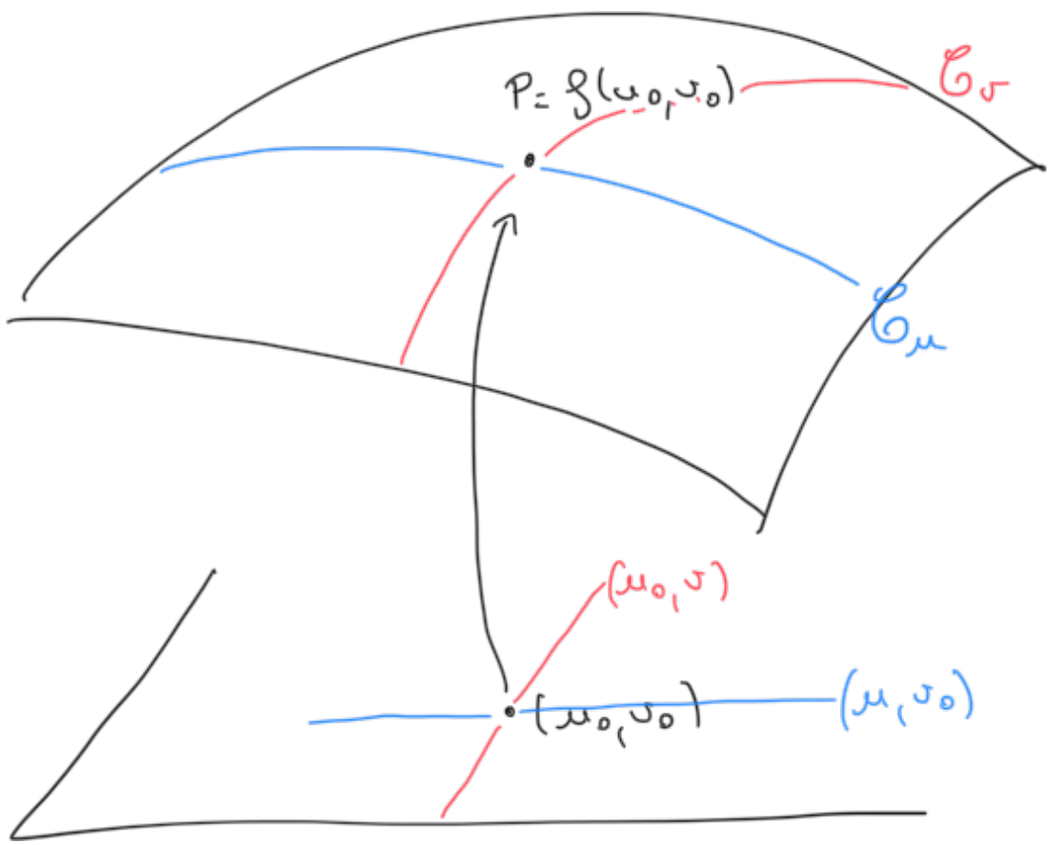
$\mathbb{R} \to \mathbb{R}^3$ surface

$$\mathscr{C}_u : u \mapsto f(u, v_0)$$

En fixant u_0 :

$$\mathscr{C}_v : v \mapsto f(u_0, v)$$

On obtient naturellement deux courbes sur la surface :



En fixant v_0 :

$$\mathbb{R} \to \mathbb{R}^3$$

$$\mathscr{C}_u : u \mapsto f(u, v_0) \xrightarrow{(\ )'} \frac{\partial f}{\partial u}$$

En fixant u_0 :

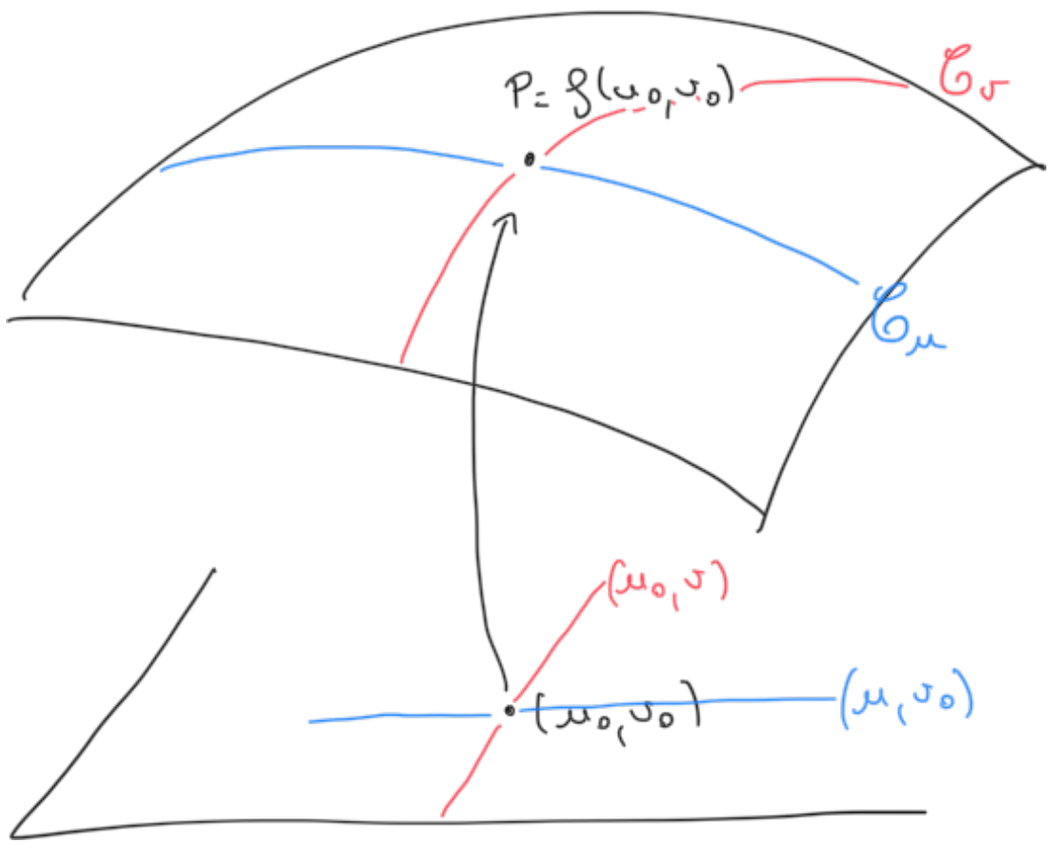
$$\mathscr{C}_v : v \mapsto f(u_0, v) \xrightarrow{(\ )'} \frac{\partial f}{\partial v}$$

tangents

On connaît la tangente de courbes …

≠ dérivative

On obtient naturellement deux courbes sur la surface :



En fixant v_0 :

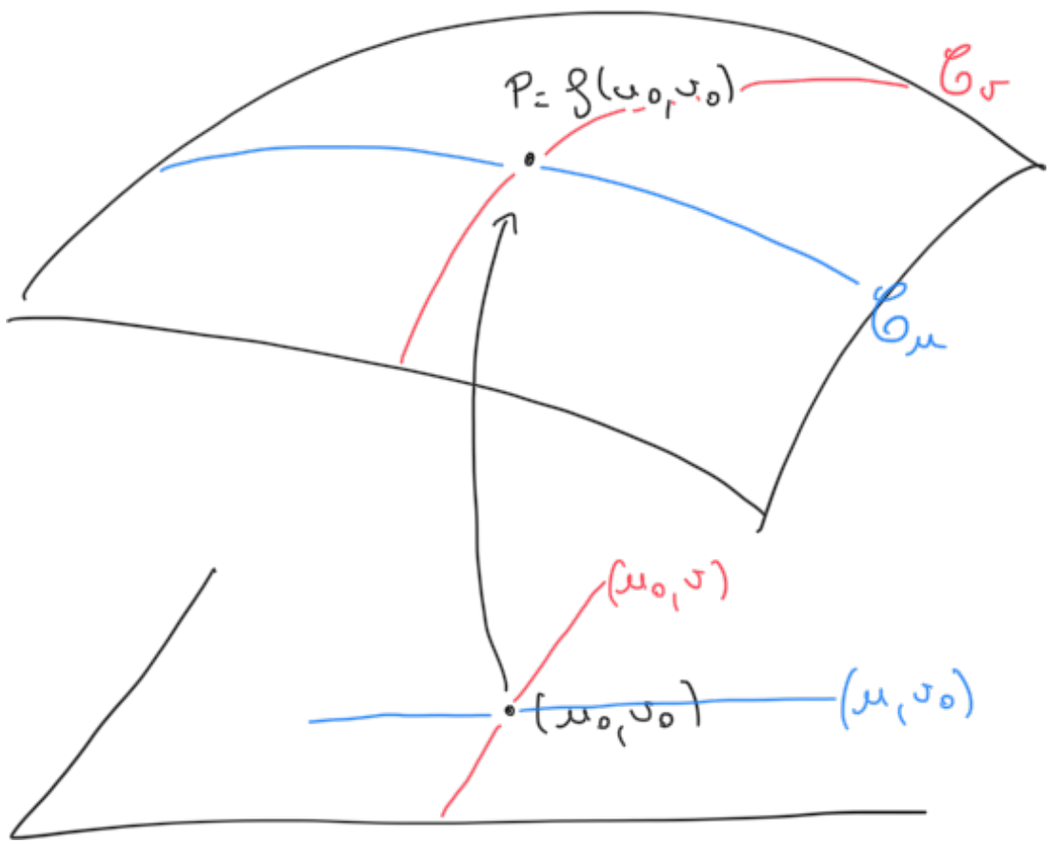
$$\mathscr{C}_u : u \mapsto f(u, v_0)$$

En fixant u_0 :

$$\mathscr{C}_v : v \mapsto f(u_0, v)$$

$(f(u, v_0))'$ $(f(u_0, v))'$

On obtient naturellement deux courbes sur la surface :



En fixant v_0 :

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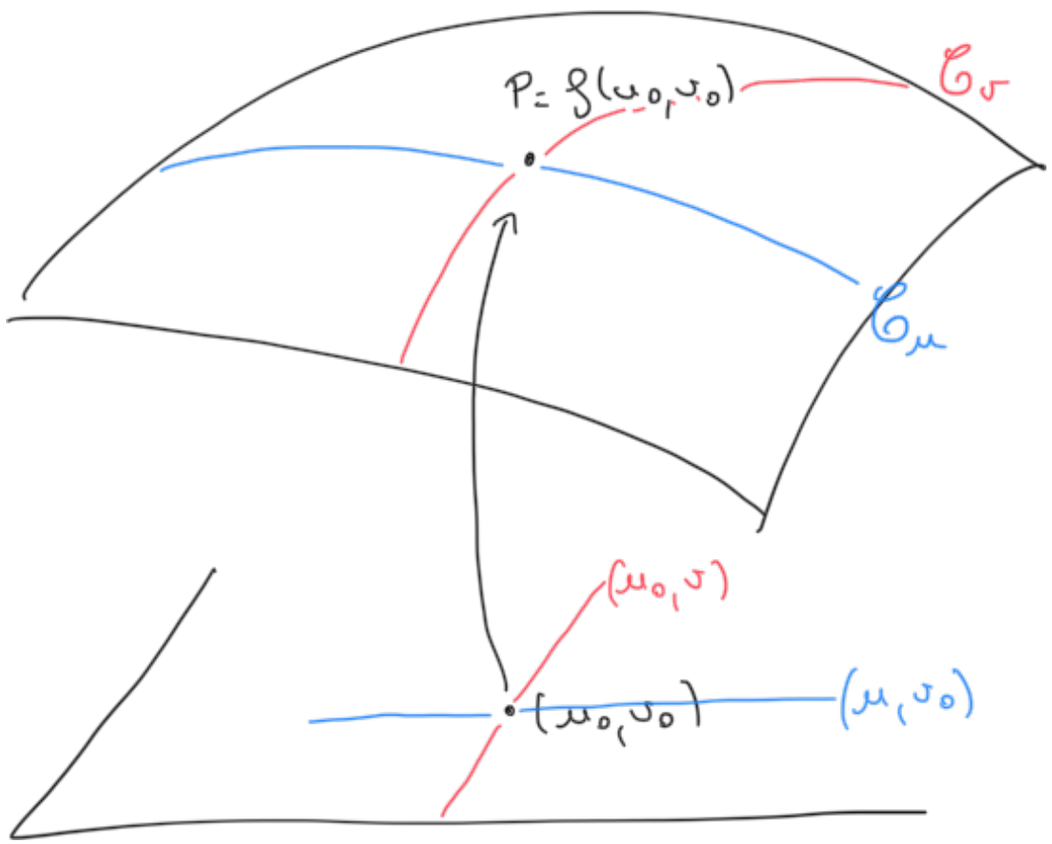
$$\mathscr{C}_v : v \mapsto f(u_0, v)$$

On connaît la tangente de courbes …

$(f(u, v_0))'$ $(f(u_0, v))'$

On obtient naturellement deux courbes sur la surface :



En fixant v_0 :

$$\mathscr{C}_u : u \mapsto f(u, v_0)$$

En fixant u_0 :

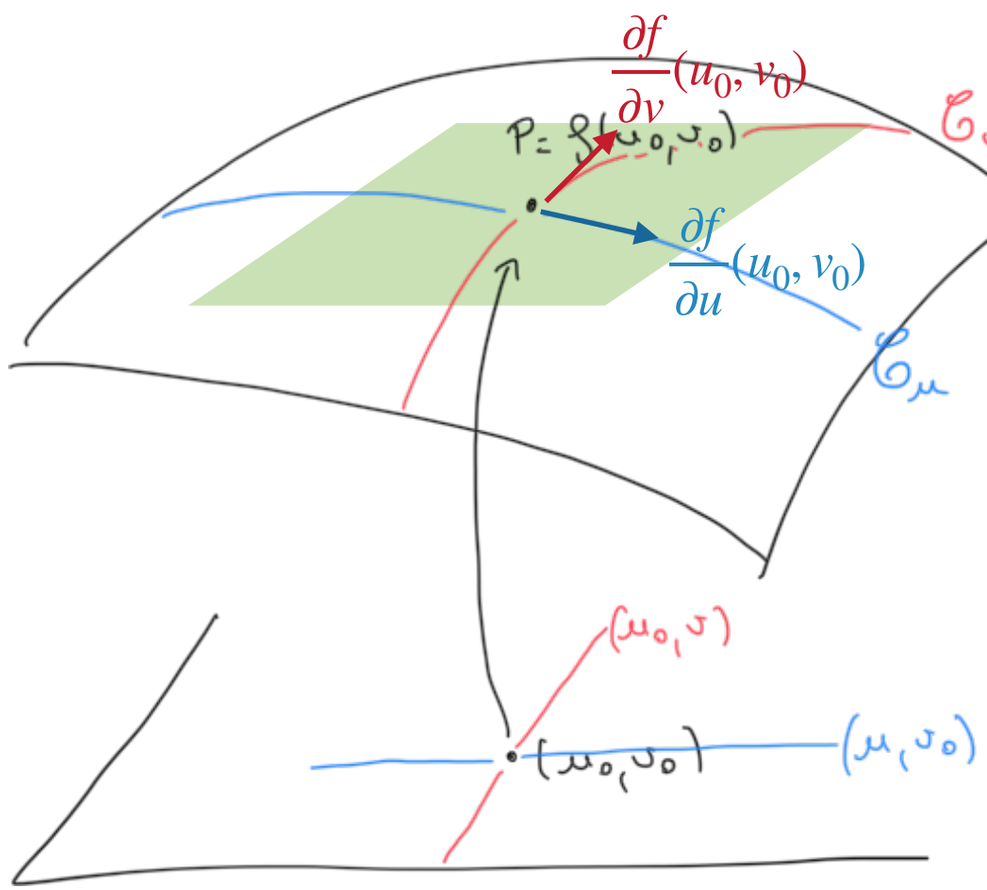
$$\mathscr{C}_v : v \mapsto f(u_0, v)$$

On connaît la tangente de courbes …

$$(f(u, v_0))' = \frac{\partial f}{\partial u}(u_0, v_0)$$

$$(f(u_0, v))' = \frac{\partial f}{\partial v}(u_0, v_0)$$

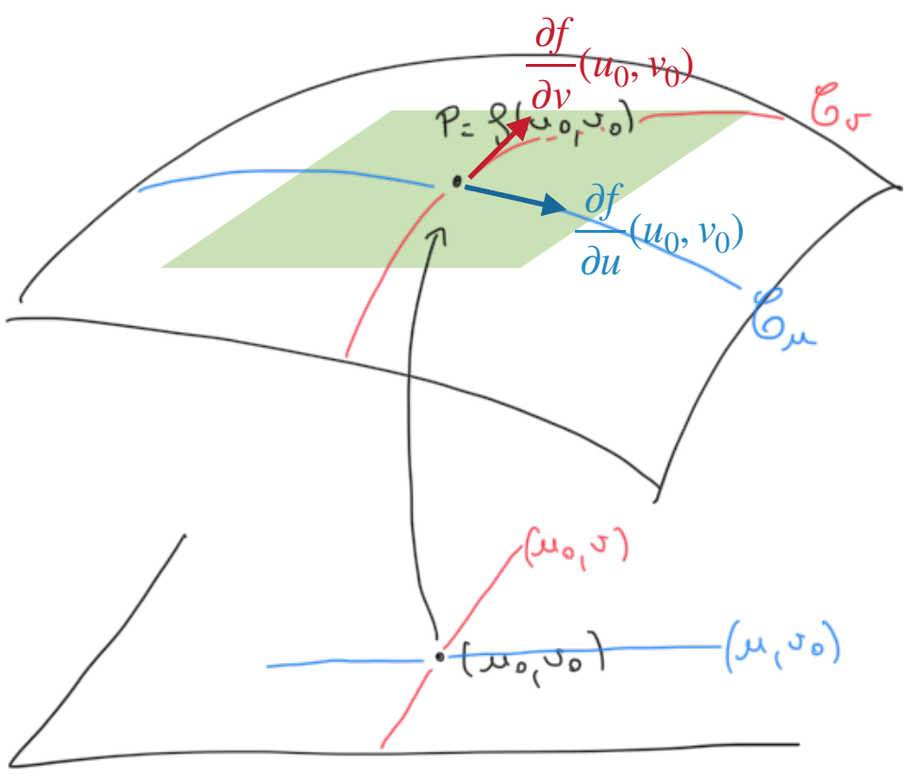
On obtient naturellement deux courbes sur la surface :



Si f différentiable en (u_0, v_0) :

- Tangentes aux courbes sur la surface forment un plan
- $\frac{\partial f}{\partial u}(u_0, v_0), \frac{\partial f}{\partial v}(u_0, v_0)$ en forment une base si …

On obtient naturellement deux courbes sur la surface :

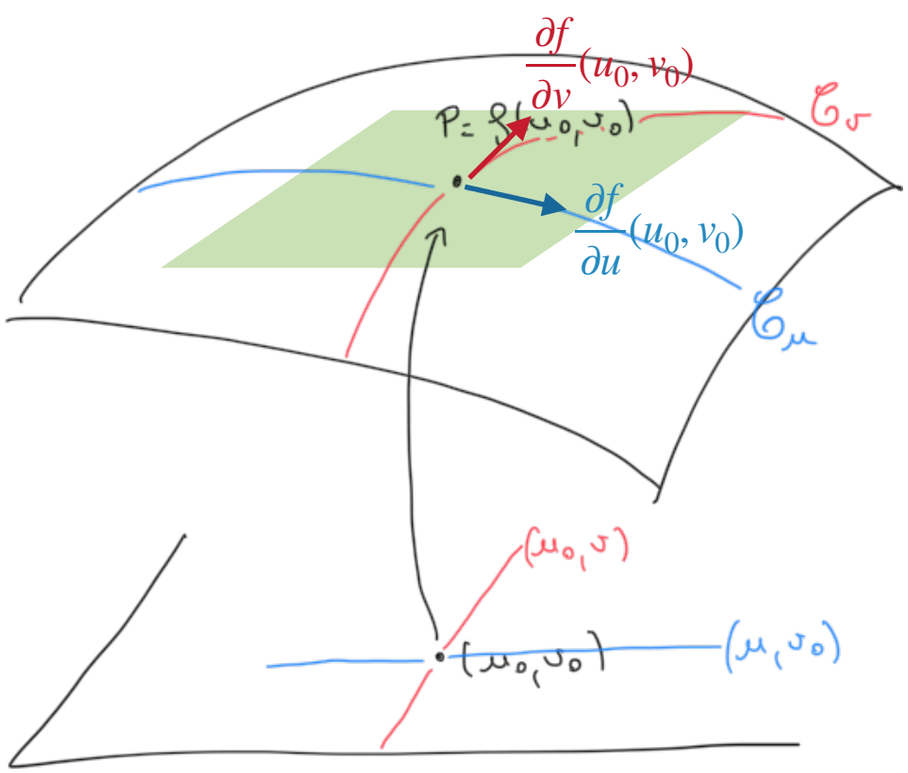


singular otherwise

$P = f(u_0, v_0)$ point régulier si :

- $\frac{\partial f}{\partial u}(u_0, v_0), \frac{\partial f}{\partial v}(u_0, v_0)$ libres, ie.
- $\frac{\partial f}{\partial u}(u_0, v_0) \wedge \frac{\partial f}{\partial v}(u_0, v_0) \neq 0$

On obtient naturellement deux courbes sur la surface :



$P = f(u_0, v_0)$ point régulier si :

- $\frac{\partial f}{\partial u}(u_0, v_0), \frac{\partial f}{\partial v}(u_0, v_0)$ libres, ie.
- $\frac{\partial f}{\partial u}(u_0, v_0) \wedge \frac{\partial f}{\partial v}(u_0, v_0) \neq 0$

Prop :
si $P = f(u_0, v_0)$ régulier, le plan tangent T_P en P a pour repère :

$$(P, \frac{\partial f}{\partial u}(u_0, v_0), \frac{\partial f}{\partial v}(u_0, v_0))$$

Son vecteur normal :

$$\overrightarrow{n_P} = \frac{\frac{\partial f}{\partial u}(u_0, v_0) \wedge \frac{\partial f}{\partial v}(u_0, v_0)}{\|\frac{\partial f}{\partial u}(u_0, v_0) \wedge \frac{\partial f}{\partial v}(u_0, v_0)\|}$$

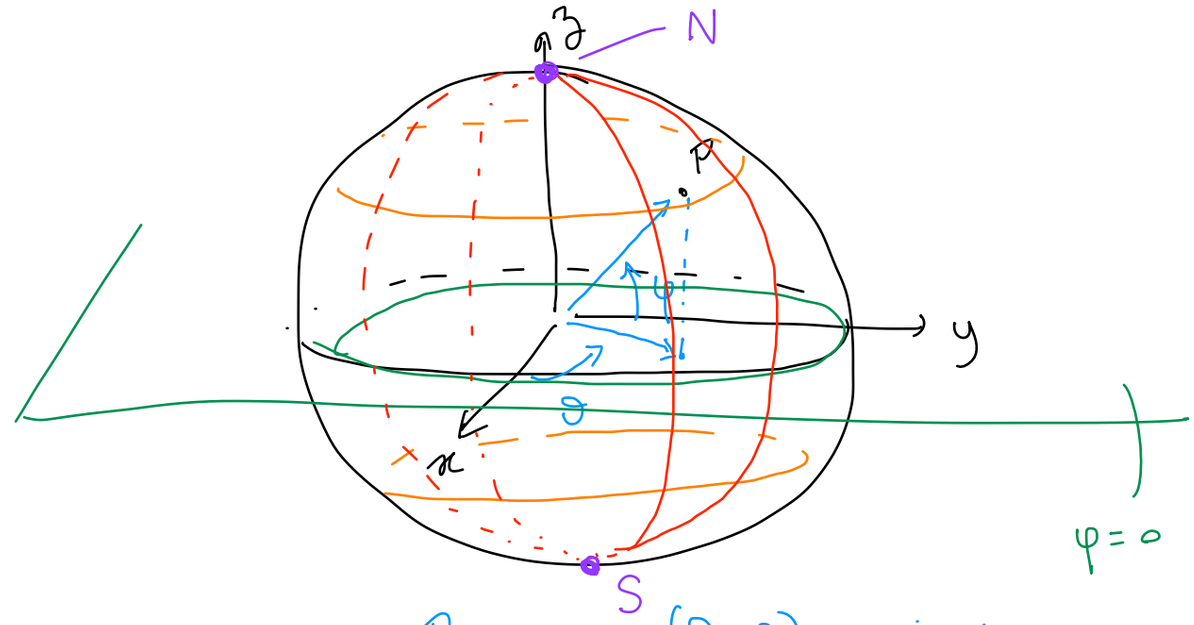
Param. eq. sphere $\longleftrightarrow$ spherical coords

ϑ, φ

$$\pm \begin{cases} x = \overbrace{\cos\vartheta \underbrace{\cos\varphi}_{1}}^{f(\vartheta,\varphi)} \\ y = \sin\vartheta \underbrace{\cos\varphi}_{1} \\ z = \sin\varphi \quad \text{—} \quad 0 \end{cases}$$

circle of rad 1

for $\varphi = 0$



$P \longleftrightarrow (\vartheta, \varphi)$ unique

φ set $\rightsquigarrow$ ϑ changes

↳ circles (latitude)

$\neq$ one another

ϑ set $\rightsquigarrow$ φ changes

↳ circles (longitude)

cross at N / S poles

$\downarrow$

$$N : \begin{cases} \varphi = \pi/2 \\ \vartheta : \text{any !} \end{cases}$$

$$S : \begin{cases} \varphi = -\pi/2 \\ \vartheta : \text{any} \end{cases}$$

$$\frac{\partial \rho}{\partial \vartheta} = \begin{pmatrix} -\sin\vartheta \cos\varphi \\ \cos\vartheta \cos\varphi \\ 0 \end{pmatrix}$$

$$\frac{\partial \rho}{\partial \varphi} = \begin{pmatrix} -\cos\vartheta \sin\varphi \\ -\sin\vartheta \sin\varphi \\ \cos\varphi \end{pmatrix}$$

at N — $\varphi = \pi/2$ | ϑ : ...

$$\frac{\partial \rho}{\partial \vartheta}(N) = \vec{0}$$

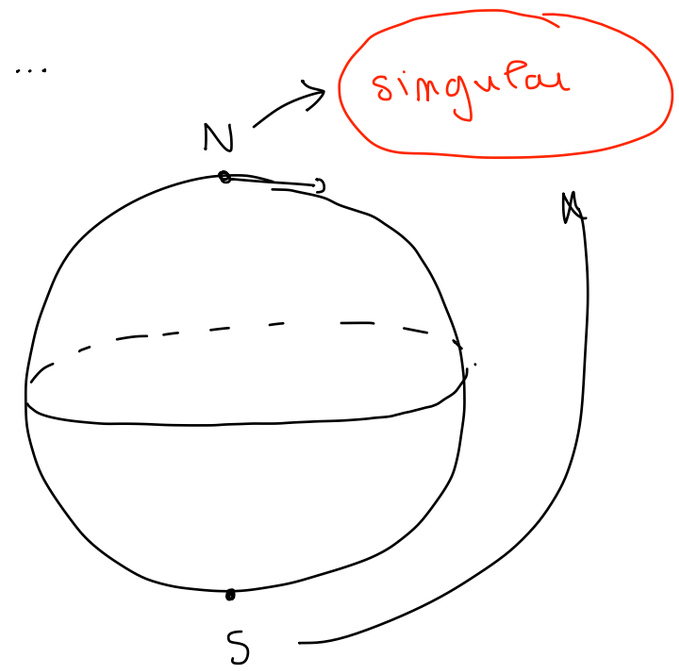
$$\frac{\partial \rho}{\partial \varphi}(N) = \begin{pmatrix} -\cos\vartheta \\ -\sin\vartheta \\ 0 \end{pmatrix} \text{ — } xoy$$

1 vect ...

$$\rho(\vartheta, \varphi) \quad \begin{vmatrix} \cos\vartheta \cos\varphi \\ \sin\vartheta \cos\varphi \\ \sin\varphi \end{vmatrix}$$

implicit

$$x^2 + y^2 + z^2 - 1 = 0$$



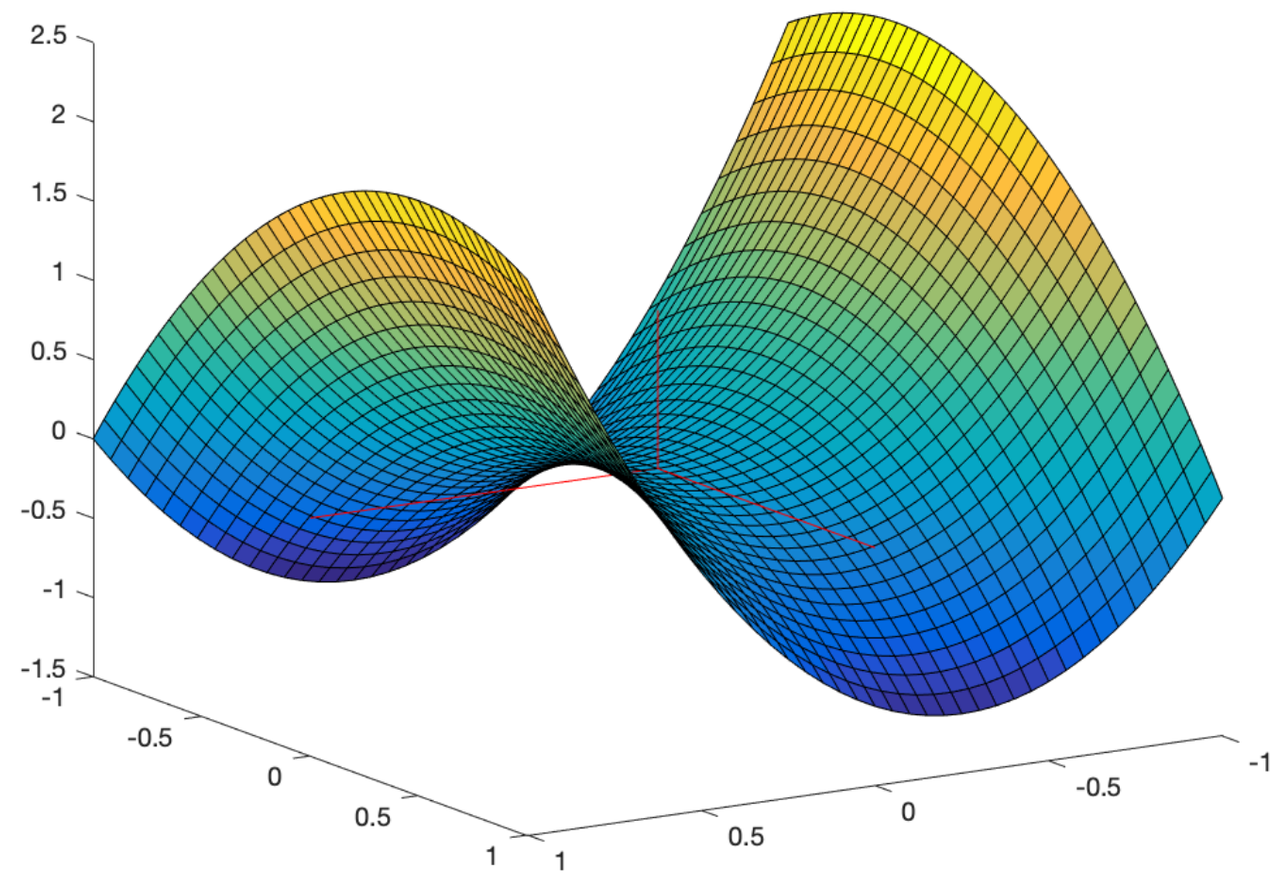
$$f : (u,v) \longmapsto \begin{pmatrix} u \\ v \\ 2u^2 - v^2 + uv \end{pmatrix}$$

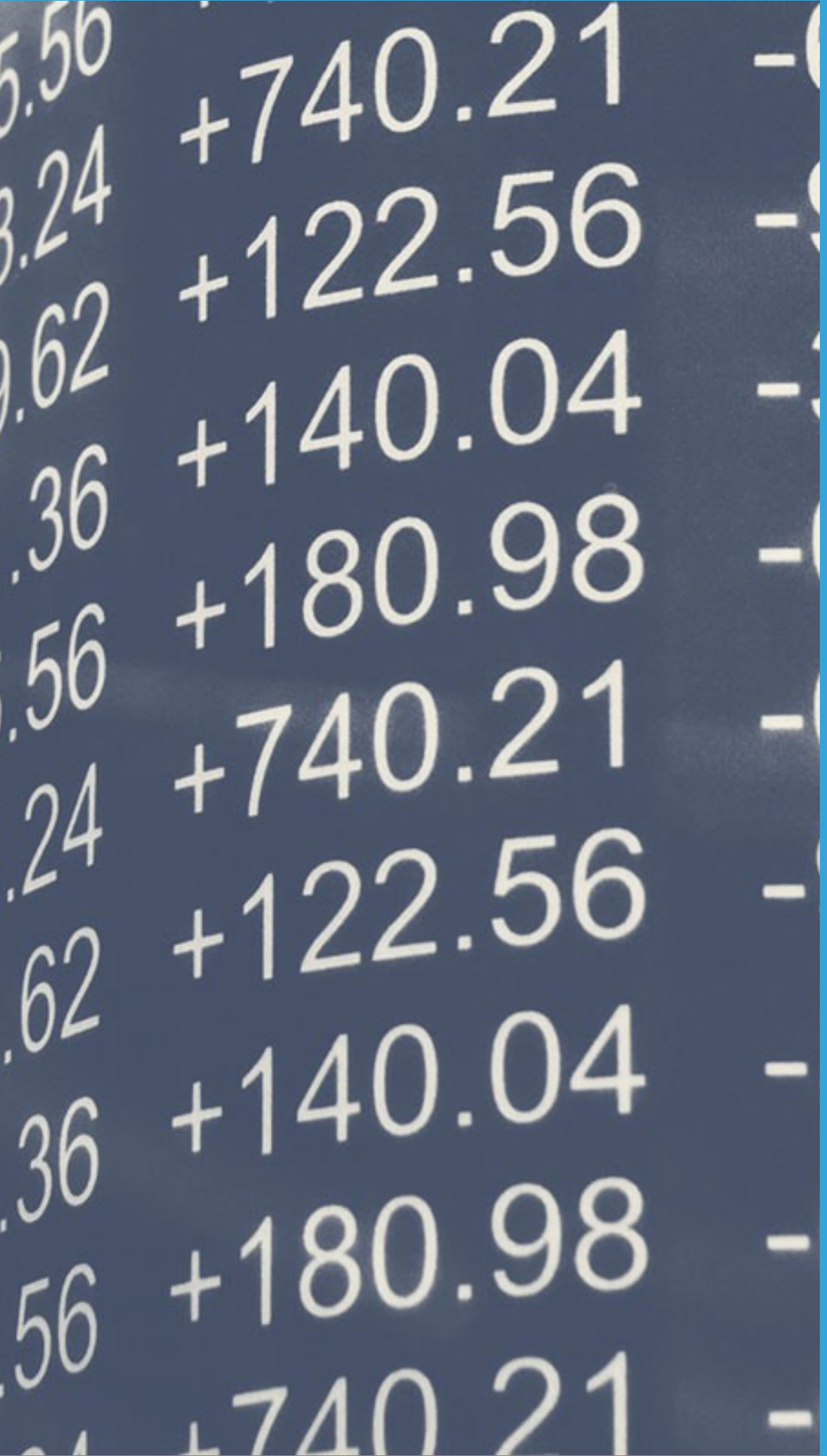
Soit $P = f(0,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\frac{\partial f}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ 2u+v \end{pmatrix} \rightsquigarrow \frac{\partial f}{\partial u}(0,0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{\partial f}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ -v+u \end{pmatrix} \rightsquigarrow \frac{\partial f}{\partial v}(0,0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Puis $\overrightarrow{n_P} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$





COURBURE

$$f : (u,v) \longmapsto \begin{pmatrix} u \\ v \\ 2u^2 - v^2 + uv \end{pmatrix}$$

Soit $P = f(0,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

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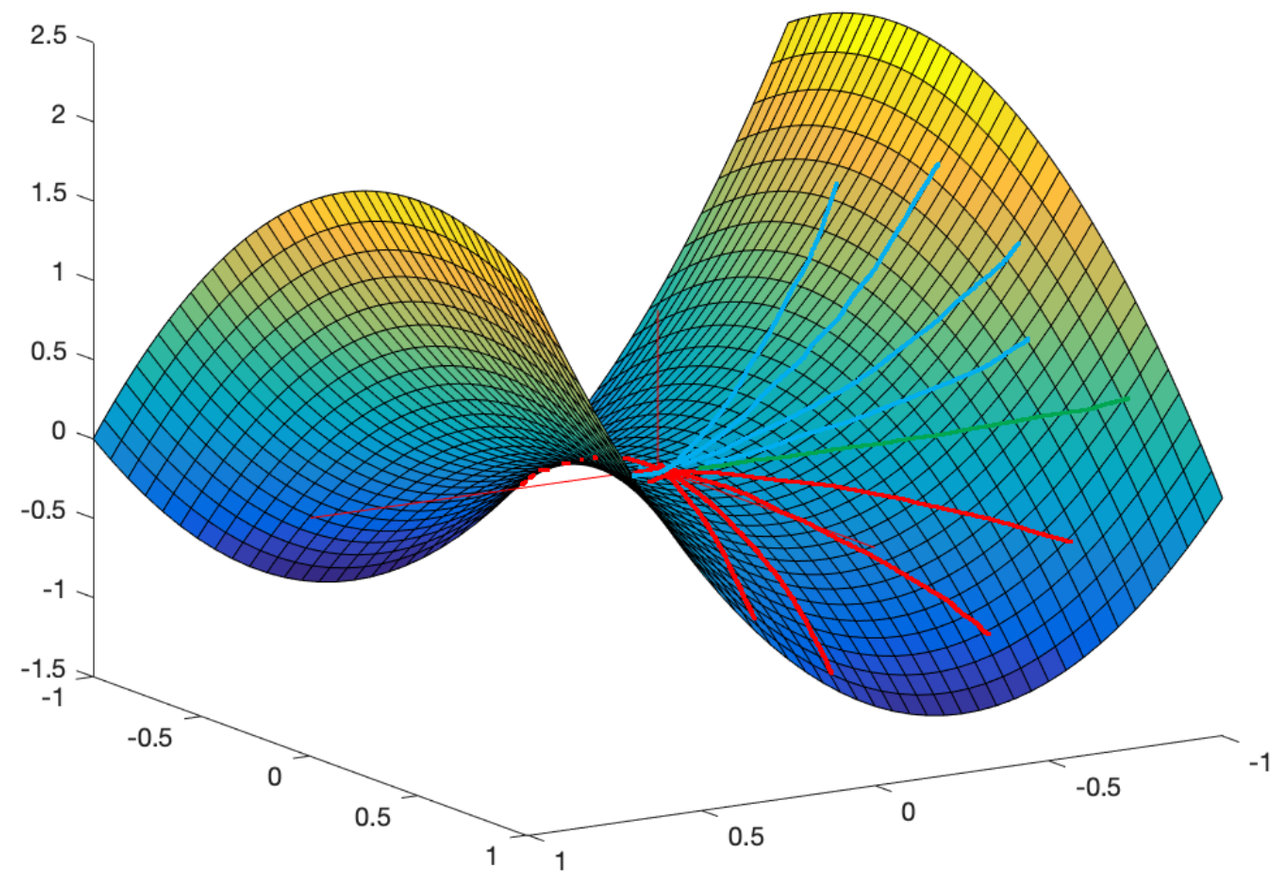
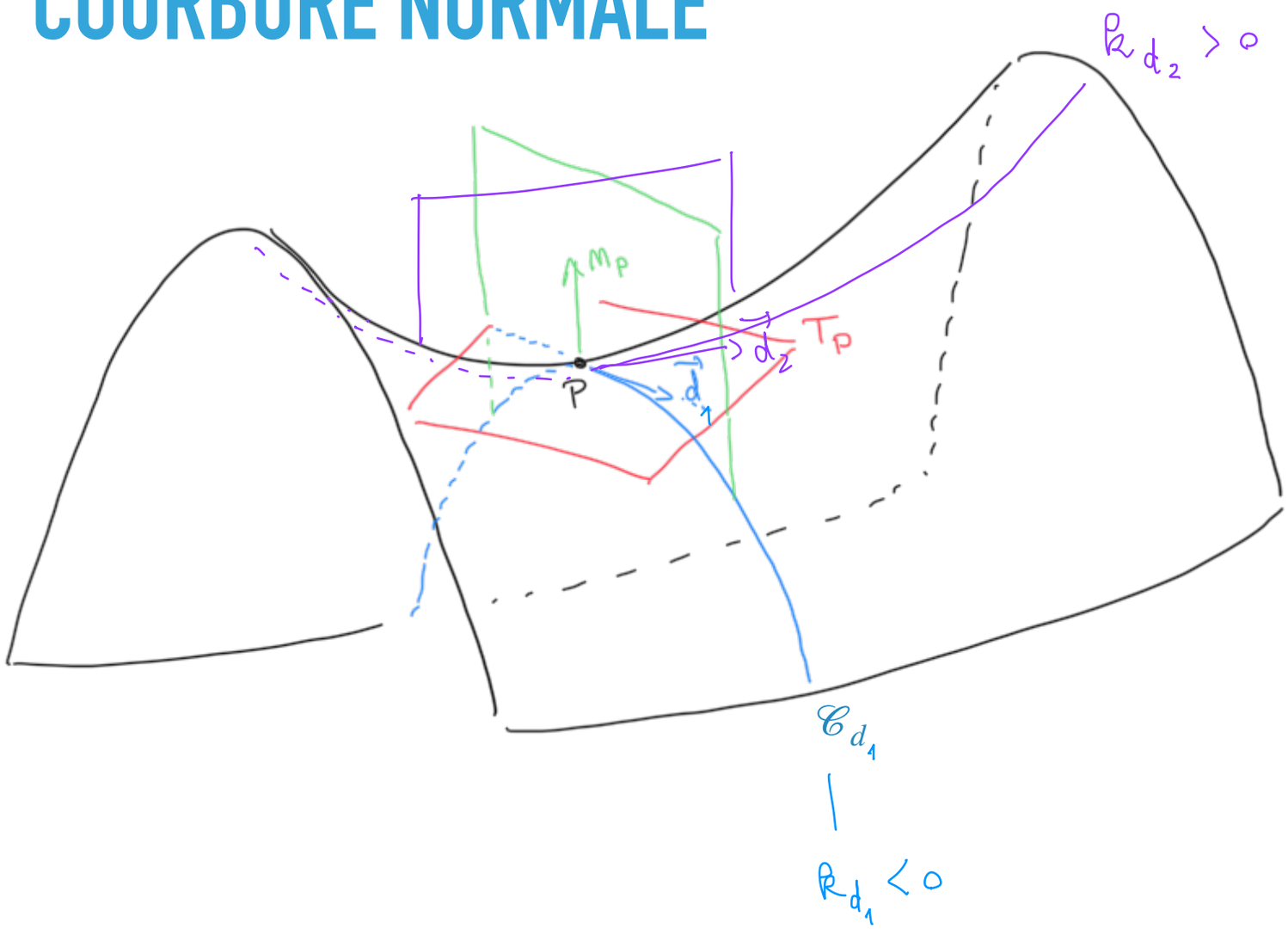
Puis $\overrightarrow{n_P} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



Pas <u>une</u> mais des courbures
(selon direction)

COURBURE NORMALE



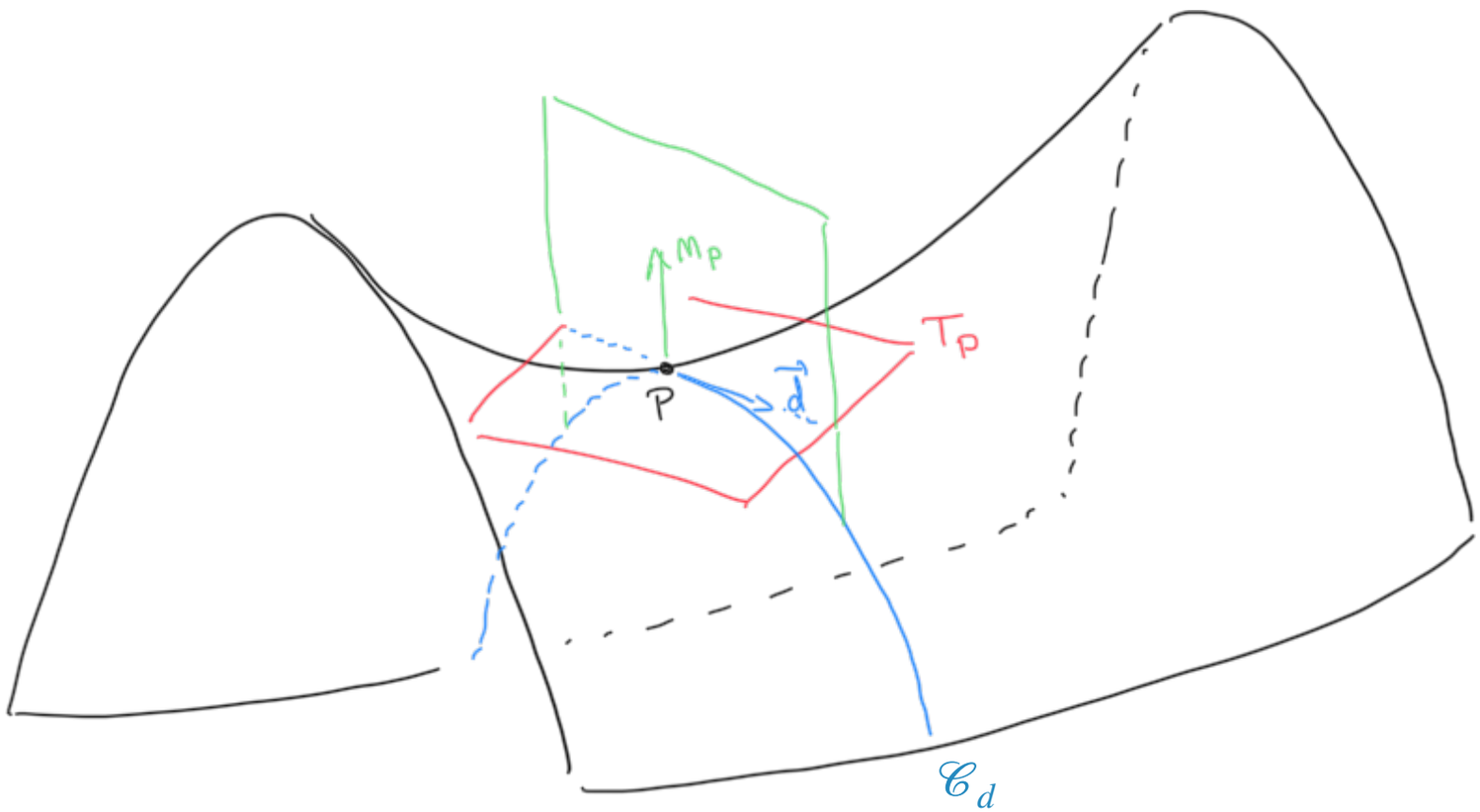
Courbure normale dans la direction $\vec{d}$:

$$k_{\vec{d}}$$

$$\vec{d}_1 \qquad \vec{d}_2$$

$$k_{\vec{d}_1} \overset{?}{=} k_{\vec{d}_2}$$

COURBURE NORMALE



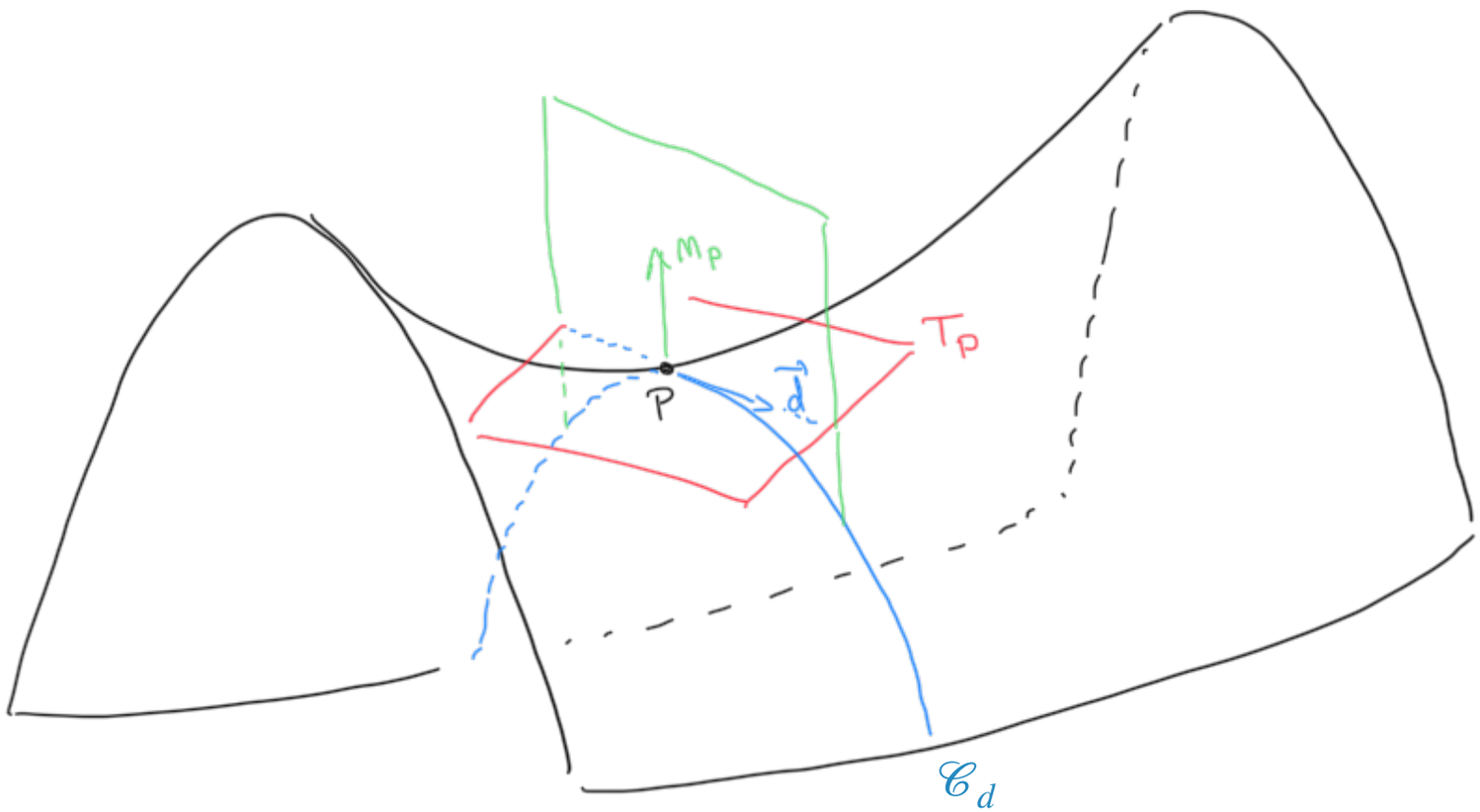
Courbure normale dans la direction $\vec{d}$:

$$k_{\vec{d}}$$

$$k_{\vec{d}} \Longleftrightarrow \frac{\text{Dérivées secondes de } f}{\text{Dérivées premières de } f}$$

renormalisée par

COURBURE NORMALE



Courbure normale dans la direction $\vec{d}$:

$$k_{\vec{d}}$$

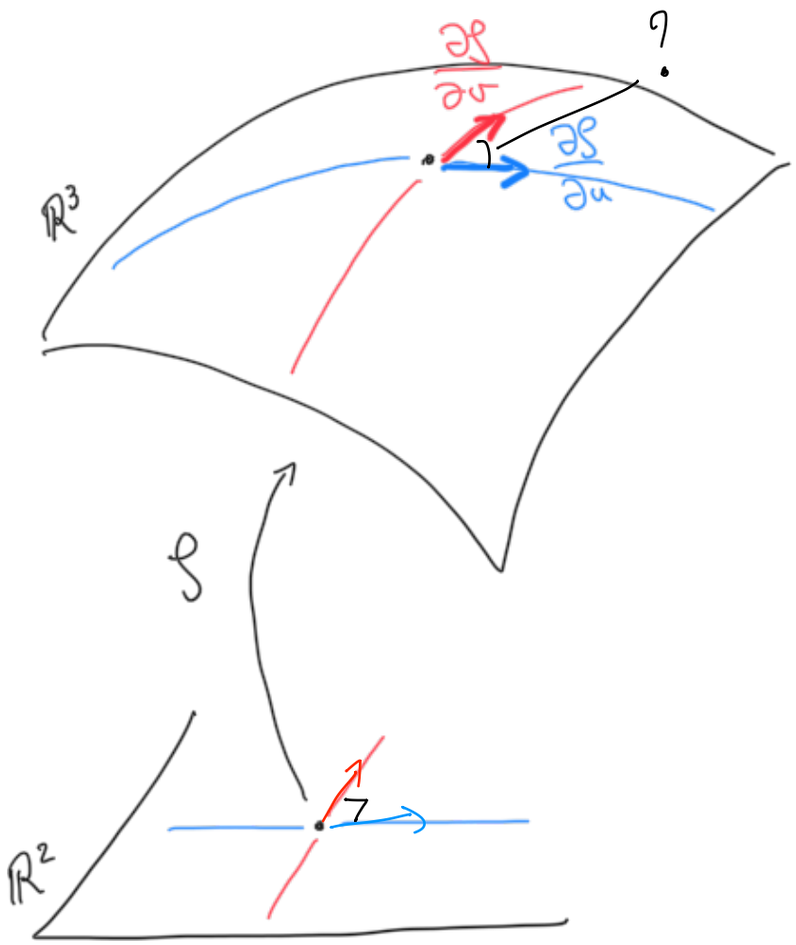
$k_{\vec{d}}$ ⟷ Dérivées secondes de f / renormalisée par Dérivées premières de f

Deuxième forme fondamentale

Première forme fondamentale

PREMIÈRE FORME FONDAMENTALE

lack of orthogonality between $\frac{\partial \rho}{\partial u} / \frac{\partial \rho}{\partial v}$



Première forme fondamentale au point P :

$$I_P = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

$$E = \langle \frac{\partial f}{\partial u}, \frac{\partial f}{\partial u} \rangle = \left\| \frac{\partial f}{\partial u} \right\|^2$$

$$F = \langle \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \rangle$$

$$G = \langle \frac{\partial f}{\partial v}, \frac{\partial f}{\partial v} \rangle = \left\| \frac{\partial f}{\partial v} \right\|^2$$

measures distortions on angles / dist the ρ mapping

Tenseur métrique de la surface

if $\frac{\partial \rho}{\partial u} \perp \frac{\partial \rho}{\partial v}$

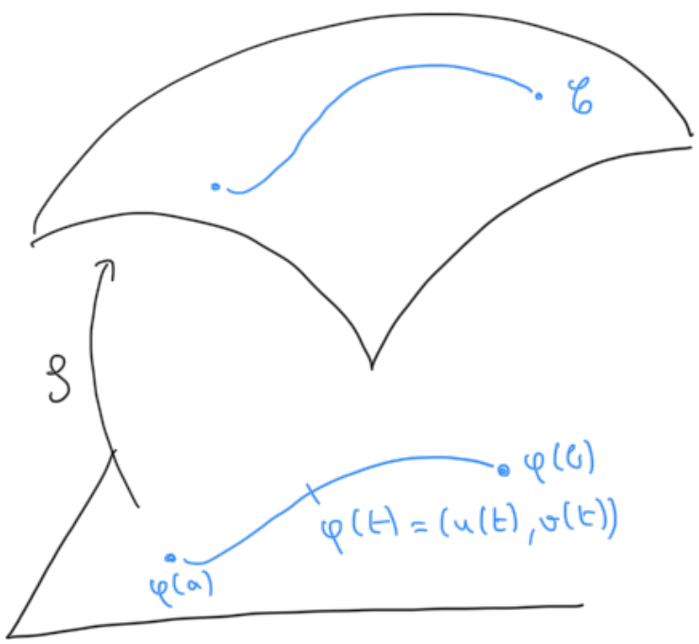
↳ $I_P \rightarrow$ diag

if $\begin{cases} \frac{\partial \rho}{\partial u} \perp \frac{\partial \rho}{\partial v} \\ \text{and } \frac{\partial \rho}{\partial u}, \frac{\partial \rho}{\partial v} \text{ unit vect} \end{cases}$

↳ $I_P = Id$

PREMIÈRE FORME FONDAMENTALE

Surfaces

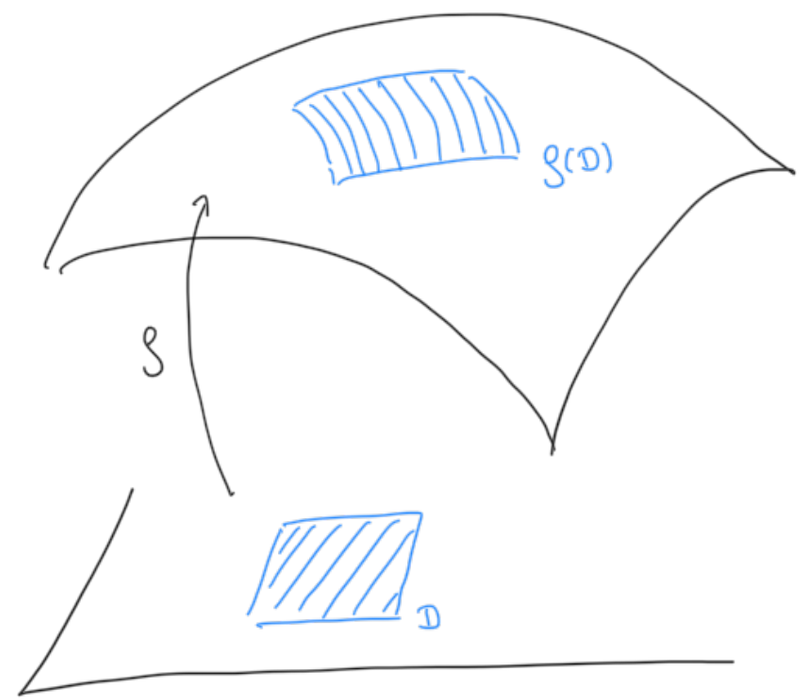


f « déforme » les longueurs

I_P mesure cette déformation

$$l(\mathscr{C}) = \int_a^b \sqrt{\left((u'(t), v'(t)) * I_P * (u'(t), v'(t))^t\right)} \, \mathrm{d}t$$

Aires



$$\mathrm{Aire}(f(D)) = \iint_D \sqrt{\det(I_P)} \mathrm{d}u \mathrm{d}v$$

Manifold (variété)

Surface ?

↓

general def. of what a surf. is

↓

generalized in dim. n

(closed) surface....

object — $X \subseteq \mathbb{R}^n$ such that

$\forall P \in X$

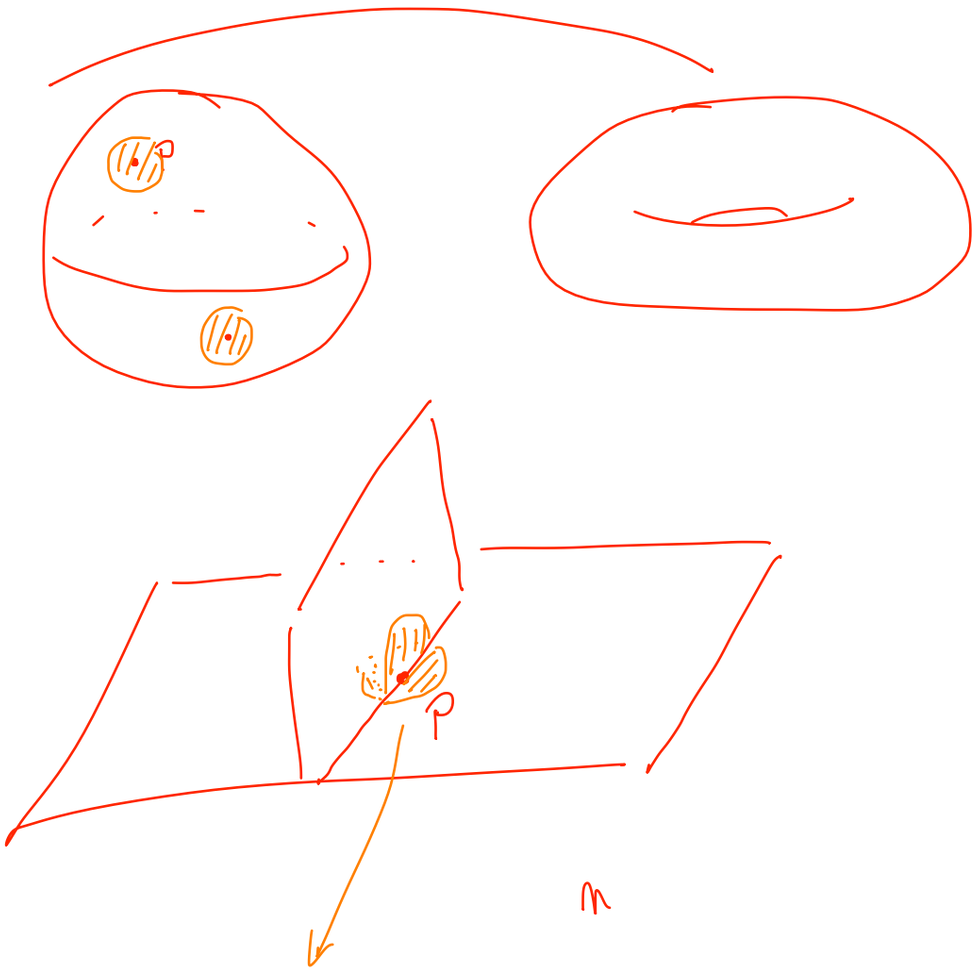
$\exists V$ neighborhood of P

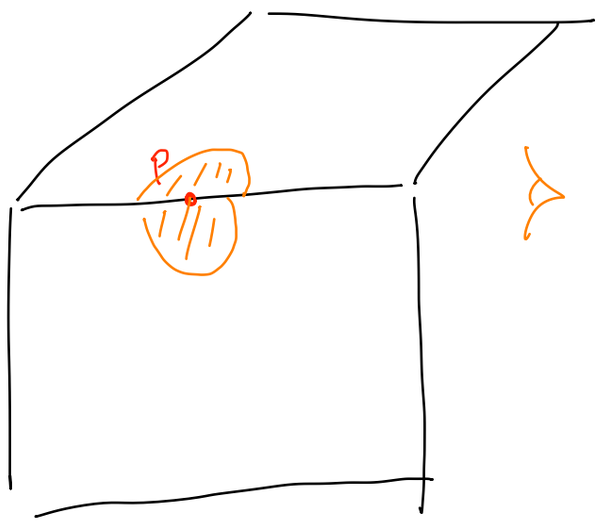
continuously deform. ← $\mathbb{R}$

disc. (dim. 2)

↓

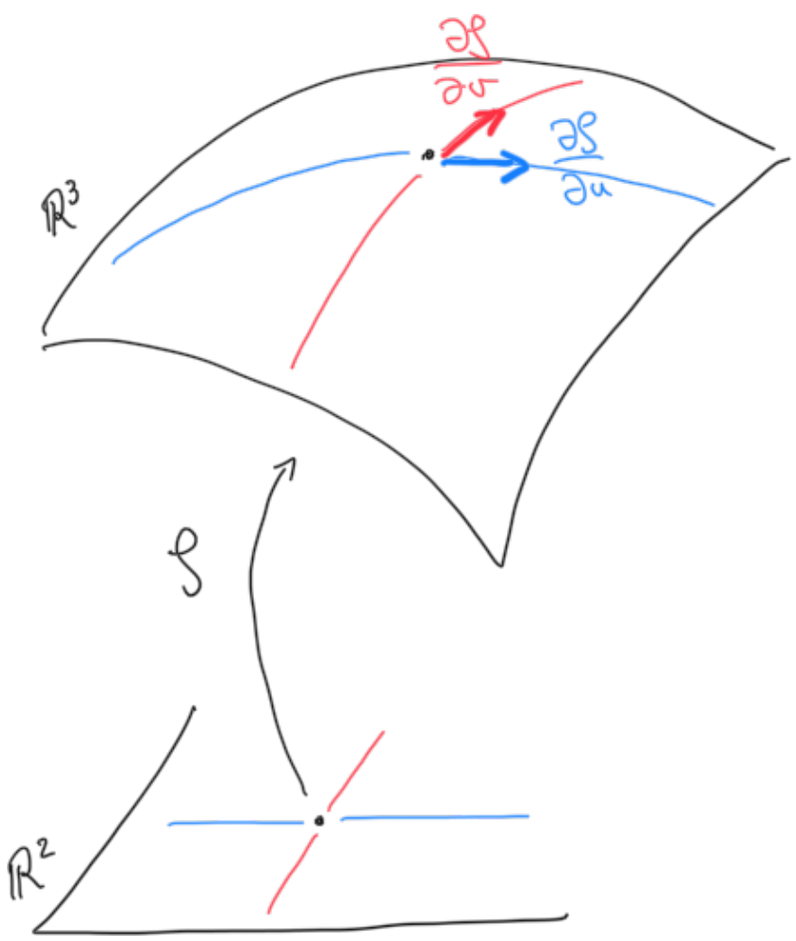
manifold dim. 2





manifold

DEUXIÈME FORME FONDAMENTALE



Deuxième forme fondamentale au point P :

$$II_P = \begin{pmatrix} l & m \\ m & n \end{pmatrix} \qquad l = \left\langle \frac{\partial^2 f}{\partial u^2}, \overrightarrow{n_P} \right\rangle$$

$$m = \left\langle \frac{\partial^2 f}{\partial u \partial v}, \overrightarrow{n_P} \right\rangle$$

$$n = \left\langle \frac{\partial^2 f}{\partial v^2}, \overrightarrow{n_P} \right\rangle$$

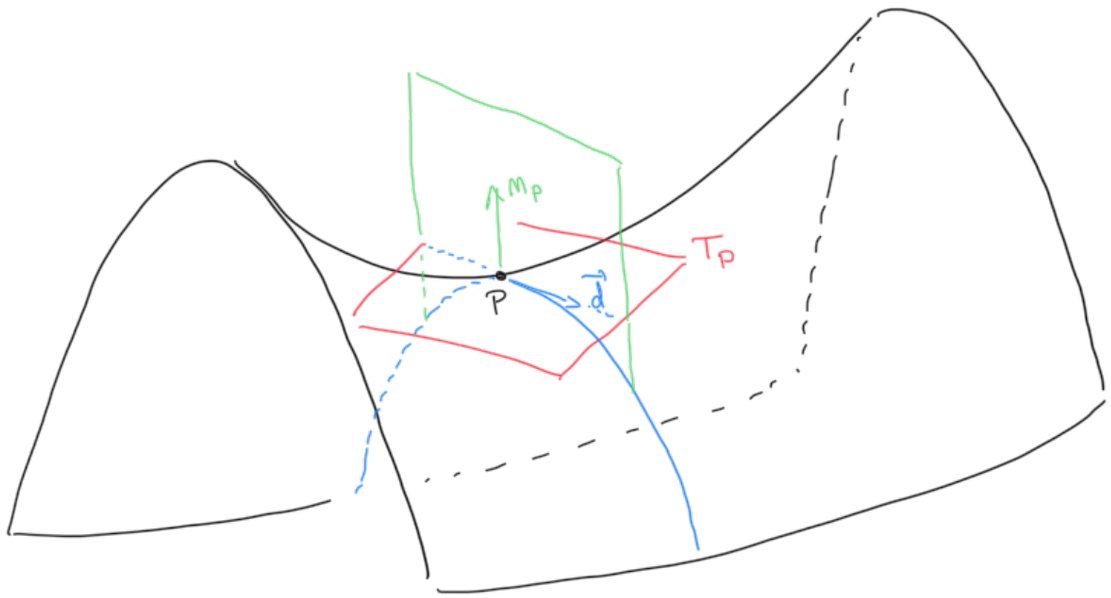
$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$\frac{\partial f}{\partial ..}$ f' : "

$\partial^2 f / \partial ..$ f'' :

$\frac{\partial^2 f}{\partial .. \partial ...} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

COURBURE



La courbure normale varie comme un polynôme de degré 2 en fonction de d :

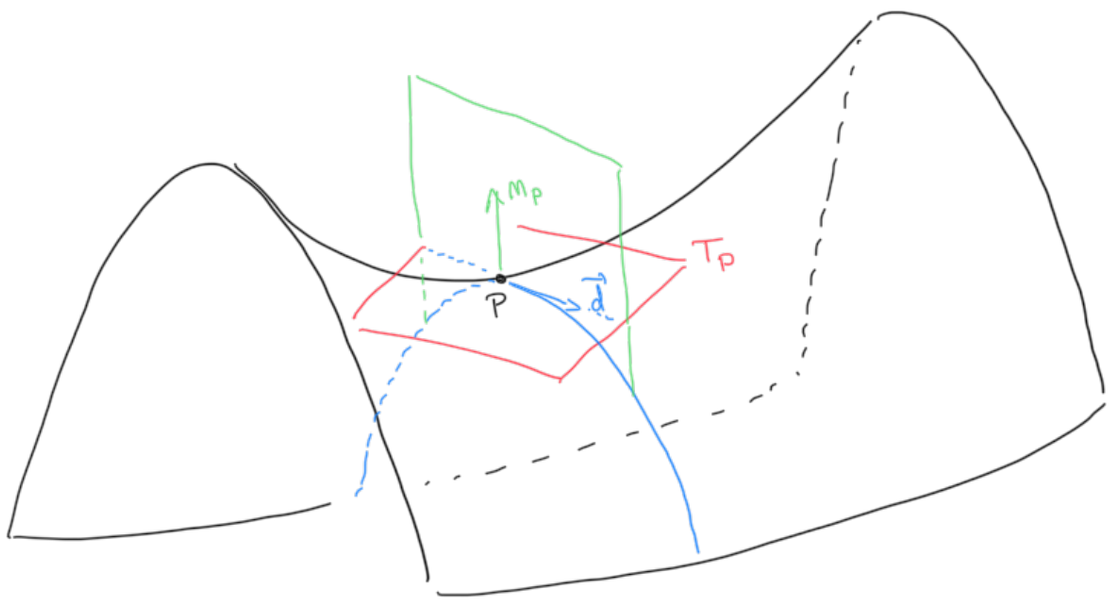
$$\boxed{k_{\vec{d}} = \vec{d}^t \cdot \kappa_P \cdot \vec{d}}$$

Tenseur de courbure au point P

$$\boxed{\kappa_P = I_P^{-1} \times II_P}$$

Dans le repère $(P, \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v})$ du plan tangent

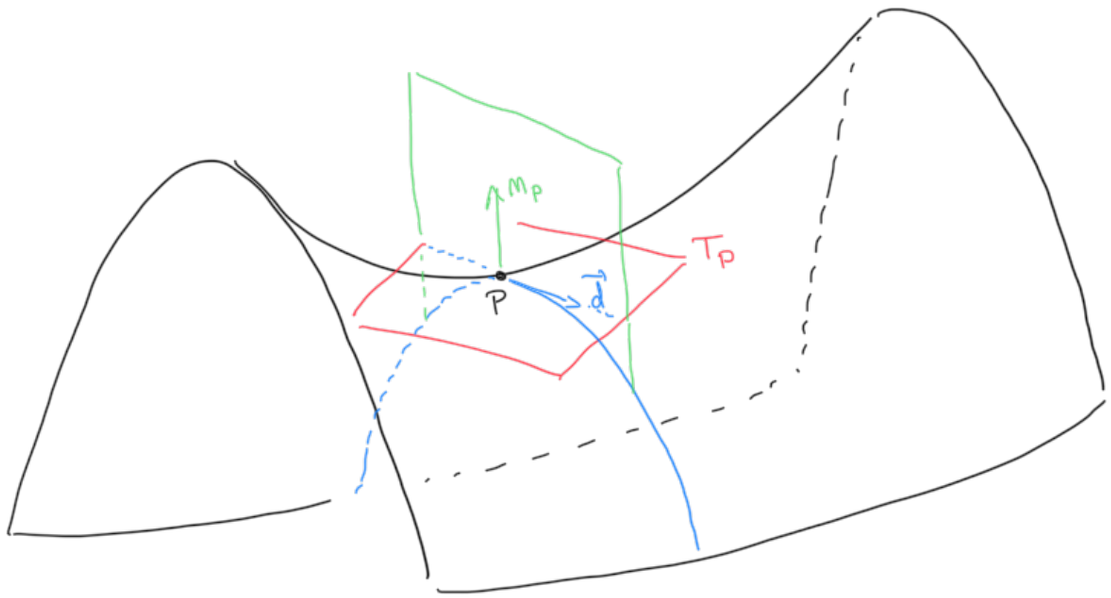
COURBURE(S)



$$\kappa_P = I_P^{-1} \times II_P$$

- Matrice 2x2 symétrique
- Donc diagonalisable en base orthonormale

COURBURE(S)



$$\kappa_P = I_P^{-1} \times II_P$$

- Matrice 2x2 symétrique
- Donc diagonalisable en base orthonormale

- Vecteurs propres : $\overrightarrow{d_1}, \overrightarrow{d_2}$ - directions principales
- Valeurs propres associées : κ_1, κ_2 - courbures principales

Extréma des courbures

K_p $\quad 2 \times 2$ $\quad$ eigen ..

K_1, K_2

$d_1, d_2)$ basis

$\downarrow$

write K_p in the basis

$\downarrow$ becomes

$$\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$$

$\underbrace{d_1, d_2}$:

$\downarrow$

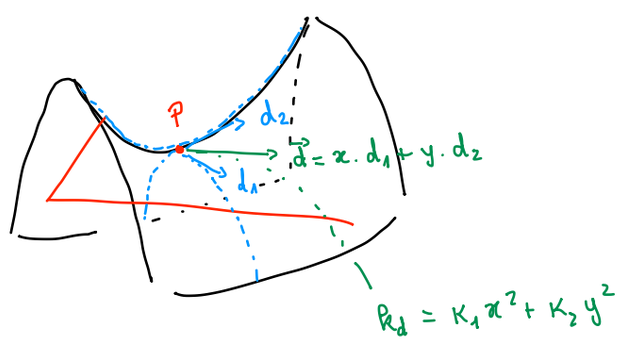
given a direction $\vec{d}$

$$k_{\vec{d}} = \vec{d}^{\,t} \times K_p \times \vec{d}$$

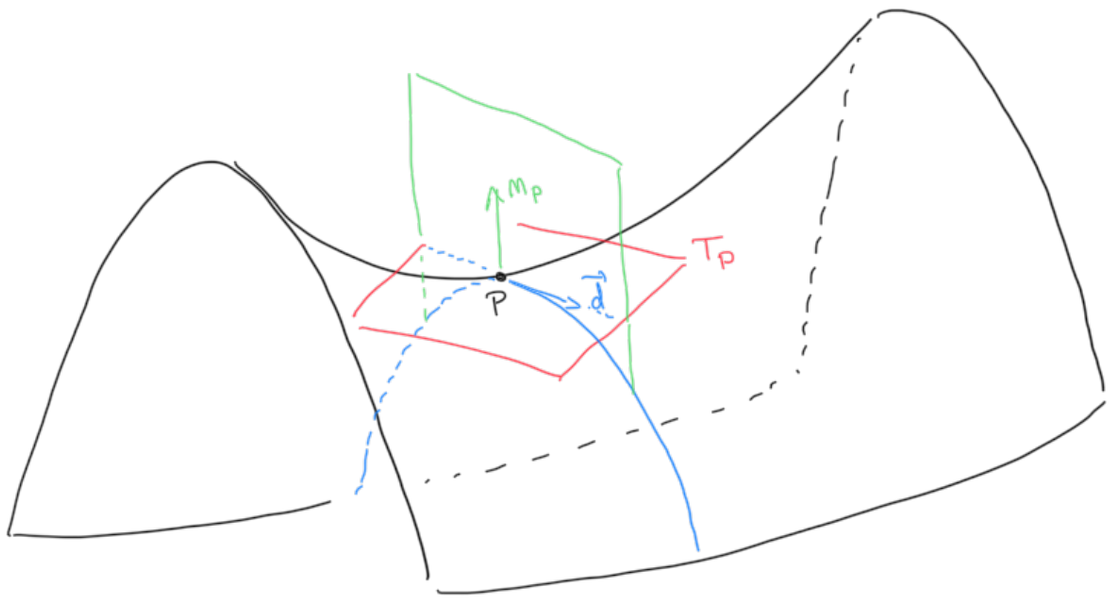
$$= \vec{d}^{\,t} \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \vec{d}$$

$\vec{d} = \begin{pmatrix} x \\ y \end{pmatrix}$ in the basis $\vec{d}_1, \vec{d}_2$

$$k_{\vec{d}} = K_1 x^2 + K_2 y^2$$



COURBURE(S)



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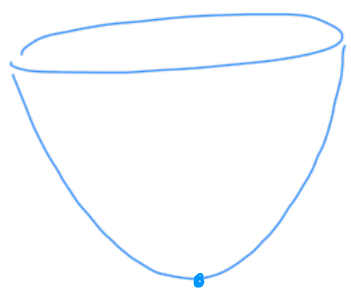
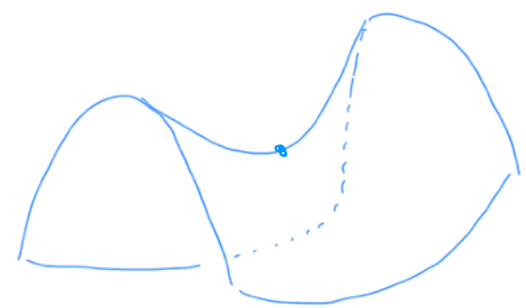
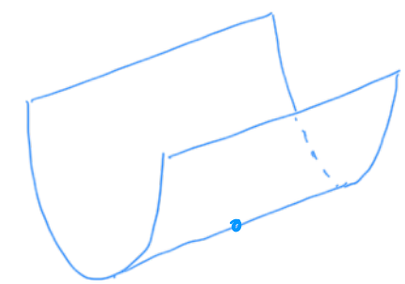
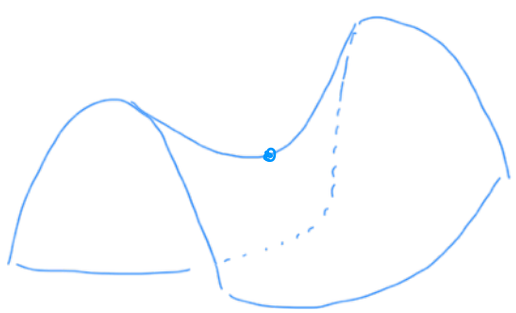
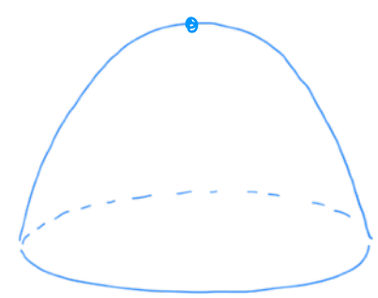
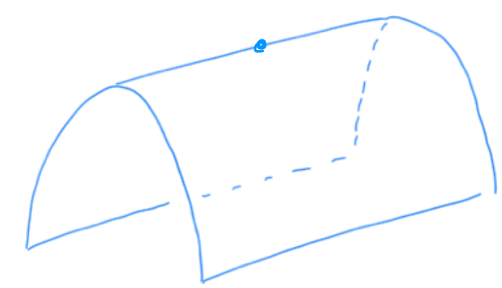
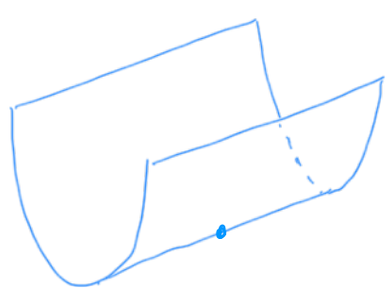
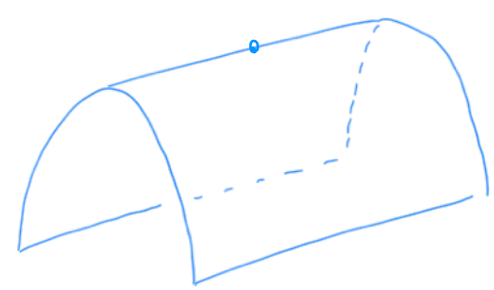
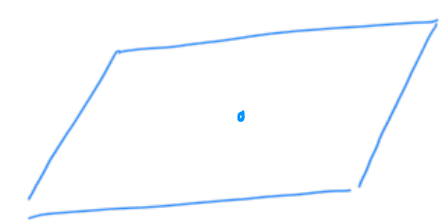
$\kappa_i > 0$: convexe selon $\overrightarrow{d_i}$

$\kappa_i = 0$: plat selon $\overrightarrow{d_i}$

$\kappa_i < 0$: concave selon $\overrightarrow{d_i}$

Extréma des courbures

COURBURE(S)

$\vec{\kappa_2}$ \ $\vec{\kappa_1}$	> 0	< 0	$= 0$
> 0			
< 0			
$= 0$			

Classification locale

COURBURE(S)

Deux mesures … difficile à manipuler (ex : carte de courbure, critère algorithmique de courbure …)

COURBURE(S)

Deux mesures … difficile à manipuler (ex : carte de courbure, critère algorithmique de courbure …)

↓

Moyenne(s)

COURBURE(S)

Deux mesures … difficile à manipuler (ex : carte de courbure, critère algorithmique de courbure …)

Moyenne(s)

Moyenne arithmétique

Moyenne géométrique — $\sqrt{K_1 K_2}$

Courbure moyenne

$$H = \frac{\kappa_1 + \kappa_2}{2}$$

Courbure gaussienne

$$K = \kappa_1 \kappa_2$$

G

approx. TP1

COURBURE(S)

Deux mesures … difficile à manipuler (ex : carte de courbure, critère algorithmique de courbure …)

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Moyenne arithmétique

Moyenne géométrique

Courbure moyenne

$$K = \frac{\kappa_1 + \kappa_2}{2}$$

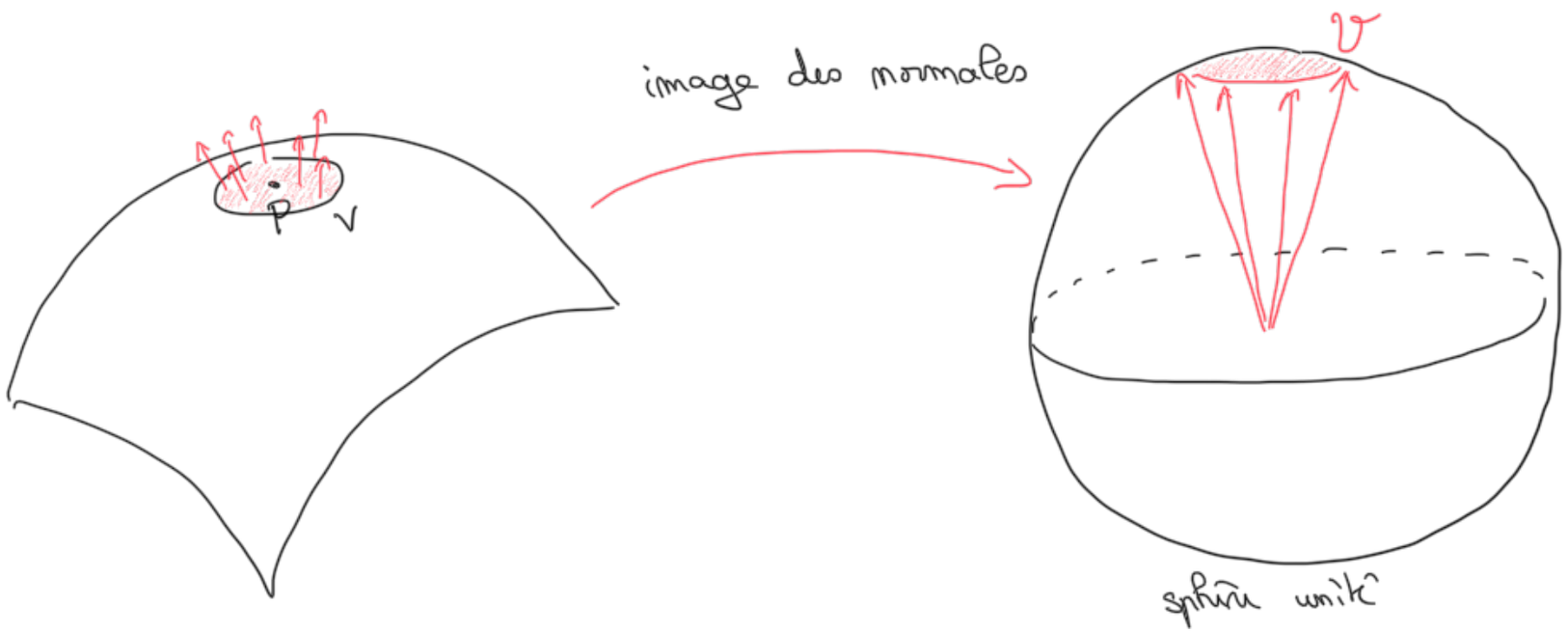
Courbure gaussienne

$$K = \kappa_1 \kappa_2$$

Introduite par Gauss bien avant la géométrie différentielle …

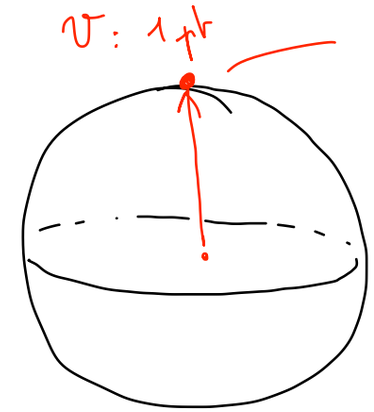
COURBURE GAUSSIENNE

Theorema Egregium de Gauss



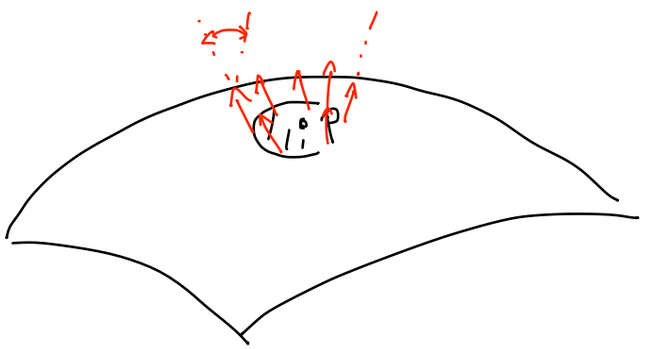
$$\lim_{V \to \{P\}} \frac{\text{Aire}(\mathscr{V})}{\text{Aire}(V)} = \kappa_P$$

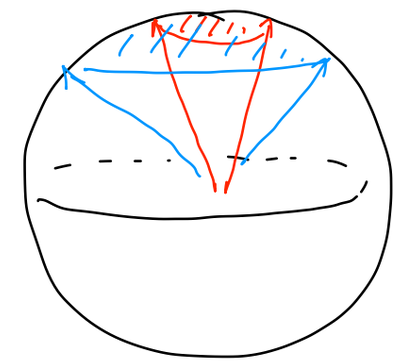
unit sphere

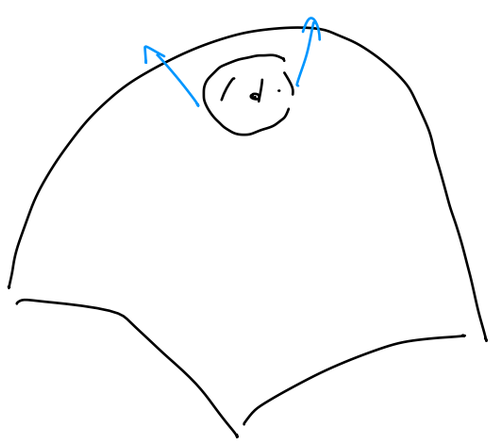


area ($\mathcal{V}$) = 0

$$K_p \cong \frac{\text{area}(\mathcal{V})}{\text{area}(V)} = 0$$



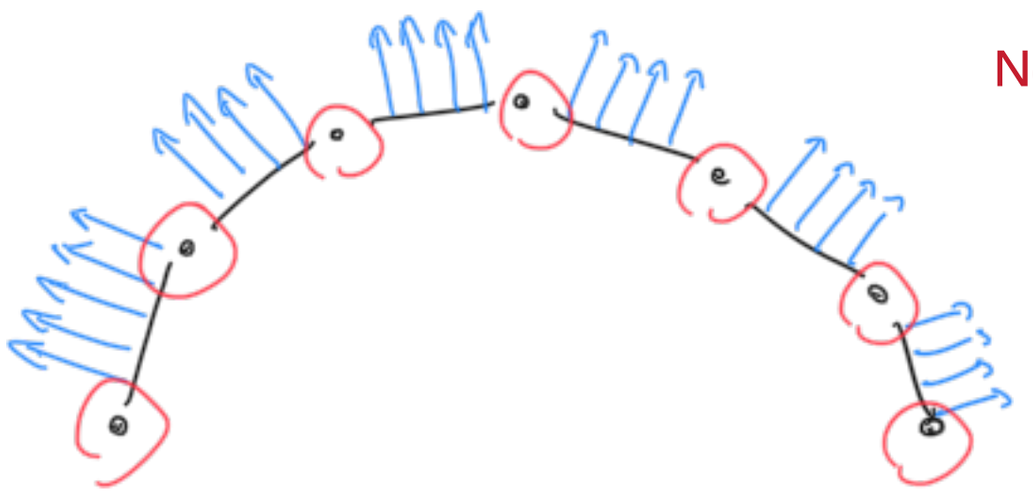




$K_p \nearrow$

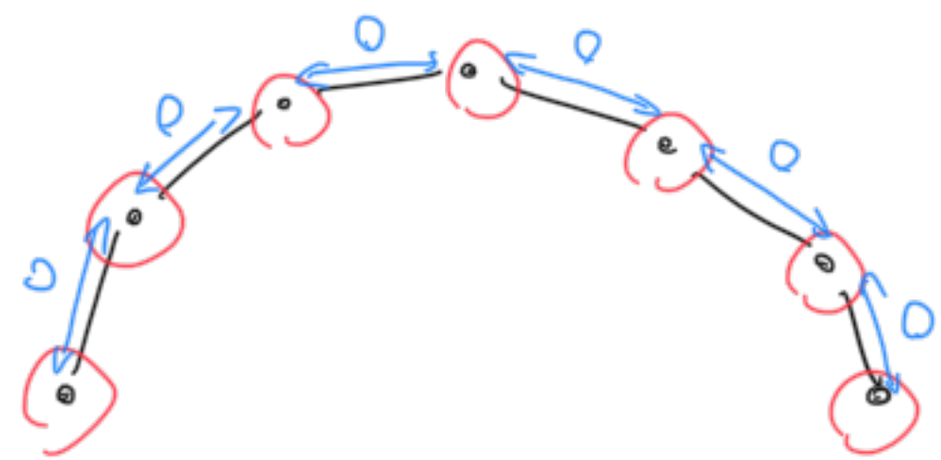
GÉOMÉTRIE DES MAILLAGES

NORMALES/COURBURES D'UN MAILLAGE ?



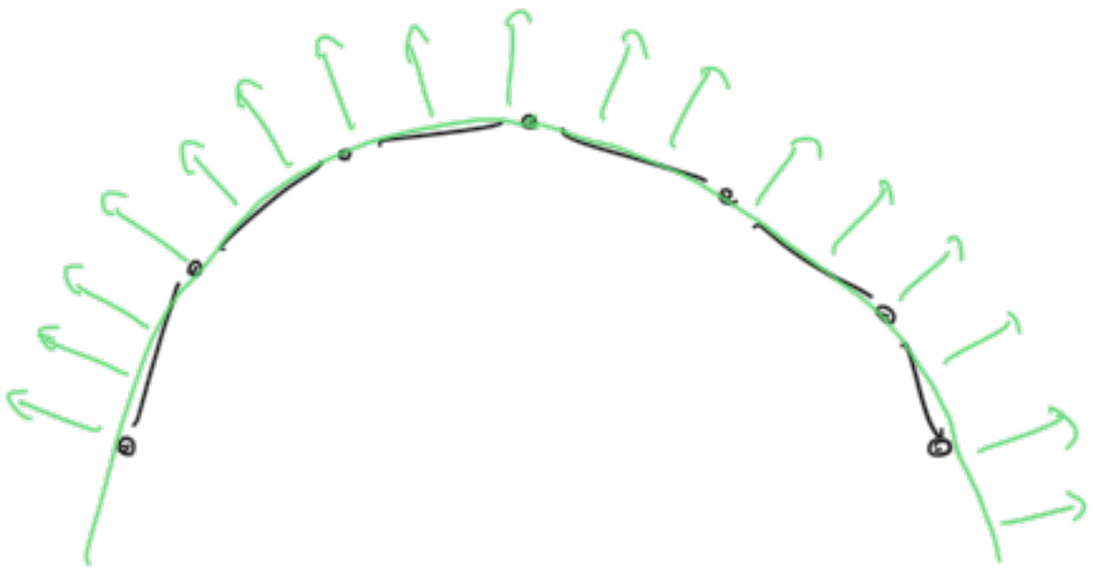
Normales

Non défini



Courbure(s)

NORMALES/COURBURES D'UN MAILLAGE ?

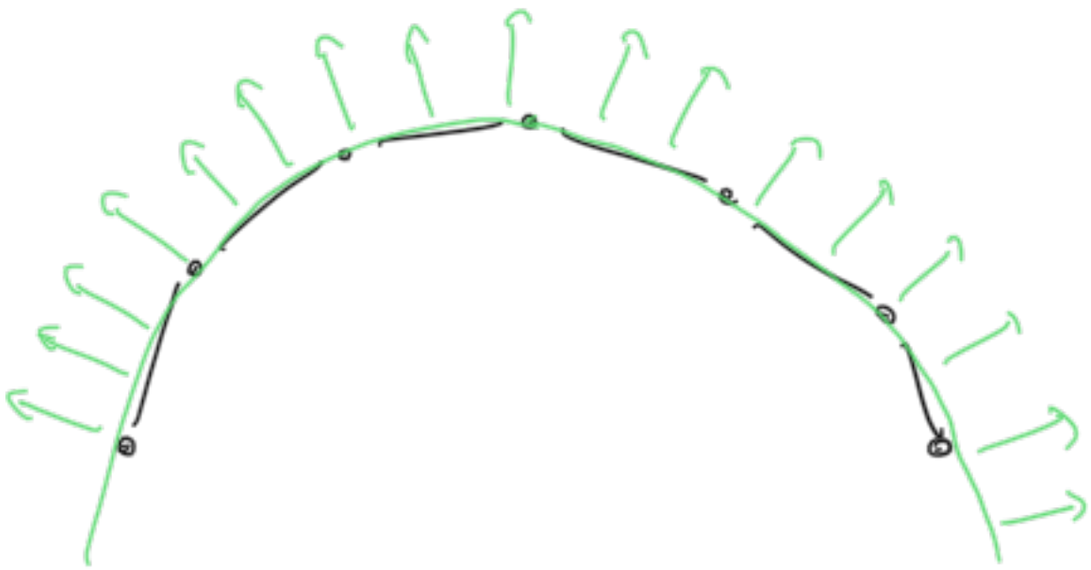


Intuition …

Le maillage est la discrétisation d'une surface continue

→ approximations / heuristiques

NORMALES/COURBURES D'UN MAILLAGE ?



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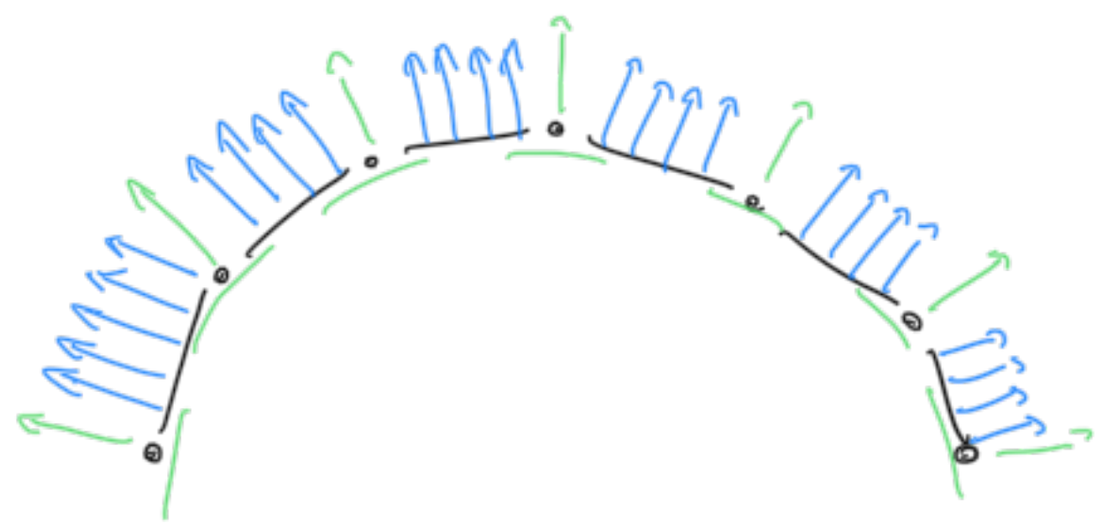
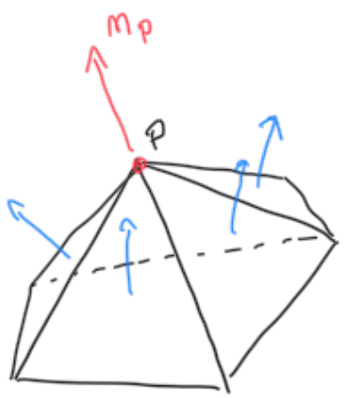
Le maillage est la discrétisation d'une surface continue



Pb : calculer des valeurs approchées cohérentes

→ approximations / heuristiques

ESTIMATION DES NORMALES

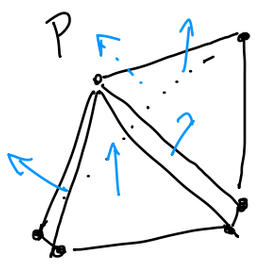


On estime généralement la normale au point P comme :

- Moyenne des normales des faces voisines
- Eventuellement pondérée par l'aire des faces

when the mesh is regular

for irregular meshes



Anisotropic

Shapes vary along the surface

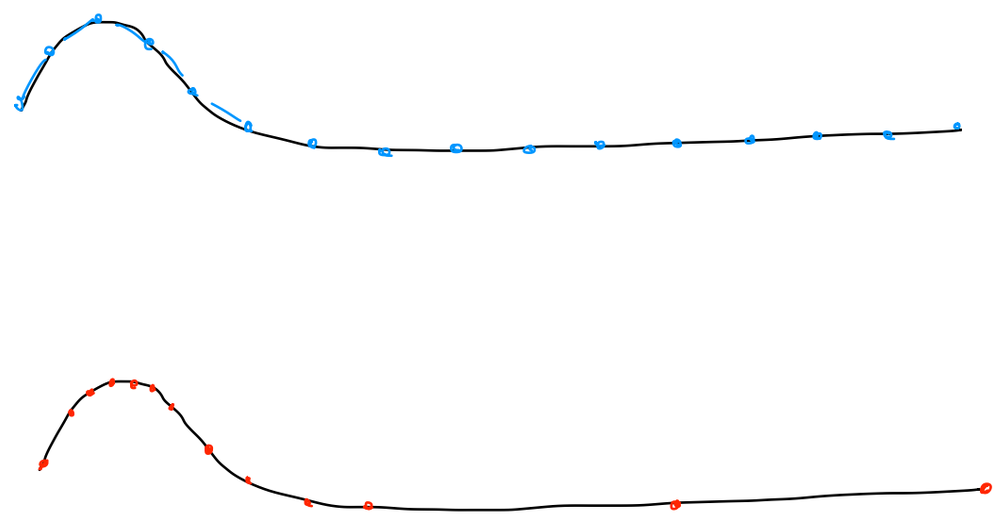
density ....

add more info. in complex areas

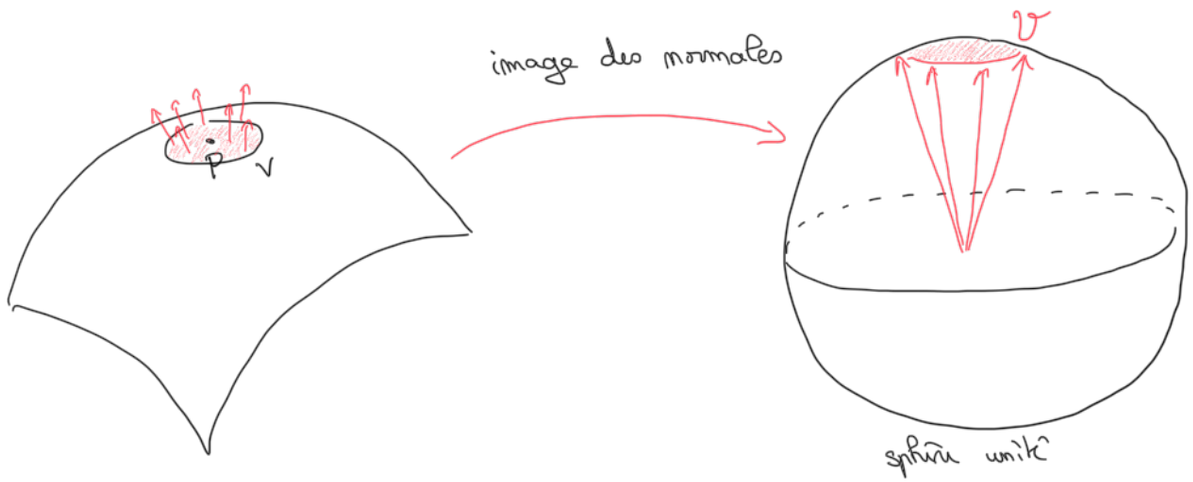
Isotropic — more efficient

similar along space

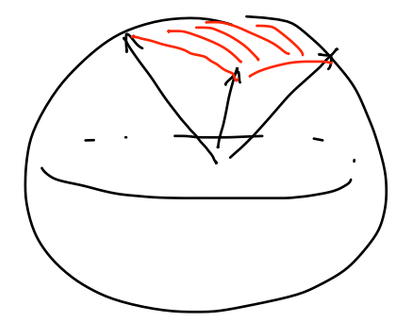
- density
- shape of triangles
- ....

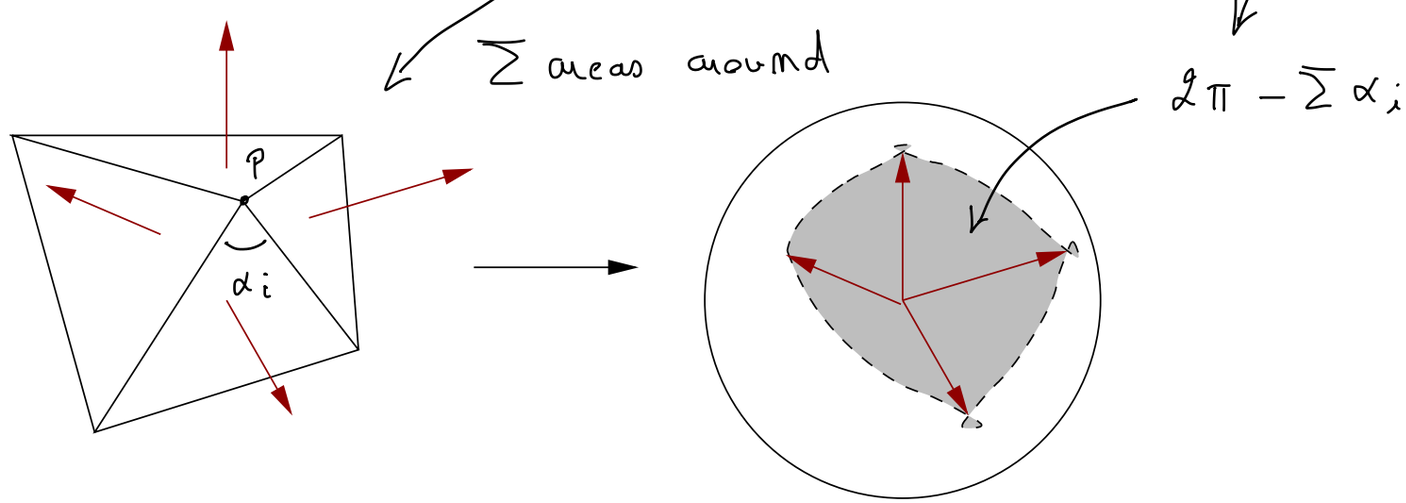


COURBURES : TH. EGREGIUM DE GAUSS ET DÉFAUT ANGULAIRE



$$\lim_{V \to \{P\}} \frac{\text{Aire}(\mathscr{V})}{\text{Aire}(V)} = \kappa_P$$





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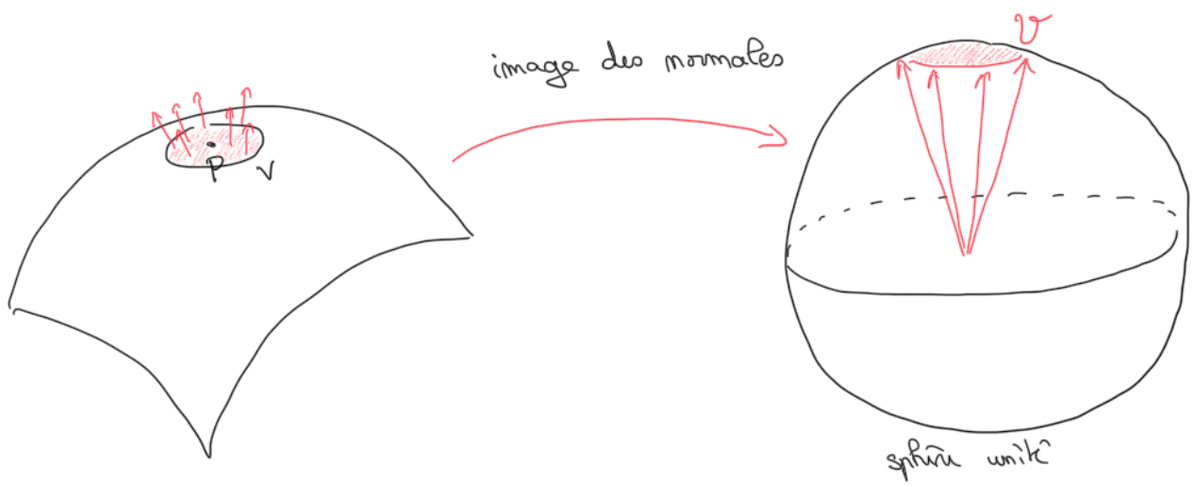

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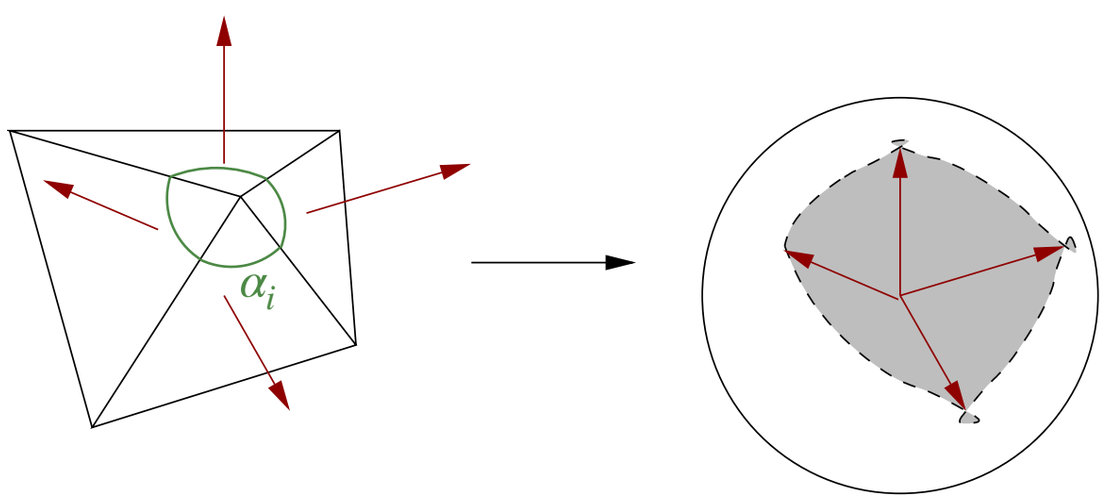

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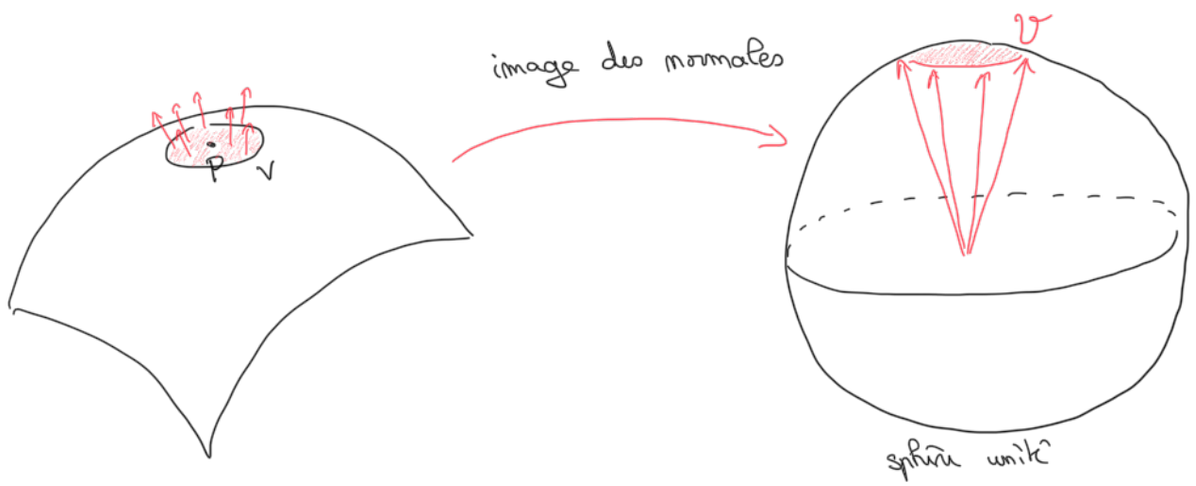

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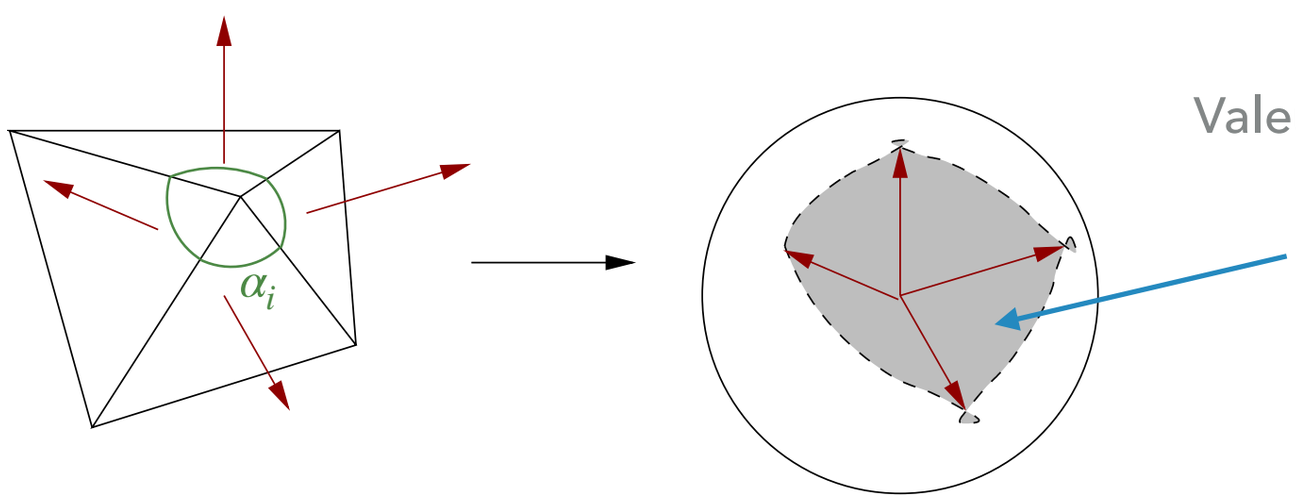
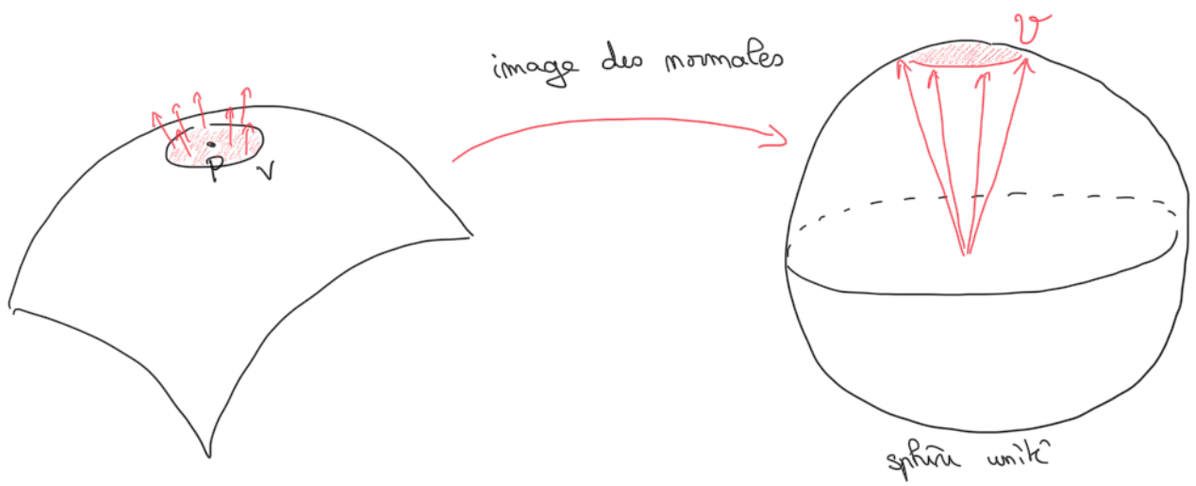


Valeur de cette aire sphérique :

$$2\pi - \sum_i \alpha_i$$

Défaut angulaire

COURBURES : TH. EGREGIUM DE GAUSS ET DÉFAUT ANGULAIRE



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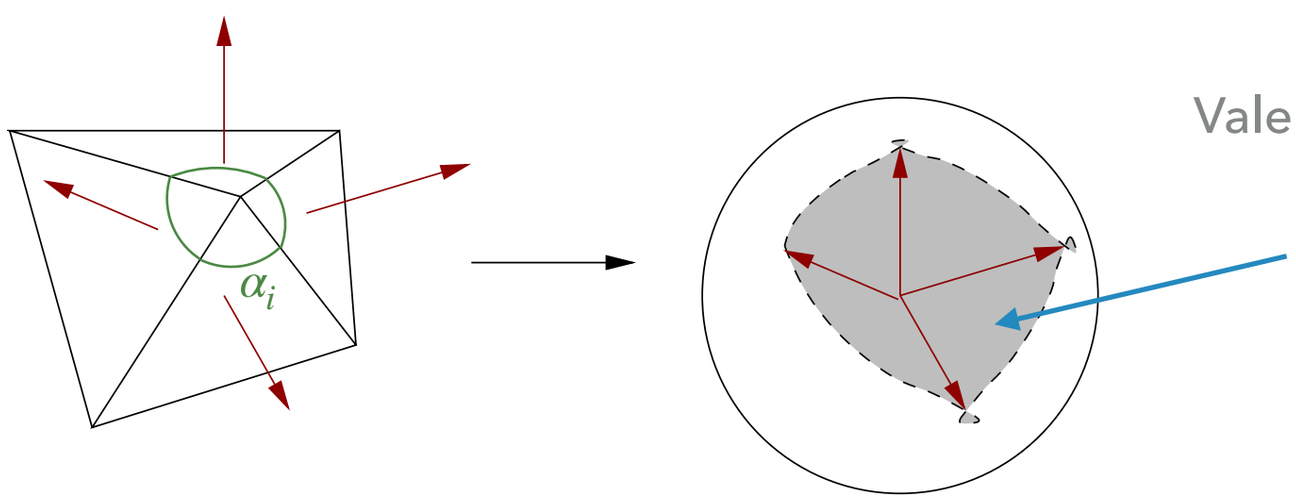

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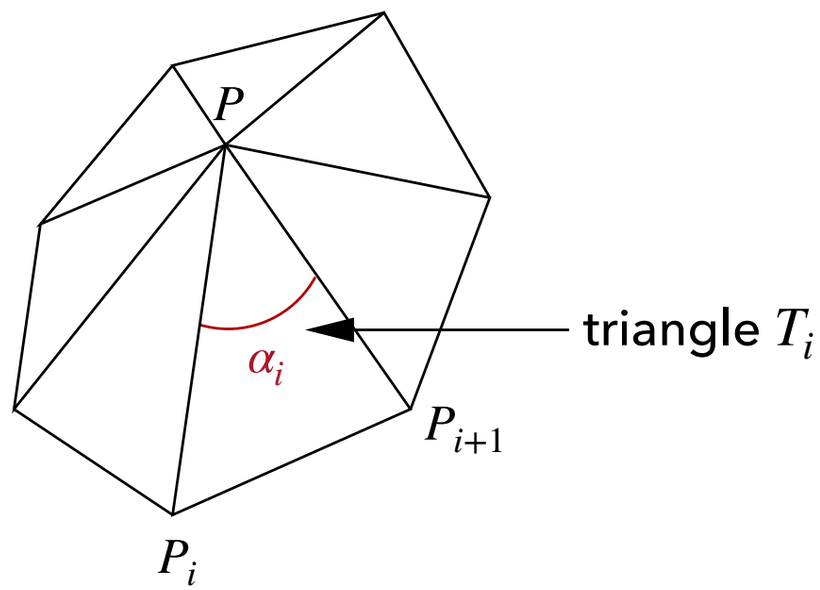
Défaut angulaire

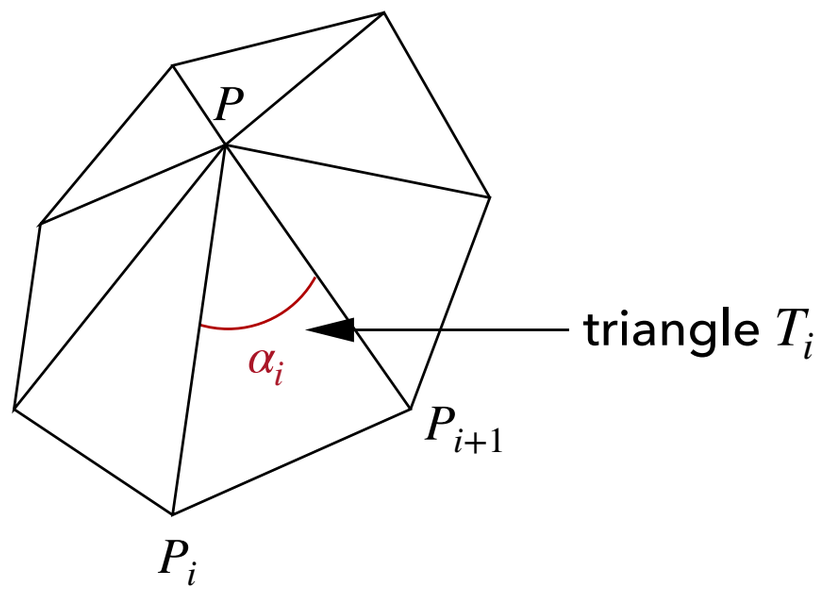
COURBURE GAUSSIENNE « CLASSIQUE » (REGGE)



Approximation de la courbure Gaussienne :

$$K_P = \frac{2\pi - \sum_i \alpha_i}{\sum_i \mathrm{Aire}(T_i)}$$ au point P

COURBURE GAUSSIENNE « CLASSIQUE » (REGGE)

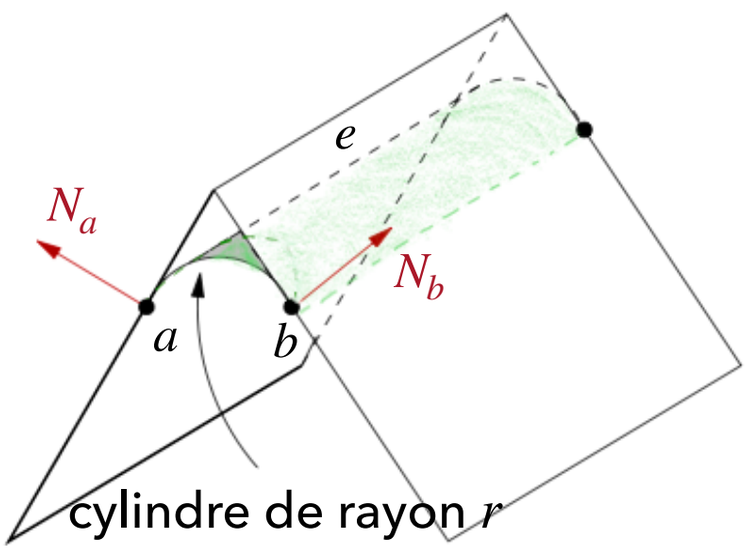


Approximation de la courbure Gaussienne :

$$K_P = \frac{2\pi - \sum_i \alpha_i}{\sum_i \mathrm{Aire}(T_i)}$$ au <u>point</u> P

- C'est la formule que vous avez implémentée au TP1
- Peu précise, dépend du maillage
- <u>Ne converge pas</u>

COURBURE MOYENNE « CLASSIQUE »

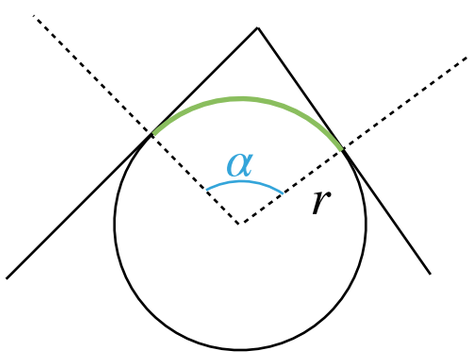


- Aire de la portion de cyclindre :

$$l(e) \cdot \alpha \cdot r$$

- Intégrale de la courbure moyenne sur cette portion :

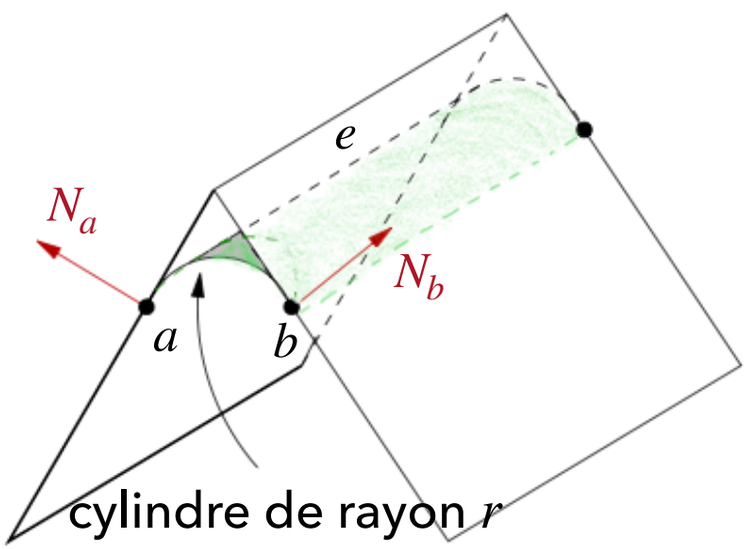
$$\frac{1}{2r} \cdot l(e) \cdot \alpha \cdot r = \frac{1}{2} \cdot l(e) \cdot \alpha$$



Approximation de la courbure moyenne :

$$H_e = \frac{1}{2} l(e) \cdot \alpha$$ le long de l'<u>arête</u> e

COURBURE MOYENNE « CLASSIQUE »



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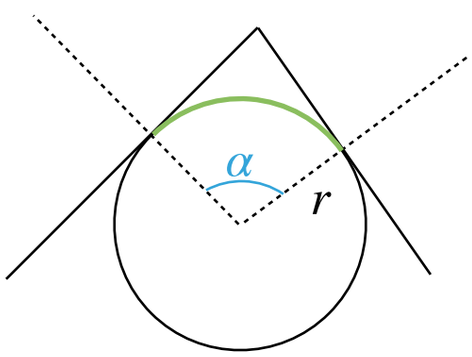

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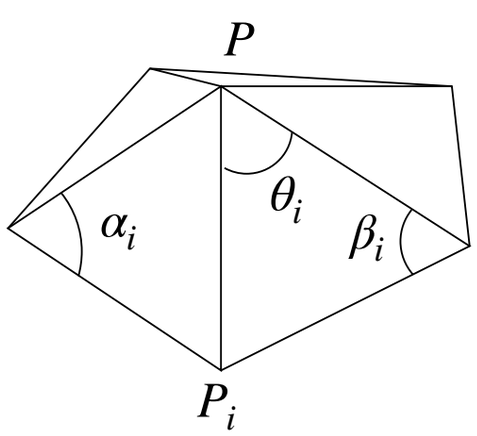
- Peu précise …
- <u>Ne converge pas</u>

APPROXIMATIONS PLUS FINES

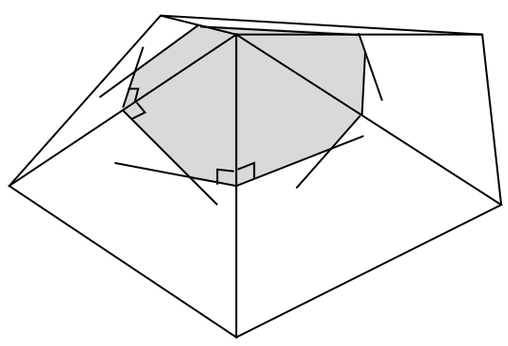
On the angular defect of triangulations and the pointwise approximation of curvatures
V. Borrelli, F. Cazals, J.-M. Morvan

Discrete Differential-Geometry Operators for Triangulated 2-Manifolds
M. Meyer, M. Desbrun, P. Schröder, and A. Barr



$$H_P \cdot N_P = \frac{1}{2\ Aire_{\text{mixte}}} \sum_i (\cot\alpha_i + \cot\beta_i)(P - P_i)$$

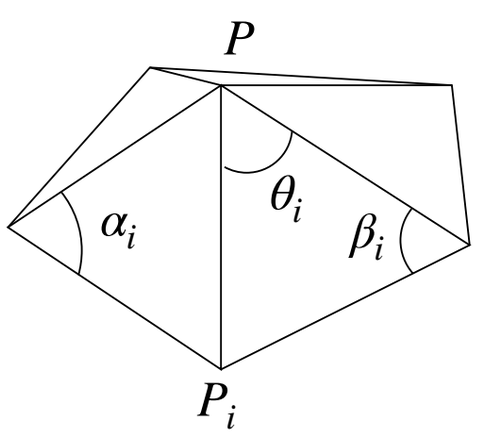
$$K(P) = \frac{2\pi - \sum_i \theta_i}{Aire_{\text{mixte}}}$$



APPROXIMATIONS PLUS FINES

On the angular defect of triangulations and the pointwise approximation of curvatures
V. Borrelli, F. Cazals, J.-M. Morvan

Discrete Differential-Geometry Operators for Triangulated 2-Manifolds
M. Meyer, M. Desbrun, P. Schröder, and A. Barr



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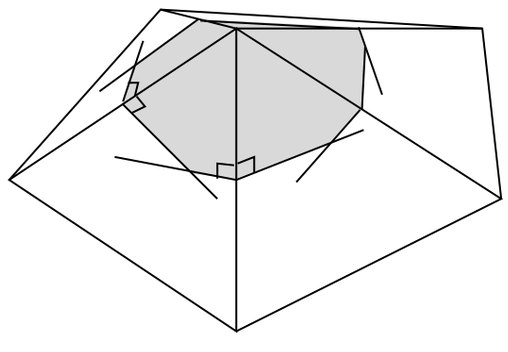

La convergences n'est pas garantie :

- dépend de la triangulation
  plus elle est régulière, meilleure est la convergence
  - idéal : régulière, valence 6
- sur une triangulation irrégulière, non garantie
  - au mieux, convergence en norme $\mathscr{L}_1$

AUTRES APPROCHES

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V. Borrelli, F. Cazals, J.-M. Morvan

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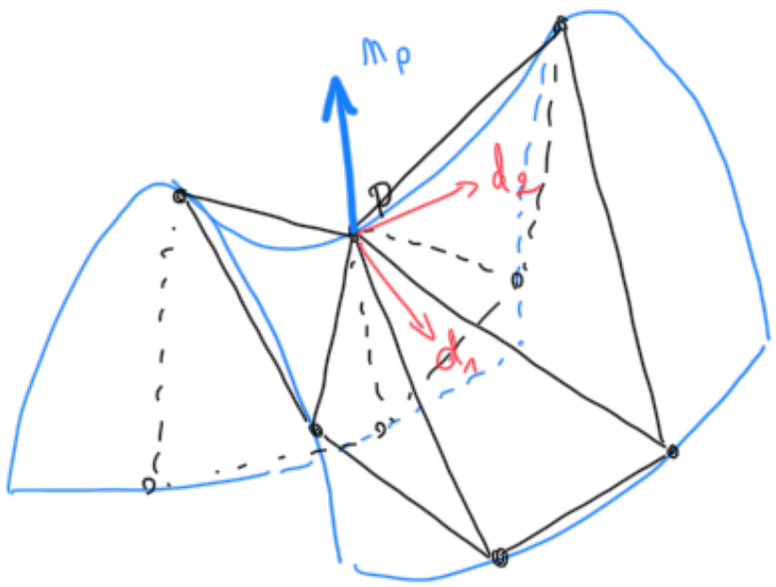
Estimating the Tensor of Curvature of a Surface from a Polyhedral Approximation
G. Taubin

Estimation de K_P (tenseur de courbure)

CALCUL VIA UNE SURFACE CONTINUE



Ajuster une surface lisse
(paramétrique ou cartésienne)
au voisinage de P

↓

Calculer normale et courbures
de la surface lisse

- Méthode plus lente mais plus stable
- La question est alors de bien approcher localement le maillage

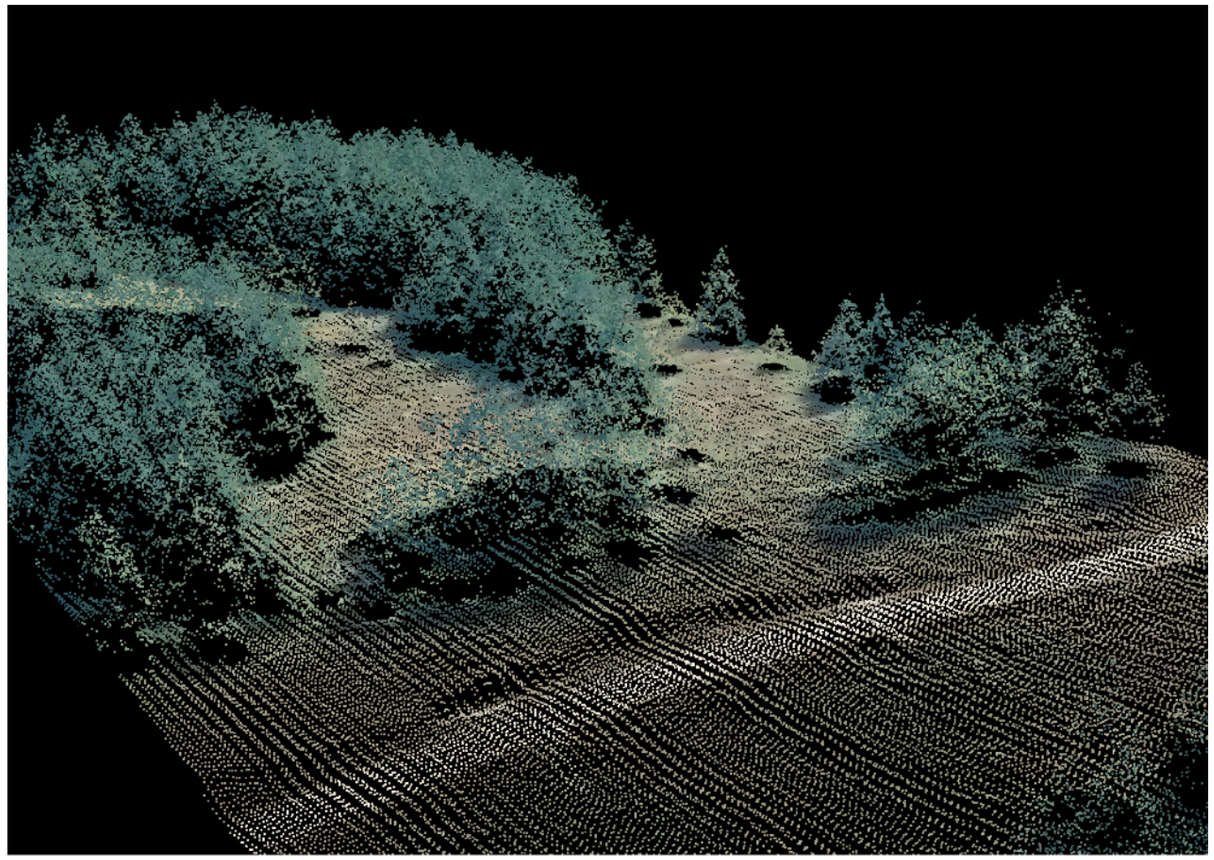
GÉOMÉTRIE D'UN NUAGE DE POINTS

GÉOMÉTRIE D'UN NUAGE DE POINTS

Sujet très actif vu le développement des capteurs de numérisation :

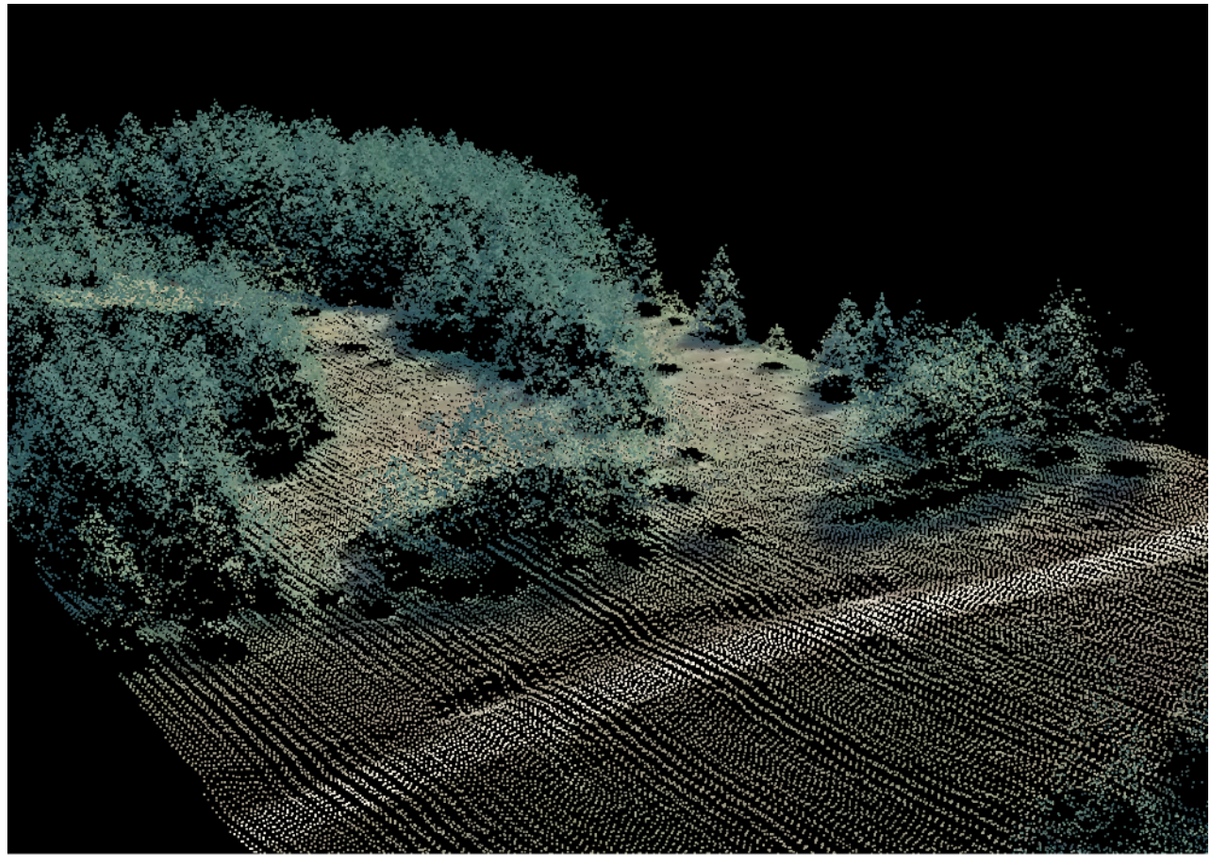
- LiDAR
- Caméras temps de vol (Kinect etc. …)
- Stéréo-photogrammétrie

T-LiDAR



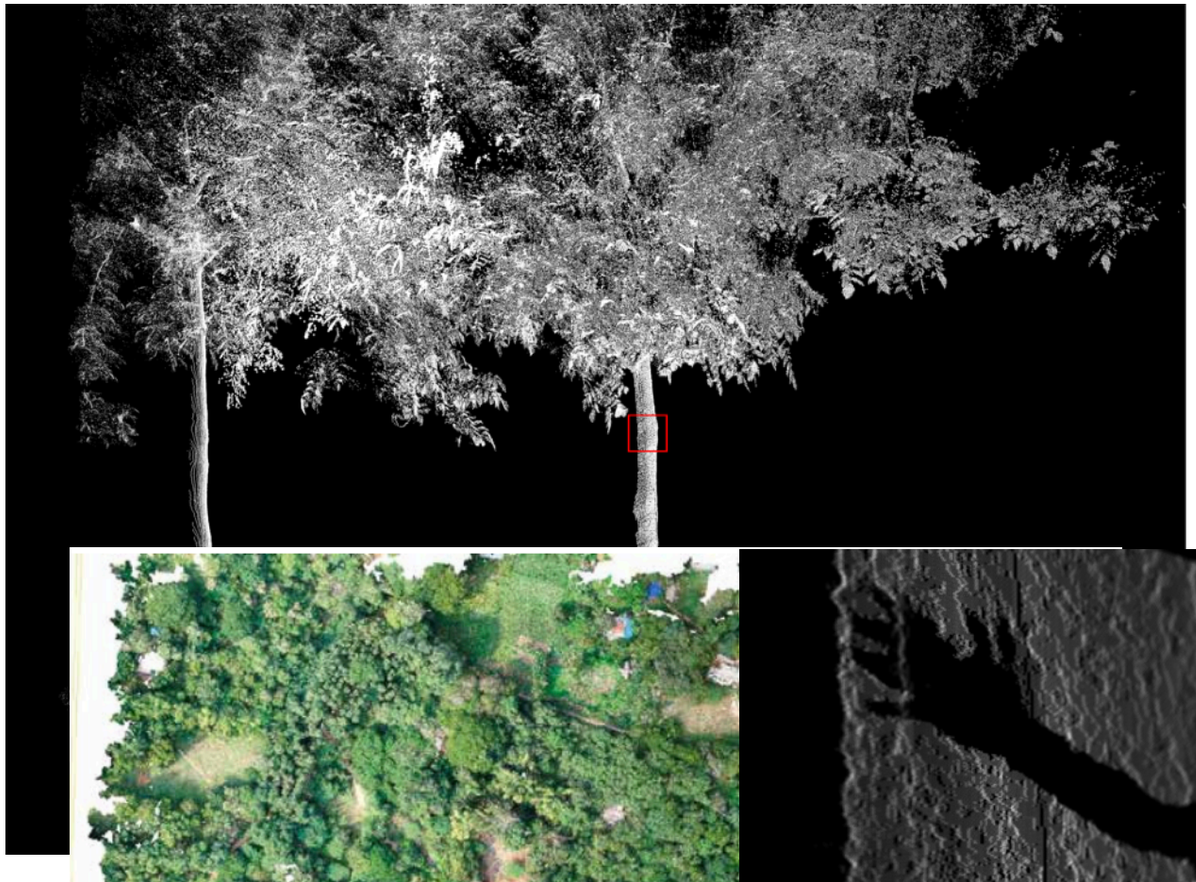
A-LiDAR

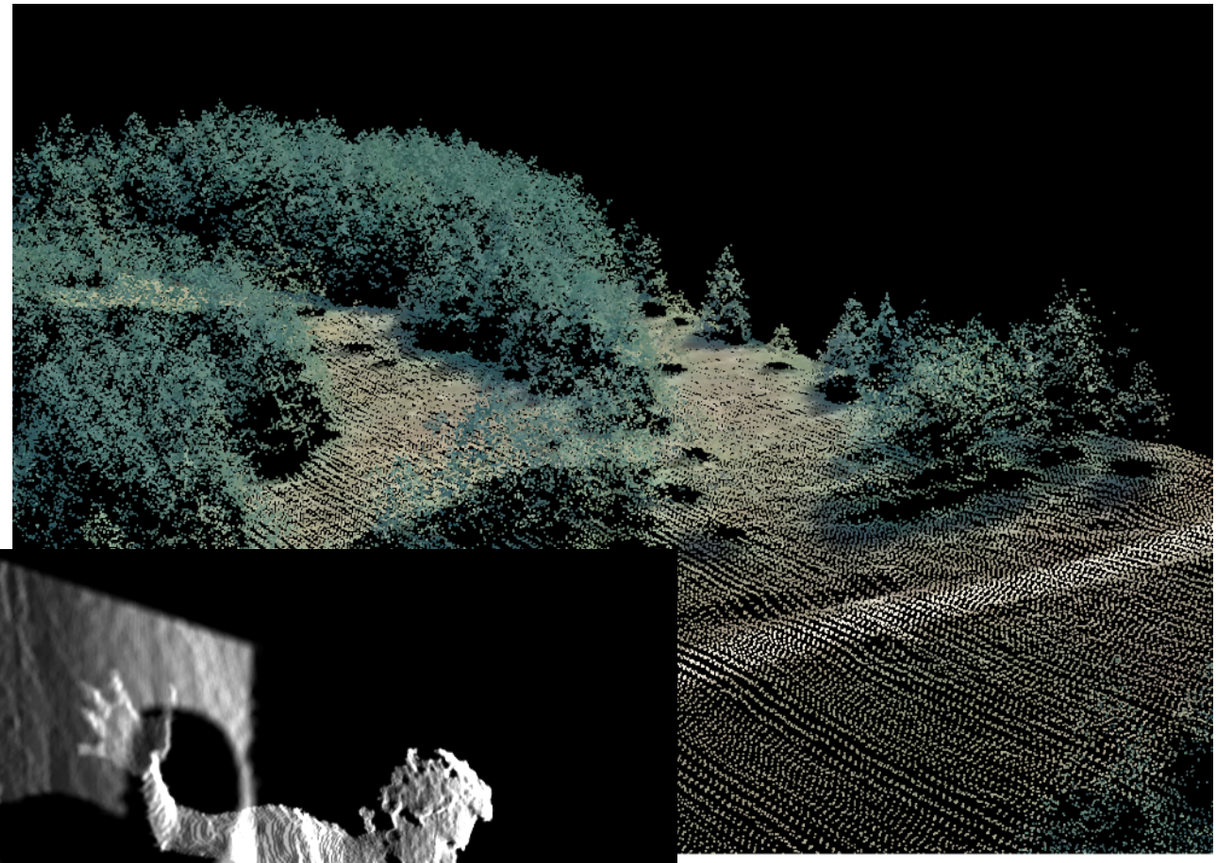
UAV Stéréo-photogrammétrie



A-LiDAR

Stéréo-photogrammétrie





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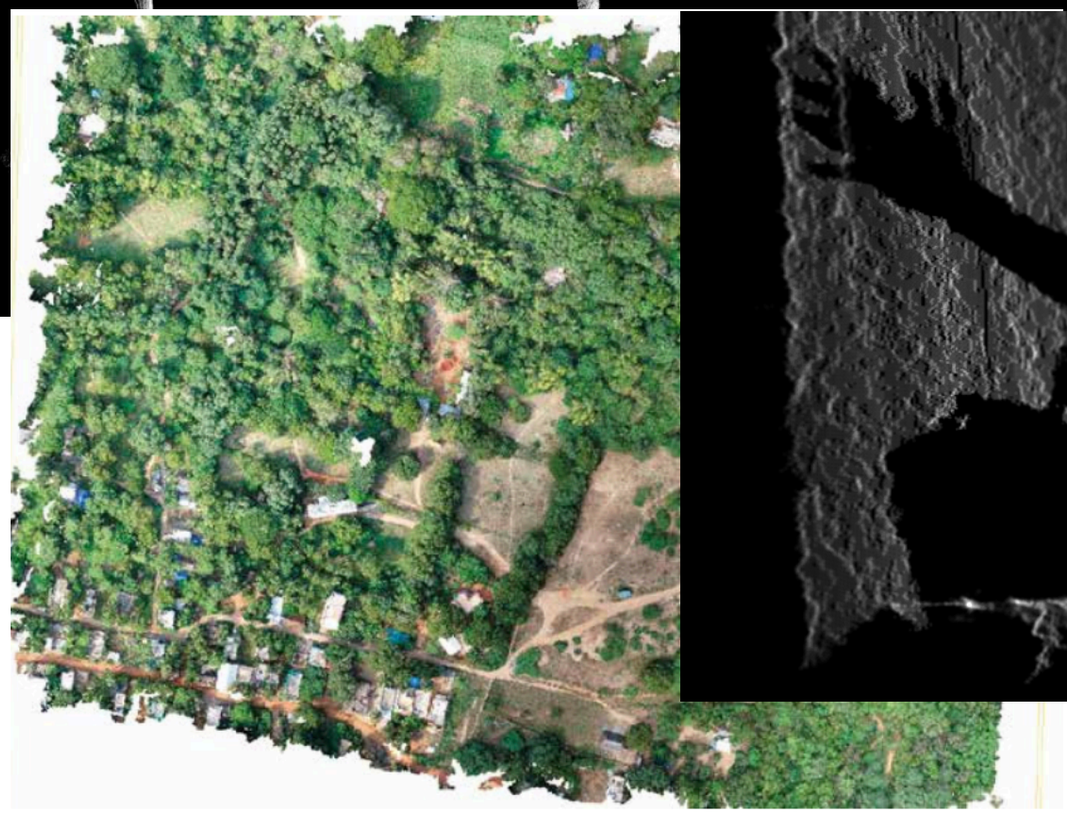
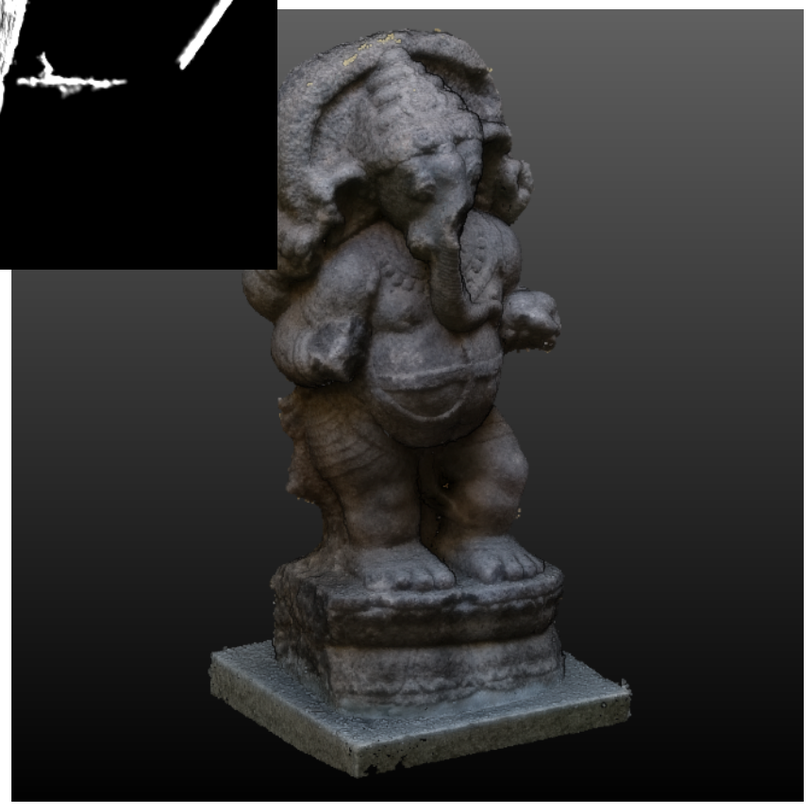


Kinect

UAV Stéréo-photogrammétrie



Stéréo-photogrammétrie

GÉOMÉTRIE D'UN NUAGE DE POINTS

Sujet très actif vu le développement des capteurs de numérisation :

- LiDAR
- Caméras temps de vol (Kinect etc. …)
- Stéréo-photogrammétrie

Comment estimer la géométrie locale directement à partir du nuage de points ?

Approches par ajustement de surfaces lisses

Approches par étude de la distribution des points

- Analyse en composantes principales

→ Cf. TP

VARIÉTÉS

VARIÉTÉS (MANIFOLDS)

Qu'est-ce qu'une surface dans $\mathbb{R}^3$?

- Maillages
- Surfaces paramétriques
- Surfaces implicites

Exemples de surfaces,
modèles surfaciques …

VARIÉTÉS (MANIFOLDS)

Qu'est-ce qu'une surface dans $\mathbb{R}^3$?

(2)-variétés :

formalisation très générale

- Maillages
- Surfaces paramétriques
- Surfaces implicites

Exemples de surfaces, modèles surfaciques …

Une <u>variété de dimension n</u> (dans $\mathbb{R}^m$) est un sous-ensemble $M \subseteq \mathbb{R}^m$ tel que tout point de M a un voisinage homéomorphisme à un sous-ensemble de $\mathbb{R}^d$.

$f : E \to F$ **<u>homéomorphisme</u>** si f bijective, continue, d'inverse continue (<u>déformation continue</u>).

VARIÉTÉS (MANIFOLDS)

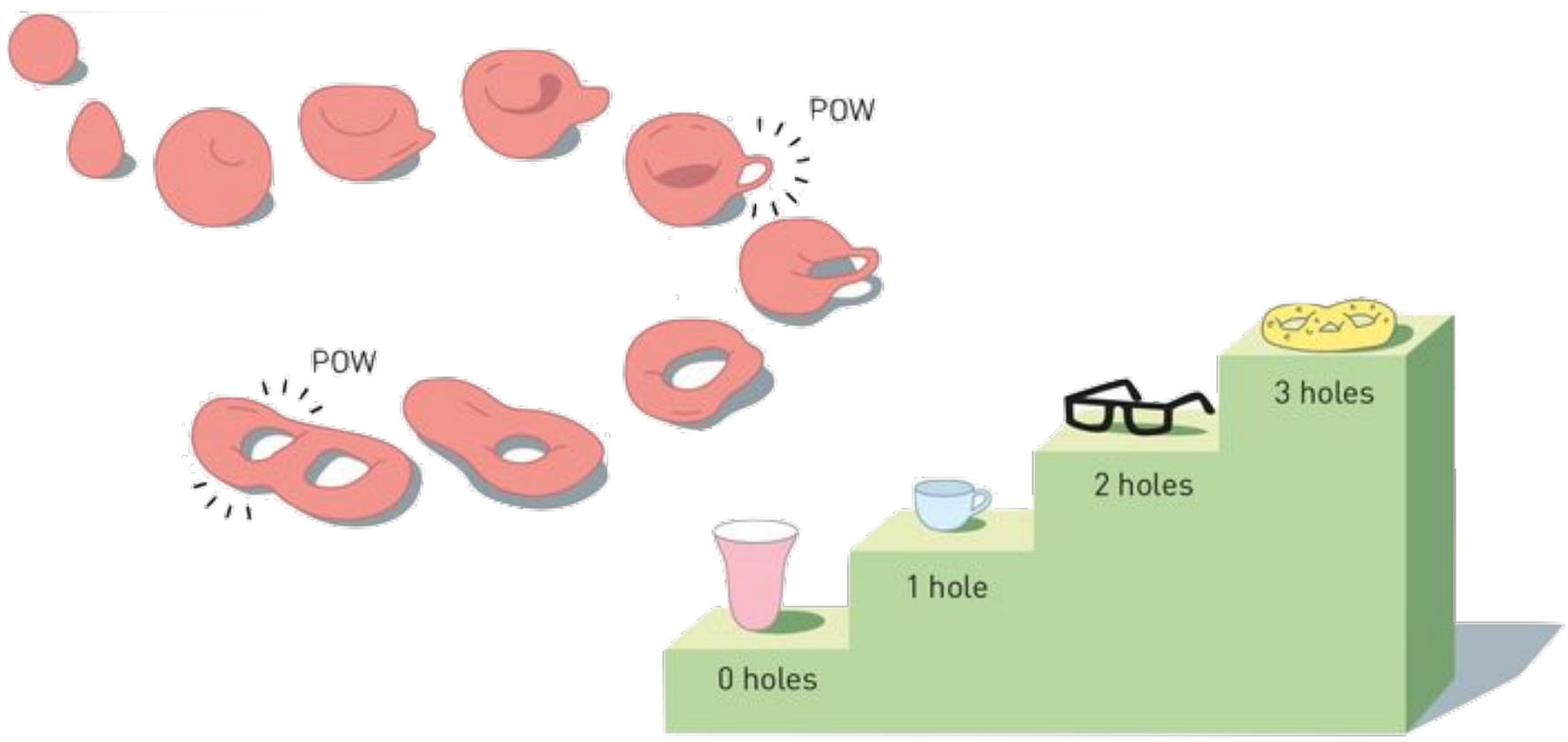
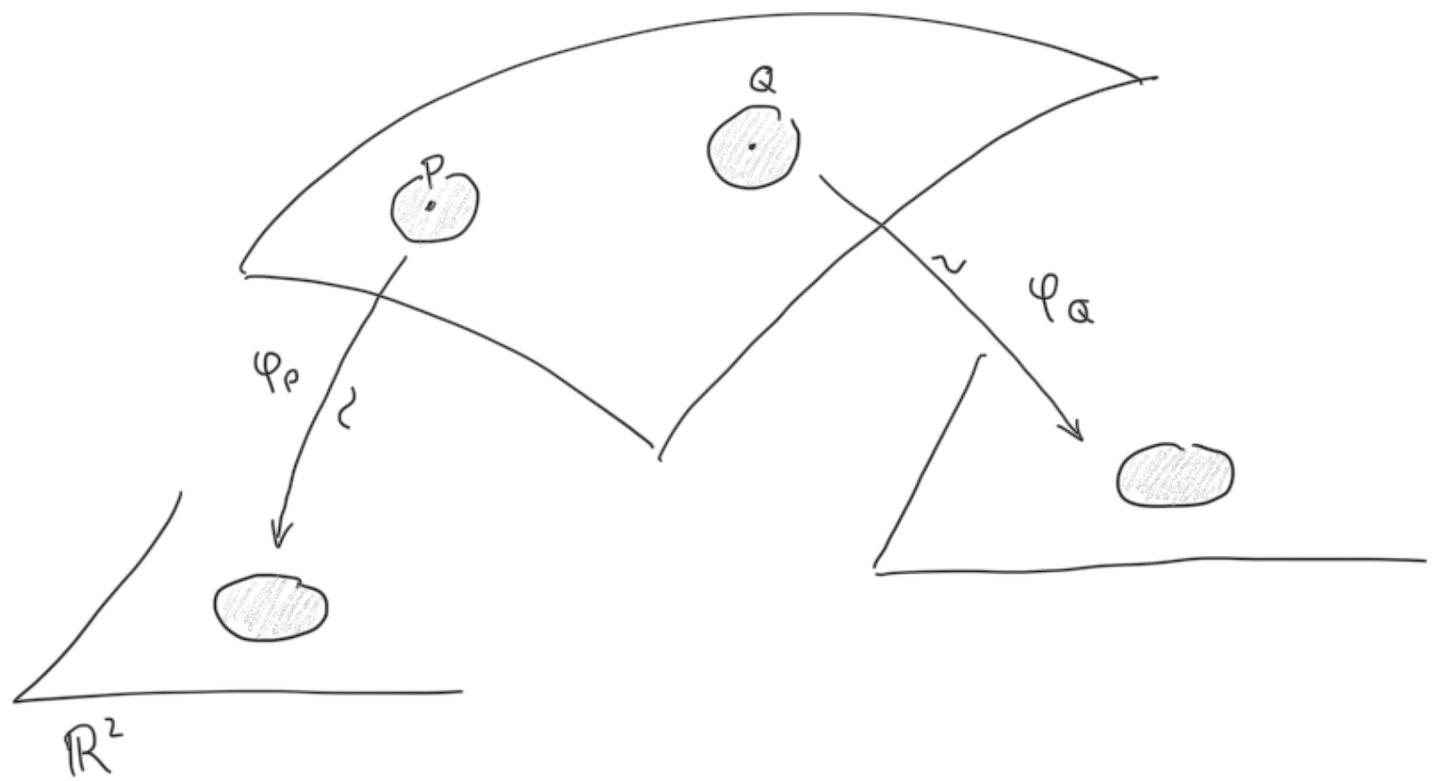


Illustration: ©Johan Jarnestad/The Royal Swedish Academy of Sciences

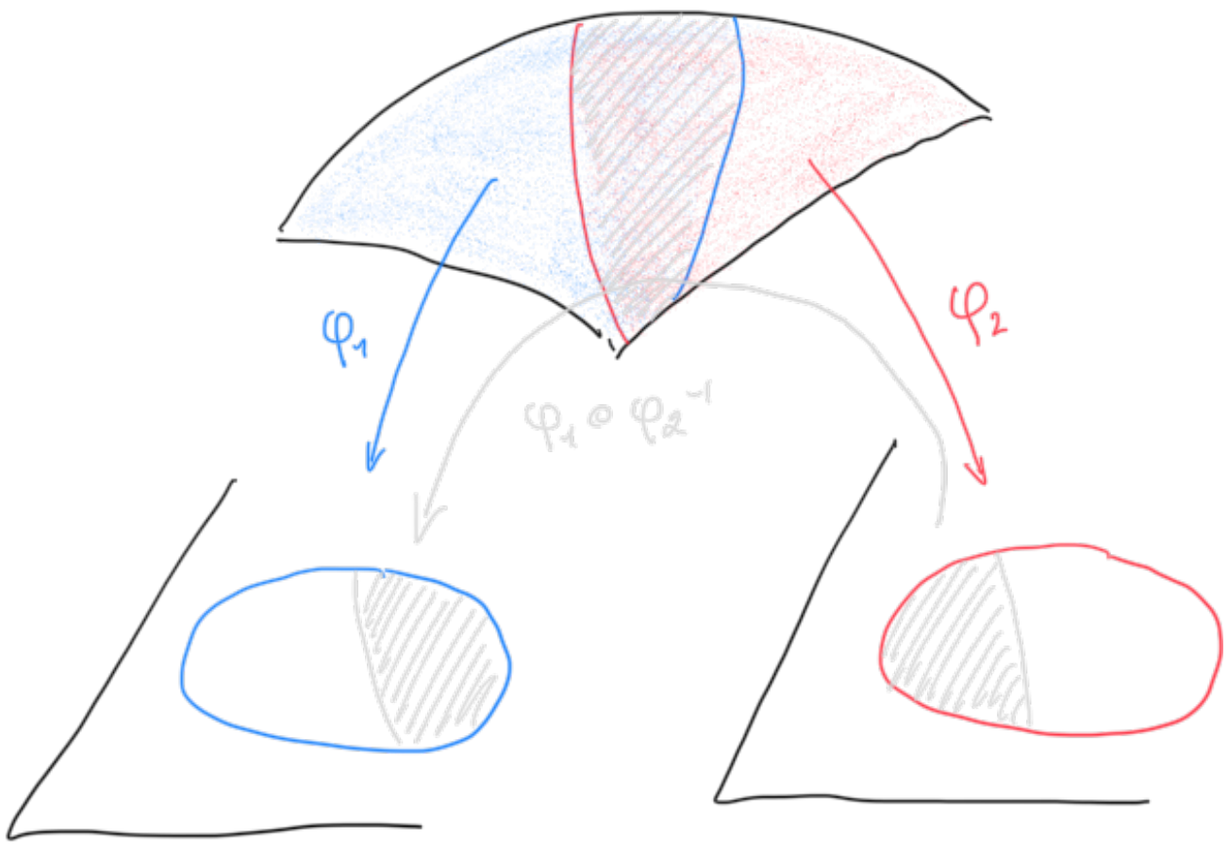
$f : E \to F$ **homéomorphisme** si f bijective, continue, d'inverse continue (déformation continue).

VARIÉTÉS (MANIFOLDS)



Une <u>variété (manifold) de dimension n</u> (dans $\mathbb{R}^m$) est un sous-ensemble $M \subseteq \mathbb{R}^m$ tel que tout point de M a un voisinage homéomorphisme à un sous-ensemble de $\mathbb{R}^d$.

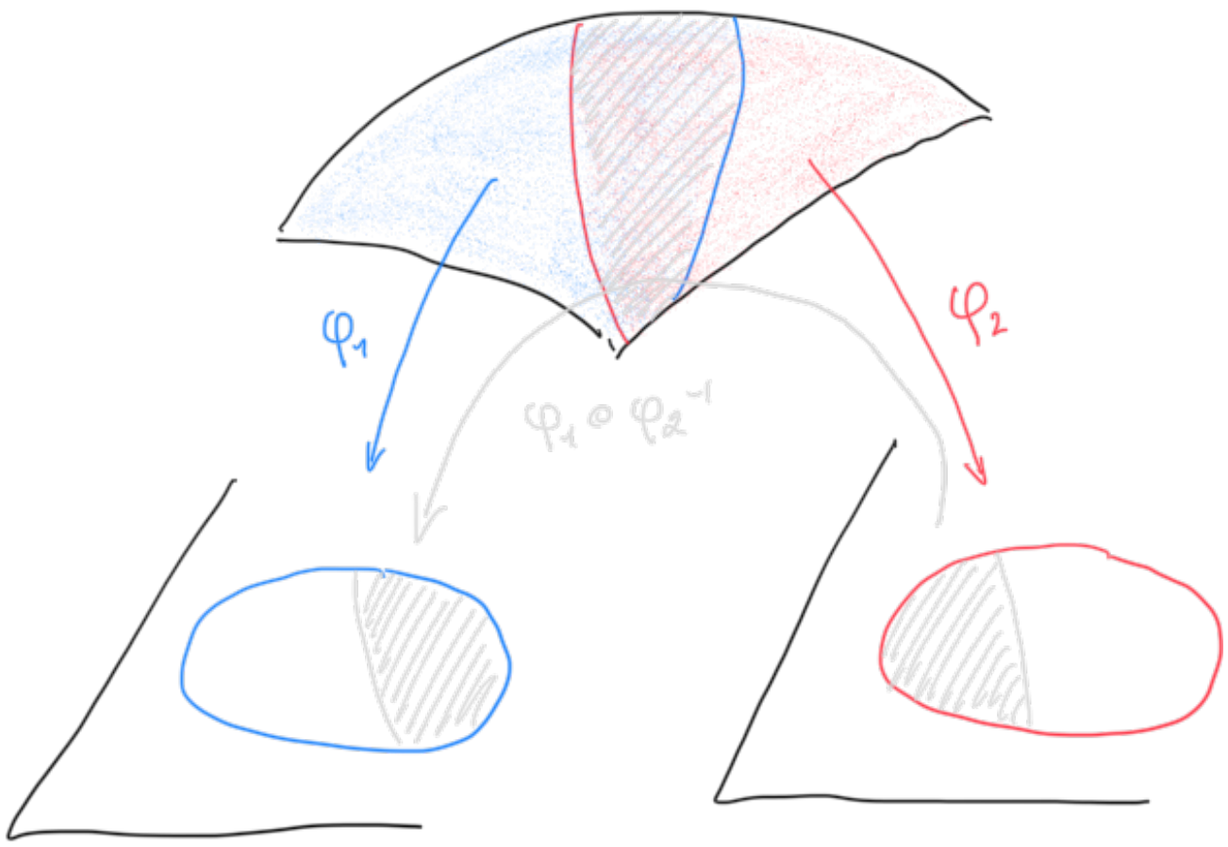
VARIÉTÉS (MANIFOLDS)



- Atlas de cartes $\{\varphi_i\,;\, i \in I\}$
- Compatibles de classe $\mathscr{C}^k$

Une <u>variété (manifold) **différentielle** de dimension n</u> (dans $\mathbb{R}^m$) est un sous-ensemble $M \subseteq \mathbb{R}^m$ tel que tout point de M a un voisinage homéomorphisme à un sous-ensemble de $\mathbb{R}^d$.

VARIÉTÉS (MANIFOLDS)



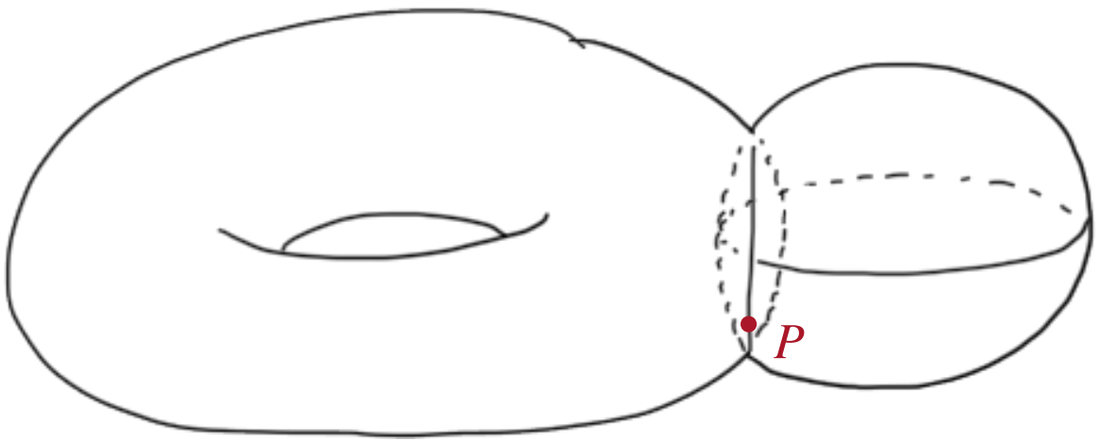
Surface lisse

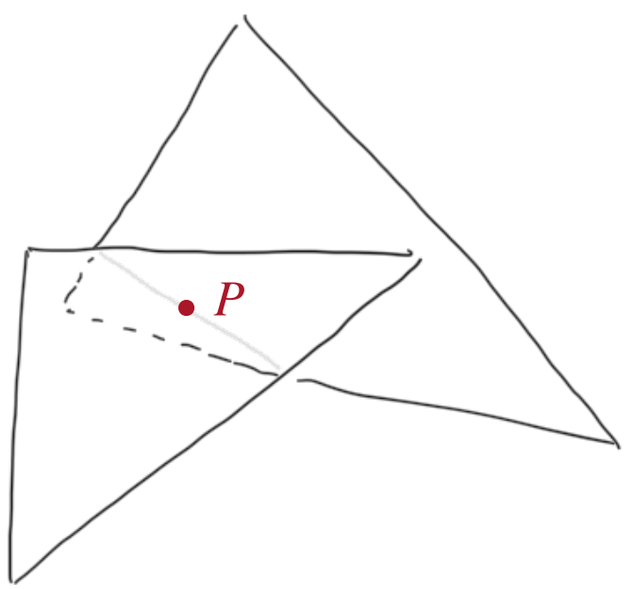
↓

2–variété différentielle

Une <u>variété (manifold) **différentielle** de dimension n</u> (dans $\mathbb{R}^m$) est un sous-ensemble $M \subseteq \mathbb{R}^m$ tel que tout point de M a un voisinage homéomorphisme à un sous-ensemble de $\mathbb{R}^d$.

VARIÉTÉS (MANIFOLDS)





Tout se généralise dans ce contexte :

- Plan tangent, normale
- Courbures …